EXACT LOCALIZED SOLUTIONS OF THE (2+1)-DIMENSIONAL COMPLEX CUBIC-QUINTIC GINZBURG-LANDAU EQUATION

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Abstract. The (2+1)-dimensional complex cubic-quintic Ginzburg-Landau equation is analytically studied. A method derived from the Painlevé test for integrability and Hirota's multilinear formalism is used. We have found that 2D propagating pulse, 2D stationary pulse, 2D stationary hole and 2D propagating hole solutions are the exact localized solutions that exist for this equation. The existence of these special solutions shows that the coefficients are constrained by certain relations. The degeneracies of this (2+1)-dimensional Cubic-quintic Ginzburg-Landau equation are examined as well as their solutions. By analyzing these degeneracies, it appears that this equation is a generalization of the nonlinear cubic-quintic Schrödinger equation and the real cubic-quintic Ginzburg-Landau equation.

KEYWORDS

(2+1)-dimensional complex cubic-quintic Ginzburg-Landau equation, Painlevé test for integrability, Hirota method, propagating pulses, stationary pulses, propagating hole and stationary hole solutions.

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1. Introduction

There is emergence of stable spatio-temporal patterns when many spatially extended systems are driven sufficiently far from their equilibrium states. In the weakly nonlinear regime, these patterns can be described by the complex cubic-quintic Ginzburg-Landau (CCQGL) equation. These include a variety of nonequilibrium phenomena, such as the dynamics of certain chemical reactions [1], binary fluid convection [2-5], parametric surface waves in fluid [6,7], Taylor-Couette flow between rotating cylinders [8-10], soliton propagation in optical fiber systems with linear and nonlinear gain and spectral filtering [11-16], pulse generation in fiber lasers with additive pulse mode-locking or nonlinear polarization rotation [17-22].

The CCQGL equation was considered in a number of publications using numerical simulations [23-28]. Perturbative analysis of the solitons of the CCQGL equation in the nonlinear Schrödinger equation (NLSE) limit was used in [23,24,29,30]. Analytic approach, which allows to obtain solutions for the (1+1)-dimensional (D) CCQGL equation was developed in [31,32]. These exact solutions include fronts, pulses, sources or sinks, seem to play important role in the selection of a final steady pattern at long times and in the evolution of periodic, quasi periodic or disordered patterns [33-35].

Patterns described by (2+1)-dimensional CCQGL equation are of special interest. In particular steady spiral solutions two dimensional (2D) objects with internal vorticity were studied in recent work [36], other novel types of nonstationary and stationary asymmetric solitons were found in [37]. 2D localized particle like solutions were found in [38]. Recently, the properties of spiral waves and other localized solutions were found [36]. New dynamical behavior that appears in the CCQGL equation as a result of the interplay between the quantized circular phase flow and bending of the spatial localized states through the spatial oscillations of their tails were described in [39]. Thus, it is obvious that numerical approach for the (2+1)-dimensional CCQGL equation has been intensively studied, but to our knowledge, much less is known about analytic solutions of this equation. Motivated by this, and since it is often put forward that a first step toward understanding complex dynamics in nonequilibrium systems is to find solution describing a basic pattern which plays crucial role in long time behavior, in the present work, we investigate analytical solutions of the (2+1)- dimensional CCQGL equation. The (2+1)-dimensional CCQGL equation reads:

$$U_t + p\Delta U = aU - q|U|^2U - \gamma|U|^4U, \tag{1}$$

where p, q and γ are in general complex that of $\chi = \chi_r + i\chi_i$, U(x,y,t) is the complex wave amplitude. In this paper, we study Eq.(1) by using a method which is derived from the Painlevé test for integrability in the sens of Weiss-Tabor-Carnevale (WTC) [40] and Hirota multi-linear method [41]. We find analytic solutions of Eq.(1) by using the method presented in [42-45] in which the process begins with painlevé analysis developed by WTC to obtain in natural way the ansatze which are then used in Hirota's method. The major contribution of our methodology lies in this use of one method to obtain the ansatze, whose application has been intuitive. This gives an algorithmic way of obtaining the ansatze, thus making the use of Hirota's metod considerably easier. The rest of paper is organized as follows. The Painlevé analysis of the (2+1)-D CCQGL equation is performed in section 2. Exact solutions of the (2+1)-D CCQGL equation are obtained in section 3. The degeneracies of (2+1)-D CCQGL equation are studied in section 4. Finally, we summarize with our conlusions in section 5.

2. PainlevÉ Analysis of the (2+1)-D CCQGL Equation

We follow the WTC procedure [40] to carry out the Painlevé analysis, according to which a given partial differential equation (PDE) is integrable, if its solutions are single valued about the movable singularity manifold. This method requires the following necessary conditions to prove the integrability:

(i) determination of leading orders of Laurent series which should be integers,

(ii) identification of powers at which the arbitrary functions can enter into the Laurent series called resonances,

(iii) verification of the existence of sufficient number of arbitrary functions at the resonance values. In order to carry out the Painlevé analysis, let us assume $U^* = V$. Eq.(1) becomes:

$$U_t + p(U_{xx} + U_{yy}) = aU - qU^2V - \gamma U^3V^2, \tag{2.a}$$

$$V_t + p^*(V_{xx} + V_{yy}) = aV - q^*V^2U - \gamma^*U^2V^3, \tag{2.b}$$

where * stands for complex conjugation . Leading-order analysis for Eqs.(2a) and (2.b) is achieved by balancing the highest-order derivative with the strongest nonlinearity. The fields U and V are now formally considered as independent variables and obey.

$$p(U_{xx} + U_{yy}) \sim -\gamma U^3V^2, \tag{3.a}$$

$$p^*(V_{xx} + V_{yy}) \sim -\gamma^* V^3U^2, \tag{3.b}$$

To determine the leding-order behavior, we expand

$$U(x, y, t) \sim U_0(x, y, t)\phi^r(x, y, t), V(x, y, t) \sim V_0(x, y, t)\phi^s(x, y, t), \tag{4}$$

where ϕ is arbitrary and analytic function which determines a singular manifold. Here r and s are leading-order exponents. Substituting Eq.(4) into Eqs.(3.a) and (3.b) and equating the dominant terms, we obtain

$$r = -\frac{1}{2} - i\alpha, s = -\frac{1}{2} + i\alpha, \tag{5}$$

where U_0, V_0 and α are solutions of the equations:

$$U_0^2V_0^2 = \frac{1}{|\gamma|^2}\left\{2\alpha(p_i\gamma_r - p_r\gamma_i) - \left(\frac{3}{4} - \alpha^2\right)(p_r\gamma_r + p_i\gamma_i)\right\}(\phi_x^2 + \phi_y^2), \tag{6}$$

$$\alpha = \frac{p_r\gamma_r + p_i\gamma_i}{p_i\gamma_r - p_r\gamma_i} \pm \left\{\frac{3}{4} + \left(\frac{p_r\gamma_r + p_i\gamma_i}{p_i\gamma_r - p_r\gamma_i}\right)^2\right\}^{\frac{1}{2}}, \alpha \neq 0. \quad (\mathbf{7})$$

We consider first the non degenerate case $p_i\gamma_r - p_r\gamma_i \neq 0$. The leading-order exponents $r = -\frac{1}{2} - i\alpha$ and $s = -\frac{1}{2} + i\alpha$ are not the integers. The non integrable (2+1)-D CCQGL equation has already failed the Painlevé test at the first step of these three steps. However, this does not forbid the existence of particular solutions, provided that they respect the singularity structure derived from the leading-order analysis.

