Gabriel-Roiter measure for $\widetilde{\mathbb{A}}_n$

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Abstract. Let $\Lambda = kQ$ be a tame hereditary algebra of type $\widetilde{\mathbb{A}}_n$ and δ be the minimal radical vector. We wish to investigate the Gabriel-Roiter measures of the indecomposable modules with dimension vector δ which turns out to be very important. Also some relevant examples are indicated.

Keywords. Tame hereditary algebras, string algebras, Gabriel-Roiter measure, defect function.

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1 Introduction

Throughout the paper, we assume that k is an algebraically closed field. For a finite dimensional k-algebra Λ, we denote by $\operatorname{mod}\Lambda$ the category of finite dimensional left Λ-modules. Let $\operatorname{ind}\Lambda$ be the full subcategory of $\operatorname{mod}\Lambda$ consisting of indecomposable Λ-modules and $\operatorname{ind}\mathcal{X} = \operatorname{ind}\Lambda \cap \mathcal{X}$ for a full subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$. We denote by $|M|$ the length of a Λ-module M. The symbol $\subset$ is used to denote proper inclusion.

By using Gabriel-Roiter measure ([10],[11]), Ringel obtained a partition of the module category for a representation infinite algebra, that is, the module category consists of take-off part, central part and landing part. Moreover, all modules landing modules are preinjective modules in the sense of Auslander and Smalø [2].

The module category of a representation infinite hereditary algebra contains preprojective modules, regular modules and preinjective modules. In this note, we shall consider tame hereditary algebras of type $\widetilde{\mathbb{A}}_n$ and show how the modules are rearranged according to Gabriel-Roiter measure. We show that all preprojective modules lies in the take-off part(Theorem 3.3). It follows that a Gabriel-Roiter submodule of a homogeneous regular module, which is not regular simple, is always given by a irreducible monomorphism (Corollary 3.4). However, we will see a stronger result which says that for a Gabriel-Roiter inclusion of homogeneous regular modules $H \subset H'$, the measure of H' is a direct successor of the measure of H, i.e. there does not exist indecomposable module whose measure lies in between (Theorem 3.9).

Let δ be the minimal radical vector for $\Lambda = k\widetilde{\mathbb{A}}_n$. We will see that the measures of indecomposable modules with dimension vector δ play an important role when comparing the measures of regular modules (Lemma 3.6). Some arguments for comparing such measures are presented (Proposition 3.11,3.13,3.14).

We first recall some definitions and properties of Gabriel-Roiter measure. The main discussions will be in section 3. Various revelent examples are indicated in section 4.

2 The Gabriel-Roiter measure

In this section, we assume that Λ is a fixed artin algebra. Let $\mathbb{N}_1=\{1,2,\cdots\}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N}_1)$ be the set of all subsets of $\mathbb{N}_1$. A total order on $\mathcal{P}(\mathbb{N}_1)$ can be defined as follows: If I,J are two different subsets of $\mathbb{N}_1$, write $I<J$ provided the smallest element in $(I\backslash J)\cup(J\backslash I)$ belongs to J. Also we write $I\ll J$ provided $I\subset J$ and for all elements $a\in I$, $b\in J\backslash I$, we have $a<b$. We say that J **starts with** I provided $I=J$ or $I\ll J$. The following statements can be easily checked:
(1) If $I\subseteq J\subseteq\mathbb{N}_1$, then $I\leq J$.
(2) If $I_1\leq I_2\leq I_3$, and I_3 starts with I_1, then I_2 starts with I_1.

For each Λ-module M, we denote by $|M|$ the length of M. Let $\mu(M)$ be the maximum of the sets $\{|M_1|,|M_2|,\cdots,|M_t|\}$ where $M_1\subset M_2\subset\cdots\subset M_t$ is a chain of indecomposable submodules of M. We call $\mu(M)$ the **Gabriel-Roiter measure** (briefly **GR measure**) of M. If M is an indecomposable Λ-module, we call an inclusion $T\subset M$ with T indecomposable a **Gabriel-Roiter inclusion** (briefly **GR inclusion**) provided $\mu(M)=\mu(T)\cup\{|M|\}$, thus if and only if every proper submodule of M has Gabriel-Roiter measure at most $\mu(T)$. In this case, we call T a **Gabriel-Roiter submodule** (briefly, **GR submodule**) of M.

We obtain the following conclusion from the above concepts, which is useful in what will follow:

Lemma 2.1. *Let X, Y and Z be indecomposable Λ-modules.*
(1) *If X is a proper submodule of Y, then $\mu(X)<\mu(Y)$.*
(2) *If $\mu(X)<\mu(Y)<\mu(Z)$ and X is a GR submodule of Z, then $|Y|>|Z|$.*

The following Main Property of Gabriel-Roiter measure is essentially due to Gabriel ([7]), and proved by Ringel ([10]) for arbitrary modules.

Main Property. *Let X, $Y_1,\cdots,Y_t$ be indecomposable modules and assume that there is a monomorphism $f:X\longrightarrow\oplus_{i=1}^{t}Y_i$. Then*
(1) $\mu(X)\leq\max\{\mu(Y_i)\}$.
(2) *If $\mu(X)=\max\{\mu(Y_i)\}$, then f splits.*
(3) *If $\max\{\mu(Y_i)\}$ starts with $\mu(X)$, then there is some j such that $\pi_j f$ is injective, where $\pi_j:\oplus_{i=1}^{t}Y_i\longrightarrow Y_j$ is the canonical projection.*

In the following proposition, we collect some basic properties of GR inclusions which will be needed in the sequel. We refer to [12] and [4] for the proof.

Proposition 2.2. *Let $\epsilon:0\longrightarrow T\stackrel{l}{\longrightarrow}M\stackrel{\pi}{\longrightarrow}M/T\longrightarrow 0$ be a short exact sequence with l a GR inclusion. Then the following statements hold:*
(1) *T is a direct summand of all proper submodules of M containing T.*
(2) *If all irreducible maps to M are monomorphisms, then l is an irreducible map.*
(3) *M/T is indecomposable.*
(4) *Any map to M/T which is not an epimorphism factors through π.*
(5) *All irreducible maps to M/T are epimorphisms.*
(6) *M/T is a factor module of $\tau^{-1}T$ and $M/T\cong\tau^{-1}T$ if and only if ϵ is an Auslander-Reiten sequence.*

The following proposition will be used in our discussion. For a proof we refer to [5].

