The hard Lefschetz theorem for concave and convex algebraic manifolds

Siegmund Kosarew

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

SFB 170 "Geometrie und Analysis"
Mathematisches Institut
der Universität
Bunsenstraße 3/5
D–3400 Göttingen

Federal Republic of Germany

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Sonderforschungsbereich 170 "Geometrie und Analysis"
Mathematisches Institut der Universität
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In this note we want to establish the hard Lefschetz theorem for the cases of concave and convex algebraic manifolds over $\mathbb{C}$. This classes of varieties admit a nice Hodge theory for the singular cohomology groups $H^n(U,\mathbb{C})$ with certain restrictions on n . "Nice" means that we have a behavior just like in the compact smooth case (see for instance $[BK]_1$, $[BK]_2$, [KK]). The results are the following

Theorem I (hard Lefschetz in the concave case). *Let* X *be an irreducible projective* $\mathbb{C}$*-scheme,* $Y \subset X$ *a closed subscheme such that* $U := X \setminus Y$ *is smooth and let* U^{an} *be the associated complex manifold. If* $\mathscr{L} \in \mathrm{Pic}(X)$ *is an ample line bundle on* X *with first Chern class* $\omega \in H^2(X^{an},\mathbb{C})$, *then there is a natural isomorphism*

$$L^r : H^{\dim X - r}(U^{an},\mathbb{C}) \longrightarrow H_c^{\dim X + r}(U^{an},\mathbb{C})$$

for each $r \geq \dim Y+1$ *which, composed with the canonical map* $H_c^{\cdot}(U^{an},\mathbb{C}) \longrightarrow H^{\cdot}(U^{an},\mathbb{C})$, *is the* r*–fold cup product with* $\omega | U^{an}$.

Moreover, this map induces bijections

$$L^r : H^j(U^{an},\Omega^i) \longrightarrow H_c^{j+r}(U^{an},\Omega^{i+r})$$

for $i+j < \operatorname{codim}(Y,X)-1$.

Theorem II (hard Lefschetz in the convex case). *Let* X *be an irreducible smooth projective* $\mathbb{C}$*–scheme,* $Y \subset X$ *an effective divisor and* $U := X \setminus Y$. *We assume that the normal bundle* $N_{Y|X}$ *of* Y *in* X *is* k*–ample in the sense of Sommese. If* $\mathscr{L} \in \operatorname{Pic}(X)$ *is an ample line bundle on* X *with characteristic class* ω , *then the* r*–fold cup product with* $\omega | U^{an}$ *induces an isomorphism*

$$L^r : H_c^{\dim X-r}(U^{an},\mathbb{C}) \longrightarrow H^{\dim X+r}(U^{an},\mathbb{C})$$

for each $r \geq k+1$.

Moreover, the induced mappings

$$L^r : H_c^j(U^{an},\Omega^i) \longrightarrow H^{j+r}(U^{an},\Omega^{i+r})$$

are bijective for $i+j \leq \dim X-k-1$.

Corollary. *In the situation of Theorem* II, *the canonical maps*

$$H^n_c(U^{an},\mathbb{C}) \longrightarrow H^n(U^{an},\mathbb{C}) ,$$

$$H^j_c(U^{an},\Omega^i) \longrightarrow H^j(U^{an},\Omega^i)$$

are injective for $n \leq \dim X{-}k{-}1$ *resp.* $i{+}j \leq \dim X{-}k{-}1$.

Some remarks to the proofs of Theorem I, II: For Theorem I we give two proofs. The first one depends on results obtained in [KK] whilst the second one, which is rather short, reduces the assertion to the hard Lefschetz theorem for intersection cohomology (compare [BBD]). Theorem II is shown by induction on k . The case $k{=}0$ follows quite easily from [N].