The general solution will be recovered under the Puiseux series as follows.

$$U(x,y,t) = \sum_{j=0}^{\infty} U_j(x,y,t)\phi^{\frac{j}{2}-\frac{1}{2}-i\alpha}(x,y,t), \quad (\mathbf{8}.\mathbf{a}),$$

$$V(x,y,t) = \sum_{j=0}^{\infty} V_j(x,y,t)\phi^{\frac{j}{2}-\frac{1}{2}+i\alpha}(x,y,t). \quad (\mathbf{8}.\mathbf{b}).$$

An usual assumption to find these solutions is to require the Puiseux series to truncate after the power $\phi^{0\pm i\alpha}$ and to find a relation between (U_1, V_1) and (U,V). This is known as " WTC truncation procedure." Thus truncated expansions for Eqs.(8.a) and (8.b) are:

$$U = U_0\phi^{-\frac{1}{2}-i\alpha} + U_1\phi^{-i\alpha}, \quad (\mathbf{9}.\mathbf{a}),$$

$$V = V_0\phi^{-\frac{1}{2}+i\alpha} + V_1\phi^{i\alpha}. \quad (\mathbf{9}.\mathbf{b}),$$

We can proceed by observing that (U_1, V_1) satisfies the same relation as (U,V) through the following equations

$$\phi^{-i\alpha} : U_{1t} + p(U_{1xx} + U_{1yy}) = aU_1 - qU_1^2V_1 - \gamma U_1^3V_1^2, \quad (\mathbf{10}.\mathbf{a})$$

$$\phi^{i\alpha} : V_{1t} + p^*(V_{1xx} + V_{1yy}) = aV_1 - q^*V_1^2U_1 - \gamma^*U_1^2V_1^3. \quad (\mathbf{10}.\mathbf{b})$$

These Eqs. (10.a) and (10.b) justify the fact that the expansion can be truncated at the level $\phi^{\mp i\alpha}$. So that by choosing $U_1 = V_1 = 0$, the resulting zeroth order of equation is easier to solve.

Thus the truncated expansions take the form:

$$U = \frac{U_0}{\phi^{\frac{1}{2}+i\alpha}}, V = \frac{V_0}{\phi^{\frac{1}{2}-i\alpha}}. \quad (\mathbf{11})$$

By observing that the plane waves

$$U(x,y,t) = R_K e^{i(K_1x+K_2y-\omega t)}, \quad (\mathbf{12})$$

with

$$\gamma_r|R_K|^4 + q_r|R_K|^2 - \{p_r(K_1^2 + K_2^2) + a\} = 0, \quad (\mathbf{13}.\mathbf{a})$$

and the dispersion relation

$$\omega(K_1, K_2) = -p_i(K_1^2 + K_2^2) + q_i|R_K|^2 + \gamma_i|R_K|^4, \quad (\mathbf{13}.\mathbf{b})$$

where K_1 and K_2 are free parameters. The general solutions of the (2+1)-D CCQGL equation can be built by the help of the trivial solutions (**12**).

So the expression (**12**) may be expressed as:

$$U = \frac{G}{F^{\frac{1}{2}+i\alpha}} e^{i(K_1x+K_2y-\Omega t)}, V = \frac{H}{F^{\frac{1}{2}-i\alpha}} e^{-i(K_1x+K_2y-\Omega t)}. \quad (\mathbf{14})$$

Expression (**14**) is the ansatz which should be used in Hirota method. Next we consider the Hirota bilinear transformation (**14**). Where G(x,y,t) and H(x,y,t) are the complex functions and F(x,y,t) is a real function. Using Eq.(14) into Eqs.(2,a) and (2.b) we obtain the following multilinearized forms:

$F\{[-i\Omega - a - p(K_1^2 + K_2^2) + \mathcal{D}_{\alpha,t} + 2ip(K_1\mathcal{D}_{\alpha,x} + K_2\mathcal{D}_{\alpha,y}) + p(\mathcal{D}_{\alpha,x}^2 + \mathcal{D}_{\alpha,y}^2)]\}(GF) - (\frac{1}{2} + i\alpha)(\frac{3}{4} + i\alpha)pG(\wedge_x + \wedge_y)(F.F) = -qHG^2F - \gamma G^3H^2,$ (**15.a**)

$F\{[i\Omega - a - p^*(K_1^2 + K_2^2) + \mathcal{D}_{\alpha,t}^* - 2ip^*(K_1\mathcal{D}_{\alpha,x}^* + K_2\mathcal{D}_{\alpha,y}^*) + p^*(\mathcal{D}_{\alpha,x}^{*2} + \mathcal{D}_{\alpha,y}^{*2})]\}(HF) - (\frac{1}{2} - i\alpha)(\frac{3}{4} - i\alpha)p^*H(\wedge_x + \wedge_y)(F.F) = -q^*GH^2F - \gamma^*G^2H^3,$ (**15.b**)

where $\mathcal{D}_{\alpha,z}$ is Hirota's modified bilinear operator defined by

$$\mathcal{D}_{\alpha,z}(GF) = \left\{\left[\frac{\partial}{\partial z} - (\frac{1}{2} + i\alpha)\frac{\partial}{\partial z'}\right] G(z)F(z')\right\}_{z=z'}, \quad (\mathbf{16.a})$$

and

$$\wedge_z(F.F) = FF_{zz} - F_z^2. \quad (\mathbf{16.b})$$

3. Exact solutions of the (2+1)-dimensional CCQGL equation

We now turn to construct some special solutions for the Eqs.(15.a) and (15.b). Different kinds of solutions are found depending of the forms of expansions chosen for F, G and H.

3.1. **2D Pulse solutions**

. We begin with the pulse like solution. We assume the following forms for G, H, and F:

$$G = \mu\epsilon^{\frac{1}{2}}g_{\frac{1}{2}}, H = \eta\epsilon^{\frac{1}{2}}h_{\frac{1}{2}}, F = 1 + \epsilon f_1 + \epsilon^2 f_2, \quad (\mathbf{17})$$

where μ and η are complex constants, and $g_{\frac{1}{2}}$, $h_{\frac{1}{2}}, f_1$ and f_2 are real valued functions. Substituting Eq.(17) into Eq.(15.a) and collecting the coefficients of different powers of ϵ $(\epsilon^{\frac{1}{2}}, \epsilon^{\frac{3}{2}}, \epsilon^{\frac{5}{2}}, \epsilon^{\frac{7}{2}}, \epsilon^{\frac{9}{2}})$ we obtain a set of five equations. Solving this set of equations, we find

$$g_{\frac{1}{2}} = h_{\frac{1}{2}} = \exp\{\frac{1}{2}(kx + qy + \omega t + \xi^{(0)})\}, \quad (\mathbf{18.a})$$

$$f_1 = d\exp\{(kx + qy + \omega t + \xi^{(0)})\}, \quad (\mathbf{18.b})$$

$$f_2 = \exp\{2(kx + qy + \omega t + \xi^{(0)})\}, \quad (\mathbf{18.c})$$

where unknowns in these expressions obey different relationships according to the value of the coefficient p_r:

Case 1: $p_r \neq 0$

This first case includes two subcases:

Subscase 1: $K_1 \neq \alpha k, K_2 \neq \alpha l$.