Proposition 2.3. *Assume that T is a GR submodule of M. Then there is an irreducible monomorphism $T \to X$ with X indecomposable and an epimorphism $X \to M$.*

Now we recall the partition obtained by using the Gabriel-Roiter measure approach. As in [10],[11], we say $I \in \mathcal{P}(\mathbb{N}_1)$ is a Gabriel-Roiter measure for Λ if there exists an indecomposable Λ-module M with $\mu(M) = I$. A measure I is said to be of **finite type** if there are only finitely many isomorphism classes of indecomposable modules with measure I. Let I and J be two measures for Λ, we say J is a **direct successor** of I if there is no measure J' with $I < J' < J$.

Theorem 2.4 ([10]). *Let Λ be a representation infinite artin algebra. Then there are Gabriel-Roiter measures I_t, I^t for Λ such that*

$$I_1 < I_2 < I_3 < \quad \cdots \quad < I^3 < I^2 < I^1$$

and such that any other measure J satisfies $I_t < J < I^t$ for all t. Moreover, all these measures I_t and I^t are of finite type.

The measures $I_t (I^t)$ are called **take-off (landing)** measures and any other measure is called a **central** measure. Indecomposable modules with GR measure I are called take-off (resp. central, landing) modules if I is a take-off (resp. central, landing) measure. It is easy to see that if J is the direct successor of I, then I is a take-off (resp. central, landing) measure if and only if J is a take-off (resp. central, landing) measure.

In [11], Ringel showed the following proposition:

Proposition 2.5. *Let Λ be a representation infinite artin algebra. Then any landing module is preinjective (in the sense of Auslander and Smalø [2]).*

The following is the **Successor Lemma** in [11].

Proposition 2.6. *Any Gabriel-Roiter measure I different from I^1 has a direct successor I'.*

There is no 'Predecessor Lemma', i.e. a GR measure different from I_1 may not have direct predecessor. In next section, an existence property of the minimal central measure will be proved. This minimal central measure does not have direct predecessor.

3 Gabriel-Roiter measure for $\widetilde{\mathbb{A}}_n$

From now on, we assume that Λ is a tame hereditary algebra of type $\widetilde{\mathbb{A}}_{n,n\geq 1}$ over an algebraically closed field k. We first recall some preliminaries. We refer to [1],[9] for details and unstated notions.

There is a decomposition of the Auslander-Reiten quiver Γ_Λ, into the preprojective part $\mathcal{P}$, the preinjective part $\mathcal{I}$ and the regular one $\mathcal{R}$, where $\mathcal{R}$ is a sum of stable tubes $\mathbb{T}_\lambda$ of ranks $r_\lambda \geq 1$, for $\lambda \in \mathbb{P}^1(k) = k \cup \{\infty\}$. A tube of rank 1 is called **homogeneous** and the ones of rank greater than 1 are called **exceptional**. Note that $\mathbb{T}_\lambda$ is exceptional for at most two $\lambda \in \mathbb{P}^1(k)$. A sequence of irreducible maps $X_1 \to X_2 \to \cdots \to X_m \to \cdots$ in Γ_Λ is called a **sectional path** if $X_i \not\cong \tau(X_{i+2})$ for each i. If X is a regular simple modules lying on a tube of rank r, there is a unique sectional path $X = X[1] \to X[2] \to \cdots \to X[r] \to \cdots$ of irreducible monomorphisms. We also call $r(X) = r$ the rank of X for an exceptional regular simple module X on a tube of rank r.

For indecomposable Λ-modules X, Y, if $\mathrm{Hom}\,(X, Y) \neq 0$ and X and Y do not belong to the same connected component of Γ_Λ, then X is preprojective or Y is preinjective.

Let $\delta = (\delta_i)_i$ be the minimal radical vector, i.e. $\delta_i = 1$ for each i. If X is a regular simple module of rank r, then $\underline{\dim}\, X[r] = \delta$. If X is of rank 1, i.e. X is a homogeneous regular simple module, we write H_i instead of $X[i]$. δ is the unique minimal vector with $\langle \delta, \delta \rangle = 0$ where $\langle -, - \rangle$ is the homological quadratic form with the property

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y \rangle = \mathrm{dimHom}\,(X, Y) - \mathrm{dimExt}^{\,1}(X, Y)$$

for any $X, Y \in \mathrm{mod}\,\Lambda$. The **defect** of a Λ-module X is defined to be $\langle \delta, \underline{\dim}\, X \rangle = -\langle \underline{\dim}\, X, \delta \rangle$. We thus get a **defect function** which is also denoted by $\delta : \delta(X) = \langle \delta, \underline{\dim}\, X \rangle$. It is well-known that an indecomposable Λ-module X is preprojective (resp. regular, preinjective) if and only if $\delta(X)$ is negative (resp. zero, positive).

Tame hereditary algebras of type $\widetilde{\mathbb{A}}$ are string algebras. For details of definitions and classification of the modules we refer to [3]. The module category of a string algebra consists of string modules and band modules. The band modules are actually homogeneous modules. For each string p, we denote by M_p the corresponding (indecomposable) string module.

Let α be an arrow and $p^{-1}\alpha q^{-1}$ be a string with p, q compositions of arrows as long as possible. Then there is an almost split sequence

$$0 \to M_{p^{-1}} \to M_{p^{-1}\alpha q^{-1}} \to M_{q^{-1}} \to 0.$$

Both the starting term and the ending term in such almost split sequences are uniserial. Any almost split sequence containing string modules with indecomposable middle term is of this kind. Therefore, for $\Lambda = k\widetilde{\mathbb{A}}_n$, the almost split sequences staring (ending) with exceptional regular simple modules are namely determined by arrows. But note that there may exist homogeneous string modules.

An indecomposable preprojective module has no proper preprojective factor module since its defect is -1. In fact, given an indecomposable preprojective module X and an epimorphism $X \xrightarrow{f} Y$ with Y preprojective, then $\delta(\ker f) = \delta(X) - \delta(Y) \geq 0$ since $\delta(X) = -1$ and $\delta(Y) < 0$. Therefore $\ker f$ has to be zero and thus $X \cong Y$. Dually, any indecomposable preinjective module has no proper preinjective submodule.

The following easy applications of the defect function will be quite often used in what will follow.

