APPLICATION OF THE CONJECTURE ON THE MANIN OBSTRUCTION TO VARIOUS DIOPHANTINE PROBLEMS

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Let X be a rational projective smooth surface over a number field k and $X(k_v) \neq \emptyset$ for any completion k_v of k. Let $\mathrm{Br}\, X = H^2_{et}(X, G_m) = \{$classes of Azumaya algebras over $X\}$ denotes the Brauer-Grothendieck group of X.

Applying to X [8, CH.VI] we obtain
the Manin obstruction: Let for any $(P_v) \in \prod_{v \in \Omega} X(k_v)$ there exists $A \in \mathrm{Br}\, X$ s.t. $\sum_{v \in \Omega} \mathrm{inv}_v A(P_v) \neq 0$. Then $X(k) = \emptyset$. (Here $A(P_v) \in \mathrm{Br}\, k_v$ denotes the specialization of A at $P_v \in V(k_v)$ ([12, CH.VI, p. 222]), $\mathrm{inv}_v : \mathrm{Br}\, k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the local invariant and Ω consists of all places of k).

Assuming that the Manin obstruction to the Hasse principle is the only one for certain families of cubic surfaces we can solve for some cubic varieties the following problems:

1) Does the Hasse principle holds for cubic three-folds?

2) (Cassels-Swinnerton-Dyer conjecture):

Let $f(x_0, \ldots, x_n)$ be a cubic form with coefficients in a field k. Suppose f has a nontrivial solution in an algebraic extension K/k of degree prime to 3. Then f also has a non-trivial solution in k.

3) Is it true that every cubic form with coefficients in a field abelian extension of $\mathbb{Q}$? (This is a particular case of the Artin's conjecture: Maximal abelian extension of $\mathbb{Q}$ is a C_1-field).

REMARK: The question: "Is the Manin obstruction to the Hasse principle for rational surfaces the only one?" was first raised by Colliot-Thélène and Sansuc in [3] and given some support via [2], [4], [5].

In view of assumptions in Theorems 1 and 2 below it makes sense to reproduce here the following result from [4]:

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THEOREM: Let a diagonal cubic surface $V \subset \mathbf{P}^3_{\mathbf{Q}}$ is given by an equation $ax^3 + by^3 + cz^3 + dt^3 = 0$, where $a,b,c,d \in \mathbf{Z}$ and x,y,z,t are homogeneous coordinates. Then for any $|a|,|b|,|c|,|d| \leq 100$ one of the following conditions holds:

1) $V(\mathbf{Q}) \neq \emptyset$;
2) there exists a prime p such that $V(\mathbf{Q}_p) = \emptyset$;
3) for any prime p , $V(\mathbf{Q}_p) \neq \emptyset$, $V(\mathbf{Q}) = \emptyset$ and the Manin obstruction non-empty.

Now we come to our results in this paper:

THEOREM 1: Let $W \subset \mathbf{P}^4_{\mathbf{Q}}$ be a cubic three-fold given by an equation $ax^3 + by^3 + cz^3 + f(u,t) = 0$, where $a,b,c \in \mathbf{Z}$, x,y,z,t,u are homogeneous coordinates and $f(u,t)$ is a homogeneous form of degree 3 in variables u,t .

Let us assume that the Manin obstruction to the Hasse principle is the only one for diagonal cubic surfaces over $\mathbf{Q}$.

Then, if W is not a cone and $W(\mathbf{Q}_v) \neq \emptyset$ for any completion $\mathbf{Q}_v$ of $\mathbf{Q}$, W containes $\mathbf{Q}$-rational points.

REMARK: For $f(u,t) = du^3 + et^3, d,e \in \mathbf{Z}$, this theorem was proved in [4] and here we only slightly modify the corresponding proof in [4].

PROOF: We will use the following result from [4].

LEMMA: Let $V \subset \mathbf{P}^3_{\mathbf{Q}}$ be a diagonal cubic surface given by an equation $ax^3 + by^3 + cz^3 + dt^3 = 0$, where $a,b,c,d \in \mathbf{Z}$ and x,y,z,t are homogeneous coordinates. Let V has points everywhere locally and let there exists a prime p such that $abc \neq 0 \bmod p$, $d = 0 \bmod p$ and $d \neq 0 \bmod p^3$. Then the Manin obstruction vanishes for V .

To prove Theorem 1, using the lemma, we must find $t_0, u_0 \in \mathbf{Z}$ such that for the cubic surface V_{t_0,u_0} , given by the equation $ax^3 + by^3 + cz^3 + f(u_0,t_0)t^3 = 0$ (where $a,b,c,f(u_0,t_0)$ are coefficients and x,y,z,t are homogeneous coordinates), the following holds:

i) V_{t_0,u_0} has points everywhere locally
ii) there exists a prime p such that $abc \neq 0 \bmod p$, $f(t_0,u_0) = 0 \bmod p$ and $f(t_0,u_0) \neq 0 \bmod p^3$.