A two-parameter family of solutions is found. K_1 is determined by a quadratic equation

$$[p_r(k^2+l^2)]K_1^2 - 2[\alpha p_r k(k^2+l^2)]K_1 + \{\alpha^2 p_r(k^2+l^2)^2 + l^2[a - (\frac{1}{4}p_r - \alpha p_i)(k^2+l^2)]\} = 0. \quad \mathbf{(19)}$$

The unknowns $\mu\eta, K_2, \omega$, and Ω are obtained as follows

$$\mu^2\eta^2 = e_1(k^2+l^2), \quad \mathbf{(20.a)}$$

$$\omega = 0, \quad \mathbf{(20.b)}$$

$$K_2 = \frac{\alpha}{l}(k^2+l^2) - \frac{k}{l}K_1, \quad \mathbf{(20.c)}$$

$$\Omega = -p_i(K_1^2+K_2^2) + (\frac{1}{4}p_i + \alpha p_r)(k^2+l^2), \quad \mathbf{(20.d)}$$

$$k^2+l^2 = R^2, \quad \mathbf{(20.e)}$$

$$R^2 = \frac{e_1 q_r^2}{d^2\left[(1-2\alpha^2)p_r - 3\alpha p_i\right]^2} > 0,, \quad \mathbf{(20.f)}$$

$$e_1 = \frac{(4-d^2)\left\{\gamma_r\left[\left(\frac{3}{4}-\alpha^2\right)p_r - 2\alpha p_i\right] + \gamma_i\left[\left(\frac{3}{4}-\alpha^2\right)p_i + 2\alpha p_r\right]\right\}}{|\gamma|^2} > 0, d \neq \{2,-2\}. \quad \mathbf{(20.g)}$$

Here d is free parameter.

This solution requires the following constraint to be satisfied

$$q_r[(1-2\alpha^2)p_i + 3\alpha p_r] - q_i[(1-2\alpha^2)p_r - 3\alpha p_i] = 0. \quad \mathbf{(21)}$$

Subcase 2:

$$K_1 = \alpha k, K_2 = \alpha l. \quad \mathbf{(22)}$$

The unknowns $\mu\eta, \omega$, k, l and Ω are defined by:

$$\mu^2\eta^2 = e_1(k^2+l^2), \quad \mathbf{(23.a)}$$

$$\omega = 0, \quad \mathbf{(23.b)}$$

$$k^2+l^2 = R^2, \quad \mathbf{(23.c)}$$

$$\Omega = \left[\left(\frac{1}{4}-\alpha^2\right)p_i + \alpha p_r\right](k^2+l^2), \quad \mathbf{(23.d)}$$

R^2, e_1 are given by Eqs.(20.f) and (20.g) respectively.

This solution is recovered under the following constraints:

$$q_r[(1-2\alpha^2)p_i + 3\alpha p_r] - q_i[(1-2\alpha^2)p_r - 3\alpha p_i] = 0, \quad \mathbf{(24.a)}$$

$$a = \left[\left(\frac{1}{4}-\alpha^2\right)p_r - \alpha p_i\right](k^2+l^2). \quad \mathbf{(24.b)}$$

Here d is free parameter. This first case in its two subcases defines the stationary pulse solutions which reads:

$$U(x,y,t) = \frac{\mu e^{\frac{1}{2}(kx+ly+\xi^{(0)})}}{\left[1+de^{(kx+ly+\xi^{(0)})}+e^{2(kx+ly+\xi^{(0)})}\right]^{\frac{1}{2}+i\alpha}} e^{i(K_1x+K_2y-\Omega t)}, \quad \textbf{(25.a)}$$

$$V(x,y,t) = \frac{\eta e^{\frac{1}{2}(kx+ly+\xi^{(0)})}}{\left[1+de^{(kx+ly+\xi^{(0)})}+e^{2(kx+ly+\xi^{(0)})}\right]^{\frac{1}{2}-i\alpha}} e^{-i(K_1x+K_2y-\Omega t)}, \quad \textbf{(25.b)}$$

Case 2: $p_r = 0$.

We find a three parameter-family of solution, parametrized by d, K_1 and K_2 where:

$$\mu^2\eta^2 = e_1'(k^2+l^2), \quad \textbf{(26.a)}$$

$$\omega = 2p_i[kK_1 + lK_2 - \alpha(k^2+l^2)], \quad \textbf{(26.b)}$$

$$\Omega = -p_i(K_1^2+K_2^2) + \frac{1}{4}p_i(k^2+l^2), \quad \textbf{(26.c)}$$

$$k^2+l^2 = R'^2, \quad \textbf{(26.d)}$$

$$R'^2 = \frac{e_1' q_r^2}{9\alpha^2 p_i^2 d^2}, \quad \textbf{(26.e)}$$

$$e_1' = \frac{(4-d^2)p_i\left[-2\alpha\gamma_r + \left(\frac{3}{4}-\alpha^2\right)\gamma_i\right]}{|\gamma|^2} > 0, d \neq \{2,-2\}. \quad \textbf{(26.f)}$$

Then α satisfies in this case the following equation

$$\alpha^2 - 2\frac{\gamma_i}{\gamma_r}\alpha - \frac{3}{4} = 0. \quad \textbf{(27)}$$

This solution imposes the following constraints:

$$(1-2\alpha^2)q_r + 3\alpha q_i = 0, \quad \textbf{(28.a)}$$

$$a = -\alpha p_i(k^2+l^2). \quad \textbf{(28.b)}$$

The propagating pulse solutions read:

$$U(x,y,t) = \frac{\mu e^{\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}}{\left[1+de^{(kx+ly+\omega t+\xi^{(0)})}+e^{2(kx+ly+\omega t+\xi^{(0)})}\right]^{\frac{1}{2}+i\alpha}} e^{i(K_1x+K_2y-\Omega t)}, \quad \textbf{(29.a)}$$

$$V(x,y,t) = \frac{\eta e^{\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}}{\left[1+de^{(kx+ly+\omega t+\xi^{(0)})}+e^{2(kx+ly+\omega t+\xi^{(0)})}\right]^{\frac{1}{2}-i\alpha}} e^{-i(K_1x+K_2y-\Omega t)}, \quad \textbf{(29.b)}$$

After some arrangements, the stationary pulse solutions (**25.a**) and (**25.b**) leads to:

$$U(x,y,t) = \frac{\mu}{\left[2\cosh(kx+ly+\xi^{(0)})+d\right]^{\frac{1}{2}}} e^{i\{(K_1-\alpha k)x+(K_2-\alpha l)y-\Omega t-\alpha\xi^{(0)}-\alpha\ln[2\cosh(kx+ly+\xi^{(0)})+d]\}}, \quad \textbf{(30.a)}$$

$$V(x,y,t)=\frac{\eta}{\left[2\cosh(kx+ly+\xi^{(0)})+d\right]^{\frac{1}{2}}}e^{-i\{(K_1-\alpha k)x+(K_2-\alpha l)y-\Omega t-\alpha\xi^{(0)}-\alpha\ln[2\cosh(kx+ly+\xi^{(0)})+d]\}}. \qquad (\mathbf{30.b})$$

In the same way, solutions (**29.a**) and (**29.b**) can be written:

$$U(x,y,t)=\frac{\mu}{\left[2\cosh(kx+ly+\omega t+\xi^{(0)})+d\right]^{\frac{1}{2}}}e^{i\{(K_1-\alpha k)x+(K_2-\alpha l)y-(\Omega+\alpha\omega)t-\alpha\xi^{(0)}-\alpha\ln[2\cosh(kx+ly+\omega t+\xi^{(0)})+d]\}}, (\mathbf{31.a})$$

$$V(x,y,t)=\frac{\eta}{\left[2\cosh(kx+ly+\omega t+\xi^{(0)})+d\right]^{\frac{1}{2}}}e^{-i\{(K_1-\alpha k)x+(K_2-\alpha l)y-(\Omega+\alpha\omega)t-\alpha\xi^{(0)}-\alpha\ln[2\cosh(kx+ly+\omega t+\xi^{(0)})+d]\}}.(\mathbf{31.b})$$