Lemma 3.1. (1) *All non-zero maps between indecomposable preprojective modules are monomorphisms.*
(2) *All irreducible maps between indecomposable preprojective modules are monomorphisms.*
(3) *Any GR inclusion between preprojective module is an irreducible map.*
(4) *Any nonzero map from an indecomposable preprojective module to a regular simple module is either injective or surjective.*
(5) *Any nonzero map from a regular simple module to a preinjective indecomposable module is either injective or surjective.*
(6) *Any GR submodule of an indecomposable preinjective module is a regular module.*

From now on, we fix an algebra $\Lambda = k\widetilde{\mathbb{A}}_n$. We denote by $\mathcal{T}$, $\mathcal{C}$, $\mathcal{L}$ the full subcategory of take-off modules, central modules and landing modules, respectively. Under our convention, they are all collections of indecomposable Λ-modules. Recall that $\mathrm{ind}\,\mathcal{P}$ (resp. $\mathrm{ind}\,\mathcal{R}$, $\mathrm{ind}\,\mathcal{I}$) are used to denote the full subcategory of indecomposable preprojective (resp. regular, preinjective) modules.

Proposition 3.2. *Let H_1 be a homogeneous regular simple module and X be a GR submodule of H_1. Then H_1/X is an injective simple module.*

Proof. Assume H_1/X is not simple. We take a GR submodule Y of H_1/X. It follows from Lemma 3.1 that Y is a regular module. Then the canonical inclusion $Y \rightarrow H_1/X$ factor through H_1 (Proposition 2.2). In particular, there is a nonzero homomorphism from Y to H_1. Thus Y is a preprojective module. This contradiction shows H_1/X is simple, thus an injective module by the dual of Lemma 3.1(2). □

Theorem 3.3. *Every indecomposable preprojective module lies in the take-off part, i.e.* $\operatorname{ind}\mathcal{P} \subset \mathcal{T}$.

Proof. It is sufficient to show that for any $X \in \operatorname{ind}\mathcal{P}$, there are only finitely many isomorphism classes of indecomposable modules which are of measures smaller than $\mu(X)$.

Since all irreducible maps in preprojective component are monomorphism, we thus have that only finitely many indecomposable preprojective modules with measures smaller than $\mu(X)$.

Let H_1 be a homogeneous regular simple module. We show that any indecomposable preprojective module has measure smaller than $\mu(H_1)$. If not, we may take $Y \in \mathcal{P}$ with $\mu(Y) > \mu(H_1)$ such that $|Y|$ is minimal. Assume that Y' is a GR submodule of Y. Since no indecomposable preprojective module has length $\sum_i \delta_i = |\delta| = |H_1|$, $\mu(Y') < \mu(H_1) < \mu(Y)$ by the minimality of Y. Therefore, $|H_1| > |Y|$ by Lemma 2.1. Thus Lemma 3.1(4) implies that $\mu(Y) < \mu(H_1)$. This is a contradiction and therefore, $\mu(M) < \mu(H_1)$ for all $M \in \operatorname{ind}\mathcal{P}$.

Now assume M is an indecomposable preinjective module. If M is sincere, then any nonzero map from H_1 to M is injective. Thus $\mu(H_1) < \mu(M)$. Note that there are only finitely many non-sincere indecomposable preinjective module, therefore, only finitely any indecomposable preinjective have measures smaller than $\mu(X)$ for the given $X \in \operatorname{ind}\mathcal{P}$.

To finish the discussion, we have to show that only finitely many exceptional regular modules are of measures smaller than $\mu(X)$. Assume that $Y[1] \rightarrow Y[2] \rightarrow Y[3] \rightarrow \cdots \rightarrow Y[t] \rightarrow \cdots$ is a sectional path in an exceptional tube. It is known that for $t \gg 1$, we may get indecomposable preprojective module X_t which is a proper submodule of Y_t such that $\lim_{t\rightarrow\infty} |X_t| = \infty$. Therefore, for t large enough, we may have $\mu(X) < \mu(X_t) < \mu(Y_t)$. Since there are at most 2 exceptional tubes, only finitely many indecomposable exceptional regular modules have measures smaller than $\mu(X)$ for the given $X \in \operatorname{ind}\mathcal{P}$. □

Corollary 3.4. *Let $H_1 \rightarrow H_2 \rightarrow H_3 \rightarrow \cdots \rightarrow$ be a path of irreducible monomorphisms with H_1 a homogeneous simple module. Then for each $i \geq 2$, H_i contains, up to isomorphism, H_{i-1} as the unique GR submodule. Therefore,*

$$\mu(H_i) = \mu(H_1) \cup \{2|\delta|, 3|\delta|, \cdots, i|\delta|\} = I \cup \{|\delta| - 1, |\delta|, 2|\delta|, \cdots, i|\delta|\}$$

where I is a take-off measure.

Proof. This is a direct consequence of the above theorem and Proposition 3.2 □

We now begin to show several interesting lemmas which can be used to compare GR measures of indecomposable modules.

Lemma 3.5. *Let $X \in \operatorname{ind}\mathcal{I} \setminus \mathcal{T}$ and Y be a GR submodule of X with $Y = Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_n \rightarrow \cdots$ a sequence of irreducible monomorphisms. Then $\mu(X) > \mu(Y_i)$ for all i.*

Proof. It is clear that Y is a regular module and therefore, all Y_i are regular modules. Since $Y_1 = Y$ is a GR submodule of X, there is an epimorphism from Y_2 to X (Proposition 2.3). It follows that

$|Y_i| > |X|$ for all $i \geq 2$. The GR submodule T of Y_2 is either in $\operatorname{ind}\mathcal{P}$ or isomorphic to Y_1. It follows that $\mu(T) < \mu(X)$ by Theorem 3.3 or the assumption. However, $\mu(T) < \mu(X) < \mu(Y_2)$ implies $|X| > |Y_2|$ which is a contradiction. We thus have $\mu(X) > \mu(Y_2)$. Continuing the induction steps, we get $\mu(X) > \mu(Y_i)$ for all i. □

Lemma 3.6. *Let $X = X[1]$ be an exceptional regular simple module of rank r. Then*
(1) If $\mu(X[r]) \geq \mu(H_1)$, then $\mu(X[j]) > \mu(H_i)$ for all $j > r, i \geq 1$.
(2) If $\mu(X[r]) < \mu(H_1)$, then $\mu(X[j]) < \mu(H_1)$ for all $j \geq 1$.

