THE CONFIGURATION OF A FINITE SET ON SURFACE

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§ 0. Introduction

Let S be a smooth surface in $\mathbb{P}^n$ and m be an integer with $n \geq m \geq 2$. For any m different points on S, if they are linearly dependent we say this set is special. Let M be the collection of all these special sets, then M is a scheme with a natural algebro–geometric structure. We can show that, when $n = 3m-2$ and S in general position, M is a finite scheme. Denote the degree of M by $\nu(s)$ which is intuitively the number of points in M possibly with multiplicities.

S.K. Donaldson posed a conjecture about this case in [2]:

"Conjecture 5. There is a universal formula for expressing $\nu(s)$ in terms of m, the Chern numbers of S, the degree of S in $\mathbb{P}^{3m-2}$, and the intersection number of the canonical class of S with the restriction of the hyperplane class."

He pointed out this enumerative problem has something to do with Yang–Mills invariants.

In this paper we give an affirmative answer for the conjecture. But the formula for expressing $\nu(s)$ is complicated for writing down explicitely though there is an algorithm for computing it.

In § 1 we explain the meaning of "general position" in the present case and give the basic construction for computing $\nu(s)$. In § 2, all of the objects considered in § 1 are lifted to some projective vectors bundle where it is comparatively easy for computation. In § 3 we construct the blowing–up which is needed for computing some Segre class and finally in § 4 we prove the main result.

Author thanks K.C. Mong for showing him this problem and thanks R. Piene for the profitable discussion with her.

§ 1.

In sequels we assume the ground field is algebraically closed with arbitrary characteristic $> m$ or characteristic 0 , where m is given as follows.

Let $m \geq 2$ be an integer and $n = 3m-2$.

Let $Y = (\mathbb{P}^n)^m$, the cross product of m times $\mathbb{P}^n$ and $X = (S)^m$ where S is a smooth surface in $\mathbb{P}^n$ which is in general position in a sense as follows.

Proposition 1.1. Let $i : S \hookrightarrow \mathbb{P}^n$ be a non–degenerate embedding then there exists an embedding $j : S \longrightarrow \mathbb{P}^{n+1}$ such that

(i) i(S) is the image of j(S) via a certain projection from $\mathbb{P}^{n+1}$ to $\mathbb{P}^n$ with a point as center; but all the hyperplanes passing the center may have a common component on j(S)

(ii) on the image of j(S) via a generic projection, every set of m points is linearly independent except for a finite number of these sets which span (m–2)–spaces.

(iii) the k–osculating space of j(S) at any point with $2 \leq k \leq m$ and any other m–k points on j(s) span a (m–1)–space.

<u>Proof</u>. Let $i^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}(1)$. We shall show, there exists an integer N_0 such that for every $N \geq N_0$ and the embedding φ determined by $\mathcal{O}(N)$, every m points on $\varphi(S)$ are linearly independent.

In fact, let Z be a subscheme of m points on S with reduced structure and J_Z be the sheaf of ideal defining Z. From the exact sequence

$$0 \longrightarrow H^0(S, J_Z(N)) \longrightarrow H^0(S, \mathcal{O}(N)) \longrightarrow H^0(S, \mathcal{O}_Z(N))$$

$$\longrightarrow H^1(S, J_Z(N)) \longrightarrow H^1(S, \mathcal{O}(N)) \longrightarrow 0$$

We see that if $H^1(S, J_{Z'}(N)) = 0$ for every (reduced) subscheme $Z' \subset Z$, then these m points are linearly independent. By Cartan–Serre Theorem B the condition is satisfied for every $N \geq N_0$ with a certain N_0. Now we have to show that N_0 can be chosen only depending on m and not on their position on S.

As a standard method we take Z as a subscheme of $\mathbb{P}^n$ and show that we may replace the ideal defining Z in $\mathbb{P}^n$ for J_Z in the above argument. But in $\mathbb{P}^n$ we can prove the above assertion directly. Then the vanishing of $H^1(S, J_Z(N))$ is independent of the position of the points.

Continue to prove the proposition.

Let $r+1 = H^0(S,\mathcal{O}(N_0))$ and $\phi : S \longrightarrow \mathbb{P}^r$ be the embedding determined by $\mathcal{O}(N_0)$. We show that for $r \geq n+2 = 3m$ a generic projection from $\mathbb{P}^r$ to $\mathbb{P}^{r-1}$ gives an embedding of S into $\mathbb{P}^{r-1}$ and preserves the independence of arbitrary m points on S. Indeed, the subscheme consisting of all the $(m-1)$–planes in $\mathbb{P}^n$ spanned by some m points on S has dimension $3m-1$ and the subscheme consisting of all the $(m-1)$–planes in $\mathbb{P}^n$ spanned by a k–osculating and any other $(m-k)$ points has dimension at most $3(m-1)$, thus a projection with a generic point as center meets our need. We proceed like this till we arrive at $\mathbb{P}^{3m-1}$. Since for $m = 2$ this proposition is true automatically we may assume $m \geq 3$. Then taking a generic point in $\mathbb{P}^{3m-1}$ as center will give a projection which preserves the independence of m points on S except for a finite number of these sets. And anyone of these exceptional sets spans a $(m-2)$–plane. The reasons for that are (i) a generic point in $\mathbb{P}^{3m-1}$ is in a finite number of all $(m-1)$–plane spanned by m points on S; (ii) a generic point in $\mathbb{P}^{3m-1}$ gives an embedding and preserves the independence of arbitrary $m-1$ points on S.

Hereafter the words "a surface in general position" means the sense of Proposition 1.1.