The solutions (**30.a**), (**30.b**), (**31.a**) and (**31.b**) have the following asymptotic behavior:

$$\lim_{\zeta\longrightarrow\pm\infty}U(\zeta)=\lim_{\zeta\longrightarrow\pm\infty}V(\zeta)=0, \qquad (\mathbf{32})$$

3.2. **2D Hole solutions**

. We now proceed to construct the 2D propagating hole solutions. To obtain these solutions, we assume the following forms for G, H and F.

$$G=\mu(1+\epsilon g_1), H=\eta(1+\epsilon h_1), F=1+\epsilon f_1+\epsilon^2 f_2, \qquad (\mathbf{33})$$

where μ,η are complex constants, h_1,g_1 are complex functions and f_1 and f_2 are real functions. Substituting Eqs.(33) into Eq.(15.a) and proceeding as in the previous manner, we obtain:

$$g_1=c\exp(kx+ly+\omega t+\xi^{(0)}), h_1=c^*\exp(kx+ly+\omega t+\xi^{(0)}), (\mathbf{34.a})$$

$$f_1=\exp(kx+ly+\omega t+\xi^{(0)}), f_2=\exp 2(kx+ly+\omega t+\xi^{(0)}).(\mathbf{34.b})$$

Whence: $(1-|c|^2)(1-4c_r+|c|^2)=0.(\mathbf{35})$

The condition (35) implies two cases:

Case 1: $c_i^2=-c_r^2+4c_r-1.(\mathbf{36})$

The unknowns are defined as follows:

$$\mu^2\eta^2=\frac{-e}{(2c_r-1)^2|\gamma|^2}(k^2+l^2), e<0, \qquad (\mathbf{37.a})$$

$$\omega=2p_i(kK_1+lK_2)-2\alpha p_i(k^2+l^2)+\frac{q_i}{2\alpha}(|c|^2-1)\mu\eta+\frac{\gamma_i}{2\alpha}(|c|^4-1)\mu^2\eta^2, \qquad (\mathbf{37.b})$$

$$\Omega=-p_i(K_1^2+K_2^2)+q_i\mu\eta+\gamma_i\mu^2\eta^2, \qquad (\mathbf{37.c})$$

$$k^2+l^2=R^2, \qquad (\mathbf{37.d})$$

$$K_2=\frac{1}{l}\left\{\left[\alpha+\frac{e(|c|^4-1)\gamma_r}{4\alpha p_r(2c_r-1)^2|\gamma|^2}\right]R^2-\frac{\sqrt{-e}(|c|^2-1)}{4\alpha p_r|2c_r-1||\gamma|}R\right\}-\frac{k}{l}K_1, \qquad (\mathbf{37.e})$$

$$R^2=\frac{-e(2c_r-1)^2|\gamma|^2[(2c_r-1)q_i+2c_iq_r]^2}{4\alpha^2e_{2r}^2}, \qquad (\mathbf{37.f})$$

$$e=\gamma_r\left[\left(\frac{3}{4}-\alpha^2\right)p_r-2\alpha p_i\right]+\gamma_i\left[\left(\frac{3}{4}-\alpha^2\right)p_i+2\alpha p_r\right]. \qquad (\mathbf{37.g})$$

Here α satisfies Eq.(7) and K_1 satisfies the following quadratic equation

$$p_r R^2 K_1^2 - 2kb_1 K_1 + b_2(k,l) = 0, \quad (\mathbf{38})$$

where

$$b_1 = p_r R \left\{ \left[\alpha + \frac{e(|c|^4-1)\gamma_r}{4\alpha p_r (2c_r-1)^2|\gamma|^2} \right] R - \frac{\sqrt{-e}(|c|^2-1)q_r}{4\alpha p_r |2c_r-1||\gamma|} \right\}, \quad (\mathbf{39.a})$$

$b_2(k,l) = p_r \left[\alpha + \frac{e(|c|^4-1)\gamma_r}{4\alpha p_r (2c_r-1)^2|\gamma|^2}\right]^2 R^4 - \frac{\sqrt{-e}(|c|^2-1)q_r}{2\alpha|2c_r-1||\gamma|}\left[\alpha + \frac{e(|c|^4-1)\gamma_r}{4\alpha p_r(2c_r-1)^2|\gamma|^2}\right] R^3 - \frac{e(|c|^2-1)^2 q_r^2}{16\alpha^2 p_r (2c_r-1)^2|\gamma|^2} R^2 +$
$l^2\left\{a + \frac{e\gamma_r}{(2c_r-1)^2|\gamma|^2}R^2 - \frac{\sqrt{-e}q_r}{|2c_r-1||\gamma|}R\right\}$.(**39.b**)

The unknown c_r should satisfies the following system of equations:

$$\begin{cases} (3c_r-1)[e_{2r}e_{4i} - e_{2i}e_{4r}] + (-c_r^2+4c_r-1)^{\frac{1}{2}}[e_{2r}e_{4r} + e_{2i}e_{4i}] = 0, (\mathbf{40.a}) \\ (4c_r-1)e_{4r} + 2\alpha(3c_r-1)e_{2r} + 2\alpha(-c_r^2+4c_r-1)^{\frac{1}{2}}e_{2i} = 0. (\mathbf{40.b}) \end{cases}$$

The unknowns remaining e_{2r}, e_{2i}, e_{4r} and e_{4i} are functions of c_r such as defined as follows

$$e_{2r} = d_{1r}c_r^2 + d_{2r}c_r + d_{3r} + (-c_r^2+4c_r-1)^{\frac{1}{2}}[-d_{1i}c_r + d_{4i}], \quad (\mathbf{41.a})$$

$$e_{2i} = d_{1i}c_r^2 + d_{2i}c_r + d_{3i} + (-c_r^2+4c_r-1)^{\frac{1}{2}}[d_{1r}c_r + d_{4r}], \quad (\mathbf{41.b})$$

$$e_{4r} = j_{1r}c_r^2 + j_{2r}c_r + j_{3r} + (-c_r^2+4c_r-1)^{\frac{1}{2}}[-j_{1i}c_r + j_{3i}], \quad (\mathbf{41.c})$$

$$e_{4i} = j_{1i}c_r^2 + j_{2i}c_r + j_{3i} + (-c_r^2+4c_r-1)^{\frac{1}{2}}[j_{1r}c_r - j_{3r}]. \quad (\mathbf{41.d})$$

The unknowns are given in Appendix.

