Ricci-Flat Kähler Metrics on Symmetric Varieties

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Dedicated to Professor Masaru Takeuchi

ABSTRACT. We prove the existence of a complete Ricci-flat Kähler metric on symmetric varieties and describe its asymptotic behavior at infinity.

1. Introduction

(1.1) By a symmetric variety we mean the complexification $G^{\mathbf{C}}/K^{\mathbf{C}}$ of a Riemannian symmetric space G/K of compact type. In this paper we study the existence of G-invariant complete Ricci-flat Kähler metric on symmetric varieties. Our main result is

Theorem. *Let $G^{\mathbf{C}}/K^{\mathbf{C}}$ be a symmetric variety. Then there exists a G-invariant complete Ricci-flat Kähler metric on $G^{\mathbf{C}}/K^{\mathbf{C}}$.*

The resulting metric has maximal volume growth (namely Euclidean volume growth). But, if $\operatorname{rank}(G/K) > 1$, curvature decay does not obey the inverse quadratic law. Some of hyper-Kähler examples of complete Ricci-flat Kähler manifolds in Theorem (1.1) were observed by several authors. Here we mention Calabi [C], Hitchin . *et al.* [HKLR] and Kronheimer [Kr].

(1.2) There are several motivations for this work.

(1) According to [AL1], all G-invariant strictly plurisubharmonic function f on $G^{\mathbf{C}}/K^{\mathbf{C}}$ are qualitatively non distinguishable, namely f is a proper exhaustion function whose critical points occur only at the unique totally real orbit G/K. It is then natural to ask how to distinguish these functions. One natural idea is to relate them with the $G^{\mathbf{C}}$-invariant top degree holomorphic form η. Indeed, if $\sqrt{-1}\partial\bar\partial P$ is a Ricci-flat complete Kähler metric in Theorem (1.1), then $\bigwedge^{\mathrm{top}}(\sqrt{-1}\partial\bar\partial P) = \eta \wedge \bar\eta$ holds.

(2) Special types of complete noncompact Ricci-flat Kähler manifolds appear in geometry as building blocks of, or, as bubbles out of, general compact Ricci-flat Kähler manifolds. So it is natural to construct fundamental building blocks for these manifolds. If we look at deformation and degeneration of hyper-Kähler metrics in compact manifolds, such objects appear naturally. A Lagrangian submanifold $M \subset X$ (in the sense of holomorphic symplectic geometry) which can be blown down to a point is a natural candidate in this category. Indeed, the possibility of being blown down yields many linear functions in a tubular neighborhood of M in X, which is isomorphic to T^*M. Hence M should have plenty of holomorphic

vector fields. More generally, let $M \subset X$ be a Lagrangean submanifold of a Kähler manifold X. Suppose that there exist a subvariety $M' \subset X$ with $\dim M' < \dim M$, a deformation J_t of complex structures of X and Kähler metrics g_t on (X, J_t) such that g_t are Ricci-flat (or more generally Ricci bounded) and that $\mathrm{Vol}(X, g_t)$ is constant and $\mathrm{Vol}(M \cup M', g_t|_{M \cup M'})$ goes to zero as $t \to 0$. Suppose moreover that the contraction $X/M \cup M'$ of $M \cup M'$ in X (with the complex structure J_0) is realized as a projective variety in some projective space. Then, although not clear at all, it seems plausible that M has many Killing vector fields with respect to some Riemannian metric. Thus, it is natural that, as a starting point, we consider T^*M with $M = G/K$ a symmetric space (note that $G^{\mathbf{C}}/K^{\mathbf{C}}$ is diffeomorphic to T^*M). We discuss in this paper only the existence problem of Ricci-flat Kähler metrics (the hyper-Kähler structure on $G^{\mathbf{C}}/K^{\mathbf{C}}$ when G/K is Hermitian symmetric will be studied in a separate paper).

(3) In [BK1,2] and [TY] the smoothness of the divisor at infinity was assumed. It is then natural to try to remove this condition. To get a good analytical feeling for the Ricci-flat metric, we start the study of this problem with examples of complements of normal crossing divisors of Fano manifolds with big symmetry. As complete Ricci-flat Kähler manifolds in [BK2] and [TY] are with inverse quadratic curvature decay, it is also natural to seek the construction of examples of Ricci-flat complete Kähler manifolds with maximal volume growth whose curvature decay does not obey the inverse quadratic law.

(1.3) The technical ingredients of this paper are the canonical compactification of symmetric varieties due to DeConcini-Procesi [DP] and the analytical technique used by Bando and the second author in [BK1,2] to prove the existence of a complete Ricci-flat Kähler metric on certain affine algebraic manifolds. In [DP] a $G^{\mathbf{C}}$-equivariant compactification of $G^{\mathbf{C}}/K^{\mathbf{C}}$ is constructed. The compactification is a Fano manifold X and the divisor at infinity consists of r ($r = \mathrm{rank}(G/K)$) smooth hypersurfaces $D = \cup_{i=1}^{r} D_i$ intersecting transversally. The G-orbits are the affine parts of $D_{i_1 \cdots i_k} = D_{i_1} \cap \cdots \cap D_{i_k}$. Each $D_{\underline{i}}$ is blown down in X to $G^{\mathbf{C}}/P_{\underline{i}}$ by a $G^{\mathbf{C}}$-equivariant birational holomorphic map $\Pi_{\underline{i}} : X \to \Pi_{\underline{i}}(X)$, where $\underline{i} = \{i_1, \cdots, i_k\}$ and $P_{\underline{i}}$ is a parabolic subgroup of $G^{\mathbf{C}}$ determined by $\underline{i}$. Using these equivariant blow downs, we construct a background metric $\Omega(d_{ij})$ which is a complete Kähler metric on $X - D$ whose Ricci curvature decays with order like $(\text{distance})^{-4-\varepsilon_n}$ with a positive constant ε_n dependeng on n only. The proof of Theorem (1.1) is done by the continuity method. This needs a background metric. Because we will work on non-compact spaces, the choice of background metrics is very important to get good *a priori* estimates. The main difficulty in the proof of Theorem (1.1) lies in the procedure of constructing a background metric, especially the difficulty caused by fact that D is not smooth if $\mathrm{rank}(G/K) > 1$. We overcome this difficulty by introducing a concept of bifurcation of Kähler potentials at $\mathrm{Sing}(D)$ (see (5.2.1) and (5.4.3)) and the existence of $G^{\mathbf{C}}$-equivariant blow downs $\Pi_{\underline{i}}$ of $D_{\underline{i}}$. Bifurcated Kähler potential will produce linearly independent $|\underline{i}|$ directions pointing to infinity near $D_{\underline{i}}$. The parameters (d_{ij}) stand for those involved in the definition of the background metric. Such parameters appear because we construct the background metric by considering bifurcation of Kähler potential at any intersection $D_{\underline{i}}$ and the bifurcation contains some degree of freedom described by (d_{ij}). The existence of blowing downs Π_i allows us to prove Theorem (1.1) by induction on the rank. Now almost the same analysis as in [BK2] can be applied to solve the Ricci-flat equation

$$(\Omega(d_{ij})+\sqrt{-1}\partial\bar{\partial}u)^n = e^{-f}\Omega(d_{ij})^n = \eta\wedge\bar{\eta},$$

where η is a $G^{\mathbf{C}}$-invariant holomorphic n-form on $G^{\mathbf{C}}/K^{\mathbf{C}}$. As the Ricci curvature of $\Omega(d_{ij})$ decays with order like $(\text{distance})^{-4-\epsilon_n}$, the function f in the above equation also decays with order like $(\text{distance})^{-2-\epsilon_n}$. This, with the technique in [BK2], allows us to get a priori decay estimates for the solution u. More precisely we will use the continuity method as in [Y1], [BK2] and [TY]. Applying the Sobolev inequalities and maximum principle together with decay estimates for f, which follow from the construction of a good background metric, we get *a priori* C^0 and decay estimates. Then, with the Bochner technique, we derive *a priori* C^2 estimates. Then we apply general technique from analysis ([GT]) together with decay estimates of f to get $C^{k,\alpha}$ estimates on appropriate $C^{k,\alpha}$ space constructed on the basis of the complete metric $\Omega(d_{ij})$. Then the Ascoli-Arzela type argument completes the continuity method to show the existence of a solution to the Ricci-flat equation with *a priori* estimates. Thus we can examine the asymptotic behavior of the Ricci-flat metric via the background metric (see (5.3.11)).

(1.4) This paper is organized as follows. In Chapter 2 we give a Riemannian geometric exposition of the Jensen-Lassalle decomposition. This decomposition is important in describing G-invariant objects defined on $G^{\mathbf{C}}/K^{\mathbf{C}}$. In Chapter 3 we describe some general properties of G-invariant Kähler metrics on $G^{\mathbf{C}}/K^{\mathbf{C}}$. In particular we have emphasized the role of the split torus and the unique totally real orbit $G/K\subset G^{\mathbf{C}}/K^{\mathbf{C}}$. Chapter 4 is concerned with Ricci-flat Kähler metrics on symmetric varieties of rank one. Some results of Borel-Hirzebruch [BH] on representation theory, sufficient for our purposes, are also explained there. Chapter 5 is the main part of this paper. Here, after reviewing the DeConcini-Procesi compactification, we construct a background metric in rank two case in detail. A similar construction is described for higher rank symmetric varieties, which is combinatorially more complicated but analytically it is quite similar to the rank two case. Here a modification of techniques in [BK2] is applied to solve the Ricci-flat equation. The main arguments of this paper can be described in a more abstract setting if we replace the group action by the abstract hypothesis on the existence of Kähler-Einstein metric on $D_{1\cdots r}$, but we did not do so as the conditions are suggested naturally in a Lie group theoretic setting. Without Kähler-Einstein condition at infinity (corresponding to $D_{1\cdots r}$) the problem becomes much harder analytically (cf. [K2]).