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1. *Comparing cohomology and intersection cohomology*

Let X denote a pure dimensional reduced complex space and $\mathscr{I}$ the intersection cohomology complex associated to the constant sheaf $\mathbb{C}_X$ on X with respect to a fixed perversity p . Adopting the notations as in the book [B], we take a stratification

$X_{\bullet} = (X_2 \supset X_3 \supset \dots)$ of X such that

$$U_k := X \backslash X_k$$

and

$$S_{m-k} := U_{k+1} \backslash U_k = X_k \backslash X_{k+1}\ , \quad m := \dim_{\mathbb{R}} X\ ,$$

is a pure real $(m-k)$–dimensional manifold or empty. Moreover, let

$$\begin{aligned} j_k &: U_k \hookrightarrow U_{k+1}\ , \\ i_k &: S_{m-k} \hookrightarrow U_{k+1} \end{aligned}$$

be the canonical inclusions.

<u>(1.1) Lemma</u>. *For the natural maps*

$$\begin{aligned} \alpha_k^\nu &: H^\nu(U_{k+1}, \mathscr{O}) \longrightarrow H^\nu(U_k, \mathscr{O})\ , \\ \beta_k^\nu &: H_c^\nu(U_k, \mathscr{O}) \longrightarrow H_c^\nu(U_{k+1}, \mathscr{O}) \end{aligned}$$

the following assertions hold:

(i) α_k^ν *is bijective for* $\nu \le p(k)$ *and injective for* $\nu = p(k)+1$,

(ii) β_k^ν *is bijective for* $\nu \ge p(k)+m-k+2$ *and surjective for* $\nu = p(k)+m-k+1$.

Proof. Part (i) has already been established in $[BK]_2$, section 3. We only mention that it is a formal consequence of the distinguished triangle (in the derived category)

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & R(j_k)_*(j_k)^* \mathcal{A} \\ & \nwarrow \quad \swarrow {\scriptstyle [1]} & \\ & (i_k)_!(i_k)^! \mathcal{A} & \end{array}$$

and the vanishing

$$\mathcal{H}^{\nu}(i_k^! \mathcal{A})_x = 0 , \quad \text{for } x \in S_{m-k} \text{ and } \nu \leq p(k)+1 .$$

For the proof of (ii), we use the triangle

$$\begin{array}{ccc} (j_k)_!(j_k)^! \mathcal{A} & \longrightarrow & \mathcal{A} \\ & {\scriptstyle [1]} \nwarrow \quad \swarrow & \\ & (i_k)_*(i_k)^*(\mathcal{A}) & \end{array}$$

and the vanishing

$$\mathcal{H}^{\nu}(\mathcal{A}) = 0 \quad \text{for } j > p(k) ,$$

see for instance [B] p. 86. The spectral sequence

$$E_2^{i,j} = H_c^i(S_{m-k}, \mathcal{H}^j(\mathcal{A})) \Rightarrow H_c^{\nu}(S_{m-k}, \mathcal{A})$$

gives now

$$H_c^{\nu}(S_{m-k}, \mathcal{A}) = 0 \quad \text{for } \nu > p(k)+m-k .$$

Since $H_c^{\nu}(U_{k+1}, (j_k)_!(j_k)^! \mathcal{A}) = H_c^{\nu}(U_k, \mathcal{A})$ for all ν, the assertion follows. ∎

(1.2) Corollary. *Let* $n_0 \geq 2$ *be an integer such that*

$$U_2 = U_3 = \dots = U_{n_0} \subset U_{n_0+1} \subset \dots .$$

Then, for the natural maps

$$\alpha^\nu : I_pH^\nu(X,\mathbb{C}) \longrightarrow H^\nu(U_{n_0},\mathbb{C})$$
$$\beta^\nu : H^\nu_c(U_{n_0},\mathbb{C}) \longrightarrow I_pH^\nu_c(X,\mathbb{C})$$

the following holds:

(i) α^ν *is bijective for* $\nu \leq p(n_0)$ *and injective for* $\nu = p(n_0)+1$,

(ii) β^ν *is bijective for* $\nu \geq p(n_0)+m-n_0+2$ *and surjective for* $\nu = p(n_0)+m-n_0+1$.