The following constraint should be respected:

$$[(2c_r-1)e_{2r} + 2(-c_r^2+4c_r-1)^{\frac{1}{2}}e_{2i}]q_r - [2(-c_r^2+4c_r-1)^{\frac{1}{2}}e_{2r} - (2c_r-1)e_{2i}]q_i = 0. \quad (\mathbf{42})$$

Case 2: $c_i^2 = 1 - c_r^2$(**43**)

We find that

$$kK_1 + lK_2 = \alpha(k^2+l^2), \quad (\mathbf{44})$$

which leads to two subcases:

Subcase 1: $K_1 = \alpha k, K_2 = \alpha l$.(**45**)

The unknowns remaining $\mu\eta, k, l, \omega, \Omega$ and c_r can be computed from the following equations

$$\mu^2\eta^2 = \frac{3e}{(2c_r-1)^2|\gamma|^2}(k^2+l^2), e > 0, \quad (\mathbf{46.a})$$

$$\omega = 0, \quad (\mathbf{46.b})$$

$$\Omega = -\alpha^2 p_i(k^2+l^2) + q_i\mu\eta + \gamma_i\mu^2\eta^2, \quad (\mathbf{46.c})$$

$$k^2 + l^2 = R'^2, \quad (\mathbf{46.d})$$

$$R'^2 = \frac{3e(2c_r-1)^2|\gamma|^2 q_r^2}{e_{2r}'^2}. \quad (\mathbf{46.e})$$

where e is given by Eq.(37.g). These two constraints should be respected

$$e'_{2r}q_i - e'_{2i}q_r = 0, \quad \textbf{(47.a)}$$

$$a = \left[\frac{3e\gamma_r}{(2c_r-1)^2|\gamma|^2} - \alpha^2 p_r\right] R'^2 - \frac{\sqrt{3e}q_r}{|2c_r-1||\gamma|}R'. \quad \textbf{(47.b)}$$

Subcase 2: $K_1 \neq \alpha k, K_2 \neq \alpha l$. The unknowns are defined as follows:

$$\mu^2\eta^2 = \frac{3e}{(2c_r-1)^2|\gamma|^2}(k^2+l^2), e > 0, \quad \textbf{(48.a)}$$

$$\omega = 0, \quad \textbf{(48.b)}$$

$$\Omega = -p_i(K_1^2+K_2^2) + q_i\mu\eta + \gamma_i\mu^2\eta^2, \quad \textbf{(48.c)}$$

$$k^2 + l^2 = R'^2, \quad \textbf{(48.d)}$$

$$K_2 = \frac{\alpha}{l}R'^2 - \frac{k}{l}K_1. \quad \textbf{(48.e)}$$

K_1 satisfies the following quadratic equation:

$$p_r R'^2 K_1^2 - 2\alpha p_r R'^2 k K_1 + \left\{\alpha^2 p_r R'^4 - l^2\left[\frac{3e\gamma_r}{(2c_r-1)^2|\gamma|^2}R'^2 + \frac{\sqrt{3e}q_r}{|2c_r-1||\gamma|}R' - a\right]\right\} = 0. \quad \textbf{(49)}$$

where e, R'^2 respect Eq.(37.g) and Eq.(46.e) respectively. This solution is obtained under the constraint (47.a).

The remaining unknown c_r in these two subcases should satisfy:

$$\begin{cases} (c_r-1)[e'_{2i}e'_{4r} - e'_{2r}e'_{4i}] + (1-c_r^2)^{\frac{1}{2}}[e'_{2r}e'_{4r} + e'_{2i}e'_{4i}] = 0, \textbf{(50.a)} \\ e'_{4r} - (c_r-1)e'_{2r} + (1-c_r^2)^{\frac{1}{2}}e'_{2i} = 0. \textbf{(50.b)} \end{cases}$$

The unknowns remaining $e'_{2r}, e'_{2i}, e'_{4r}$ and e'_{4i} are functions of c_r such as defined as follows

$$e'_{2r} = d'_{1r}c_r^2 + d'_{2r}c_r + d'_{3r} + (1-c_r^2)^{\frac{1}{2}}[-d'_{1i}c_r + \frac{1}{2}d'_{1i}], \quad \textbf{(51.a)}$$

$$e'_{2i} = d'_{1i}c_r^2 + d'_{2i}c_r + d'_{3i} + (1-c_r^2)^{\frac{1}{2}}[d'_{1r}c_r - \frac{1}{2}d'_{1r}], \quad \textbf{(51.b)}$$

$$e'_{4r} = j'_{1r}c_r^2 + j'_{2r}c_r + j'_{3r} + (1-c_r^2)^{\frac{1}{2}}[-j'_{1i}c_r + j'_{3i}], \quad \textbf{(51.c)}$$

$$e'_{4i} = j'_{1i}c_r^2 + j'_{2i}c_r + j'_{3i} + (1-c_r^2)^{\frac{1}{2}}[j'_{1r}c_r - j'_{3r}]. \quad \textbf{(51.d)}$$

The remaining unknowns are given in Appendix.

This second case determines the 2D stationary hole solution.

The 2D propagating hole solutions obtained in the first case can be written:

$$U(x,y,t) = \frac{\mu[1+(c_r+ic_i)e^{(kx+ly+\omega t+\xi^{(0)})}]}{\left[1+e^{(kx+ly+\omega t+\xi^{(0)})}+e^{2(kx+ly+\omega t+\xi^{(0)})}\right]^{\frac{1}{2}+i\alpha}}e^{i(K_1x+K_2y-\Omega t)}, \quad \textbf{(52.a)}$$

$$V(x,y,t)=\frac{\eta[1+(c_r-ic_i)e^{(kx+ly+\omega t+\xi^{(0)})}]}{\left[1+e^{(kx+ly+\omega t+\xi^{(0)})}+e^{2(kx+ly+\omega t+\xi^{(0)})}\right]^{\frac{1}{2}-i\alpha}}e^{-i(K_1x+K_2y-\Omega t)}. \quad (\mathbf{52.b})$$

After some arrangements these solutions lead to:

$$U(x,y,t)=\frac{\mu(c_r+ic_i)e^{\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}+\mu e^{-\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}}{\left[1+e^{(kx+ly+\omega t+\xi^{(0)})}+e^{-(kx+ly+\omega t+\xi^{(0)})}\right]^{\frac{1}{2}}}\times$$
$$e^{i\{(K_1'x+K_2'y-\Omega' t)-\alpha\xi^{(0)}-\alpha\ln[2\cosh(kx+ly+\omega t+\xi^{(0)})+1]\}}, (\mathbf{53.a})$$

$$V(x,y,t)=\frac{\eta(c_r-ic_i)e^{\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}+\eta e^{-\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}}{\left[1+e^{(kx+ly+\omega t+\xi^{(0)})}+e^{-(kx+ly+\omega t+\xi^{(0)})}\right]^{\frac{1}{2}}}\times$$
$$e^{-i\{(K_1'x+K_2'y-\Omega' t)-\alpha\xi^{(0)}-\alpha\ln[2\cosh(kx+ly+\omega t+\xi^{(0)})+1]\}}, (\mathbf{53.b})$$

Where

$$K_1'=K_1-\alpha k, K_2'=K_2-\alpha l, \Omega'=\Omega+\alpha\omega. \quad (\mathbf{54})$$

It can be easily easily seen that the solutions (53.a) and (53.b) have the following asymptotic behavior for $\zeta=kx+ly+\omega t+\xi^{(0)}$

$$\lim_{\zeta\longrightarrow+\infty}U(\zeta)=\mu(c_r+ic_i)e^{i[k_1x+l_1y-\Omega_1't-2\alpha\xi^{(0)}]}, \quad (\mathbf{55.a})$$

$$\lim_{\zeta\longrightarrow+\infty}V(\zeta)=\eta(c_r-ic_i)e^{-i[k_1x+l_1y-\Omega_1't-2\alpha\xi^{(0)}]}, \quad (\mathbf{55.b})$$

$$\lim_{\zeta\longrightarrow-\infty}U(\zeta)=\mu e^{i[k_2x+l_2y-\Omega_2't]}, \quad (\mathbf{55.c})$$

$$\lim_{\zeta\longrightarrow-\infty}V(\zeta)=\eta e^{-i[k_2x+l_2y-\Omega_2't]}, \quad (\mathbf{55.d})$$

with

$$k_1=K_1'-\alpha k, l_1=K_2'-\alpha l, \Omega_1'=\Omega'-\alpha\omega, \quad (\mathbf{56.a})$$

$$k_2=K_1'+\alpha k, l_2=K_2'+\alpha l, \Omega_2'=\Omega'+\alpha\omega. \quad (\mathbf{56.b})$$