It is our pleasure to dedicate this paper to Professor Masaru Takeuchi. The second named author is grateful to Professor Takeuchi for introducing him to Kähler-Einstein manifolds when he was a graduate student.

Notations and Standard Definitions on Symmetric Varieties

Let G be a compact connected semisimple Lie group and K a closed subgroup such that there exists an involutive automorphism θ of G with $(G^\theta)_0 \subset K \subset G^\theta$. Then the homogeneous space G/K with a G-invariant Riemannian metric is a Riemannian symmetric space of compact type. It is known that this has nonnegative sectional curvature. Let $\underline{g}$ denote the Lie algebra of G, *e.t.c.*. We set $\underline{l} = \{X \in \underline{g};\ d\theta(X) = -X\}$. Hereafter we write simply θ instead of $d\theta$ for the differential. The Cartan decomposition $\underline{g} = \underline{k} + \underline{l}$ is the decomposition of $\underline{g}$ into ± 1 eigenspaces w.r.to the involutive automorphism θ. Then $\underline{l}$ is naturally identified with the tangent space of G/K at the origin $\xi_0 = eK$. There is a noncompact semi-simple Lie group G' with Lie algebra $\underline{g}' = \underline{k} + i\underline{l}$. The homogeneous space G'/K with a G'-invariant Riemannian metric is a noncompact symmetric space called the noncompact dual of G/K. It is known that G'/K has nonpositive sectional curvature. The rank of the symmetric space is the dimension of a maximal flat totally geodesic submanifold or equivalently the dimension of a maximal abelian subspace of $\underline{l}$. Set $r = \mathrm{rank}(G/K)$. Let $G^{\mathbf{C}}$ be the complexification of G. Then $G^{\mathbf{C}}$ is a reductive algebraic group. Let $K^{\mathbf{C}}$ be the complexification of K in $G^{\mathbf{C}}$. Hereafter we assume for simplicity that $K^{\mathbf{C}}$ is the largest subgroup of $G^{\mathbf{C}}$ with the same Lie algebra. (If it is not the case, there exists a largest subgroup $\widetilde{K}^{\mathbf{C}}$ with the same Lie algebra as $K^{\mathbf{C}}$ and so there is a finite covering $G^{\mathbf{C}}/K^{\mathbf{C}} \to G^{C}/\widetilde{K}^{\mathbf{C}}$. We will study geometry and analysis on these spaces which are equivariant under this covering.) The noncompact complex homogeneous space $G^{\mathbf{C}}/K^{\mathbf{C}}$ is the **symmetric variety** associated to the symmetric space G/K. This contains G/K and G'/K as real manifolds. In Chapter 2 we consider $G^{\mathbf{C}}/K^{\mathbf{C}} = (G'/K)^{\mathbf{C}}$ and in other chapters $G^{\mathbf{C}}/K^{\mathbf{C}} = (G/K)^{\mathbf{C}}$. We extend θ to an involutive automorphism of $G^{\mathbf{C}}$ and denote it again as θ. Then we have the Cartan decomposition $\underline{g}^{\mathbf{C}} = \underline{k}^{\mathbf{C}} + \underline{l}^{\mathbf{C}}$ and $\underline{l}^{\mathbf{C}}$ is naturally identified with the tangent space of $G^{\mathbf{C}}/K^{\mathbf{C}}$ at the origin $\xi_0 = eK^{\mathbf{C}}$ with the induced complex structure J. Let $\underline{a}$ be a maximal abelian subspace of $i\underline{l}$. (In Chapter 2 we will change real and imaginary parts and write it as $i\underline{a}$.) Then the exponential map is a diffeomorphism of $\underline{a}$ and the corresponding abelian subgroup A of $G^{\mathbf{C}}$: $\underline{a} \cong A \cong \mathbf{R}^r$. Throughout the paper we assume that G/K is irreducible.

2. The Jensen-Lassalle Decomposition

(2.1) In this section we describe the Jensen-Lassalle decomposition ([FJ], [L]) from a Riemannian geometric viewpoint.

(2.2) First we prove a decomposition theorem for non-compact semi-simple Lie groups by a Riemannian geometric method. Later we will apply it to the complexification of a compact group.
Let $M = G/K$ be a Riemannian symmetric space of non-compact type, i.e., with nonpositive sectional curvature, and set $\xi_0 = eK \in M$. Let σ be an involution of G such that $\sigma(K) \subset K$. Take a point $p \in M$. Now $M = G/K$ is a simply connected Riemannian manifold with nonpositive sectional curvature. Hence any two points are joined by a unique geodesic on M. Thus p and $\sigma(p)$ lie in a unique geodesic joining p and $\sigma(p)$. Let x be its mid point. Then we have $s_x(p) = \sigma(p)$ and $s_x(x) = x$, where s_x is the geodesic symmetry with center x. Hence we have $\sigma(x) = x$. As K is a maximal compact subgroup, we have the Cartan decomposition $G = KP$. As this decomposition is unique, we have a unique $a \in P$ such that $x = a \cdot \xi_0$. Now $\sigma(x) = x$ implies $\sigma(a) = a$. The geodesic through p and $\sigma(p)$ is of the form

$$\gamma(t) = a \cdot \exp(tX) \cdot \xi_0.$$

Then we have $s_x(\gamma(t)) = a\cdot\exp(-tX)\cdot\xi_0$ and $\sigma(\gamma(t)) = a\cdot\exp(t\sigma(X))\cdot\xi_0$. Since any two points are joined by a unique geodesic on a complete Riemannian manifold with nonpositive sectional curvature and σ is an isometry, we have $s_x(\gamma(t)) = \sigma(\gamma(t))$. We thus have $\sigma(X) = -X$ and so

$$p = \gamma(-1) = a \cdot \exp(\widetilde{X}) \cdot \xi_0$$

with $\sigma(\widetilde{X}) = -\widetilde{X}$ and with $a \cdot \xi_0$ the mid point of p and $\sigma(p)$. Let $\underline{g} = \underline{k} + \underline{p}$ be the Cartan decomposition at the Lie algebra level. The tangent space $T_{\xi_0}M$ is naturally identified with $\underline{p}$. Then σ is an involution which maps $\underline{k}$ to itself. Hence it maps $\underline{k}^\perp = \underline{p}$ into $\underline{p}$. Let $\underline{p}^\pm$ be the ± 1 eigenspace w.r.to σ. We thus have $G = \exp(\underline{p}^+) \cdot \exp(\underline{p}^-) \cdot K$ and so

$$G = K \cdot \exp(\underline{p}^-) \cdot \exp(\underline{p}^+).$$

This decomposition is unique. Indeed, suppose $p = ae^{-X} \cdot \xi_0 = a'e^{-X'} \cdot \xi_0$ with $a, a' \in \exp \underline{p}^+$ and $X, X' \in \underline{p}^-$. Then the geodesics $ae^{tX} \cdot \xi_0$ and $a'e^{tX'} \cdot \xi_0$ pass through p at $t = -1$ and both are mapped to the reverse geodesic via σ. So at $t = 1$ both geodesics pass through $\sigma(p)$. Therefore $a \cdot \xi_0 = a' \cdot \xi_0$ is the mid point of p and $\sigma(p)$. Hence $a = a'$. Now from the uniqueness of the Cartan decomposition we have $X = X'$. Thus the above decomposition is unique.

(2.3) Now let K be a compact semi-simple Lie group and σ an involutive automorphism of K. We then have the Cartan decomposition $\underline{k} = \underline{l} + \underline{m}$ and K/L is a compact symmetric space. Let $\underline{p} = i\underline{k} = i\underline{l} + i\underline{m}$ and extend σ linearly on $\underline{p}$. Then

$\underline{p}^+ = i\underline{l}$ and $\underline{p}^- = i\underline{m}$. Now we apply (2.2) to the non-compact symmetric space $K^{\mathbf{C}}/K$. It follows that

$$K^{\mathbf{C}} = K \cdot \exp(i\underline{m}) \cdot \exp(i\underline{l}) \quad \text{(unique)}.$$

If $\underline{a}$ is a maximal abelian subspace in $\underline{m}$, then $\cup_{l \in L} l\underline{a}l^{-1} = \underline{m}$. Therefore we have

$$\begin{aligned} K^{\mathbf{C}} &= K \cdot \exp(i\underline{m}) \cdot \exp(i\underline{l}) \quad \text{(unique)} \\ &= K \cdot (\cup_{g \in L} g \exp(i\underline{a}) g^{-1}) \exp(i\underline{l}) \\ &= K \cdot \exp(i\underline{a}) \cdot L^{\mathbf{C}}. \end{aligned}$$

At the last stage uniqueness does not hold. But the $\underline{a}$-part is determined in the following sense: Let $N_K(A)$ and $Z_K(A)$ be the normalizer and the centralizer of $A = \exp(\underline{a})$ in K. Then $W = N_K(A)/Z_K(A)$ is the Weyl group of the symmetric space K/L which is a finite group operating on $\underline{a}$ by conjugation. It follows that the $\underline{a}$-part of the above decomposition is determined up to the action of W. Thus we have proved

Theorem ([FJ], [L]). *Let K/L be a compact symmetric space with K a compact semi-simple Lie group, $\underline{k} = \underline{l} + \underline{m}$ the Cartan decomposition and $\underline{a}$ a maximal abelian subspace in $\underline{m}$. Then*

$$K^{\mathbf{C}} = K \cdot \exp(i\underline{a}) \cdot L^{\mathbf{C}}$$

with $\underline{a}$-component determined in the following sense: $k_0 e^H \cdot g = k_0' e^{H'} \cdot g'$ implies $H' = wH$ for some $w \in W$. Equivalently, the correspondence

$$k_0 e^H \cdot g \mapsto H$$

defines a diffeomorphism

$$K \backslash K^{\mathbf{C}}/L^{\mathbf{C}} \cong W \backslash \underline{a}.$$

The decomposition in Theorem (2.3) is called the Jensen-Lassalle decomposition, which is a sort of polar decomposition ($\underline{a}$-part is the radial part). Theorem (2.3) will be important in the description of the geometry of the symmetric variety $K^{\mathbf{C}}/L^{\mathbf{C}}$ and its compactification. The following corollary is true for general homogeneous spaces ([M]) but is a direct consequence of Theorem (2.3) if G/K is a symmetric space.