Then by the Lemma $V_{t_0,u_0}(Q) \neq \emptyset$ i.e. $V(Q) \neq \emptyset$.

In order to construct V_{t_0,u_0} with properties i), ii) let us choose for every prime $q|3abc$ a point $(x_q, y_q, z_q, t_q, u_q) \in W(Z_q)$ (with integer coordinates not all equal to $0 \bmod q$). Let us also choose a prime p not dividing $3abc$ and $t_p, u_p \in Z$ such that

$$(*) \qquad \begin{aligned} f(t_p,u_p) &= 0 \bmod p \\ f(t_p,u_p) &\neq 0 \bmod p^3 \end{aligned}$$

(Such choice is possible, since for all but finitely many primes p $f(u,t) \neq c_p \cdot \ell_p(u,t)^3 \bmod p$, where $\ell_p(u,t)$ is a linear form in variables u,t and $c_p \in Z_p$).

Then $t_0, u_0 \in Z$, such that $(t_0,u_0) = (t_q,u_q) \bmod q^2$ for all $q|3abc$ or $q = p$, are just integers that we are looking for. (Indeed, i) follows from the Hensel lemma and ii) from (*)). This proves Theorem 1.

<u>THEOREM 2</u>: Let us assume that the Manin obstruction is the only one for diagonal cubic surfaces over Q. Let $V \subset P^3_Q$ be given by an equation $ax^3 + by^3 + cz^3 + dt^3 = 0$, where $a,b,c,d \in Z$ and x,y,z,t are homogeneous coordinates. Let K/Q be a finite extension such that $V(K) \neq \emptyset$ and $[K:Q]$ prime to 3.

Then $V(Q) \neq \emptyset$.

<u>PROOF</u>: We will use

<u>Fact 1</u>: $V(Q_v) \neq \emptyset$ for any completion Q_v of Q.

This follows from the theorem by Coray [6] stating that the Cassels-Swinnerton-Dyer conjecture holds for cubic surfaces over local fields and from the observation that for any place v of Q there exists a place w of K lying above v such that $[K_w : Q_v]$ prime to 3.

The following result was proved in [4].

<u>Fact 2</u>: If there exists a rational prime $p|3abcd$ such that V is not Q_p-rational, then the Manin obstruction vanishes for V/Q.

In view of Fact 2 we can assume further that for all primes

$p \nmid 3abcd$ V is Q_p-rational. Then for any completion K_w of K, $A \in Br(V \otimes K_w)$ and $P, P' \in V(K_w)$ $A(P) = A(P')$ (for w at which $V \otimes K_w$ has bad reduction this follows from K_w-rationality of V and for "good" places of K - from [1]). Therefore (since $V(K) \neq \emptyset$) for any $A' \in Br\, V \otimes K$ and $(P_w) \in \prod_w V(K_w)$ we have

$$(*) \qquad \sum_w inv_w\, A'(P_w) = 0 \quad .$$

This implies for any $(P_v) \in \prod_v V(Q_v)$ and $A \in Br\, V$:

$$(**) \qquad [K:Q](\sum inv_v\, A(P_v)) = \sum_v \sum_{w|v} inv_w\, Res_{Q_v/K_w}(A(P_v)) =$$

$$= \sum_w inv_w (Res_{Q/K} A)(P_w) = 0 \quad .$$

Here (P_w) is the image of (P_v) under the diagonal map $\prod_v V(Q_v) \longrightarrow \prod_w V(K_w)$, $Res_{Q_v/K_w} : Br\, Q_v \longrightarrow Br\, K_w$ and $Res_{Q/K} : Br\, V \longrightarrow Br\, V \otimes K$ are restriction maps. (The first equality in (**) follows from the global class-field theory and the last one from (*)).

By [4] the Manin obstruction for V/Q is provided by $A \in Br\, V$ such that $3A \in Br\, Q$ and, since $[K:Q]$ prime to 3, (**) implies that $\sum_v inv_v\, A(P_v) = 0$. Thus the Manin obstruction vanishes for V, proving the theorem.

<u>THEOREM 3</u>: Let V be a cubic surface over a number field k . For any finite extension of fields K/k such that $V(K) = \emptyset$ and $V \otimes K$ has points everywhere locally let us assume that the Manin obstruction non-empty for $V \otimes K$. Then V containes a rational point in some abelian extension of k .

<u>PROOF</u>: Since the Hasse principle holds for singular cubic surfaces, we can assume that V is smooth. Using the theorem of Lang [7] that a maximal unramified extension of a completion k_v of k is C_1 , we can choose a finite abelian extension K/k such that $V(K_w) \neq \emptyset$ for all completions K_w of K .