This result shows that solutions (**53.a**) and (**53.b**) connect two waves with different wave vectors. The solution $U(x,y,t)$ (Eq.(53.a)) behaves like a homogeneous wave of amplitude $\mu(c_r+ic_i)$ and wave vector $\vec{k}=(k_1,l_1)$ as $\zeta\longrightarrow+\infty$; for $\zeta\longrightarrow-\infty$, it corresponds to a wave of amplitude μ and wave vector $\vec{k'}=(k_2,l_2)$. If we assume that μ is real, the parameter $\mu(c_r+ic_i)$ can then be written as $\mu(c_r+ic_i)=\mu|c|exp(i\theta)$

where

$$\cos(\theta)=\frac{c_r}{\sqrt{c_r^2+c_i^2}}, \quad (\mathbf{58.a})$$

$$\sin(\theta)=\frac{c_i}{\sqrt{c_r^2+c_i^2}}. \quad (\mathbf{58.b})$$

Eqs.(58.a) and (58.b) represent the phase shift $\theta\equiv\arg(\mu c,\mu)$ between the two asymptotic waves. The 2D stationary hole solutions read:

$$U(x,y,t)=\frac{\mu(c_r+ic_i)e^{\frac{1}{2}(kx+ly+\xi^{(0)})}+\mu e^{-\frac{1}{2}(kx+ly+\xi^{(0)})}}{\left[1+e^{(kx+ly+\xi^{(0)})}+e^{-(kx+ly+\xi^{(0)})}\right]^{\frac{1}{2}}}e^{i\{(K_1'x+K_2'y-\Omega't)-\alpha\xi^{(0)}-\alpha\ln[2\cosh(kx+ly+\xi^{(0)})+1]\}},\quad (\mathbf{59}.\mathbf{a})$$

$$V(x,y,t)=\frac{\eta(c_r-ic_i)e^{\frac{1}{2}(kx+ly+\xi^{(0)})}+\eta e^{-\frac{1}{2}(kx+ly+\xi^{(0)})}}{\left[1+e^{(kx+ly+\xi^{(0)})}+e^{-(kx+ly+\xi^{(0)})}\right]^{\frac{1}{2}}}e^{-i\{(K_1'x+K_2'y-\Omega't)-\alpha\xi^{(0)}-\alpha\ln[2\cosh(kx+ly+\xi^{(0)})+1]\}}.\quad (\mathbf{59}.\mathbf{b})$$

We should recall that these solutions are obtained in the case 2 with its two subcases.

4. The degeneracies of the (2+1)-dimensional CCQGL equation

The singularity structure of the (2+1)-D CCQGL equation degenerates in the limiting case of $\alpha = 0$, and correspond to $I_m\left(\frac{2p}{\gamma}\right) = 0$. This put the following restriction on the allowed coefficients in this equation,

$$p_r\gamma_i - p_i\gamma_r = 0. \quad (\mathbf{60})$$

The constraint (60) has to be solved by respecting the leading-order balance (**3**.**a**) and (**3**.**b**). Leading-order analysis then leads to:

$$U_0^2V_0^2 = -\frac{3}{4|\gamma|^2}(p_r\gamma_r + p_i\gamma_i)(\phi_x^2+\phi_y^2). \quad (\mathbf{61})$$

The special solutions of the degeneracies of the CCQGL equation may be obtained if we proceed in a similar way as presented in the previous section.

4.1. **2D Pulses**

. The pulse solutions read

$$U(x,y,t)=\frac{\mu}{[2\cosh(kx+ly+\omega t+\xi^{(0)})+d]^{\frac{1}{2}}}e^{i(K_1x+K_2y-\Omega t)}, \quad (\mathbf{62}.\mathbf{a})$$

$$V(x,y,t)=\frac{\eta}{[2\cosh(kx+ly+\omega t+\xi^{(0)})+d]^{\frac{1}{2}}}e^{-i(K_1x+K_2y-\Omega t)}, \quad (\mathbf{62}.\mathbf{b})$$

and necessary respect $p_r(kK_1 + lK_2) = 0$. We distinguish two cases:

Case 1: $p_r \neq 0$ A discrete set of stationary pulses is found in a co-dimension one subspace of parameter space defined by

$$p_r\gamma_i - p_i\gamma_r = 0$$

The unknowns are fixed by:

$$\mu^2\eta^2 = \frac{3(4-d^2)(p_r\gamma_r+p_i\gamma_i)}{4|\gamma|^2}(k^2+l^2) > 0, d \neq \{2,-2\}, \quad (\mathbf{63}.\mathbf{a})$$

$$\omega = 0, \quad (\mathbf{63}.\mathbf{b})$$

$$K_2 = -\frac{k}{l}K_1, \quad (\mathbf{63}.\mathbf{c})$$

$$\Omega = -p_i(K_1^2+K_2^2) + \frac{1}{4}p_i(k^2+l^2), \quad (\mathbf{63}.\mathbf{d})$$

$$k^2 + l^2 = R_0^2, \quad (\mathbf{63}.\mathbf{e})$$

$$R_0^2 = \frac{3(4-d^2)(p_r\gamma_r+p_i\gamma_i)}{4|\gamma|^2}\frac{q_r^2}{p_r^2d^2}, \quad (\mathbf{63}.\mathbf{f})$$

$$K_1^2 = l^2 \left[\frac{1}{4} - \frac{a}{p_r R_0^2} \right] > 0. \qquad (\mathbf{63}.\mathbf{g})$$

d is free parameter.

Restrictions (**60**) include the case of (2+1)-dimensional real CQ Ginzburg-Landau equation (RCQGL: $p_i = q_i = \gamma_i = 0$):

$$U_t + p_r(U_{xx} + U_{yy}) = aU - q_r|U|^2U - \gamma_r|U|^4U. \qquad (\mathbf{64})$$

Unknowns are defined by:

$$\mu^2\eta^2 = \frac{3(4-d^2)p_r}{4\gamma_r}(k^2+l^2), d \neq \{2,-2\}, \qquad (\mathbf{65}.\mathbf{a})$$

$$\omega = 0, \qquad (\mathbf{65}.\mathbf{b})$$

$$K_2 = -\frac{k}{l}K_1, \qquad (\mathbf{65}.\mathbf{c})$$

$$\Omega = 0, \qquad (\mathbf{65}.\mathbf{d})$$

$$k^2 + l^2 = R_1^2, \qquad (\mathbf{63}.\mathbf{e})$$

$$R_1^2 = \frac{3(4-d^2)}{4p_r\gamma_r}\frac{q_r^2}{d^2}, \qquad (\mathbf{65}.\mathbf{f})$$

$$K_1^2 = l^2 \left[\frac{1}{4} - \frac{a}{p_r R_1^2} \right] > 0. \qquad (\mathbf{65}.\mathbf{g})$$

with $(4-d^2)p_r\gamma_r > 0$. d is free parameter.