Corollary. *The correspondence*

$$K \times_L i\underline{m} \to K^{\mathbf{C}}/L^{\mathbf{C}}, \quad (k_0, v) \mapsto k_0 e^v \cdot \xi_0$$

defines a G-equivariant diffeomorphism from the cotangent bundle of K/L to $K^{\mathbf{C}}/L^{\mathbf{C}}$.

(2.4) The importance of the abelian part $\underline{a}$ in the study of geometry and analysis on symmetric varieties is evident from Theorem (2.3). For instance, applying this, the following theorem is proved in [AL1].

Theorem. *There exists a natural one-to-one correspondence between the space of all G-invariant plurisubharmonic functions (resp. G-invariant strictly plurisubharmonic functions) on $K^{\mathbf{C}}/L^{\mathbf{C}}$ and the space of all W-invariant convex functions (resp. W-invariant strictly convex functions) on $i\underline{a}$.*

3. Invariant Kähler Metrics on Symmetric Varieties

(3.1) In this section we discuss general properties of G-invariant Kähler metrics on a symmetric variety $G^{\mathbf{C}}/K^{\mathbf{C}}$ associated to the Riemannian symmetric space G/K of compact type.

(3.2) Let $n = \dim_{\mathbf{C}} G^{\mathbf{C}}/K^{\mathbf{C}}$.

Lemma. *Suppose K is connected. Then there exists, up to a multiplicative constant, a unique $G^{\mathbf{C}}$-invariant holomorphic n-form η on the symmetric variety $G^{\mathbf{C}}/K^{\mathbf{C}}$.*

Proof. Let $n = \dim(G/K)$. Now K operates on $\bigwedge^n T_{\xi_0}(G/K)$ which is 1-dimensional. Since K is compact and connected, it operates trivially in any real 1-dimensional space. Hence K operates trivially on $\bigwedge^n T_{\xi_0}(G/K) \otimes_{\mathbf{R}} \mathbf{C} = \bigwedge^n (G^{\mathbf{C}}/K^{\mathbf{C}})$, so by analytic continuation $K^{\mathbf{C}}$ operates trivially on $\bigwedge^n(G^{\mathbf{C}}/K^{\mathbf{C}})$. Hence any non-zero element of $\bigwedge^n T_{\xi_0}(G^{\mathbf{C}}/K^{\mathbf{C}})$ extends to give a $G^{\mathbf{C}}$-invariant holomorphic n-form. *q.e.d.*

Corollary. *Let $G^{\mathbf{C}}/K^{\mathbf{C}}$ be any symmetric variety. Then there exists, up to a multiplicative constant, a unique Ricci-flat volume form on $G^{\mathbf{C}}/K^{\mathbf{C}}$.*

Proof: There exists a covering $G^{\mathbf{C}}/K_1^{\mathbf{C}} \to G^{\mathbf{C}}/K^{\mathbf{C}}$ such that K_1 is connected in G. Although a holomorphic n-form does not descend to $G^{\mathbf{C}}/K^{\mathbf{C}}$, the Ricci-flat volume form $\eta \wedge \bar{\eta}$ certainly does. *q.e.d.*

(3.3) Let $\tilde{f}$ be a W-invariant strictly convex function on A and f the G-invariant strictly plurisubharmonic function on $G^{\mathbf{C}}/K^{\mathbf{C}}$ defined by

$$f(g_0 \exp(H) \cdot \xi_0) = \tilde{f}(H).$$

Let $\omega = \sqrt{-1}\partial\bar{\partial} f$ be the associated Kähler form.

Lemma. *Let $\tilde{f}$ and f be as above. Assume that $f > 0$. If $\|d\tilde{f}\| \leq c\tilde{f}$ with some positive constant c outside of a compact set, then the Kähler metric $\omega = dd^c f$ is complete.*

Proof. For any orthonormal system $(e_1, \cdots, e_{2n})$ of tangent vectors we have $df = \sum_{i=1}^{2n} df(e_i) e_i^*$ and $d^c f = \sum_{i=1}^{2n} df(e_i) J e_i^*$. It is shown in [AL] that f attains its minimum value along the totally real orbit G/K and it has no other critical points. So df never vanishes outside of G/K. Thus we can set $e_1 = \operatorname{grad} f / \|\operatorname{grad} f\|$ outside of G/K. We can extend this to an orthonormal system $(e_1, Je_1, \cdots, e_n, Je_n)$ at a point p. Then we have

$$\begin{aligned} dd^c f &= \sum_{i=1}^{2n} e_i^* \wedge J e_i^* \geq e_1^* \wedge J e_1^* \\ &= \frac{df \wedge d^c f}{\|df\|^2} \geq 2c^{-1} \frac{df \wedge d^c f}{f^2}. \end{aligned}$$

Let $c(t)$ be a differentiable curve parametrized by its arclength t. It follows that

$$\begin{aligned} t &= \int_0^t \sqrt{g(c'(t), c'(t))} dt \geq c^{-1} \int_0^t \frac{\|df(c'(t))\|}{f(c(t))} dt \\ &\geq c^{-1} \int_0^t \frac{d}{dt} \log f(c(t)) = c^{-1}(\log f(c(t)) - \log f(c(0))). \end{aligned}$$

It is shown in [AL1] that any G-invariant plurisubharmonic function is a proper exhaustion function. If $c(t)$ is a divergent curve, then $f(c(t)) \to \infty$ as $t \to \infty$. Thus $c(t)$ is defined on the whole half line $[0, \infty)$. Hence $\omega = dd^c f$ is complete. *q.e.d.*

(3.4) Let $\omega = dd^c f$ be as in (3.3).

Lemma (cf.[AL2]). *The totally real orbit G/K and the orbit of the non-compact torus A are orthogonal relative to ω.*

Proof. Let $p \in A \subset G^{\mathbf{C}}/K^{\mathbf{C}}$ be a point on the totally real orbit G/K. Let $\pi : G^{\mathbf{C}} \to G^{\mathbf{C}}/K^{\mathbf{C}}$ be the projection. We claim that it suffices to prove $\pi^*\omega = 0$ on $G \cdot p$ (the G-orbit of p in $G^{\mathbf{C}}/K^{\mathbf{C}}$). Indeed, take two tangent vectors $X, Y \in T_p(G^{\mathbf{C}}/K^{\mathbf{C}})$ such that $X \in T_p(G \cdot p)$ and $Y \in A \cdot p$. Let g be the Riemannian metric defined by ω. Then $g(X, Y) = \omega(X, JY)$ and both X and JY are tangent to the G-orbit. So we get the claim. Now we work on $G^{\mathbf{C}}$. Set $f^* = \pi^* f$. We show that $\alpha = 0$ on $G \cdot p$. Indeed, let X be any vector tangent to $G \cdot p$. Then we have $\alpha(X) = df(JX)$. Now f is a G-invariant plurisubharmonic function. It is shown in [AL1] that f assumes its minimum on the totally real orbit. Hence $df^* = 0$ on $G \cdot p$. This implies $\alpha = 0$ on $G \cdot p$. Let $i : G \cdot p \to G^{\mathbf{C}}$ be the inclusion. Then

$$i^*\pi^*\omega = i^*\pi^* dd^c f = i^* dd^c f^* = di^*(d^c f^*) = d\alpha = 0.$$

q.e.d.

(3.5) Let τ be the complex conjugation on $G^{\mathbf{C}}/K^{\mathbf{C}}$. Suppose that a Kähler metric ω on $G^{\mathbf{C}}/K^{\mathbf{C}}$ has a $< G, \tau >$-invariant Kähler potential f. Then

Lemma. *The totally real orbit G/K in $G^{\mathbf{C}}/K^{\mathbf{C}}$ is totally geodesic.*

Proof. Theorem (2.3) implies that τ is an anti-holomorphic involution

$$g(\exp H) \cdot \xi_0 \mapsto g(\exp(-H)) \cdot \xi_0.$$

We show that this is an isometry. It suffices to show $g(\tau_* u, \tau_* u) = g(u, u)$ for $u \in T_p(G \cdot p)$ orthogonal to the A_c-orbit. Let $\widetilde{u}$ be a smooth extension of u. As f is invariant under τ, we have

$$\begin{aligned} g(\tau_* u, \tau_* u) &= dd^c f(\tau_* u, J\tau_* u) \\ &= -\tau_* u \cdot df(\widetilde{u}) - Ju \cdot df(\tau_*(J\widetilde{u})) - df(\tau_*(J[\widetilde{u}, J\widetilde{u}])) \\ &= -u \cdot df(\widetilde{u}) - Ju \cdot df(J\widetilde{u}) - df(J[\widetilde{u}, J\widetilde{u}]) \\ &= g(u, u). \end{aligned}$$

Hence τ is an isometry. As the totally real orbit G/K is pointwise fixed by the isometry τ, it is totally geodesic. *q.e.d.*

(3.6) It is a natural question to ask whether Lemma (3.4) holds for any G-orbit $G \cdot p$ and A-orbit through $p \in A$. Although these orbits are always transversal, the answer turns out to be not true. Namely, let $p \in A$ and suppose p is *not* on the totally real orbit G/K. Then the G-orbit $G \cdot p$ and $A \cdot \xi_0$ is *not* (in general) orthogonal.