Let $p = (p_1,\dots,p_m) \in Y$ and $p_i = (z_{i0},\dots,z_{in})$ be the homogeneous coordinates of p_i in $\mathbb{P}^n$. We say p is a <u>special point</u> if $\mathrm{rk}(z_{ij}) \leq m-1$ namely, $p_1,\dots,p_m$ are in the same hyperplane of $\mathbb{P}^n$. The ideal generated by the m–minors of (z_{ij}) defines a subscheme $G \subset \mathbb{P}^n$ which represents all of the special points in $\mathbb{P}^n$.

<u>Lemma 1.2</u>. G is a variety with codimension $2m$.

<u>Proof</u>. Let $H_i = q_i^* \mathcal{O}_{\mathbb{P}^n}(1)$ where q_i is the ith projection from Y to $\mathbb{P}^n$, and $\varphi_i : \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}^{n+1}$ be the canonical embedding of the universal line bundle into the

trivial bundle. Therefore on Y we have a homomorphism

$$\varphi = \sum_{i=1}^{m} q_i^* \varphi_i : H_1^{-1} \oplus \dots \oplus H_m^{-1} \longrightarrow \mathcal{O}_Y^{n+1} .$$

We recall that in [1] or [5], a generic determinantal variety $M_k(m,n)$ is the locus of matrices of rank at most k and the ideal for defining M_k in $M(m,n) \simeq A^{mn}$ is generated by the $(k+1)\times(k+1)$ minors. The present situation is essentially the case of a generic determinal variety.

Indeed, over a point $p \in Y$, φ is represented by the matrix (z_{ij}), and the m–minors defines a variety M_{m-1} on vector bundle $H_1^{-1} \oplus \dots \oplus H_m^{-1}$ with codimension $2m$. On the other hand, every m–minor is homogeneous with respect to each row of it and thus there is a scheme, which is exactly G, with $q^{-1}(G) = M_{m-1}$ where $q : H_1^{-1} \oplus \dots \oplus H_m^{-1} \longrightarrow Y$ is the structure morphism. By the faithful flatness of $q|_{M_{m-1}}$ we have shown G is a variety with codimension $2m$.

<u>Remark 1.3</u>. G can be described by the desingularization of M_{m-1}, that means, if letting $\tilde{M}_{m-1} = \{(A,W) \in (\bigoplus_{i=1}^{m} H_i^{-1}) \times \mathbb{P}(\oplus H_i) \mid A \cdot w = 0\}$, then $\tilde{M}_{m-1}$ is mapped by the projection onto M_{m-1} properly, and by the another projection, $\tilde{M}_{m-1}$ is mapped onto a subvariety $\overline{G}$ of $P = \mathbb{P}(\oplus H_i)$, which is defined by the degeneracy $D_{m-1}(\psi)$ of ψ and where ψ is the composition of the canonical homomorphism $\mathcal{O}_P(-1) \longrightarrow \oplus H_i^{-1}$ and φ. It is clear that, the projection from P to Y maps $\overline{G}$ onto G.

We shall use this description in § 2.

Usually the next step should be the computation for the intersection of G and X, but in the present case this intersection $V = G \times_Y X$ has an excess part i.e. they meet in a higher dimensional subscheme than that in the general case. Therefore we have to exclude the "bad" points from $X \cdot G$ which is caused by the excess part.

Lemma 1.4.

(i) $V = V_0 \coprod V_1$, where V_0 is the finite subscheme representing the special points on Y and V_1 is a connected subscheme.

(ii) As a scheme–theoretic union, $V_1 = \bigcup_{0<i<j<m} S_{ij} \cup \bigcup_{i<j<k} S_{ijk}^{a_3} \cup \ldots \cup S_{1\ldots m}^{a_m}$ with multiplicities $a_\ell \geq 1$ (Since the symmetry of $s_{i_1,\ldots,i_\ell}$ with respect to its subscripts in V, every multiplicity for $s_{i_1\ldots i_\ell}$ is same.), where $s_{i_1\ldots i_\ell}$ is the image of the mapping

$$\Delta_{i_1\ldots i_\ell} \times (\mathrm{id})^{m-\ell} : S^{m-\ell+1} \longrightarrow S^m$$

and which is isomorphic to $S^{m-\ell+1}$ under this mapping where $\Delta_{i_1\ldots i_\ell}$ is the diagonal morphism for the $i_1,\ldots,i_\ell$–th factors.

(iii) a_ℓ only depends on m for every $2 \leq \ell \leq m$.

Proof. Let $p \in V$, then $\mathrm{rk}(z_{ij}(p)) \leq m-1$. If $p_1,\ldots,p_m$, the components of p, are m different points in $\mathbb{P}^n$, then by Proposition 1.1 they span a linear space of dimension m–2, i.e. $\mathrm{rk}(z_{ij}(p)) = m-1$, and the number of such p's is finite. Denote this finite scheme by V_0. The other points of V must have at least two of $\{p_1,\ldots,p_m\}$ being a same point and the inverse statement is valid too. Therefore, they form a subscheme V_1 supporting on $\cup S_{ij}$. (i) follows.

Before starting the proof of (ii) and (iii) we make some conventions. As done above we still fix a same coordinate system in each factor of Y, and for the coordinates $(z_{k_0},\dots,z_{k_n})$ of a point $p_k \in \mathbb{P}^n$, sometimes we take it as the affine coordinates and thus mention the Kähler differential of p_k, denoted by $D^1 p_k$. We use D^ℓ to denote the ℓ-th Kähler differential.

We see from the proof of Lemma 1.3, V is defined in G by ideal $\mathfrak{a}$ generated by the m–minors of matrix

$$\begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix} = \begin{bmatrix} z_{10}(p_1), \dots, z_{1n}(p_1) \\ \cdots\cdots \\ z_{m0}(p_m), \dots, z_{mn}(p_m) \end{bmatrix}$$

for $p \in X$.