Case 2: $p_r = 0$

A three-parameter family of propagating pulses is found for the cubic-quintic nonlinear Schrödinger equation ($p_r = q_r = \gamma_r = a = 0$):

$$-iU_t + p_i(U_{xx} + U_{yy}) = -q_i|U|^2U - \gamma_i|U|^4U. \qquad (\mathbf{66})$$

The unknowns $\omega, \Omega, \mu\eta$, k and l are parametrized by K_1, K_2 and d

$$\mu^2\eta^2 = \frac{3(4-d^2)p_i}{4\gamma_i}(k^2+l^2), d \neq \{2,-2\}, \qquad (\mathbf{67}.\mathbf{a})$$

$$\omega = 2p_i(kK_1 + lK_2), \qquad (\mathbf{67}.\mathbf{b})$$

$$\Omega = -p_i(K_1^2 + K_2^2) + \frac{1}{4}p_i(k^2+l^2), \qquad (\mathbf{67}.\mathbf{c})$$

$$k^2 + l^2 = R_2^2, \qquad (\mathbf{67}.\mathbf{d})$$

$$R_2^2 = \frac{3(4-d^2)}{4p_i\gamma_i}\frac{q_i^2}{d^2}, \qquad (\mathbf{67}.\mathbf{f})$$

with $(4-d^2)p_i\gamma_i > 0$.

4.2. **2D Propagating hole solutions**

. These solutions read:

$$U(x,y,t)=\frac{\mu(c_r+ic_i)e^{\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}+\mu e^{-\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}}{\left[1+e^{(kx+ly+\omega t+\xi^{(0)})}+e^{-(kx+ly+\omega t+\xi^{(0)})}\right]^{\frac{1}{2}}}e^{i\{(K_1x+K_2y-\Omega t)\}}, \qquad (\mathbf{68}.\mathbf{a})$$

$$V(x,y,t)=\frac{\eta(c_r-ic_i)e^{\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}+\eta e^{-\frac{1}{2}(kx+ly+\omega t+\xi^{(0)})}}{\left[1+e^{(kx+ly+\omega t+\xi^{(0)})}+e^{-(kx+ly+\omega t+\xi^{(0)})}\right]^{\frac{1}{2}}}e^{-i\{(K_1x+K_2y-\Omega t)\}}, \qquad (\mathbf{68}.\mathbf{b})$$

The unknowns parameters $k, l, \omega, \mu\eta, K_1, K_2, \Omega, c_r$ and c_i are defined by

$$\mu^2\eta^2=\frac{e_1'}{(2c_r-1)^2|\gamma|^2}(k^2+l^2), \qquad (\mathbf{69}.\mathbf{a})$$

$\omega=-\frac{1}{2c_i}\{[(2c_r-1)p_i+2c_ip_r](k^2+l^2)+2[(2c_r-1)p_r-2c_ip_i](kK_1+lK_2)+2(2c_r-1)q_i\mu\eta+4(2c_r-1)\gamma_i\mu^2\eta^2\}$, (**69.b**)

$$\Omega=-p_i(K_1^2+K_2^2)+q_i\mu\eta+\gamma_i\mu^2\eta^2, \qquad (\mathbf{69}.\mathbf{c})$$

$$k^2+l^2=R_0'^2, \qquad (\mathbf{69}.\mathbf{d})$$

$$K_2=\frac{1}{l}(e_2R_0'^2+e_3R_0')-\frac{k}{l}K_1, \qquad (\mathbf{69}.\mathbf{e})$$

$$c_r^2+c_i^2=1. \qquad (\mathbf{69}.\mathbf{f})$$

K_1 satisfies the following quadratic equation

$$p_rR_0'^2K_1^2-2p_r[e_2R_0'^2+e_3R_0']kK_1+\left\{p_r[e_2R_0'^2+e_3R_0']^2-l^2\left[\frac{e_1'\gamma_r}{(2c_r-1)^2|\gamma|^2}R_0'^2+\frac{\sqrt{e_1'}q_r}{|2c_r-1||\gamma|}R_0'-a\right]\right\}=0. \qquad (\mathbf{70})$$

The remaining unknowns are given by

$$e_1'=\frac{9}{4}(p_r\gamma_r+p_i\gamma_i)>0, \qquad (\mathbf{71}.\mathbf{a})$$

$$R_0'^2=e_1'\frac{4(2c_r-1)^2|\gamma|^2[(c_r-2)q_i+c_iq_r]^2}{\{(5-4c_r)|\gamma|^2[(c_r-2)p_i+c_ip_r]+4e_1'[(c_r-2)\gamma_i+c_i\gamma_r]\}^2}, \qquad (\mathbf{71}.\mathbf{b})$$

$$e_2=\frac{2e_1'[(2c_r-1)\gamma_i+2c_i\gamma_r]}{3(2c_r-1)|\gamma|^2p_r}-\frac{[(4c_r-5)p_i+4c_ip_r]}{6p_r}, \qquad (\mathbf{71}.\mathbf{c})$$

$$e_3=\frac{(2c_r-1)\sqrt{e_1'}[(2c_r-1)q_i+2c_iq_r]}{3|2c_r-1||\gamma|p_r}. \qquad (\mathbf{71}.\mathbf{d})$$

These solutions are obtained under the following constraints

$$p_r\gamma_i-p_i\gamma_r=0, \qquad (\mathbf{72}.\mathbf{a})$$

$$[(5-4c_r)|\gamma|^2p_i+4e_1'\gamma_i]q_r-[(5-4c_r)|\gamma|^2p_r+4e_1'\gamma_r]q_i=0. \qquad (\mathbf{72}.\mathbf{b})$$

Restrictions (**60**) include the case of CQ nonlinear Schrödinger equation Eq.(66) the parameters are defined as:

$$\mu^2\eta^2 = \frac{9p_i}{4(2c_r-1)^2\gamma_i}(k^2+l^2), \qquad (\mathbf{73.a})$$

$$\omega = \frac{2c_i}{2c_r-1}p_i(k^2+l^2)+2p_i(kK_1+lK_2), \qquad (\mathbf{73.b})$$

$$\Omega = -p_i(K_1^2+K_2^2)+q_i\mu\eta+\gamma_i\mu^2\eta^2, \qquad (\mathbf{73.c})$$

$$k^2+l^2 = \frac{9(2c_r-1)^2q_i^2}{4(7-2c_r)^2}\frac{1}{p_i\gamma_i}, \qquad (\mathbf{73.d})$$

$$c_r^2+c_i^2 = 1. \qquad (\mathbf{73.e})$$

with $p_i\gamma_i > 0$, K_1 and K_2 are free parameters. Restrictions (**60**) also include the case of RCQGL equation (Eq.(64)). The parameters can be computed from:

$$\mu^2\eta^2 = \frac{9p_r}{4(2c_r-1)^2\gamma_r}(k^2+l^2), \qquad (\mathbf{74.a})$$

$$\omega = 0, \qquad (\mathbf{74.b})$$

$$\Omega = 0, \qquad (\mathbf{74.c})$$

$$K_2 = \frac{-c_i}{l(2c_r-1)}(k^2+l^2)-\frac{k}{l}K_1, \qquad (\mathbf{74.d})$$

$$k^2+l^2 = R_1'^2, \qquad (\mathbf{74.e})$$

$$R_1'^2 = \frac{9(2c_r-1)^2q_r^2}{4(7-2c_r)^2}\frac{1}{p_r\gamma_r}, \qquad (\mathbf{74.f})$$

$$c_r^2+c_i^2 = 1. \qquad (\mathbf{74.g})$$

With $p_r\gamma_r > 0$.