Example. We examine this on the 2-dimensional affine quadric M defined by the equation $z_1^2 + z_2^2 + z_3^2 = 1$ in $\mathbf{C}^3$. This is a symmetric variety associated to $S^2 = SO_3/SO_2$. Then the totally real orbit is the 2-sphere given by the equation $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ in M. The restriction of the flat metric of $\mathbf{C}^3$ is an $SO(3)$-invariant complete Kähler metric on M. Take a point $p = {}^t(\sqrt{2}, i, 0) \in M$ outside of the totally real orbit. As a vector tangent to the A-orbit through p, we can take the vector $v_A = {}^t(0,0,1)$ obtained by applying the infinitesimal transformation defined by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

and as a vector tangent to the G-orbit we can take the vector $v_G = {}^t(0,0,-\sqrt{2})$ obtained by the infinitesimal transformation defined by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

These tangent vectors v_A and v_G are not orthogonal. This can be understood by looking at a picture (see Figure 1). The intersection with M and the sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = r^2$ is a 2-sphere when $r = 1$ but if $r > 1$ the intersection is an S^1-bundle on S^2 (this is $P_3(\mathbf{R})$) which is realized in S^5 as a 3-dimensional submanifold and S^2 is realized as the locus of the center of the S^1 fibers. Tangent vectors not orthogonal to the A-direction will appear as a direction tangent to the newly born S^1-direction.

(3.7) It is also a natural question whether the A-orbit $A \cdot \xi_0$ is totally geodesic or not. The answer turns out to be *no*. This is essentially because G-orbits are not orthogonal to A-orbits at points $p \in A$ outside of the totally real orbit. Indeed, pick such a point p and two vectors $X, Y \in T_p(A)$. Extend X, Y smoothly as tangent vector fields of the A-orbit. Let $Z \in \underline{g}$ and let V be the infinitesimal transformation defined by Z. Then V is tangent to the G-orbit. Extend X, Y to smooth vector fields in a neighborhood of p by applying $\exp(tZ)$ which maps $A \cdot \xi_0$ to $(\exp(tX))A \cdot \xi_0$ isometrically. Let α be the second fundamental form of $A \cdot \xi_0$. Then at $p \in A$ we have

$$\begin{aligned} 2(\alpha(X,Y),V) &= g(\nabla_X Y, V) + g(\nabla_Y X, V) \\ &= Xg(Y,V) - g(Y, \nabla_X V) + Yg(X,V) - g(X, \nabla_Y V). \end{aligned}$$

As X and Y are extended by the 1-parameter subgroup generated by V, we have $[V,X] = [V,Y] = 0$. This implies $\nabla_X V = \nabla_V X$ and $\nabla_Y V = \nabla_V Y$ and so $g(Y, \nabla_X V) + g(X, \nabla_Y V) = Vg(X,Y)) = 0$. Hence we have

$$2(\alpha(X,Y),V) = Xg(Y,V) + Yg(X,V).$$

If V is not orthogonal to the A-orbit, this certainly does not vanish identically.

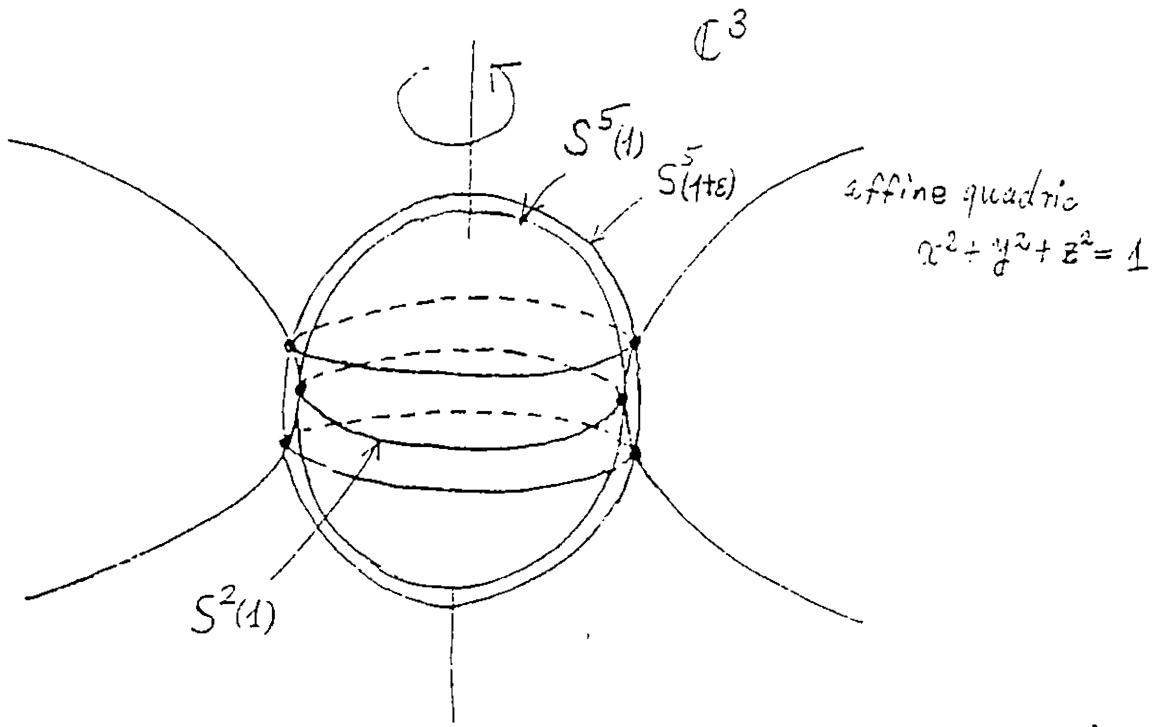

FIGURE 1. Creation of new direction near the totally real orbit

4. Ricci-Flat Kähler Metrics on Symmetric Varieties of Rank One

(4.1) We describe the structure of symmetric varieties of rank one and their compactifications explicitly. The compact symmetric spaces of rank one are the n-spheres, the projective spaces over the real, complex and quaternion fields and the Cayley projective plane.

As symmetric spaces G/K, these are $S^n = SO(n+1)/SO(n)$, $\mathbf{P}_n(\mathbf{R}) = SO(n+1)/S(O(n-1) \times O(1))$, $\mathbf{P}_n(\mathbf{C}) = SU(n+1)/S(U(n) \times U(1))$, $\mathbf{P}_n(\mathbf{H}) = Sp(n+1)/Sp(1) \times Sp(n)$ and $\mathbf{P}_2(\mathbf{C}ay) = F_4/\mathrm{Spin}(9)$. Geometric models of the natural complexifications $G^{\mathbf{C}}/K^{\mathbf{C}}$ of the above spaces are the affine quadric $A\mathbf{Q}_n$ in $\mathbf{C}^n$, the complement of the smooth quadric $\mathbf{Q}_{n-1}$ in $\mathbf{P}_n(\mathbf{C})$, the complement of the hypersurface $H = \{([z],[w]);\ \sum z_i w_i = 0\} \subset \mathbf{P}_n(\mathbf{C}) \times \mathbf{P}_n(\mathbf{C})$, the complement of the subvariety formed by degenerate two-planes in $\mathrm{Grass}_{2,2n}(\mathbf{C})$ (with respect to a fixed symplectic form on $\mathbf{C}^{2n}$), respectively. Here is a list of $G^{\mathbf{C}}$-equivariant compactifications of rank one symmetric varieties ([Ah]). All of them are of the form $H^{\mathbf{C}}/P$, $H^{\mathbf{C}}$ being a reductive (in fact semi-simple) algebraic group and P a parabolic subgroup of $H^{\mathbf{C}}$. In the following table, M denotes a Riemannian symmetric space of compact type, $X = H^{\mathbf{C}}/P$ the equivariant compactification of the associated symmetric variety where P is the corresponding parabolic subgroup. Up to conjugacy, parabolic subgroups P of a simple group are in one to one correspondence with subsets $P(\Sigma)$ of the nodes of the Dynkin diagram (i.e., with the subsets of the set of simple roots). In the following table, we indicate the simple root (the node of the Dynkin diagram) omitted by the corresponding parabolic subgroup.

Let us review the computation of $H^2(H^{\mathbf{C}}/P)$ following [BH] (H being a compact Lie group and P any parabolic subgroup of $H^{\mathbf{C}}$). Recall that if $H^{\mathbf{C}}$ is a reductive

TABLE 1. Compactification of rank one symmetric varieties

M	X	G^C	$P(\Sigma)^c$
S^{2n}	$Q_{2n}(C)$	SO_{2n+2}	first node
S^{2n+1}	$Q_{2n+1}(C)$	SO_{2n+1}	first node
$P_n(R)$	$P_n(C)$	SL_{n+1}	first node
$P_n(C)$	$(P_n(C))^2$	$(SL_{n+1})^2$	first and last nodes
$P_n(H)$	$\mathrm{Grass}_{2,2n}(C)$	SL_{2n}	second node
$P_2(Cay)$	E_6/P	E_6	first node

group and B a Borel subgroup, $T^{\mathbf{C}} \subset B$ a Cartan subgroup and S the corresponding simple system of roots, then a parabolic subgroup $(B \subset)P$ is completely specified by choosing a subsystem $\pi \subset S$. Denoting the root group corresponding to a root α by U_α, one knows that P is generated by U_α, $U_{-\alpha}$ and B $(\alpha \in \pi)$. Let $\{\omega_\alpha, \alpha \in S\}$ be the set of fundamental dominant weights and $\{\rho_\alpha\}$ the irreducible representations of $H^{\mathbf{C}}$ with highest weight ω_α. Choosing a norm invariant under a maximal compact subgroup H of $H^{\mathbf{C}}$ and a highest weight vector v_α, let ϕ_α be the function on $H^{\mathbf{C}}$ defined by $\phi_\alpha(g) = \rho_\alpha(g)v_\alpha$ $(g \in H^{\mathbf{C}})$. Now by [AL2], if $\alpha \in S-\pi$, the nonnegative (1,1)-form