We are going to compute the multiplicity of any point Q of $S_{ij} \setminus \bigcup_k S_{ijk}$ in V. The differential of $\mathfrak{a}$ is generated by the m–minors of $\begin{bmatrix} Q_1 \\ DQ_1 \\ \vdots \\ Q_m \end{bmatrix}$. By Proposition 1.1 (iii), we see that the matrix is non–degenerated at Q. Therefore, Q has multiplicity 1 in V and so does S_{ij}. Moreover, since Q is an arbitrary point in $S_{ij} \setminus \cup S_{ijk}$ we deduce that there does not exist any embedded component over $S_{ij} \setminus \cup S_{ijk}$.

Now suppose $Q \in S_{123} \setminus \bigcup_{k \geq 4} S_{123k}$. We shall compute the multiplicity of Q in the scheme defined by $\mathfrak{a}|_{S_{12}}$. Noticing that, the ideal defining S_{12} is generated by the 2–minors of $\begin{bmatrix} z_{10}, \dots, z_{1n} \\ z_{20}, \dots, z_{2n} \end{bmatrix}$ and the restriction of them to S_{12} gives the generators of $\Omega_{S_{12}}$, then $\mathfrak{a}|_{S_{12}}$ is generated by (m–1)–minors of the matrix

$$\begin{bmatrix} P_1 \\ DP_3 \\ \vdots \\ P_m \end{bmatrix}$$

over S_{12}. Therefore the differentials of these generators at Q are the $(m-2)$–minors of

$$\begin{bmatrix} Q_1 \\ D^2Q_1 \\ Q_4 \\ \vdots \\ Q_m \end{bmatrix} .$$

By Proposition 1.1 again we see the matrix is non–degenerated, and thus Q has multiplicity 1 in $\alpha|_{S_{12}}$. This means the multiplicity of Q in V equals the multiplicity of any point $Q' = (Q_1', Q_1', Q_1', Q_4, \ldots, Q_m) \in M_{m-2}(m,n)$ in $M_{m-1}(m,n)$, and thus it only depends on m.

With the same trick we work with $S_{i_1,\ldots,i_\ell}$ inductively and then get our conclusion for (ii) and (iii).

Note. We can prove that $a_\ell = \ell - 2$ for $\ell \geq 3$.

Proposition 1.5. As a 0–cycle,

$$[V_0] = X \cdot G - (c(N_X Y)|_{V_1} \cap s(V_1, G))_0 \in A_0(V) ,$$

where $X \cdot G$ is the intersection cycle of X and G in Y, c is the Chern operator, $N_X Y$

is the normal bundle of X in Y, $s(V_1,G)$ is the Segre class of V_1 in G, $A_*(V)$ is the Chow ring of V and $(\ )_0$ denotes the 0–part of a cycle in the bracket.

All of these symbols and their meaning can be found in [5].

Proof. Since $i : X \hookrightarrow Y$ is a regular embeddimg, then by the definition of the refined Gysin morphism [5] we have

$$\begin{aligned} i^! \cdot G &= X \cdot G \\ &= (c(N_X Y)|_V \cap s(V,G))_0 \\ &= (c(N_X Y)|_{V_0} \cap s(V_0,G))_0 + (c(N_X Y)|_{V_1} \cap s(V_1,G))_0 . \end{aligned}$$

By Lemma 1.4, V_0 is the scheme of special points on $(S)^m$ and then, $(c(N_X Y)|_{V_0} \cap s(V_0,G))_0$ gives the cycle $[V_0]$.

Definition. $\nu(S) = \frac{1}{m!} \deg [V_0]$.

Because of symmetry of the special points on $(S)^m$ with respect to its components, the definition gives the number of special points on S .

§ 2.

Though it is easy to compute $X \cdot G$ but it seems difficult to compute $s(V_1,G)$. So we would like to lift all of the objects in consideration up to certain (projective) vector bundles.

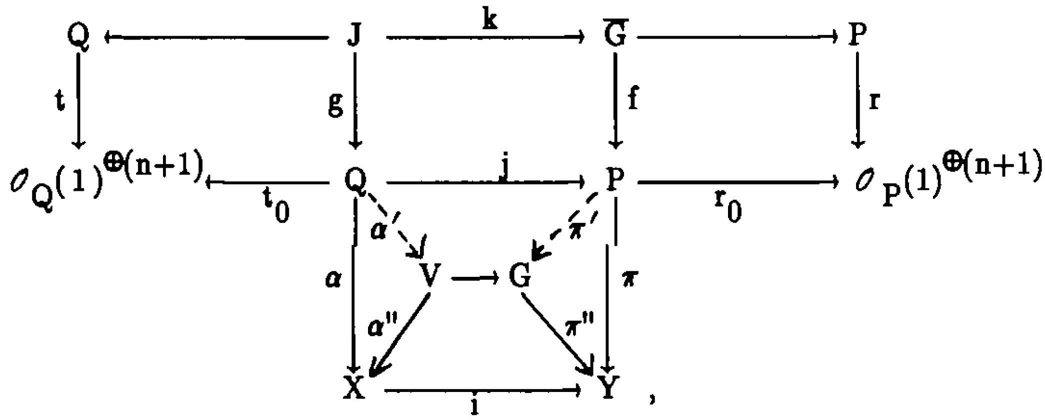
From Remark 1.3, we see $\overline{G} = D_{m-1}(\psi)$, where ψ is a composition of morphisms:

$$\psi : \mathcal{O}_P(-1) \longrightarrow H_1^{-1} \oplus \ldots \oplus H_m^{-1} \longrightarrow \mathcal{O}_P^{\oplus(n+1)} .$$

ψ induces a section $r : P \longrightarrow \mathcal{O}_P(1)^{\oplus(n+1)}$ and $\overline{G}$ is exactly the 0–locus of r.

Therefore we have a diagram as follows:

(*)



where r_0 is the 0–section of P in $\mathcal{O}_P(1)^{\oplus(n+1)}$, $Q = i^*P$, and every square with solid lines in (*) is a fiber product.