K_1 is solution of the following quadratic equation
$(2c_r-1)^2p_rR_1'^2K_1^2+2[p_rc_i(2c_r-1)R_1'^2]kK_1+\left\{c_i^2p_rR_1'^4-l^2\left[\frac{9}{4}p_rR_1'^2+\frac{3}{2}q_r|2c_r-1|\sqrt{\frac{p_r}{\gamma_r}}R_1'-(2c_r-1)^2a\right]\right\}$
$=0.(\mathbf{75})$

5. CONCLUSION

Based on the (2+1)-dimensional complex cubic-quintic Ginzburg-Landau (CCQGL) equation, we have investigated exact analytic localized solutions. Using a method derived from a combination of the methodology of Painlevé test for integrability due to Weiss-Tabor-Carnevale and the Hirota multilinear methodology, we have found that a number of exact solutions exist for this equation provided that coefficients are bound by special relations. The set of solutions include particular types of localized structures such as: 2D propagating pulse, 2D stationary pulse, 2D propagating hole and 2D stationary hole solutions. It appears that this combined methodology is a very powerful method, and we hope that the method may also be used to other (2+1)-dimensional family of Ginzburg-Landau equation like, such as the coupled GL equation with waves that travelled right and left.

By analysing the degeneracies of the (2+1)-D CCQGL equation, it appears that this equation is a generalization of the (2+1)-D CQ NLS and of the (2+1)-D Real CQ GL equation.

6. APPENDIX

The coefficients related to Eqs.(41.a) - (41.d) are:

$$d_{1r} = 2\left[(p_r - 2\alpha p_i)|\gamma|^2 - \frac{2e\gamma_i}{\alpha}\right], d_{1i} = 2\left[(p_i + 2\alpha p_r)|\gamma|^2 + \frac{2e\gamma_r}{\alpha}\right], \qquad (\mathbf{A.1})$$

$$d_{2r} = 2\left\{[\alpha(p_i + 2\alpha p_r) - (p_r - 2\alpha p_i)]\,|\gamma|^2 + e\left(2\gamma_r + \frac{\gamma_i}{\alpha}\right)\right\}, \qquad (\mathbf{A.2.a})$$

$$d_{2i} = -2\left\{[\alpha(p_r - 2\alpha p_i) + (p_i + 2\alpha p_r)]\,|\gamma|^2 - e\left(2\gamma_i - \frac{\gamma_r}{\alpha}\right)\right\}, \qquad (\mathbf{A.2.b})$$

$$d_{3r} = -\frac{1}{2}\{[2\alpha(p_i + 2\alpha p_r) - (p_r - 2\alpha p_i)]|\gamma|^2 + 4e\gamma_r\}, \qquad (\mathbf{A.3.a})$$

$$d_{3i} = \frac{1}{2}\{[2\alpha(p_r - 2\alpha p_i) + (p_i + 2\alpha p_r)]|\gamma|^2 - 4e\gamma_i\}, \qquad (\mathbf{A.3.b})$$

$$d_{4r} = -|\gamma|^2(p_r - 2\alpha p_i), d_{4i} = |\gamma|^2(p_i + 2\alpha p_r), \qquad (\mathbf{A.4})$$

$$j_{1r} = 4\alpha\left\{-2|\gamma|^2\left[\left(\frac{3}{4} - \alpha^2\right)p_r - 2\alpha p_i\right] + e\left(2\gamma_r + \frac{\gamma_i}{\alpha}\right)\right\}, \qquad (\mathbf{A.5.a})$$

$$j_{1i} = 4\alpha\left\{-2|\gamma|^2\left[\left(\frac{3}{4} - \alpha^2\right)p_i + 2\alpha p_r\right] + e\left(2\gamma_i - \frac{\gamma_r}{\alpha}\right)\right\}, \qquad (\mathbf{A.5.b})$$

$$j_{2r} = 4\alpha\left\{3|\gamma|^2\left[\left(\frac{3}{4} - \alpha^2\right)p_r - 2\alpha p_i\right] - e\left(3\gamma_r - \frac{\gamma_i}{\alpha}\right)\right\}, \qquad (\mathbf{A.6.a})$$

$$j_{2i} = 4\alpha\left\{3|\gamma|^2\left[\left(\frac{3}{4} - \alpha^2\right)p_i + 2\alpha p_r\right] - e\left(3\gamma_i + \frac{\gamma_r}{\alpha}\right)\right\}, \qquad (\mathbf{A.6.b})$$

$$j_{3r} = 4\alpha\left\{-|\gamma|^2\left[\left(\frac{3}{4} - \alpha^2\right)p_r - 2\alpha p_i\right] + e\gamma_r\right\}, \qquad (\mathbf{A.7.a})$$

$$j_{3i} = 4\alpha \left\{ -|\gamma|^2 \left[\left(\frac{3}{4} - \alpha^2 \right) p_i + 2\alpha p_r \right] + e\gamma_i \right\}. \qquad (\mathbf{A.7.b})$$

The coefficients related to Eqs.(51.a) - (51.d) are defined by

$$d'_{1r} = 2(p_r - 2\alpha p_i)|\gamma|^2, d'_{1i} = 2(p_i + 2\alpha p_r)|\gamma|^2, \qquad (\mathbf{A.8})$$

$$d'_{2r} = 2[\alpha(p_i + 2\alpha p_r) - (p_r - 2\alpha p_i)]|\gamma|^2, d'_{2i} = -2[\alpha(p_r - 2\alpha p_i) + (p_i + 2\alpha p_r)]|\gamma|^2, \qquad (\mathbf{A.9})$$

$$d'_{3r} = -\frac{1}{2}\{[2\alpha(p_i + 2\alpha p_r) - (p_r - 2\alpha p_i)]|\gamma|^2 - 12e\gamma_r\}, \qquad (\mathbf{A.10.a})$$

$$d'_{3i} = \frac{1}{2}\{[2\alpha(p_r - 2\alpha p_i) + (p_i + 2\alpha p_r)]|\gamma|^2 + 12e\gamma_i\}, \qquad (\mathbf{A.10.b})$$

$$j'_{1r} = -4|\gamma|^2 \left[\left(\frac{3}{4} - \alpha^2 \right) p_r - 2\alpha p_i \right], \qquad (\mathbf{A.11.a})$$

$$j'_{1i} = -4|\gamma|^2 \left[\left(\frac{3}{4} - \alpha^2 \right) p_i + 2\alpha p_r \right], \qquad (\mathbf{A.11.b})$$

$$j'_{2r} = 6 \left\{ |\gamma|^2 \left[\left(\frac{3}{4} - \alpha^2 \right) p_r - 2\alpha p_i \right] + e\gamma_r \right\}, \qquad (\mathbf{A.12.a})$$

$$j'_{2i} = 6 \left\{ |\gamma|^2 \left[\left(\frac{3}{4} - \alpha^2 \right) p_i + 2\alpha p_r \right] + e\gamma_i \right\}, \qquad (\mathbf{A.12.b})$$

$$j'_{3r} = -2 \left\{ |\gamma|^2 \left[\left(\frac{3}{4} - \alpha^2 \right) p_r - 2\alpha p_i \right] + 3e\gamma_r \right\}, \qquad (\mathbf{A.13.a})$$

$$j'_{3i} = -2 \left\{ |\gamma|^2 \left[\left(\frac{3}{4} - \alpha^2 \right) p_i + 2\alpha p_r \right] + 3e\gamma_i \right\}. \qquad (\mathbf{A.13.b})$$

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