$$\omega_\alpha = \frac{1}{2\pi} dd^c \log \|\phi_\alpha\|^2$$

descends to a (1,1)-form on $H^{\mathbf{C}}/P$. We denote this by ω_α again. For a root α let L_α be the subgroup generated by the root groups $U_{\pm\alpha}$. Let $\xi_0 = eP$. If $\alpha \in S - \pi$ then $P_\alpha = L_\alpha \cdot \xi_0 \cong \mathbf{P}_1(\mathbf{C})$. A computation shows that

$$\int_{P_\alpha} \omega_\beta = \delta_{\alpha\beta}.$$

Let L be the maximal reductive subgroup of P containing $T^{\mathbf{C}}$. The group $B \cap L$ is a Borel subgroup of L and the simple system of roots of the pair $(B \cap L, T^{\mathbf{C}})$ defines the corresponding subsystem π of S. We have $P = L \cdot \mathrm{Ru}(P)$ where $\mathrm{Ru}(P)$ is the unipotent radical of P and hence a cell (the Lie algebra of $\mathrm{Ru}(P)$ is generated by the root vectors X_α with α a positive root which is not supported by π, i.e., not orthogonal to all roots in $S-\pi$). We have $\pi_1(P) = \pi_1(L)$. Finally $L = Z(L)L'$, the derived group L' being semi-simple. Hence $\pi_1(L) = \pi_1(Z(L)) = \mathbf{Z}^{\dim Z(L)}$. Now $\dim(Z(L))$ is equal to the cardinality of $S - \pi$. In our case $H^{\mathbf{C}}$ is semi-simple. As $\pi_1(H^{\mathbf{C}}) = \pi_2(H^{\mathbf{C}}) = 0$ we have $H^2(H^{\mathbf{C}}/P) = \pi_2(H^{\mathbf{C}}/P) = \pi_1(P)$. Hence

$H^2(H^{\mathbf{C}}/P) = \pi_1(P) = \mathbf{Z}^{\mathrm{card}(S-\pi)}$. Thus the forms $\{\omega_\alpha\}_{\alpha\in S-\pi}$ are independent generators of $H^2_{\mathrm{dR}}(H^{\mathbf{C}}/P)$. Let χ be a character of P and

$$\begin{aligned} L_\chi &= H^{\mathbf{C}} \times_\chi \mathbf{C} \\ &= (H^{\mathbf{C}} \times \mathbf{C})/\{(g,v) \sim (gx, \chi(x)^{-1}v);\ x \in P\} \end{aligned}$$

the corresponding line bundle. The character χ of the Cartan subgroup $T^{\mathbf{C}}$ is a character of P if and only if $(\chi, \alpha) = 0$ for $\forall\alpha \in \pi$. Set $\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}$. Then we have the Chern class formula for L_χ:

$$c_1(L_\chi) = \sum_{\alpha\in S-\pi} (-\chi, \check{\alpha})[\omega_\alpha].$$

Hence if $(-\chi, \check{\alpha}) > 0$ for all $\alpha \in S - \pi$, then L_χ is ample. Let χ be the sum of all roots in the unipotent radical $\mathrm{Ru}(P)$ of P. Then the Chern class formula for $H^{\mathbf{C}}/P$ tells us that the anticanonical bundle is given by

$$K^{-1}_{H^{\mathbf{C}}/P} = L_{-\chi}$$

and we have from [BH, p.512] that $(\chi, \check{\alpha}) > 0$ for $\forall\alpha \in S - \pi$ and $(\chi, \alpha) = 0$ for $\forall\alpha \in \pi$. Hence

$$c_1(H^{\mathbf{C}}/P) = \sum_{\alpha\in S-\pi} (\chi, \check{\alpha})[\omega_\alpha].$$

Thus the anticanonical bundle of $H^{\mathbf{C}}/P$ is ample. On the other hand, if K is connected any $G^{\mathbf{C}}$-invariant holomorphic n-form on $G^{\mathbf{C}}/K^{\mathbf{C}}$ extends meromorphically to the compactification $X = H^{\mathbf{C}}/P$. Hence $c_1(X) = a[D]$ with some $a \in \mathbf{Z}$ (if K is connected). The computation of a can be done if we write $[D]$ in terms of $[\omega_\alpha]$ and compute the values of $P_\alpha \cdot D$ and $P_\alpha \cdot c_1(X)$ for a suitable α. The computation is simple in the classical case. The case that $M = \mathbf{P}_n(\mathbf{R})$ is essentially the same as $M = S^n$ because the double covering $S^n \to \mathbf{P}_n(\mathbf{R})$ extends to the double covering $A\mathbf{Q}_n \to \mathbf{P}_n(\mathbf{C}) - \mathbf{Q}_{n-1}$. For the exceptional cases, we refer to the description of the roots given in [Sp] and apply [ABS, Lemma 2.1] (we have $a > 1$ simply because E_6 contains A_5 as a subsystem). In any case we have $a > 1$.

(4.2) We quote the following existence theorem for complete Ricci-flat Kähler metrics.

Theorem ([BK2],[TY]). *Let X be a Fano manifold and D a smooth hypersurface in X. Suppose that $c_1(X) = \alpha[D]$ with $\mathbf{Q} \ni \alpha > 1$ and that D admits a Kähler-Einstein metric. Then $X - D$ admits a complete Ricci-flat Kähler metric of asymptotically flat geometry in the sense of* [BK2].

(4.3) Let $G^{\mathbf{C}}/K^{\mathbf{C}}$ be a symmetric variety of rank one and $X = H^{\mathbf{C}}/P$ its compactification as in (4.1). Then X is a Fano manifold and $D = H^{\mathbf{C}}/P - G^{\mathbf{C}}/K^{\mathbf{C}}$ is a smooth hypersurface. In (4.1) we have shown that $c_1(X) = \alpha[D]$ with $\alpha > 1$. Hence we have the following theorem.

TABLE 2. The value of a

$c_1(X) = \alpha[D]$		
X	$\dim X$	α
$SO(2n+2, C)/P$	$2n$	$2n$
$SO(2n+1, C)/P$	$2n+1$	$2n+1$
$SL(n+1, C)/P$	n	$(n+1)/2$
$SL(n+1, C) \times SL(n+1, C)/P$	$2n$	$n+1$
$SL(2n, C)/P$	$2(2n-2)$	$2n$
E_6/P	16	$\alpha > 1$

Theorem. *There exists a G-invariant complete Ricci-flat Kähler metric on any rank one symmetric variety. The resulting metric has asymptotically flat geometry in the sense of* [BK2].

Proof. We follow [BK2]. First we construct a background metric ω such that
(i) ω is a complete Kähler metric on $X - D$,
(ii) $\omega^n = e^f \eta \wedge \bar{\eta}$ ($\eta \wedge \bar{\eta}$ being a $G^{\mathbf{C}}$-invariant Ricci-flat volume form as in Corollary (3.2)) with $f = O(\|\sigma\|^2)$, where $D = (\sigma)_0$ and $\|\cdot\|$ is a suitable Hermitian metric on $\mathcal{O}_X(D)$.
This implies that there exists a $q < n$ such that $f \in L^q$ with respect to the metric ω. Indeed, we have $\alpha > n+1$ for symmetric varieties of rank one (in fact, by [KO], we always have $\alpha \leq n+1$ for any pair of (X, D) where X is a Fano manifold and D is a divisor with $c_1(X) = \alpha[D]$, and $\alpha = 1$ if and only if (X, D) is isomorphic to the hyperplane section $(\mathbf{P}^n, \mathbf{P}^{n-1})$). On the other hand, in order that

$$\int |f|^q \omega^n \leq \int_0^* \frac{dr}{r^{2\alpha - 2q - 1}} < \infty$$

holds, we must have $q \geq \alpha - 1$. As $\alpha < n+1$, we certainly have a $q < n$ such that $f \in L^q$.

Now a compact group G operates on $X - D$. It is easy to see that one can construct the above ω so that
(iii) ω is G-invariant.
Here one remark is in due. In [BK2], we used a solution of the following equation to construct ω with the property that $f := \log \frac{\omega^n}{\eta \wedge \bar{\eta}} = O(\|\sigma\|^2)$:

$$\Delta_{\bar{g}} s + (\alpha - 2)s = s_0,$$

with s unknown and s_0 a given and G-invariant, where the Laplacian operates on sections of the conormal bundle of D, and $c_1(X) = \alpha[D]$. But this equation is clearly

G-invariant and we get a G-invariant solution. Thus we get such ω satisfying the above three conditions.