Denote $V_1 \times_G \overline{G}$ by $J_1 \subset J$, which is $(\alpha' g)^{-1}(V_1)$.

<u>Lemma 2.1</u>. (i) $X \cdot G = (\alpha' g)_*(Q \cdot \overline{G})$,

(ii) $s(V_1, G) = (\alpha' g)_* s(J_1, \overline{G})$

<u>Proof</u>. By Remark 1.3, $(\pi f)_* \overline{G} = G$. Since j is a regular embedding with codim j = codim i, then by using Excess Intersection Theorem in [5] we have

$$X \cdot G = i^! G = i^!(\pi' f)_* \overline{G} = (\alpha' g)_* i^! \overline{G}$$
$$= (\alpha' g)_* j^! \overline{G} \ .$$

(i) has been proved. As for (ii), we claim first that $\overline{G}$ is birationally isomorphic to G. Since $\overline{G}$ and G both are varieties and the morphism from $\overline{G}$ to G is surjective, it is enough to show that for a generic point $p \in G$, $(\pi' f)^{-1}(p)$ is a single point.

In fact, if p is a point in G such that the matrix corresponding to p has rank $m-1$ and $p_1, \dots, p_m$ are different then the kernel of $\varphi(p)$ has dimension 1 and thus the degeneracy of ψ in $\pi^{-1}(p)$ is a single point. The claim is true.

We see from [5] the Segre class is birationally invariant and thus

$$(\alpha' g)_* s(J_1, \overline{G}) = s(V_1, G) \ .$$

We wish to transfer the objects further into the left square in $(*)$.

<u>Lemma 2.2</u>. (i) $Q \cdot G = [Q]^2 \in A_0 J$
(ii) $(g^* c(N_Q P) \cap s(J_1, \overline{G}))_0 = (g^* c(N_Q(\mathcal{O}_Q(1)^{\oplus(n+1)}) \cap s(J_1, Q))_0$.

<u>Proof</u>. For (i), since $\operatorname{codim}_P \overline{G} = 2m$, then r and r_0 intersect properly at $\overline{G}$ and thus $N_{\overline{G}} P = f^* N_P(\mathcal{O}_P(1)^{\oplus(n+1)})$. Therefore,

$$\begin{aligned} Q\cdot\overline{G} = j^{!}\cdot\overline{G} = j^{!}\cdot(c_{n+1}(\mathcal{O}_P(1)^{\oplus(n+1)}) \cap [P]) \\ = c_{n+1}(\mathcal{O}_Q(1)^{\oplus(n+1)}) \cap i^{!}[P] \\ = c_{n+1}(\mathcal{O}_Q(1)^{\oplus(n+1)}) \cap [Q] \\ = [Q]^2 \ . \end{aligned}$$

Proof of (ii). Since j and f both are regular embeddings, then

$$g^*c(N_QP) \cap s(J_1,Q) = k^*c(N_{\overline{G}}P) \cap s(J_1,Q) \ .$$

Additionally,

$$\begin{aligned} k^*c(N_{\overline{G}}P) = k^*f^*c(\mathcal{O}_P(1)^{\oplus(n+1)}) = g^*j^*c(\mathcal{O}_P(1)^{\oplus(n+1)}) \\ = g^*c(\mathcal{O}_Q(1)^{\oplus(n+1)}) \ , \end{aligned}$$

hence the conclusion follows.

Lemma 2.1 and 2.2 tell us $[V_0] = (\alpha g)_*([Q]^2 - (g^*c(N_Q\mathcal{O}(1)^{\oplus(n+1)})|_{J_1} \cap s(J_1,Q))_0)$. So hereafter we always work with the left square in $(*)$.

Let $i^*\mathbf{H}_\ell = H_\ell$ then $Q = \mathbb{P}(\oplus H_i)$ and J is the 0–locus of section t induced by r .

For computing $s(J_1,Q)$ we have to know more about the structure of J_1 .

Let us denote $\alpha'^{-1}(s_{ij})$ by Q_{ij}, then it is easy to see

$Q_{ij} = \mathbb{P}(H_1 \oplus \ldots \oplus H_i \oplus \ldots \oplus H_i \oplus \ldots \oplus H_m)$. Denote $g^{-1}(Q_{ij})$ by W_{ij} . From Lemma 1.4, W_{ij} is exactly the degeneracy of the restriction of ϕ to Q_{ij} . In other words every point of W_{ij} is an 1–dimensional subspace of $H_1^{-1} \oplus \ldots \oplus H_i^{-1} \oplus \ldots \oplus H_i^{-1} \oplus \ldots \oplus H_m^{-1}$ which is the kernel of $\varphi|_{Q_{ij}}$ (fiberwisely). But $\varphi|_{Q_{ij}}$ is represented fiberwisely by matrix (z_{ij}) and thus an 1–dimensional subspace if it is contained in $H_i^{-1} \oplus H_i^{-1}$ must be the diagonal subspace i.e. the image of $H_i^{-1} \longrightarrow H_i^{-1} \oplus H_i^{-1}$ with $h \longmapsto (h,h)$.

Therefore W_{ij} is the image of $\mathbb{P}(H_1 \oplus \ldots \oplus H_i \oplus \ldots \oplus \hat{H}_j \oplus \ldots \oplus H_m) \longrightarrow \mathbb{P}(H_1 \oplus \ldots \oplus H_i \oplus \ldots \oplus H_i \oplus \ldots \oplus H_m)$ induced by the diagonal homomorphism.

As a conclusion we have

<u>Lemma 2.3</u>. $J_1 = \bigcup_{1 \le i < j \le m} W_{ij} \cup W_{ijk}^{a_3} \cup \ldots \cup W_{12\ldots m}^{a_m}$, where $W_{i_1 \ldots i_\ell}$ will be defined in the beginning of § 3.