(1.3) Proposition. *Let* X *be a pure dimensional reduced complex space and* $Y \subset X$ *a closed complex subspace such that* $X \setminus Y$ *is smooth. Then we have for the natural maps* *)

$$\alpha^\nu : IH^\nu(X,\mathbb{C}) \longrightarrow H^\nu(X\setminus Y,\mathbb{C}) ,$$
$$\beta^\nu : H^\nu_c(X\setminus Y,\mathbb{C}) \longrightarrow IH^\nu_c(X,\mathbb{C})$$

the following assertions:

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*) Here we take the middle perversity.

(i) α^ν *is bijective for* $\nu \leq \mathrm{codim}_{\mathbb{C}}(Y,X)-1$ *and injective for* $\nu = \mathrm{codim}_{\mathbb{C}}(Y,X)$,

(ii) β^ν *is bijective for* $\nu \geq \dim_{\mathbb{C}} X + \dim_{\mathbb{C}} Y + 1$ *and surjective for* $\nu = \dim_{\mathbb{C}} X + \dim_{\mathbb{C}} Y$.

Proof. We take a complex–analytic Whitney stratification $X_\cdot$ such that $Y = X_{n_0}$ with $n_0 = 2\,\mathrm{codim}_{\mathbb{C}}(Y,X)$. Since the middle perversity is given here by $p(k) = (k-2)/2$, the assertion follows from (1.2). ■

2. *Proof of the hard Lefschetz theorem in the concave case*

Our first proof goes by induction with respect to $\dim Y$. So let us assume $\dim Y = 0$. In this case we take a resolution

$$\pi : \tilde{X} \longrightarrow X$$

where $\tilde{X}$ is smooth and proper over $\mathbb{C}$ and π is an isomorphism outside Y . Let $E := \pi^{-1}(Y)$ denote the exceptional divisor. Moreover, we fix an ample divisor D' on $\tilde{X}$ such that $\mathrm{supp}(D') = \pi^{-1}(\mathrm{supp}(D)) \cup E$ and denote by $\eta \in H^2(\tilde{X},\mathbb{C})$ the class of D'.*) For simplicity we assume that $\eta | U = \omega | U$. Then there is a natural commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_E^{n-r}(\tilde{X}) & \longrightarrow & H^{n-r}(\tilde{X}) & \longrightarrow & H^{n-r}(U) & \longrightarrow & 0 \\ & & \downarrow \eta^r \cup & (I) & \downarrow \eta^r \cup & & & & \\ 0 & \longleftarrow & H^{n+r}(E) & \longleftarrow & H^{n+r}(\tilde{X}) & \longleftarrow & H_c^{n+r}(U) & \longleftarrow & 0 \end{array}$$

*) We may assume $\mathcal{L} \cong \mathcal{O}_X(D)$ with an effective divisor D .

if $r \geq 1$. The two vertical maps are bijective. This follows from [N] Prop. (5.1), (6.1). By the commutativity of (I), we obtain immediately a projection $H^{n+r}(\tilde{X}) \xrightarrow{p} H_c^{n+r}(U)$ whose composition with the natural map $H_c^{n+r}(U) \longrightarrow H^{n+r}(U)$ is the usual restriction from $\tilde{X}$ to U . By construction $p \circ (\eta^r U)$ factorizes over $H^{n-r}(U)$ which gives us our desired bijection

$$L^r : H^{n-r}(U) \xrightarrow{\sim} H_c^{n+r}(U) \ , \quad r \geq 1 \ .$$

Now assume $\dim(Y) > 0$. Let D be very ample, $\mathscr{L} = \mathcal{O}_X(D)$ (which is not a restriction) and $X \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$ the induced embedding. We fix a general hyperplane section X' of X such that for $Y' := X' \cap Y$, $U' := X' \setminus Y'$ the following holds

(i) U' is smooth,

(ii) $\dim Y' = \dim Y - 1$, $\operatorname{codim}(Y',X') = \operatorname{codim}(Y,X)$,

(iii) the restriction map $H^\nu(U) \longrightarrow H^\nu(U')$ is bijective for $\nu \leq \operatorname{codim}(Y,X) - 1$.