Then we consider the following Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^{-f}\omega^n.$$

It is proved in [BK2] that this equation has a solution with a priori decay estimates for u. In particular, we have the estimate of the form

$$\sup|u| \leq C(\|f\|_q, \|f\|_\infty)$$

where the right hand side is a quantity depending only on $(X - D, \omega)$, $\|f\|_q$ and $\|f\|_\infty$, where $\|f\|_q$ is some L^q-norm of f with $q < n$. Let $g \in G$. Then $u_g(*) = u(g(*))$ is also a solution of the same equation. Since ω is G-invariant, u_g has the same decay condition as the original u. Now we consider $\omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u$ as a background metric. Then $u_g - u$ satisfies the Monge-Ampère equation

$$(\omega_u + \sqrt{-1}\partial\bar{\partial}(u_g - u))^n = \omega_u^n (= \eta \wedge \bar{\eta}).$$

Since u satisfies a priori estimates, $\omega + \sqrt{-1}\partial\bar{\partial}u$ is a complete Kähler metric asymptotically equal to ω. Now the a priori estimates in [BK2] applied to the above Monge-Ampère equation show $u_g - u \equiv 0$. *q.e.d.*

(4.4) We give an explicit invariant Ricci-flat Kähler metric on the affine quadric $A\mathbf{Q}_n = SO_{n+1}(\mathbf{C})/SO_n(\mathbf{C})$. Suppose that $A\mathbf{Q}_n$ is defined by the equation $(z, z) = z_1^2 + z_2^2 + \cdots + z_n^2 = 1$ in $\mathbf{C}^{n+1}$. It follows from Lemma (3.2) that the $SO_{n+1}(\mathbf{C})$-invariant holomorphic n-form on it is given by

$$\eta = \sum_{i=1}^{n+1}(-1)^{i-1} z_i dz_1 \wedge \cdots \wedge \hat{dz_i} \wedge \cdots \wedge dz_{n+1}.$$

Introduce a radial variable x on $A\mathbf{Q}_n$ by putting

$$\sinh x = \left(\frac{\|z\|^2 - 1}{2}\right)^{\frac{1}{2}}.$$

Then x is $SO_{n+1}(\mathbf{R})$-invariant. We seek an invariant Ricci-flat Kähler metric in the form $\sqrt{-1}\partial\bar{\partial}\psi(z)$, where $\psi(z)$ depends on x only. Set $\psi(z) = g(x)$. Then the Ricci-flat condition $(\sqrt{-1}\partial\bar{\partial}\psi(z))^n = K\eta \wedge \bar{\eta}$ becomes

$$g''(x)(g'(x))^{n-1} = K(\sinh 2x)^{n-1}$$

where K is an arbitrary positive constant. One can solve this equation explicitly. The result is

$$\psi(z) = g(x) \quad \text{with} \quad g'(x) = \left(\int_0^x nK(\sinh 2t)^{n-1} dt\right)^{\frac{1}{n}}.$$

It is easy to verify the condition in Lemma (3.3). So the metric $\sqrt{-1}\partial\bar{\partial}\psi(z)$ is complete.

Stenzel [St] constructed Ricci-flat Kähler metrics on rank one symmetric varieties from a view point somewhat different from ours.

5. Ricci-Flat Kähler Metrics on Symmetric Varieties of Higher Rank

5.1. Review of DeConcini-Procesi Compactification

(5.1.1) For general information on Lie group theory, we refer to [H1] and [Sp1]. Let G/K be a compact symmetric space and $G^{\mathbf{C}}/K^{\mathbf{C}}$ be the associated symmetric variety. We assume that K is connected and $K^{\mathbf{C}}$ is the largest subgroup of $G^{\mathbf{C}}$ with Lie algebra $\underline{k}^{\mathbf{C}}$. The Killing form on $\underline{g}^{\mathbf{C}}$ is a $\mathbf{C}$-linear symmetric bilinear form $(\cdot,\cdot)$ which is invariant under θ and $\mathrm{Ad}(G^{\mathbf{C}})$ and negative definite on $\underline{g}$. We extend $i\underline{a}$ to a Cartan subalgebra $\underline{t}$ of $\underline{g}$. Now $\underline{t}^{\mathbf{C}} = \underline{t}_0^{\mathbf{C}} + \underline{t}_1^{\mathbf{C}}$ ($\underline{t}_1 = i\underline{a}$) according to the (± 1)-eigenspaces of θ and it is θ-stable. So θ operates on roots. The roots take real values on $i\underline{t}_1$. We call this induced set Σ of root system the restricted roots. Let $v_1, \cdots, v_r$ be a basis of $i\underline{t}_1$ and $v_{r+1}, \cdots$ a basis of $i\underline{t}_0$. Set $\omega_1, \cdots, \omega_r, \omega_{r+1}, \cdots$ to be the dual basis. The lexicographic ordering induces an order on the roots, namely, $\omega = \sum a_i \omega_i$ is positive if the first nonzero a_i is positive. Now if $v \in i\underline{t}_1$ then $\theta(v) = -v$. Suppose α is a positive root and $\alpha^\theta \neq \alpha$, i.e., $\alpha(v_i) \neq 0$ for some $v_i \in i\underline{t}_1$. Then the first nonzero $\alpha(v_i)$ is positive. So there is a system of positive roots Φ^+ such that $\alpha \in \Phi^+$ and $\alpha \not\equiv 0$ on $i\underline{t}_1$ imply $\alpha^\theta \in \Phi^-$. We have a decomposition

$$\Phi^\pm = \Phi_0^\pm \cup \Phi_1^\pm$$

where $\Phi_0^\pm = \{\alpha \in \Phi^\pm;\, \alpha|_{\underline{t}_1^{\mathbf{C}}} = 0\}$ and $\Phi_1^\pm = \Phi^\pm - \Phi_0^\pm$. We fix such a system of positive roots. Let B be the corresponding Borel subgroup and U the unipotent subgroup generated by the root subgroups U_α ($\alpha \in \Phi_1^-$), $\Delta = \Delta_0 \cup \Delta_1$ the set of simple roots, where

$$\Delta_0 = \{\alpha \in \Delta;\, \alpha|_{\underline{t}_1^{\mathbf{C}}} = 0\}, \quad \Delta_1 = \Delta - \Delta_0.$$

Set

$$\Delta_0 = \{\beta_1, \cdots, \beta_k\}, \quad \Delta_1 = \{\alpha_1, \cdots, \alpha_j\}.$$

The choice of the positive roots Φ^+ induces a choice of positive roots Σ^+ in the restricted root system and the set Δ_1 maps onto the simple roots in Σ^+. One can order $\alpha_1, \cdots, \alpha_j$ so that $\alpha_j - \alpha_j^\theta$ are mutually distinct for $j \leq r$ and for all $i > l$ there is an index $s \leq r$ such that $\alpha_i - \alpha_i^\theta = \alpha_s - \alpha_s^\theta$. We call $\overline{\alpha}_s = \frac{1}{2}(\alpha_s - \alpha_s^\theta)$ ($s \leq r$) the restricted simple roots.

The fundamental weights form the dual basis of the simple coroots

$$\{\check{\beta}_1, \cdots, \check{\beta}_k; \check{\alpha}_1, \cdots, \check{\alpha}_j\},$$

where $\check{\lambda} = \frac{2\lambda}{(\lambda,\lambda)}$, with respect to the Killing form. Let

$$\omega_1, \cdots, \omega_j; \zeta_1, \cdots, \zeta_k$$

be the set of the fundamental weights, where

$$(\omega_s, \check{\beta}_t) = 0, \quad (\omega_s, \check{\alpha}_t) = \delta_{st}$$

and similarly for ζ_s's. Then by [DP,pp.5-6] there exists a permutation $\widetilde{\theta}$ of order 2 on the indices $\{1, \cdots, j\}$ determined by

$$\omega_i^\theta = -\omega_{\widetilde{\theta}(i)}.$$

Definition. *A dominant weight $\lambda = \sum n_i \omega_i$ is special (resp. regular) if $n_i = n_{\widetilde{\theta}(i)}$ or equivalently $\lambda^\theta = -\lambda$ (resp. $\forall n_i \neq 0$).*

(5.1.2) Let V be an irreducible representation of $G^{\mathbf{C}}$ with highest weight λ. If P is a parabolic subgroup which is the stabilizer of the line in $\mathbf{P}(V)$ generated by the highest weight vector, $G^{\mathbf{C}}/P$ is the unique closed orbit in $\mathbf{P}(V)$. The pull-back of $\mathcal{O}(1)$ on $G^{\mathbf{C}}/P$ via the projection $G^{\mathbf{C}}/B \to G^{\mathbf{C}}/P$ is a holomorphic line bundle L on $G^{\mathbf{C}}/B$ and the space $H^0(G^{\mathbf{C}}/B, L)^*$ the dual of the space of the global sections is isomorphic to the representation space V. Conversely suppose we are given a weight λ. Then λ exponentiates to a character of the Cartan subgroup $T^{\mathbf{C}}$ (corresponding to $\underline{t}$) and this canonically extends to a character of B. We then have a holomorphic line bundle L_λ on $G^{\mathbf{C}}/B$. If λ is dominant, then the Borel-Weil theory [B] implies that $H^i(G^{\mathbf{C}}/B, L_{-\lambda}) = 0$ for $i > 0$ and $V_\lambda = H^0(G^{\mathbf{C}}/B, L_{-\lambda})^*$ is a finite dimensional irreducible representation of $G^{\mathbf{C}}$ with highest weight λ. We are interested in vectors fixed by $K^{\mathbf{C}}$ and their expressions in weight vectors. These are described in (5.1.3)-(5.1.5).

(5.1.3)

Proposition ([DP]). *Let λ be a dominant weight and V_λ the corresponding representation of $G^{\mathbf{C}}$ with highest weight λ. Let V_λ^K be the vector subspace of the $K^{\mathbf{C}}$-invariant vectors. Then $\dim V_\lambda^K \leq 1$ and $V_\lambda^K \neq 0$ only if λ is special.*

(5.1.4) For $\mu \in (\underline{t}_1^{\mathbf{C}})^*$, define $\widetilde{\mu} \in (\underline{t}^{\mathbf{C}})^*$ by setting $\widetilde{\mu}|_{\underline{t}_0^{\mathbf{C}}} = 0$. Helgason [H2, Section 3] determined all special weights λ with $V_\lambda^K \neq 0$:

Theorem ([H2]). *The set Λ_1 of all special weights λ with $V_\lambda^K \neq 0$ is given by*

$$\Lambda_1 = \{\widetilde{\mu};\ \mu \in (\underline{t}_1^{\mathbf{C}})^*,\ \frac{(\mu, \overline{\alpha})}{(\overline{\alpha}, \overline{\alpha})} \in \mathbf{Z}_{\geq 0}\,(\forall \overline{\alpha} \in \Sigma^+)\}.$$

Therefore Λ_1 is contained in the positive lattice generated by ω_i (if $\widetilde{\theta}(i) = i$) and $\frac{1}{2}(\omega_i - \omega_{\widetilde{\theta}(i)})$ (if $\widetilde{\theta}(i) \neq i$). Hence, if $r = \operatorname{rank}(G, K)$,

$$\Lambda_1 = \{\sum_{i=1}^r n_i \mu_i\}$$

with $\forall n_i \geq 0$ and $\mu_i = \omega_i$ or $2\omega_i$ (resp. $\frac{1}{2}(\omega_i - \omega_{\widetilde{\theta}(i)})$ or $\omega_i - \omega_{\widetilde{\theta}(i)}$). The weight $\sum_{i=1}^r n_i \mu_i \in \Lambda_1$ is regular if and only if $\forall n_i > 0$.