<u>Lemma 2.4</u>. W_{ij} is a divisor on Q_{ij} and the corresponding inverse sheaf is $H_i^{-1} \otimes \mathcal{O}(1)$.

<u>Proof</u>. It is a standard fact from § 8 of Ch. II in [3] .

§ 3.

In this section we shall reconstruct the blowing–up of Q with respect to J_1 . For that we make an observation of S_{ij} , Q_{ij} and W_{ij} .

(**) (1) S_{ij} (resp. Q_{ij}, W_{ij}) is smooth for all $1 \leq i < j \leq m$

(2) (a) Let $S_{ij} \cap S_{jk} = S_{ijk}$, which is defined as the image of $\Delta_{ijk} \times (id)^{m-3} : (S)^{m-2} \longrightarrow (S)^m$ where Δ_{ijk} is the diagonal mapping with respect to the ith, jth and kth factors.

(b) Let $Q_{ij} \cap Q_{jk} = Q_{ijk}$, which is defined as $(\Delta_{ijk} \times (id)^{m-3})^* Q$.

(c) Let $W_{ij} \cap W_{jk} = W_{ijk}$, which is defined as the image of $\mathbb{P}(H_1 \oplus \ldots \oplus H_i \oplus \ldots \oplus \hat{H}_j \oplus \ldots \oplus \hat{H}_k \oplus \ldots \oplus H_m) \longrightarrow Q_{ijk}$ induced by $H_i^{-1} \longrightarrow H_i^{-1} \oplus H_i^{-1} \oplus H_i^{-1}$ with $h \longmapsto (h,h,h)$.

All of the intersections in (a), (b) and (c) are proper and every S_{ijk} (resp. Q_{ijk}, W_{ijk}) is smooth.

In a similar way we can define $S_{i_1 \ldots i_k}$ (resp. $Q_{i_1 \ldots i_k}$, $W_{i_1 \ldots i_k}$) for $4 \leq k \leq m$ if necessary. We call k the length of $S_{i_1 \ldots i_k}$ (resp. $Q_{i_1 \ldots i_k}$, $W_{i_1 \ldots i_k}$).

(3) (a) $S_{i_1 \ldots i_k} \simeq (S)^{m-k+1}$ in an obvious way.

(b) Under the isomorphism of (a), $W_{i_1 \ldots i_k} \simeq Q_{m-k+1}$ which denotes the space constructed in (*) with $m-k+1$ replacing m.

Let $\beta : B \longrightarrow Q$ be the blowing–up of Q with respect to J_1. We are going to reconstruct β.

In the following construction we shall use some basic facts about blowing–up. Let us list them below.

(A) If $V, W \subset Q$ are two algebraic subschemes, then in $B\ell_{V\cap W}Q$ $\tilde{V}\cap\tilde{W} = \phi$, where $B\ell_{V\cap W}Q$ denotes the blowing–up of Q with respect to $V\cap W$ and $\tilde{V}$, $\tilde{W}$ denote the strict transforms of V and W respectively under this blowing–up.

(B) Besides the assumptions in (A) there is a subscheme $U \subset V\cap W$. Then in $B\ell_U Q$ $\tilde{V}\cap\tilde{W} = B\ell_U(W\cap V)$.

(C) If $V_1,\ldots,V_\ell \subset Q$ meet properly, that is, $\mathrm{codim}_Q(V_{i_1}\cap\ldots\cap V_{i_k}) = \sum_{t=1}^{k} \mathrm{codim}\, V_{i_t}$ for every $k \le \ell$, then $B\ell_{V_1\cup\ldots\cup V_\ell}Q \longrightarrow Q$ can be realized step by step. Each step is a blowing–up with respect to a strict transform of some V_i.

In particular, if $V_1,\ldots,V_\ell$ are disjoint we can get $BL_{V_1\cup\ldots\cup V_\ell}Q$ by blowing up along all V_i simultaneously.

Our reconstruction is divided into some steps.

(R_m): Blowing Q up along $W_{12\ldots m}$ we arrive in $\beta_m : B_m \longrightarrow Q$ and denote the exceptional divisor of β_m by $W'_{12\ldots m}$. B_m is smooth. Since any two of $\{W_{i_1\ldots i_{m-1}}\}$ intersect at $W_{12\ldots m}$ then by (A) their strict transforms $\{W'_{i_1\ldots i_{m-1}}\}$ are disjoint.

(R_{m-1}): Blowing B_m up along all $\{W'_{i_1\ldots i_{m-1}}\}$ simultaneously we arrive in $\beta_{m-1} : B_{m-1} \longrightarrow B_m$ by using (C). Let $\beta'_{m-1} = \beta_m\beta_{m-1}$, $W''_{i_1\ldots i_{m-1}}$ be the exceptional divisors, and $W''_{1\ldots m}$, $W''_{i_1\ldots i_k}$ with $k \le m-2$ be the strict transforms of $W'_{1\ldots m}$ and $W'_{i_1\ldots i_k}$ respectively. The situation of

$\{W''_{i_1 \ldots i_{m-2}}\}$ is different from that of $\{W'_{i_1 \ldots i_{m-1}}\}$ in (R_m).

In fact, if $W_{i_1 \ldots i_{m-2}}$ and $W_{j_1 \ldots j_{m-2}}$ meet at $W_{1 \ldots m}$ or $W_{k_1 \ldots k_{m-1}}$ they are disjoint by (A), but they may intersect elsewhere properly. The later case happens if and only if $\{i_1, \ldots, i_{m-2}\} \cap \{j_1, \ldots, j_{m-2}\} = \phi$. Taking into account the situation when we blow B_{m-1} up we should go by several steps from (C) though we write them down in a single step (R_{m-2}).