These properties can be achieved, compare $[BK]_2$ section 3. By induction hypothesis, we have an isomorphism

$$L^r : H^{\dim X' - r}(U') \xrightarrow{\sim} H_c^{\dim X' + r}(U')$$

for $r \geq \dim Y' + 1$ (where we take $\mathscr{L}' := \mathscr{L} | X'$ as an ample line bundle). Now we consider the composition

$$H^{\dim X-r}(U) \xrightarrow{\ a\ } H^{\dim X-r}(U') \underset{\sim}{\xrightarrow{L^{r-1}}} H_c^{\dim X+r-2}(U') \xrightarrow{\ b\ } H_c^{\dim X+r}(U)$$

for $r \geq \dim Y+1$. By property (iii), the maps a and b are bijective for this range and consequently we get an isomorphism on U

$$L^r : H^{\dim X-r}(U) \longrightarrow H_c^{\dim X+r}(U) \ .$$

The interpretation of L^r as an r–fold cup product with $\omega|U$ is seen also by induction and using the natural commutative diagram (the horizontal maps are Gysin homomorphisms)

$$\begin{array}{ccc} H_c^{\nu-2}(U') & \longrightarrow & H_c^{\nu}(U) \\ \downarrow & & \downarrow \\ H^{\nu-2}(U') & \longrightarrow & H^{\nu}(U) \\ \uparrow & \nearrow_{\omega\,U} & \\ H^{\nu-2}(U) & & \end{array} \ .$$

Our second proof is based on the following commutative diagram $(n := \dim X)$

$$\begin{array}{ccc} H^{n-r}(U,\mathbb{C}) & \overset{L^r}{\dashrightarrow} & H_c^{n+r}(U,\mathbb{C}) \\ \alpha\uparrow & & \downarrow\beta \\ IH^{n-r}(X,\mathbb{C}) & \underset{\omega^r U}{\longrightarrow} & IH^{n+r}(X,\mathbb{C}) \end{array} \ .$$

By the choice of r and (1.3), the maps α and β are bijective. So our Lefschetz theorem is equivalent to that in intersection cohomology in the appropriate range.

The second statement in Theorem I can be verified by taking into account the fact that the differentials $d_r^{i,j}$ in the spectral sequence

$$E_1^{i,j} = H^j(U,\Omega^i) \Rightarrow H^{i+j}(U,\mathbb{C})$$

are zero for $r \geq 1$, $i+j < \operatorname{codim}(Y,X)-1$ and, by duality, also those of

$${}_cE_1^{i,j} = H_c^j(U,\Omega^i) \Rightarrow H_c^{i+j}(U,\mathbb{C})$$

for $r \geq 1$, $i+j > \dim X + \dim Y + 1$. Moreover, ω induces in a natural way a cohomology class in $H^1(U,\Omega^1)$ which we denote again by $\omega | U$. This class is algebraic, so all $d_r^{1,1}$ vanish on it and therefore $\omega^r U\_$ is compatible with the two spectral sequences (which carry a multiplicative structure). The Hodge filtration is respected by L^r

$$L^r : F^s(H^{n-r}) \longrightarrow F^{s+r}(H_c^{n+r}), \quad s \geq 0$$

modulo shift by r. Obviously, it suffices to show that this map is bijective for $r > \dim Y+1$ and all s. Now this may be seen by induction on $\dim Y$ as above (where $Y = \emptyset$ is the first step here) and the calculation in section 4 of [KK] together with the weak Lefschetz result in $[BK]_2$ Prop. (3.1.4). ■