(5.1.5) Now we are ready to express the vector in V_χ fixed by $K^{\mathbf{C}}$ in terms of weight vectors.

Proposition ([DP]). *Let χ be a regular special weight contained in Λ_1, V_χ the corresponding irreducible representation of $G^{\mathbf{C}}$ with highest weight χ and h a nonzero element of V_χ fixed under $K^{\mathbf{C}}$. Then h is unique up to scalar multiplication and can be normalized to be*

$$h = v_\chi + \sum z_i$$

with v_χ a highest weight vector in V_χ and z_i's weight vectors having distinct weights of the form $\chi - 2\sum_{s=1}^r n_s\overline{\alpha}_s$, n_i nonnegative integers. Moreover one can assume that the vectors $z_1, \cdots, z_r$ have weight $\chi - 2\overline{\alpha}_1, \cdots, \chi - 2\overline{\alpha}_r$.

(5.1.6) In (5.1.6)-(5.1.8) we describe the DeConcini-Procesi compactification [DP] of symmetric varieties. Our description is quite restricted but we describe all its properties which are necessary to prove the existence of a Ricci-flat Kähler metric. Let $\mu_1, \cdots, \mu_r$ ($r = \mathrm{rank}(G/K)$) be dominant weights as in Theorem (5.1.4). Let (ρ_i, V_{μ_i}) be the corresponding irreducible representations and $[h_i]$ the $K^{\mathbf{C}}$-invariant line in V_{μ_i}. Let $\chi = \sum_{i=1}^r \mu_i$. Then the representation $\otimes_{i=1}^r V_{\mu_i}$ contains the highest weight χ and the line l generated by $\otimes_{i=1}^r h_i$ is fixed under $K^{\mathbf{C}}$. As $K^{\mathbf{C}}$ is a maximal group with the same Lie algebra, if $K^{\mathbf{C}}$ fixes a line l, then $\mathrm{Stab}(l)$ is either $K^{\mathbf{C}}$ or $G^{\mathbf{C}}$. Now $G^{\mathbf{C}}$ operates non-trivially and so the stabilizer of l is just $K^{\mathbf{C}}$.

Definition. *The DeConcini-Procesi compactification (or the canonical compactification) of the symmetric variety $G^{\mathbf{C}}/K^{\mathbf{C}}$ is the closure of the $G^{\mathbf{C}}$-orbit of $[\otimes_{i=1}^r h_i]$ under the projectivized representation*

$$(\mathbf{P}(\otimes_{i=1}^r \rho_\chi), \mathbf{P}(\otimes_{i=1}^r V_{\mu_i})).$$

This is contained in the image of the the natural imbedding of the product representation

$$(\prod_{i=1}^r \mathbf{P}(\rho_{\mu_i}), \prod_{i=1}^r \mathbf{P}(V_{\mu_i}))$$

into $\mathbf{P}(\otimes_{i=1}^r V_{\mu_i})$.

Let X be the DeConcini-Procesi compactification of $G^{\mathbf{C}}/K^{\mathbf{C}}$. They prove in [DP, Theorem 3.1, Lemma 4.1] that X has the following properties.
(i) X is a unique $G^{\mathbf{C}}$-equivariant compactification.
(ii) $D := X - G^{\mathbf{C}}/K^{\mathbf{C}}$ consists of r smooth hypersurfaces $D_1, \cdots, D_r$ with at worst simple normal crossings.
(iii) $Y := \cap_{i=1}^r D_i$ is the unique closed $G^{\mathbf{C}}$-orbit in X which is isomorphic to $G^{\mathbf{C}}/P$ where P is the parabolic subgroup associated to the set Δ_0, i.e., the parabolic subgroup generated by B and $U_{\pm\alpha}$ ($\alpha \in \Delta_0$), which is the stabilizer of the line generated by the highest weight vector $\otimes_{i=1}^r v_{\mu_i}$ corresponding to the highest weight $\chi = \sum_{i=1}^r \mu_i$.
(iv) the $G^{\mathbf{C}}$-orbit are just

$$O_{i_1,\cdots,i_k} = D_{i_1} \cap \cdots \cap D_{i_k} - \cup_{i \neq i_1,\cdots,i_k} D_{i_1} \cap \cdots \cap D_{i_k} \cap D_i$$

All these properties are examined by looking at the operation of $e^{\underline{a}^{\mathbf{C}}} = A^{\mathbf{C}}$. Indeed, let $h = \otimes_{i=1}^r h_{\mu_i}$. Then, it follows from Proposition (5.1.5) that $t \in A^{\mathbf{C}}$ operates on h as

$$th = t^\chi v_\chi + \sum_{i=1}^r t^{\chi - 2\overline{\alpha}_i} v_i + \sum_{i>r} t^{\chi - 2\sum_{s=1}^r n_s^{(i)}\overline{\alpha}_i} v_i.$$

In affine coordinates this is

$$v_X + \sum_{i=1}^{r} t^{-2\overline{\alpha}_i} v_i + \sum_{i>r} t^{-2\sum_{s=1}^{r} n_s^{(i)} \overline{\alpha}_s} v_i.$$

We have (essentially from the above observations) the following description of an affine covering of X "centered at infinity" Y: There exists an open affine set V in X such that
(a) $V \cong \mathbf{C}^r \times U$ (U being the unipotent group generated by U_α for $\alpha \in \Phi_1^-$),
(b) $(\mathbf{C}^r \times U) \cap (X - G^{\mathbf{C}}/K^{\mathbf{C}})$ is characterized by the condition that at least one of the first r coordinates is zero,
(c) $X = \cup_{g \in G^{\mathbf{C}}} g(\mathbf{C}^r \times U)$ gives an affine covering of X. In particular, U is a cell and gives a local coordinate system for Y.

(5.1.7) We give here the canonical bundle formula for X. It follows from the description of $H^2(G^{\mathbf{C}}/P)$ ($Y = G^{\mathbf{C}}/P$) in Chapter 4 and the injectivity of the Chern class map $H^1(Y, \mathcal{O}^*) \to H^2(Y; \mathbf{Z})$ that $\operatorname{Pic}(Y)$ is isomorphic to the lattice spanned by the fundamental weights relative to the simple roots in Φ_1. On the other hand, it is proved in [DP, Proposition 8.1] that the restriction

$$\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$$

is injective. Note that each special (resp. regular special) weight defines a homogeneous line bundle (resp. homogeneous ample line bundle) on Y. Therefore $\operatorname{Pic}(X)$ is identified with a sublattice L of the above lattice. Thus the set of all regular special weights is contained in L and each regular special weight λ defines a projective imbedding of X and hence Y in $\mathbf{P}(V_\lambda)$. For each $\lambda \in L$ write L_λ for the corresponding line bundle on X. The line bundle $L_{-\lambda}$ on X (and Y) is the hyperplane bundle $\mathcal{O}(1)$ with respect to the imbedding of X (and Y) in $\mathbf{P}(V_\lambda)$. It follows from the explicit description of the natural embedding $T^{\mathbf{C}} \to T^{\mathbf{C}} \cdot p$ that $2\overline{\alpha}_i = \alpha_i - \alpha_i^\theta \in L$ and $L_{-2\overline{\alpha}_i} \cong \mathcal{O}(D_i)$ ([DP, Corollary 8.2]). Set $\mu = \sum_{\alpha \in \Phi_1^+} \alpha$. Since μ is a regular special weight ([DP, Lemma 6.1]), we have $\mu \in L$. Then we have

Lemma. *The anticanonical bundle K_X^{-1} of X is isomorphic to the line bundle corresponding to $-(\mu + \sum_{i=1}^{r}(\alpha_i - \alpha_i^\theta)) \in L$.*

Remark. By [DP, Proposition 8.4], any line bundle corresponding to $-(\mu + \gamma)$ is ample for any dominant weight $\gamma \in L$. Hence the DeConcini-Procesi compactification X is a Fano manifold, i.e., K_X^{-1} is ample.

Proof of Lemma (5.1.7). The result is a consequence of the following five facts:
(i) the adjunction formula, i.e.,

$$K_X^{-1}|_Y = K_Y^{-1} \otimes \otimes_{i=1}^{r} [D_i]|_Y,$$

(ii) $Y = G^{\mathbf{C}}/P$ where P is a parabolic subgroup generated by B and $U_{\pm\alpha}$ ($\alpha \in \triangle_0$),
(iii) the Chern class formula on Y:

$$c_1(Y) = \sum_{\alpha \in \triangle - \triangle_0} (\chi, \check{\alpha})[\omega_\alpha],$$

χ being the sum of all roots in the unipotent radical of P,
(iv) the injectivity of the Chern class map $H^1(Y,\mathcal{O}^*) \to H^2(Y;\mathbf{Z})$, which is a consequence of $H^1(Y;\mathbf{Z}) = 0$ (since Y is a Fano manifold),
(v) the injectivity of $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$.
Indeed, (i),(ii) and (iii) imply that the line bundle $K_X^{-1}|_Y$ has the correct Chern class and so, by (iv), corresponds to the correct weight. The result now follows from (v). *q.e.d.*

(5.1.8) Let X and D_i's be as in (5.1.7). Then in X the only irreducible $G^{\mathbf{C}}$-stable subvarieties are of the form $D_{i_1 i_2 \cdots i_k} = D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ for $\{i_1, i_2, \cdots, i_k\} \subset \{1, 2, \cdots, r\}$. DeConcini and Procesi [DP, pp.17-19 and Theorem 5.2] determined the structure of the $G^{\mathbf{C}}$-stable subvarieties in X. This result will be most important in our analysis of Monge-Ampère equation, because it allows us to use induction on $r = \mathrm{rank}(G/K)$. Let μ_i's be as in Theorem (5.1.4).