Continuing in this way, suppose we have arrived in (R_k), i.e. $\beta_k : B_k \longrightarrow B_{k+1}$. Let $\beta'_k : B_k \longrightarrow Q$ be the composition of $\{\beta_\ell\}$, $\ell = m$, $m-1, \ldots, k$. We denote the "strict transform" of $W_{i_1 i_2 \ldots i_\ell}$ under β'_k still by $W'_{i_1 \ldots i_\ell}$, where the "strict transform" means that we take the usual strict transform of $W_{i_1 \ldots i_\ell}$ successively under each β_ℓ, $\ell = m, \ldots, k$ if it is not a center of β_ℓ, and take its inverse image if it is a center of β_ℓ.

Now the relation between $W'_{i_1 \ldots i_{k-1}}$ and $W'_{j_1 \ldots j_{k-1}}$ is divided into different cases:

(***)

(i) If $k-1 < \#\{i_1, \ldots, i_{k-1}, j_1, \ldots, j_{k-1}\} < 2(k-1)$ they are disjoint. Since in this case $W_{i_1 \ldots i_{k-1}} \cap W_{j_1 \ldots j_{k-1}} = W_{s_1 \ldots s_\ell}$ for some $\ell \geq k$ and thus $W_{s_1 \ldots s_\ell}$ is a center in step (R_ℓ), from (A) the assertion follows. This is true for two variables with different length too.

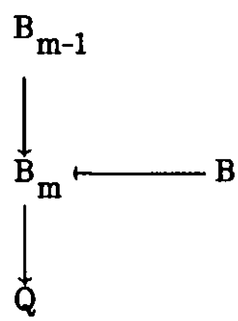
(ii) If $\#\{i_1, \ldots, i_{k-1}, j_1, \ldots, j_{k-1}\} = 2(k-1)$ they intersect properly.

(iii) If $\#\{i_1, \ldots, i_{k-1}, j_1, \ldots, j_{k-1}\} = k-1$, they coincide.

Finally we arrive at (R_2): $\beta' = \beta'_2 : B' = B_2 \longrightarrow Q$.

<u>Proposition 3.2</u>. $B \simeq B'$ over Q .

<u>Proof</u>. By the universal property of blowing–up we have a unique morphism from B' to B over Q taking $\sum [\beta^{-1}(W_{ij})] + \sum [\beta^{-1}(W^{a_3}_{ijk})] + \dots [\beta^{-1}(W^{a_m}_{1\dots m})]$ to $\sum [\beta'^{-1}(W_{ij})] + \dots + [\beta'^{-1}(W^{a_m}_{1\dots m})]$. We need to show there is a morphism from B to B' over Q which is the inverse of the above morphism. Indeed, since $\beta^{-1}(W_{1\dots m})$ is a divisor then we have a unique morphism from B to B_m over Q . In the following diagram



we see that each $W'_{i_1 \dots i_{m-1}}$ has a divisor as inverse image in B , then using Lemma 3.1 again there exists a unique morphism from B to B_{m-1} over B_m and hence over Q . Inductively we have got a unique morphism from B to B' over Q and which meets our requirement.

§ 4.

In this section we shall compute the Segre class $s(J_1,Q)$ and prove the main theorem. In the following computation we shall constantly use some new facts about blowing up.

(D) Let $V, W \subset Q$ be three smooth varieties and $V \cap W$ be smooth too. Let $\pi : B \longrightarrow Q$ be the blowing–up of Q with respect to W, then

(i) If $V \cap W \subset V$ is a proper subvariety of V, then $\pi^* N_V Q \simeq N_{V'} B$, where V' is the strict transform of V under π.

(ii) If $W \subset V$, $N_{V'} B \simeq (\pi^* N_V Q) \otimes \mathcal{O}(-1)|_{V'}$.

In B constructed in § 3, let $\hat{W}_{i_1 \ldots i_k}$ be the strict transform of $W_{i_1 \ldots i_k}$ in a sense we explained in § 3. By the definition of $s(J_1, Q)$ it is

$$\beta_* \sum_{k=1} (-1)^{k-1} (\sum [\beta^{-1} W_{ij})] + \ldots + [\beta^{-1} W_{1 \ldots m}^{a_m}] =$$

$$= \sum_{k=1} (-1)^{k-1} \beta_* (\sum_{i<j} \hat{W}_{ij} + b_3 \sum_{i<j<k} \hat{W}_{ijk} + .. + b_\ell \sum_{i_1 < \ldots < i_\ell} \hat{W}_{i_1 .. i_\ell} + .. b_m \hat{W}_{12..m})^k ,$$

where $b_\ell = a_\ell + \frac{\ell(\ell-1)}{2}$.

Proposition 4.1. Let M be a monomial of variables $\hat{W}_{i_1 \ldots i_k}$ with $2 \leq k \leq m$, then $\beta_* M$ is a cycle in which each term can be written as some Chern classes of the normal bundles of $W_{i_1 \ldots i_k}$ in Q or of $W_{j_1 \ldots j_t}$ in $W_{j_1 \ldots j_s}$ with $s < t$ acting on some $W_{h_1 \ldots h_r}$ or $W_{h_1 \ldots h_r} \cap \ldots \cap W_{k_1 \ldots k_\ell}$ with disjoint subscripts.

Proof. We shall prove this inductively.

Assume $m = 2$, then M is simply the form $\hat{W}_{12}^i$ if $i \leq 2$ then $\beta_* \hat{W}^i = 0$; if $i \geq 3$, $\beta_* \hat{W}_{12}{}^{i+3} = (-1)^i \frac{1}{c(N_{W_{12}} Q)_i} \cap [W_{12}]$ so the assertion is true in this case.

Now suppose the assertion is true for the cases $\leq m-1$.