Proof. We first note that $\underline{\dim}\, X[r] = \delta$ and $|X[r+1]| < |X[r]| + |\delta| = 2|\delta|$. If $\mu(X[r]) \geq \mu(H_1)$, then

$$\mu(X[r+1]) \geq \mu(X[r]) \cup \{|X[r+1]|\} > \mu(X[r]) \cup \{2|\delta|, \cdots, i|\delta|\} \geq \mu(H_1) \cup \{2|\delta|, \cdots, i|\delta|\} = \mu(H_i).$$

Therefore, $\mu(X[j]) > \mu(X[r+1]) > \mu(H_i)$ for all $j > r$ and $i \geq 1$.

Now we prove (2) by showing $\mu(X_{r+1}) < \mu(H_1)$. Then using the same argument, we may show that $\mu(X_j) < \mu(H_1)$ for all j. Assume for a contradiction that $\mu(X[r+1]) > \mu(H_1)$. Let Y be a GR submodule of $X[r+1]$. Then Y is either a preprojective module or isomorphic to X_r. In any case, we have inequalities $\mu(Y) < \mu(H_1) < \mu(X[r+1])$ by Theorem 3.3 or assumption. Thus $|H_1| > |X_{r+1}|$ by Lemma 2.1. This contradiction shows that $\mu(X[r+1]) < \mu(H_1)$. □

Lemma 3.7 ([5]). *If Y is a GR submodule of X, then $\underline{\dim}\, X/Y \leq \delta$.*

Lemma 3.8. *Let H_1 be a homogeneous regular simple module. Assume that $X \in \operatorname{ind}\mathcal{I}$ with $\mu(X) > \mu(H_1)$. Then $\mu(X) > \mu(H_i)$ for all i.*

Proof. We assume for a contradiction that $\mu(X) < \mu(H_j)$ for some j. Since there is no indecomposable preinjective module with length $s|\delta|$ for all natural number s, we get an index i such that $\mu(H_i) < \mu(X) < \mu(H_{i+1})$. It follows that $|X| > |H_{i+1}|$ Assume Y is a GR submodule of X. Then by Lemma 3.5, Y can be assumed to be an exceptional regular module. By Lemma 3.7, $|Y| > |X| - |\delta| > |H_{i+1}| - |\delta| = |H_i|$, i.e. $Y = T[j]$ for some exceptional regular module T of rank r and some $j > r$. If $\mu(Y) < \mu(H_i)$, then $|H_i| > |X|$ since Y is a GR submodule of X. But this can not happen since $|X| > |H_{i+1}|$. Thus $\mu(Y) > \mu(H_i)$ and Lemma 3.6 implies $\mu(T[r]) \geq \mu(H_1)$. It follows that $\mu(X) > \mu(Y) = \mu(T[j]) > \mu(H_s)$ for all $s \geq 0$. This is a contradiction. Therefore, we have $\mu(X) > \mu(H_i)$ for all i. □

The following theorem is a direct consequence of the above lemmas.

Theorem 3.9. *Let H_1 be a homogeneous regular simple module. Then $\mu(H_i)$ is a direct successor of $\mu(H_{i-1})$ for all $i \geq 2$.*

Proof. Let X be an indecomposable module with $\mu(H_i) < \mu(X) < \mu(H_{i+1})$ for some $i \geq 1$. In particular, $\mu(H_1) \leq \mu(X)$. Then by Theorem 3.3 and Lemma 3.8, X is an exceptional regular module. Now, the theorem follows from Lemma 3.6. □

Remark. We will see in a successive publication ([6]) that Theorem 3.3 and Theorem 3.9 also hold for tame hereditary algebras of type $\widetilde{\mathbb{D}}_n$ and $\widetilde{\mathbb{E}}_{6,7,8}$. But the proofs are more complicate although the ideals are almost the same.

Proposition 3.10. *There exists a minimal central measure, i.e. a measure I such that J is a take-off measure whenever $J < I$. The minimal central measure I has no direct predecessor.*

Proof. We first claim that a non-sincere indecomposable is either a take-off module or with measure $> \mu(H_1)$ where H_1 is a homogeneous regular simple module. Namely, if X is a GR submodule of H_1 and M is a non-sincere indecomposable module with $\mu(X) < \mu(M) < \mu(H_1)$, then $|M| > |H_1|$. This is a contradiction. Therefore, either $\mu(M) \leq \mu(X)$ and M is thus a take-off module, or $\mu(M) \geq \mu(H_1)$. It follows that a preinjective module can not possess minimal GR measure since a sincere indecomposable preinjective module has measure $> \mu(H_1)$. For each exceptional regular simple X, let $i_X \geq 0$ be the minimal index such that $X[i_X]$ is a central module. It is clear that $\{X[i_X]|X$ is an exceptional regular simple module$\}$ is a finite set. Then $I = \min_X\{\mu(X[i_X]), \mu(H_1)\}$ is the minimal central measure.

The minimal central measure has no direct predecessor since a measure and its direct predecessor (successor) are of the same type. □

We have seen in Lemma 3.6 that the measures of indecomposable modules with dimension vector δ play an important role when comparing the measures of regular modules. Now many questions can be raised. For instance: Does there exist exceptional regular simple X (say with rank r) such that $\mu(X[r]) < \mu(H_1)$ [or $\geq \mu(H_1)$]? If $\mu(X[r]) \geq \mu(H_1)$, is $\mu(X[j+1])$ a direct successor of $\mu(X[j])$ for $j \geq r$ and, is $X[1] \rightarrow X[2] \rightarrow \cdots \rightarrow X[r] \rightarrow X[r+1] \rightarrow \cdots$ always a chain of GR inclusions? In what will follow, we try to go further in this direction.