3. *Proof of the hard Lefschetz theorem in the convex case*

We proceed by induction on k. In the case $k=0$, the complement U of Y in X is a 1–convex complex manifold, so it has a compact exceptional analytic subset $E \subset U$. From this we may conclude that the natural maps between cohomology groups (with $\mathbb{C}$–coefficients)

$$H_E^{n-r}(U) \longrightarrow H_c^{n-r}(U) \ ,$$
$$H^{n+r}(U) \longrightarrow H^{n+r}(E)$$

are bijective for every $r \geq 1$ $(n = \dim U)$. In fact the first map is the Poincaré dual of the second one. For this we have the identifications with de Rhan cohomology

$$\begin{array}{ccc} H^\nu(U,\mathbb{C}) & \xrightarrow{\sim} & H^\nu(U,\Omega_U^\cdot) \\ \downarrow & & \downarrow \varphi^\nu \\ H^\nu(E,\mathbb{C}) & \xrightarrow{\sim} & H^\nu(E,(\Omega_U^\cdot)^{\wedge E}) \end{array} .$$

Now φ^ν is bijective for $\nu \geq n+1$ by a spectral sequence argument together with the fact that

$$H^j(U,\Omega_U^i) \longrightarrow H^j(\hat{U},(\Omega_U^i)^{\wedge E})$$

is an isomorphism for all i and $j \geq 1$. The result of [N] Prop. (6.1) tells us that

$$\omega^r U : H_E^{n-r}(U) \longrightarrow H^{n+r}(E)$$

is always bijective which implies immediately the assertion.

Now let $k \geq 1$. We want to use induction by taking "good" hyperplane sections D on X with $\mathscr{L} \cong \mathcal{O}_X(D)$. We consider the natural commutative diagram

$$\begin{array}{ccc} H_c^{n-r}(U) & \xrightarrow{\omega^r U} & H^{n+r}(U) \\ a\downarrow & & \uparrow b \\ H_c^{n-r}(D_U) & \xrightarrow[\omega^{r-1}U]{} & H^{n+r-2}(D_U) \end{array}$$

where $D_U := U \cap D$ and a is the restriction map with the Poincaré dual b . It is no restriction to assume that $N_{Y \cap D | D}$ is (k–1)–ample on $Y \cap D$ (compare $[BK]_2$ proof of (5.2)) and so $\omega^{r-1}U$ is bijective by induction. Moreover we have the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H_c^{n-r}(U) & \longrightarrow & H^{n-r}(X) & \longrightarrow & H^{n-r}(Y) \\ & a\downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & H_c^{n-r}(D_U) & \longrightarrow & H^{n-r}(D) & \longrightarrow & H^{n-r}(Y \cap D) \end{array}$$

which has exact lines by $[BK]_2$ Prop. (5.2). Since $r \geq 2$, the map $H^{n-r}(X) \longrightarrow H^{n-r}(D)$ is bijective and $H^{n-r}(Y) \longrightarrow H^{n-r}(Y \cap D)$ is still injective, see [GNPP] p. 85, Cor. 3.12 (iii). Consequently a and also b are bijections which gives the first assertion of Theorem II.

The second part of the statement can also be verified by induction on k . The case k=0 follows from [F] (1.6). The induction step is achieved by the same argument which was used in section 2, together with the E_1–degeneration results. ■

Proof of the corollary. This is a trivial consequence of the commuting diagram

$$\begin{array}{ccc} H^{\nu}_{c}(U) & \xrightarrow{\ \omega^{r}\cup\ } & H^{\nu+2r}(U) \\ & \searrow \qquad \nearrow_{\omega^{r}\cup} & \\ & H^{\nu}(U) & \end{array}$$

(similarly for the second arrow in the assertion) together with Theorem II. ■

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