Given a monomial M on B, we arrange the variables in M by their length. If the first non–trivial variables in M is $\hat{W}^s_{i_1 \dots i_\ell} \cdot \dots \cdot \hat{W}_{j_1 \dots j_\ell}$ then if $\{i_1, \dots, i_\ell\}, \dots, \{j_1, \dots, j_\ell\}$, are not disjoint this intersection will be zero by (***). This fact is also true for the intersection of any two variables with the same length. Therefore we may assume that any two variables appearing in M with same length have disjoint indices; for two variables with different length, for example $\hat{W}_{i_1 \dots i_k}$, $\hat{W}_{j_1 \dots j_r}$ with $r > k$, if $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_r\}$ are not disjoint and $\{i_1 \dots i_k\} \subset \{j_1 \dots j_r\}$ then the intersection of them must be zero by (***). Therefore, without loss of generality we write down M as

$$\hat{W}^{i_\ell}_{12\dots\ell} \cdots \hat{W}^{j_\ell}_{s+1\dots s+\ell} \cdot \hat{W}^{i_t}_{12\dots\ell\dots t} \cdots \hat{W}^{j_t}_{s+1\dots s+\ell\dots s+t} \cdots \cdot \hat{W}^{i_m}_{12\dots m} \, .$$

Since $W_{12\dots\ell}$ has been blown up in step (R_ℓ) $(\beta_{\ell+1})_* \dots (\beta_2)_* M$ does not change its shape on B_ℓ and in abuse of notations, we use the same expression as in B.

Because $\{1,2,\dots,\ell\}, \dots, \{s+1,\dots,s+\ell\}$ are disjoint, $W'_{1\dots\ell}, \dots, W'_{s+1\dots s+\ell}$ meet properly on $B_{\ell-1}$, where W' denotes the strict transform of W in $B_{\ell-1}$. Therefore by (C) in § 3 β_ℓ can be realized by successive blowing–ups, each time taking a $W'_{1\dots\ell}$ as center. On the other hand $\hat{W}_{1\dots\ell\dots t} = \beta^*_\ell W'_{1\dots\ell\dots t}$ for every variable with a longer length. So

$$\beta_{\ell*} M = \epsilon \, s_{h_1}(N_{W'_{12\dots\ell}} B_{\ell-1}) \cap [W'_{1\dots\ell}] \dots s_{h_k}(N_{W'_{s+1\dots s+\ell}} B_{\ell-1}) \cap [W'_{s+1\dots s+\ell}]$$

$$\cdot \, {W'}^{i_t}_{12\dots\ell\dots t} \cdots {W'}^{j_t}_{s+1\dots s+t} \cdots {W'}^{i_m}_{12\dots m} \, ,$$

where $\epsilon = (-1)^{i_\ell - 1 + \ldots + j_\ell - 1}$, $h_1 = i_\ell - 3(\ell-1), \ldots, h_k = j_\ell - 3(\ell-1)$. (Note, since codim $W_{1\ldots\ell} = 3(\ell-1)$, for every $1 < i_\ell < 3(\ell-1)$ $\beta_{\ell *} M = 0$. We always exclude this trivial case).

In the expression, $[W'_{1\ldots\ell}]\cdots[W'_{s+1\ldots s+\ell}] = [W'_{1\ldots\ell} \cap \ldots \cap W'_{s+1\ldots s+\ell}]$ since they meet properly. Using the isomorphism of (3) (b) in (**), we have $W_{1\ldots\ell} \cap \ldots \cap W_{s+1\ldots s+\ell} \simeq Q_{m-k(\ell-1)}$ where k is the number of $W_{1\ldots\ell}, \ldots, W_{s+1\ldots s+\ell}$ appearing in M and $W'_{1\ldots\ell} \cap \ldots \cap W'_{s+1\ldots s+\ell}$ corresponds the blowing–up of $Q_{m-k(\ell-1)}$ with respect to its own J_1 (Intuitively what we are doing is simply replacing $1, \ldots, \ell$–th factor of $(S)^m$ (resp. $\oplus H_i^{-1}$) with their diagonal. Thus we return to the original situation but replacing m with $m-k(\ell-1)$). At the same time $W'_{1\ldots\ell\ldots t}$ is identified with $\hat{W}_{1\ldots t-\ell+1}$ and so on.

On the other hand from (D) in this section we have

$$s_h(N_{W'_{1\ldots\ell}} B_{\ell-1}) = \sum_{i=0}^{h} (-1)^{h-i} \begin{bmatrix} e+h \\ e+i \end{bmatrix} s_i(\beta'^{*}_{\ell-1} N_{W_{1\ldots\ell}} Q)(-\sum_j W'_{1\ldots\ell j} - \sum_{js} W'_{1\ldots\ell js\ldots})$$

where the last factor on the right side is the exceptional divisor of the blowing–up of $Q_{m-k(\ell-1)}$ with respect to its J_1 , and $e+1$ is the rank of N i.e., $e = 3(\ell-1)-1$.

Therefore except for $M = \hat{W}^{\ell}_{12\ldots m}$ we use the inductive hypothesis to deduce our conclusion. And $\beta_* \hat{W}^{\ell+3(m-1)}_{1\,.\,.\,.\,m} = \epsilon\, s_\ell(N_{W_{1\ldots m}} Q) \cap [W_{1\ldots m}]$, $\epsilon = (-1)^{\ell+3m}$.

<u>Theorem 4.2</u>. $\nu(S)$ can be expressed by a polynomial of the Chern number of S , the de-

gree of S in $\mathbb{P}^{3m-2}$ and the intersection number of the canonical class of S with the restriction of the hyperplane class; the coefficients and the degree of the polynomial depend only on m .