Proposition 3.11. *Let $\mathbb{T}$ be an exceptional tube of rank $r > 1$ and $X_1, X_2 \cdots X_r$ be the regular simple modules on $\mathbb{T}$. Let H_1 be a homogeneous simple module. Then there exists $1 \leq j \leq r$ such that $\mu(X_j[r]) \geq \mu(H_1)$.*

Proof. We first claim that for any indecomposable projective Λ-module P, there is an index $1 \leq j \leq r$ such that $\mathrm{Hom}\,(P, X_j[r-1]) = 0$. If not, we take an indecomposable projective module P_t such that $\dim\mathrm{Hom}\,(P_t, X_i[r-1]) = (\underline{\dim}\, X_i[r-1])_t \geq 1$ for all $1 \leq i \leq r$. If $r = 2$, then $1 = (\underline{\dim}\, X_i[2])_t = (\underline{\dim}\, X_1[1])_t + (\underline{\dim}\, X_2[1])_t \geq 2$, a contradiction. Now assume $r \geq 3$. From the exact sequences

$$0 \rightarrow X_i[r-1] \rightarrow X_{i+1}[r-2] \oplus X_i[r] \rightarrow X_{i+1}[r-1] \rightarrow 0,$$

we obtain that $(\underline{\dim}\, X_i[r-2])_t \geq 1$ for all $1 \leq i \leq r$. It follows from induction that $(\underline{\dim}\, X_i[2])_t \geq 1$ for all $1 \leq i \leq r$. Thus the following inequalities holds:

$$\begin{cases} (\underline{\dim}\, X_1)_t + (\underline{\dim}\, X_2)_t \geq 1 \\ (\underline{\dim}\, X_2)_t + (\underline{\dim}\, X_3)_t \geq 1 \\ \qquad \cdots \\ (\underline{\dim}\, X_{r-1})_t + (\underline{\dim}\, X_r)_t \geq 1 \\ (\underline{\dim}\, X_r)_t + (\underline{\dim}\, X_1)_t \geq 1 \end{cases}$$

Adding all the inequalities up, we get $2\delta_t = 2\sum_{i=1}^{r}(\underline{\dim}\, X_i)_t \geq r$. In particular $1 = \delta_t \geq \frac{r}{2} > 1$ which is a contradiction.

Now we assume that Y is a GR submodule of H_1. Then Y is preprojective, say $Y = \tau^{-m}P$ for some indecomposable projective module P. Note that $\mathrm{Hom}\,(Y, X_i[r-1]) \cong \mathrm{Hom}\,(P, \tau^m X_i[r-1]) \cong \mathrm{Hom}\,(P, X_s[r-1])$ where $m \equiv i - s \pmod r$. Since for P there is an index $1 \leq j' \leq r$ such that $\mathrm{Hom}\,(P, X_{j'}[r-1]) = 0$, we obtain an index j such that $\mathrm{Hom}\,(Y, X_j[r-1]) = 0$. On the other hand $\mathrm{Hom}\,(Y, X_j[r]) \neq 0$ and the image of a nonzero map $f \in \mathrm{Hom}\,(Y, X_j[r])$ is a submodule of $X_j[r]$. However, $\mathrm{Hom}\,(Y, X_j[r-1]) = 0$ implies that f is either injective or surjective. Therefore, f is a monomorphism since $|Y| < |H_1| = |X_j[r]|$. It follows that $\mu(X_j[r]) > \mu(Y)$. If $\mu(X_j[r]) < \mu(H_1)$, then $|X_j[r]| > |H_1| = |\delta|$ since Y is a GR submodule of H_1, a contradiction. Thus $\mu(X_j[r]) \geq \mu(H_1)$. □

Lemma 3.12. *Let X be an exceptional regular simple module of rank r such that $\mu(X[r]) < \mu(H_1)$, then there is and index $i \geq r$ such that the irreducible monomorphism $X[i] \rightarrow X[i+1]$ is not a GR inclusion.*

Proof. Let T be a GR submodule of H_1. Then $\mu(X[r]) < \mu(T)$ since $\mu(X[r]) < \mu(H_1)$. Assume that all inclusions $X[r] \rightarrow X[r+1] \rightarrow X[r+2] \rightarrow \cdots$ are GR inclusions. Then

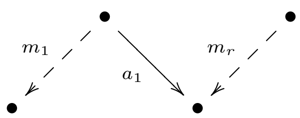
$$\mu(X[j]) = \mu(X[r]) \cup \{|X[r+1]|, \cdots |X[j-1]|, |X[j]|\}$$

for all $j \geq r$. It follows that $\mu(T) > \mu(X[j])$ for all $j > r$, since $|X[j]| > |T|$. In particular, there are infinitely many measures I with $I < \mu(T)$. We thus obtain a contradiction, since T is a preprojective module and thus a take-off module. □

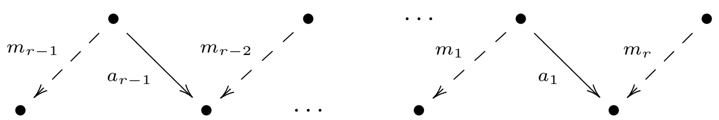
Proposition 3.13. *Let $X, \tau X, \tau^2 X, \cdots, \tau^{r-1} X$ be the regular simple modules on an exceptional tube $\mathbb{T}$ of rank r. If $\tau^i X$ is not simple for each $1 \leq i \leq r$ and $|X|$ is maximal, then $\mu(X[r]) \geq \mu(H_1)$.*

Proof. For each $m > 0$, we denote by $\overset{m}{\dashrightarrow}$ the string of arrows $\xrightarrow{c_1}\xrightarrow{c_2} \cdots \xrightarrow{c_m}$ of length m. Then the strings corresponding to the regular simple module $\tau^{-i} X$ for each i can be denoted by $\overset{m_i}{\dashrightarrow}$. Note that $m_i = |\tau^{-i} X| - 1 > 0$ since $\tau^i X$ is not simple for all $1 \leq i \leq r$.

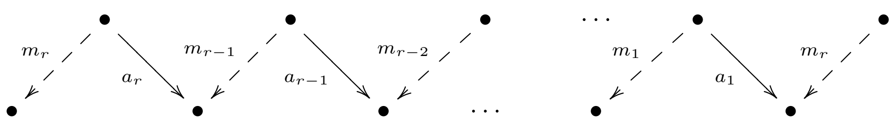
It is easily seen that $X[2]$ is a string



Since $|\tau^{-1} X| < |X|$, we can tell that X is in fact a GR submodule of $X[2]$. Continue the step we get a GR inclusion $X \subset X[2] \subset \cdots \subset X[r]$. Clearly $X[r]$ is characterized by string



Then $X[r+1]$ is characterized by the string

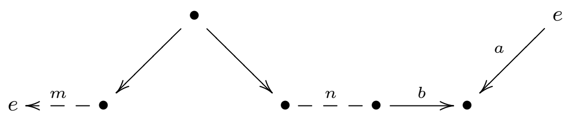


The string $\overset{m_r}{\dashrightarrow}$ on the right hand corresponds to submodule X