Further, let L/k be a finite Galois extension such that all lines of V are defined over L . Then ([8,Ch VI])

$H^1(\mathrm{Gal}(L/K), \mathrm{Pic}(V \otimes L)) \simeq H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Pic}\,\bar{V}) = \frac{\mathrm{Br}\,V}{\mathrm{Br}\,k}$ (here $\bar{k}$ is an algebraic closure of k and $\bar{V} = V \otimes \bar{k}$).

Let $d = \# \frac{\mathrm{Br}\,V}{\mathrm{Br}\,k}$ and let a finite abelian extension M/K be such that $d \mid [M:K]$, $M \cap L = k$ and M/k is an abelian extension.

Now the theorem follows from

<u>LEMMA</u>: The Manin obstruction vanishes for $V_M = V \otimes_k M$.

<u>PROOF</u>: Since $\mathrm{Gal}(M/K)$ acts trivially on lines of V_M we have that the restriction map
$\mathrm{Res}_{K/M} : H^1(\mathrm{Gal}(\bar{k}/K), \mathrm{Pic}\,\bar{V}) \longrightarrow H^1(\mathrm{Gal}(\bar{k}/M), \mathrm{Pic}\,\bar{V})$ is the epimorphism.
Therefore $\mathrm{Br}\,V_M = \mathrm{Res}_{K/M}(\mathrm{Br}\,V \otimes K) \cdot \mathrm{Br}\,M$, where $\mathrm{Res}_{K/M} : \mathrm{Br}\,V \otimes K \longrightarrow \mathrm{Br}\,V_M$ is the restriction map.

Now let the diagonal map $\prod_v V(K_v) \longrightarrow \prod_w V(M_w)$ maps some $(P_v) \in \prod_v V(K_v)$ into $(P_w) \in \prod_w V(M_w)$ and let $A = (\mathrm{Res}_{K/M} A') \cdot a \in \mathrm{Br}\,V_M$, where $A' \in \mathrm{Br}\,V \otimes K$ and $a \in \mathrm{Br}\,M$. Then, applying the global class-field theory we obtain:

$$\sum_w \mathrm{inv}_w\, A(P_w) = \sum_w \mathrm{inv}_w \left[(\mathrm{Res}_{K/M} A') \cdot a \right] (P_w) =$$

$$= \sum_w \mathrm{inv}_w (\mathrm{Res}_{K/M} A')\,(P_w) = \sum_v \sum_{w \mid v} \mathrm{Res}_{K_v/M_w} A'(P_v) =$$

$$= [M:K] \sum_v \mathrm{inv}_v A'\,(P_v) = 0 \quad .$$

(The last equality follows from this fact that $d \cdot A' \in \mathrm{Br}\,K$ and $d \mid [M:K]$) . This completes the proof of the lemma (and Theorem 3).

<u>REMARK</u>: Assuming that the Manin obstruction is the only one for any rational smooth projective surface X/k ($[k:\mathbb{Q}] < \infty$), one can prove that $X(K) \neq \emptyset$ for some abelian extension K/k .

The proof of this fact closely follows to the proof of Theorem 3. In particular, in order to apply as above the Lang's theorem, one can use the following Manin's conjecture (which was recently proved by Colliot-Thélène): If X/k is a rational projective smooth surface and k is C_1 , then $X(k) \neq \emptyset$.

Finally, I would like to mention the following generalization of the Manin's conjecture, which was also proposed by Colliot-Thélène: (in the above assumptions on X) if the cohomological dimension of k : 1 then $X(k) \neq \emptyset$.

<u>REFERENCES</u>

[1] S. BLOCH, On the Chow groups of certain rational surfaces. Ann. Sci. Ec. Norm. Sup., 4-e serie, 1981, 14 : 1, p.p. 41-59.

[2] J.-L. COLLIOT-THÉLÈNE, D. CORAY, J.-J. SANSUC, Descente et principe de Hasse pour certaines variêtês rationnelles. J. reine angew. Math., 1980, 320, p. 150-191.

[3] J.-L. COLLIOT-THÉLÈNE, J.-J. SANSUC, La descente sur les varietes rationnelles. In: Journées de geometrie algebrique d'Angers, 1979, p.p. 223-237.

[4] J.-L. COLLIOT-THÉLÈNE, D. KANEVSKY, J.-J. SANSUC, Arithmêtique des surfaces cubiques diagonales, preprint 1985.

[5] J.-L. COLLIOT-THÉLÈNE, J.-J. SANSUC, Sir P. SWINNERTON-DYER, Intersections de deux quadriques et surfaces de Châtelet. C.R. Ac. Sci., Paris, 1984, ser.I, 298 : 16, p.p. 377-380.

[6] D. CORAY, Algebraic points on cubic hypersurfaces, Acta Arithmetica XXX (1976) p.p. 267-296.

[7] S. LANG, On quasi algebraic closure, Annals of Math., vol. 55, No. 2 , 1952, p.p. 373-390.

[8] Yu. I. MANIN, Cubic forms, 1973.

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