Proof. We have proved in § 2 that

$$m!\nu(S) = \deg(\alpha g)_*([Q]^2 - g^*((1+c_1(\mathcal{O}_Q(1)))^{n+1} \cap s(J_1,Q))_0) .$$

Now

$$\begin{aligned}(\alpha g)_*[Q]^2 &= \alpha_* c_{n+1}(\mathcal{O}(1))^{n+1} \cap [Q] \\ &= \left[\frac{1}{(1-h_1)\dots(1-h_m)}\right]_{2m} \cap [(S)^m]\end{aligned}$$

where $h_i = c_1(H_i)$. Hence

$$\deg(\alpha g)_*[Q]^2 = \deg(h_1^2 \dots h_m^2) = d^m .$$

From Proposition 4.1 we see that $s(J_1,Q)$ is a combination of some Chern classes of certain normal bundles acting on $[W_{1\dots\ell}]$ for some ℓ or $W_{1\dots\ell} \cap \dots \cap W_{k\dots k+r}$ with disjoint subscripts. In fact in the proof of Proposition 4.1 we have shown that the Chern classes which act on $[W_{1\dots\ell}]$ are $s(N_{W_{i_1\dots i_r}} W_{i_1\dots i_s})$ restricted to $W_{1\dots\ell}$, where $s < r$ and $\{i_1,\dots,i_r\} \subset \{1,2,\dots,\ell\}$. But $[W_{12\dots s}] = (c_1(\mathcal{O}(1))-h_1)^{s-1} \cap [Q_{12\dots s}]$ (as a subscheme it is a complete intersection in $Q_{12\dots s}$) , and $c(N_{Q_{12\dots s}} Q) \simeq c(\alpha^* \Omega_S^{*\oplus s-1})$. Hence $c(N_{W_{1\dots r}} W_{1\dots s}) = (1+c_1(\mathcal{O}(1))-h_1)^{r-s} c(\alpha^* \Omega_s^*)^{r-s}$, and

$$(\alpha g)_*\left[q^*(1+c_1(\mathcal{O}(1))^{3m-1}\cap\left[\frac{1}{c(\alpha^* N_{W_{1\dots t}}Q)}\right]_{i_1}\cdots\left[\frac{1}{c(\alpha^* N_{W_{1\dots r}}W_{1\dots s})}\right]_{i_\ell}\cap[W_{1\dots\ell}]\right]_0$$

$$=(\alpha)_*\left[(1+c_1(\mathcal{O}(1))\right]^{3m-1}\left[\frac{1}{(1+c_1(\mathcal{O}(1))-h_1)c(\alpha^*\Omega_S^*)}\right]_{i_1}\cdots$$

$$\cdots\left[\frac{1}{(1+c_1(\mathcal{O}(1))-h_1)^r c(\alpha^*\Omega^*)^r}\right]_{i_k}(c_1(\mathcal{O}(1))-h_1)^{\ell-1}\cap[Q_{1\dots\ell}])_0$$

where i_k , r we write them at random since this has nothing to do with our proof.

Developing the expression and taking the 0–part we find the general term of it (neglecting coefficients for the time being) is

$$\alpha_*\left[c_1(\mathcal{O}(1))^{(m-1)+r}L(h_1,h_1^2,K,K^2,h_1K,c_2(S))\cap[Q_{1\dots\ell}]\right]_0$$

where L is a linear combination with integer coefficients. For the constant term we have

$$\alpha_*\left[c_1(\mathcal{O}(1))^{(m-1)+2(m-\ell+1)}\cap[Q_{1\dots\ell}]\right]=$$

$$=\left[\frac{1}{(1-h_1)^\ell(1-h_{\ell+1})\dots(1-h_m)}\right]_{2(m-\ell+1)}\cap(S)^{m-\ell+1}=(\ell+1)h_1^2h_{\ell+1}^2\dots h_m^2$$

and thus the degree is $(\ell+1)d^{m-\ell+1}$.

For the term ah_1+bK we have

$$\alpha_*(c_1(\mathcal{O}(1))^{(m-1)+2(m-\ell)+1}(ah_1+bK)\cap[Q_{1\dots\ell}])$$

$$= \left[\frac{1}{(1-h_1)^{\ell}\ldots(1-h_m)}\right]_{2(m-\ell)+1}(ah_1+bK) \cap (S)^{m-\ell+1}$$

$$= \ell(ah_1^2+bh_1K)h_{\ell+1}^2\ldots h_m^2$$

and thus the degree is $a\ell d^{m-\ell+1} + b\ell(h_1K)d^{m-\ell}$.

Finally for the term of linear combination $ah_1^2 + bK^2 + e\cdot c_2(S)$. We have in the same way

$$(ah_1^2 + bK^2 + e\, c_2(S))h_{\ell+1}^2\ldots h_m^2$$

and the degree is $(ah_1^2 + bK^2 + e\, c_2(S))d^{m-\ell}$.

As for the coefficients in the expression for $m!\nu(s)$ they come from the coefficients in the self–intersection of the exceptional divisor on B and from the coefficients in some Chern class formula. All of them only depend on m .

The computation for other possible terms is similar, so the theorem follows.

Remark 4.3. We can write this formula with a little bit more precisely,

$$m!\nu(S) = d^m + F_1 d^{m-1} + F_2 d^{m-2} + \ldots + F_m$$

where F_k is a polynomial in variables $hK, K^2, c_2(S)$ of degree at most $[\frac{k}{2}]$.

Example 1. The case m=2 .

Then we have $\nu(S) = 0$, but the computation (like we did in the proof of Theorem) gives

$$2\nu(S) = d^2 - 10d - 5hK + c_2(S) - K^2 \ .$$

Therefore $\nu(S) = 0$ is simply the well–known condition for a smooth surface embedded in $\mathbb{P}^4$.

Example 2. The case m=3 .

The computation for this simple case is a little complicated:

$$6\nu(S) = d^3 - 138d^2 - d(165(hK) + 105(K^2 - c_2) + 56392)$$

$$- 138104(hK) - 105723K^2 + 116159c_2 \ .$$

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