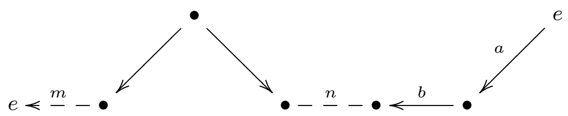
Since $|X| \geq \tau^i X$ for all $1 \leq i < r$, we see that any indecomposable proper submodule of $X[r+1]$ which does not contain X (in other word, any submodule string not containing $\overset{m_r}{\dashrightarrow}$ on the right hand) has measure $< \mu(X)$. Therefore, the chain of irreducible monomorphisms $X = X[1] \rightarrow X[2] \rightarrow \cdots \rightarrow X[r] \rightarrow X[r+1] \rightarrow \cdots$ is a chain of GR inclusions. Thus, $\mu(X[r]) \geq \mu(H_1)$ by Lemma 3.12. □

Proposition 3.14. *Let $X = X[1]$ be an exceptional regular simple module of rank r and H_1 be a homogeneous regular simple module. If X is simple, then $\mu(X[j]) < \mu(H_1)$ for all $j \geq 1$.*

Proof. It is sufficient to prove $\mu(X[r]) < \mu(H_1)$ by Lemma 3.6. Assume for a contradiction that $\mu(X[r]) \geq \mu(H_1)$. It is easily seen that$X[r+1]$ is determined by a string of one of the forms



or



where e is the vertex corresponding to the simple module X.

We only deal with the first case since the second case is similar. Let X_L be the indecomposable module determined by string $e \prec \overset{m}{-} - \bullet$ and X_R be the indecomposable module determined by string $\bullet - \overset{n}{-} - \bullet \xrightarrow{b} \bullet \xleftarrow{a} e$. Clearly, $X[r]$ is the string module obtained by deleting the vertex e on the right hand, and both X_L and X_R are proper submodule of $X[r]$.

Note that $\mu(X[r]) \geq \mu(H_1)$ implies that $X[r]$ is the unique GR submodule of $X[r+1]$. Thus $\mu(X_L) \geq \mu(X_R)$ and X_L is one of the terms in a GR filtration of $X[r]$. In particular, $\mu(X[r]) = \mu(X_L) \cup \{d_1, d_2 \cdots\}$ where $d_1 \geq 2$.

On the other hand, if we identify the two vertices corresponding to e, we will get a band, thus a band module which is a homogeneous regular simple module. It is easy to see that the indecomposable module determined by string $\bullet \xleftarrow[a]{} e \prec \overset{m}{-} - \bullet$ is a submodule of this band module. In particular, $\mu(H_1) > \mu(X_L) \cup \{|X_L| + 1\}$.

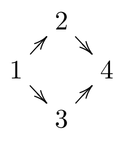
We thus have $\mu(X_L) < \mu(X_L) \cup \{|X_L| + 1\} < \mu(H_1) \leq \mu(X[r]) = \mu(X_L) \cup \{d_1, d_2 \cdots\}$ with $d_1 \geq 2$. This contradiction shows $\mu(X[r]) < \mu(H_1)$. □

4 Examples

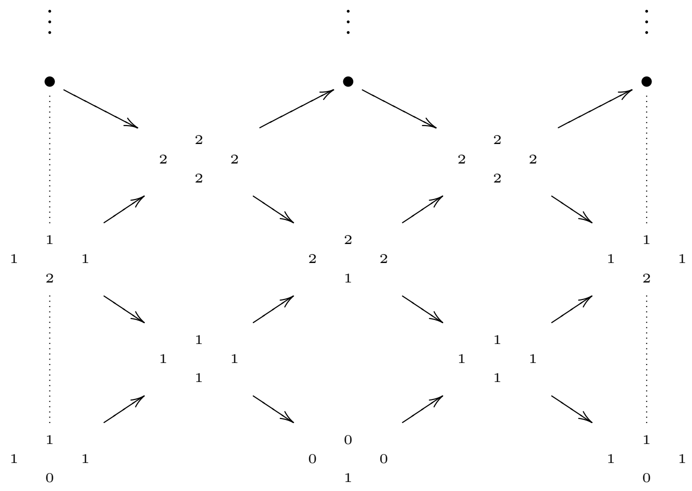
In this section, various revelent examples will be presented to show some phenomena.

Example 1. *A string module may not contain a string GR submodule.*

Let $\Lambda = k\widetilde{\mathbb{A}}_3$ with the following orientation:

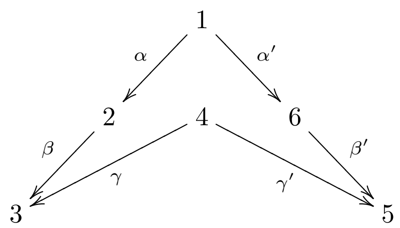


The following is one of the exceptional components of the AR quiver (the other one is "symmetric":



Set $X_1 = \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 0 & \end{smallmatrix}$, $X_2 = \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}$ and $M = \begin{smallmatrix} & 2 & \\ 2 & & 1 \\ & 2 & \end{smallmatrix}$. We want to calculate the GR submodules of the indecomposable preinjective module M. So we study the indecomposable regular modules which are of length smaller than $|M|$. First of all, because M is sincere, all homogeneous regular simple modules are submodule of M and thus, $\mu(M) > \mu(H_1) = \{1, 2, 3, 4\}$. By Proposition 3.14, we have $\mu(X_2[i]) < \mu(H_1)$ since X_2 is simple. It follows that none of them are GR submodules of M. On the other hand, because there does not exist an epimorphism from $X_1[3]$ to M, $X_1[2]$ can not be a GR submodule of M (Proposition 2.3). Therefore, the GR submodules of M are the homogeneous regular simple modules. They are all band modules.

Example 2. *In general, $\mu(H_1)$ is not the minimal central measure.*
Let $\Lambda = k\widetilde{\mathbb{A}}_5$ with the following orientation:



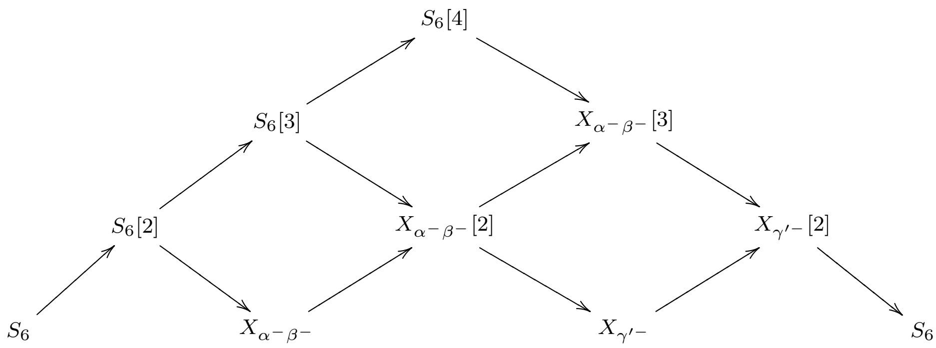
Easy calculation shows that the indecomposable preprojective module X corresponding to string $\beta'^{-}\gamma\gamma^{-}\beta$ is the unique indecomposable preprojective module with measure $\{1, 2, 4, 5\}$, and all other non-sincere preprojective modules have measures smaller than $\mu(X)$. It follows that any homogeneous regular simple module H_1 has measure $\mu(H_1) = \{1, 2, 4, 5, 6\}$.

There are the only two indecomposable preprojective modules with length 7 and

$$\mu\begin{pmatrix} & & 1 & & \\ & 1 & 1 & 1 & \\ 2 & & & & 1 \end{pmatrix} = \{1, 2, 4, 7\} = \mu\begin{pmatrix} & & 1 & & \\ & 1 & 1 & 1 & \\ 1 & & & & 2 \end{pmatrix} < \mu(X)$$

All the other sincere indecomposable preprojective modules are with lengths greater than 8 and with measures lying between $\mu(X)$ and $\mu(H_1)$, i.e, of the form $\{1, 2, 4, 5, b_1 \cdots b_m\}$ with $b_1 \geq 8$.

Now consider the exceptional tube of rank 3 containing simple module S_6.



We thus have

$$\mu(S_6[2]) = \{1, 2, 4\}, \quad \mu(S_6[3]) = \{1, 2, 4, 6\}, \quad \mu(S_6[4]) = \{1, 2, 4, 5, 7\}$$

It follows that $\mu(S_6[4])$ is larger than all the measures of preprojective modules. Thus all $S_6[j], j \geq 4$, are central modules. But note that $\mu(S_6[j]) < \mu(H_1)$ for all $j \geq 1$ and $\mu(S_6[3])$ is a take off measure. Since

$$\mu(X_{\alpha^-\beta^-}[2]) = \mu(X_{\gamma'^-}[2]) = \{1, 2, 3\} > \mu(H_1),$$

we have that $\mu(S_6[4])$ is the minimal central measure.

Example 3. *Contrary to Proposition 3.11, there may not exist exceptional regular simple module X (say with rank r) such that $\mu(X[r]) < \mu(H_1)$.*

Let $\Lambda = k\widetilde{\mathbb{A}}_3$ with sink-source orientation.

The length of an indecomposable preprojective module is an odd number. The GR measure is $\{1, 3, 5, \cdots, 2m+1\}$ for each indecomposable preprojective module of length $2m+1$.

The GR measure of any homogeneous regular simple module H_1 is $\mu(H_1) = \{1, 3, 4\}$ and thus, $\mu(H_i) = \{1, 3, 4, 8, 12, \cdots, 4i\}$. There are two exceptional tubes of rank 2 and $\mu(X) = \{1, 2\} > \mu(H_1)$ for any exceptional regular simple module X. It follows that the irreducible monomorphism $X[i] \to X[i+1]$ is actually a GR inclusion for each i. Therefore, $\mu(X[i]) = \{1, 2, 4, 6, \cdots, 2i\}$.

For any non-simple preinjective module Y, there exists an exceptional regular simple module X and a monomorphism from X to Y. Thus $\mu(Y) > \mu(X) > \mu(H_1)$. In particular the GR submodules of Y are exceptional regular modules. If $X[i]$ is a GR submodule of Y for some exceptional regular module X and quasi-length i, then there is an epimorphism from $X[i+1]$ to Y. It follows that $|Y| = |X[i]| + 1$ since $|X[i+1]/X[i]| = 2$ and the GR measure of a non-simple indecomposable preinjective module of length $2m+1$ is $\{1, 2, 4, 6, \cdots, 2m, 2m+1\}$.

It is easy to calculate the more general cases. Let $\Lambda = k\widetilde{\mathbb{A}}_n$ (n is odd) with sink-source orientation (radical square zero).

The take-off measures are of the forms $\{1, 3, 5, 7, 9, \cdots, 2m+1\}$, and the take-off part contains all preprojective modules and all simple injective modules.

The central part contains exactly all regular modules. The measures of homogeneous modules are of the forms $\{1, 3, 5, \cdots, n, n+1, 2(n+1), 3(n+1), \cdots, m(n+1)\}$. The GR measures for exceptional regular modules are of the forms $\{1, 2, 4, 6, 8, \cdots, 2m\}$.

The landing measures are of the form $\{1, 2, 4, 6, \cdots, 2m, 2m+1\}$, and the landing modules are exactly all non-simple preinjective modules. The GR submodules of a landing module are exceptional regular modules.

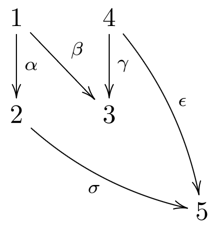
Remark. In this example, we also see the following:
(1) Each indecomposable module has, up to isomorphism, only finitely many GR submodules.

(2) The central part contains no preinjective modules. And $\mu(H_1)$ is the minimal central measure.

(3) The regular simple modules together with the simple injective are the only GR factor modules. Thus, any non-simple GR factor module has indecomposable middle term, i.e. if $Y \subset X$ is a GR inclusion and $0 \to \tau(X/Y) \to M \to X/Y \to 0$ is an almost split sequence, then M is indecomposable.

Example 4. *Let X be an exceptional regular simple module of rank r. If $\mu(X[r]) \geq \mu(H_1)$, then the sectional path of irreducible monomorphisms $X[r] \to X[r+1] \to \cdots \to X[m] \to \cdots$ is obvious a chain of GR inclusions. But different from the case of homogeneous modules (Theorem 3.9), $\mu(X[m])$ is not necessary to be the direct successor of $\mu(X[m-1])$ for $m > r$.*

Let $\Lambda = k\widetilde{\mathbb{A}}_4$ with the following orientation:



We consider the modules X_γ, X_β corresponding to the strings γ and β respectively. Then X_γ and X_β are exceptional regular simple module in different exceptional tubes of rank 2 and 3, respectively. Then

$$\mu(X_\gamma[2]) = \mu(X_\beta[3]) = \mu(H_1) = \{1, 2, 4, 5\}$$

Thus the sectional paths

$$X_\gamma[2] \to X_\gamma[3] \to \cdots \to X_\gamma[s] \to \cdots$$

and

$$X_\beta[3] \to X_\beta[4] \to \cdots \to X_\beta[t] \to \cdots$$

are both chains of GR inclusions. An easy calculation shows that

$$\mu(X_\gamma[3]) = \mu(X_\beta[4]) = \{1, 2, 4, 5, 7\}$$

and

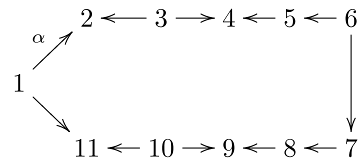
$$\mu(X_\gamma[4]) = \{1, 2, 4, 5, 7, 10\} < \{1, 2, 4, 5, 7, 9\} = \mu(X_\beta[5])$$

Thus $\mu(X_\beta[5])$ is not a direct successor of $\mu(X_\beta[4])$.

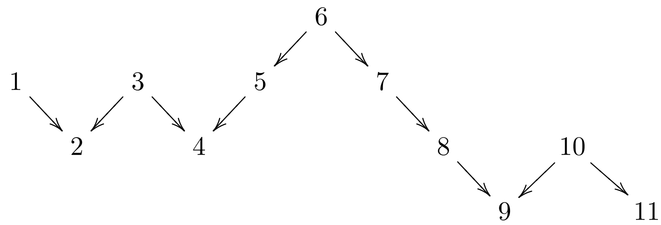
This example also implies that there are preinjective central modules whose measures are smaller than the measures of some regular modules. Let $Y = \tau I_5$ which is a string module determined by string $\gamma^{-1}\beta\alpha^{-1}\sigma^{-1}\epsilon\gamma^{-1}\beta$. Then both $(X_\gamma[3])$ and $(X_\beta[4])$ are GR submodule of Y and $\mu(Y) = \{1, 2, 4, 5, 7, 8\}$. Thus $\mu(Y) > \mu(X_\beta[j])$ and $\mu(Y) > \mu(X_\gamma[i])$ for all j, i by Lemma 3.5. Note that $\mu(Y) < \mu(X_\epsilon[2]) = \{1, 2, 3\}$.

Example 5. *The converse of Proposition 3.14 is not true in general, i.e. $\mu(X[r]) < \mu(H_1)$ does not imply X is simple.*

Let $\Lambda = \widetilde{\mathbb{A}}_{10}$ with the following orientation:



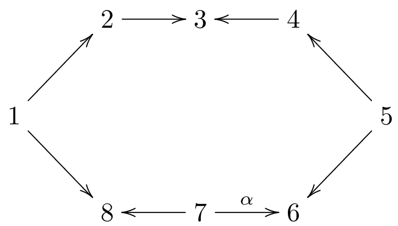
The GR measure of any homogeneous simple module H_1 is $\mu(H_1) = \{1, 2, 3, 5, 7, 9, 10, 11\}$. Let X_α be the string module defined by $1 \xrightarrow{\alpha} 2$ which is of length 2. It is an exceptional regular simple module of rank 5. Then $X_\alpha[5]$ is given by a string



and $\mu(X_\alpha[5]) = \{1, 2, 3, 5, 8, 10, 11\} < \mu(H_1)$.

Example 6. *Let X be an exceptional regular simple module of rank r, then $\mu(X[r]) \geq \mu(H_1)$ does not imply that the sequence of irreducible monomorphisms $X[1] \to X[2] \to \cdots \to X[r]$ is a GR filtration.*

Let $\Lambda = k\widetilde{\mathbb{A}}_7$ with the following orientation:

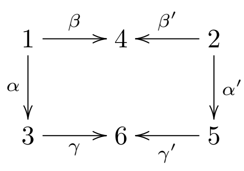


A homogeneous regular simple module H_1 has measure $\mu(H_1) = \{1, 2, 3, 5, 7, 8\}$.

Let X_α be the string module corresponds to arrow α. It is an exceptional regular simple of rank 4. The measures of $\mu(X_\alpha[i])$ can be easily calculated. Namely, $\mu(X_\alpha) = \{1, 2\}$,$\mu(X_\alpha[2]) = \{1, 2, 4, 5\}$, $\mu(X_\alpha[3]) = \{1, 2, 3, 5, 6\} > \mu(H_1)$. Thus $X_\alpha[3]$ is a GR submodule of $X_\alpha[4]$ and $\mu(X_\alpha[4]) = \{1, 2, 3, 5, 6, 8\}$. Here $\mu(X_\alpha[4]) > \mu(H_1)$ and $X_\alpha \to X_\alpha[2] \to X_\alpha[3]$ is not a chain of GR inclusions.

Example 7. *Let X be an exceptional regular simple module of rank r. Then $X[j]$ is a GR submodule of some preinjective module does not implies $X \to X[2] \to \cdots \to X[j] \to \cdots$ is a chain of GR inclusions.*

Let $\Lambda = k\widetilde{\mathbb{A}}_5$ with the following orientation:



It is easily seen that $\mu(H_1) = \{1,2,3,5,6\}$. Let $M = \tau I_6 = \begin{smallmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{smallmatrix}$ and X_β be the indecomposable module determined by arrow β. Thus X is an exceptional regular simple of rank 3. Then $\mu(X_\beta) = \{1,2\}$, $\mu(X_\beta[2]) = \{1,2,4,5\}$ and $\mu(X_\beta[3]) = \{1,2,3,5,6\} = \mu(H_1)$. Then $X_\beta[3]$ is a GR submodule of $X_\beta[4]$ and thus $\mu(X_\beta[4]) = \{1,2,3,5,6,8\}$. Easy calculation shows that $X_\beta[4]$ is a submodule, thus a GR submodule, of M. But $\mu(X_\beta[3]) = \mu(H_1)$ and $X_\beta \rightarrow X_\beta[2] \rightarrow X_\beta[3]$ is not a chain of GR inclusions.

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