Theorem ([DP]). *Let $\{i_1, i_2, \cdots, i_k\}$ be a subset of $\{1, 2, \cdots, r\}$ and let $D_{i_1 \cdots i_k}$ be the corresponding $G^{\mathbf{C}}$-stable subvariety of X. Let $P_{i_1 \cdots i_k}$ be the parabolic subgroup associated to the weight $\mu_{i_1 i_2 \cdots i_k} = \mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_k}$ (i.e., generated by B and the root subgroups $U_{\pm\alpha}$ for roots α with $(\alpha, \mu_{i_1 i_2 \cdots i_k}) = 0$). Then there is a $G^{\mathbf{C}}$-equivariant fibration*

$$\Pi_{i_1 i_2 \cdots i_k} : D_{i_1 i_2 \cdots i_k} \to G^{\mathbf{C}}/P_{i_1 i_2 \cdots i_k},$$

with fibers isomorphic to the canonical compactification of the symmetric variety

$$L_{i_1 i_2 \cdots i_k}/K_{i_1 i_2 \cdots i_k}$$

of rank $r-k$, where $L_{i_1 i_2 \cdots i_k}$ is the semi-simple part of the Levi subgroup of $P_{i_1 i_2 \cdots i_k}$ and $K_{i_1 i_2 \cdots i_k} = L_{i_1 i_2 \cdots i_k} \cap K^{\mathbf{C}}$.

In Theorem (5.1.8) the fiber of the fibration $\Pi_{i_1 i_2 \cdots i_k}$ in the $G^{\mathbf{C}}$-orbit $O_{i_1 i_2 \cdots i_k}$ is the symmetric variety $L_{i_1 i_2 \cdots i_k}/K_{i_1 i_2 \cdots i_k}$. If $\{i_1, i_2, \cdots, i_k\} = \{1, 2, \cdots, r\}$ then $P_{1,2 \cdots r} = P$, i.e., the parabolic subgroup generated by B and $U_{\pm\alpha}$ for $\alpha \in \Delta_0$.

The proof of Theorem (5.1.8) in [DP, Theorem 5.2] goes as follows. Set $\mu_1 = \mu_{i_1} + \cdots + \mu_{i_k}$ and $\mu_2 = \mu_{j_1} + \cdots + \mu_{j_l}$, where $j_1, \cdots, j_l$ are the complement of $i_1, \cdots, i_k$. Then the canonical compactification X is imbedded in $\mathbf{P}(V_{\mu_1}) \times \mathbf{P}(V_{\mu_2})$. Let Π_1 be the projection onto the first factor. It is $G^{\mathbf{C}}$-equivariant and maps onto the closure of the orbit of the point $[h_{\mu_1}] \in \mathbf{P}(V_{\mu_1})$. It is shown in [DP,Lemma 5.1] that $\Pi_1(D_{i_1 i_2 \cdots i_k})$ is equal to the unique closed orbit in $\Pi_1(X)$, i.e., $G^{\mathbf{C}}/P_1$ with P_1 the parabolic subgroup stabilizing the line generated by the highest weight vector in V_{μ_1}. This is the fibration stated in Theorem (5.1.8). Let $\overline{A}^{\mathbf{C}}$ be the compactification of the maximal torus $A^{\mathbf{C}}$ in X. If we set $\overline{A}^{\mathbf{C}}_{i_1 \cdots i_k} = D_{i_1 \cdots i_k} \cap \overline{A}^{\mathbf{C}}$, then its G-translations are the compactification of the maximal torus of fibers $L_{i_1 \cdots i_k}/K_{i_1 \cdots i_k}$. This shows that the rank of the fiber is $r - k$.

Proposition. *The holomorphic map $\Pi_1 : X \to \Pi_1(X)$ is a $G^{\mathbf{C}}$-equivariant blowing down of the subvariety $D_{i_1 i_2 \cdots i_k}$ along the fiber of the fibration $\Pi_1|_{D_{i_1 i_2 \cdots i_k}}$:*

$D_{i_1 i_2 \cdots i_k} \to G^{\mathbf{C}}/P_1$. In particular, Π_1 induces an isomorphism on the complement of $D_{i_1 i_2 \cdots i_k}$ and maps $D_{i_1 i_2 \cdots i_k}$ to the subvariety $G^{\mathbf{C}}/P_1$ in $\Pi_1(X)$.

Proof. The holomorphic map Π_1 is induced from the projection $\mathbf{P}(V_{\mu_1}) \times \mathbf{P}(V_{\mu_2}) \to \mathbf{P}(V_{\mu_1})$ of projective representations of $G^{\mathbf{C}}$. Hence $\Pi_1 : X \to \Pi(X)$ is $G^{\mathbf{C}}$-equivariant. We examine the fiber of the map $\Pi_1 : X \to \Pi_1(X)$ outside of the unique closed orbit $G^{\mathbf{C}}/P_1 \subset \Pi_1(X)$. In doing so it suffices to examine the fiber passing through a point $h_{j_1 j_2 \cdots j_l}$ in each $G^{\mathbf{C}}$-orbit $O_{j_1 j_2 \cdots j_l}$. We use $V = \mathbf{C}^r \times U$ as coordinate system. Then we may assume that the point $h_{j_1 j_2 \cdots j_r}$ in the $G^{\mathbf{C}}$-orbit $O_{j_1 j_2 \cdots j_l}$ has $\mathbf{C}^r$-coordinates $z_j = 0$ if $j \in \{j_1, j_2, \cdots, j_l\}$ and $z_j = 1$ if otherwise. Now it is clear that the fiber of $\Pi_1 = \Pi_{i_1 i_2 \cdots i_k}$ passing through $h_{j_1 j_2 \cdots j_l}$ consists of one point if $\{i_1, i_2, \cdots i_k\} \neq \{j_1, j_2, \cdots, j_l\}$. *q.e.d.*

(5.1.9)
Example. The most familiar example of the canonical compactification of symmetric varieties appears in rank two case. It is the space X of pairs (C, C') of a plane quadric $C \subset \mathbf{P}_2$ and its dual C' in the dual projective space $\mathbf{P}_2^*$. The space of smooth plane quadrics is the symmetric variety $SL_3(\mathbf{C})/SO_3(\mathbf{C})$ and X is its canonical compactification. The space of all quadratic equations defined on $\mathbf{P}_2$ is naturally identified with $\mathbf{P}_5$ and the space of singular quadrics are parametrized by a cubic hypersurface $D_1 = \mathrm{Sym}^2(\mathbf{P}_2)$, which is singular along the Veronese surface (the image of the diagonal) representing double lines. Blowing up the Veronese surface yields the canonical compactification X and the hypersurface D_2 as the exceptional divisor of the blow up. Let D_1 denote the strict transform of the original D_1. It is easy to check $D_1 \cap D_2 = \mathbf{P}(T\mathbf{P}_2)$ which has two $SL_3(\mathbf{C})$-equivariant fibrations, one is dual to the other:

$$\mathbf{P}_1^* \to \mathbf{P}(T\mathbf{P}_2) \to \mathbf{P}_2 \quad \text{and} \quad \mathbf{P}_1 \to \mathbf{P}(T\mathbf{P}_2) \to \mathbf{P}_2^*$$

defined by $(l, p) \mapsto p$ and $(l, p) \mapsto [l]$, respectively, where $p \in \mathbf{P}_2$ is a point and l a line through p determining a direction in $T_p\mathbf{P}$. The fiber structures $\Pi_i : D_i \to G^{\mathbf{C}}/P_i$ on D_i's are interpreted geometrically via the above fibrations as follows. The fiber of the fibration $D_1 \to G^{\mathbf{C}}/P_1 \cong \mathbf{P}_2$ is $\mathbf{P}_2$ which parametrizes those singular conics $C \subset \mathbf{P}_2$ whose double point lies in a fixed point (note that $\mathbf{P}_1 \times \mathbf{P}_1/\mathbf{Z}_2 = \mathbf{P}_2$, where $\mathbf{Z}_2$ operates as $(x, y) \mapsto (y, x)$). Similarly, the fiber of the fibration $D_2 \to G^{\mathbf{C}}/P_2 \cong \mathbf{P}_2$ is $\mathbf{P}_2$ which parametrizes those singular dual conics $C' \in \mathbf{P}^*$ whose double point lies on a fixed point. The fibers in D_i intersects D_j $(i \neq j)$ along a smooth conic. It is geometrically clear that all $SL_3(\mathbf{C})$-orbits are $X - (D_1 \cup D_2)$, $D_1 - (D_1 \cap D_2)$, $D_2 - (D_1 \cap D_2)$ and $D_1 \cap D_2$.

(5.1.10) Since $\mu = \sum_{\alpha \in \Phi_1^+} \alpha$ is a regular special weight, we see that

$$\mu = \sum_{i=1}^{r} d_i(\alpha_i - \alpha_i^{\theta})$$

with $0 < \forall d_i \in \mathbf{Q}$.

Let $(z_1, \cdots, z_r, w_{r+1}, \cdots, w_n)$ be coordinates on the affine part $\mathbf{C}^r \times U$ of X. Here $n = \dim X$ and $r = \operatorname{rank}(G/K)$. As $L_{-2\overline{\alpha}_i} \cong \mathcal{O}_X(D_i)$, Lemma (5.1.7) implies

$$c_1(X) = \sum_{i=1}^{r}(1+d_i)[D_i] \in H^2(X;\mathbf{Q})$$

where $[D_i]$ is the Poincaré dual of D_i. Hence if K is connected then d_i are integers (cf. Lemma (3.2)). We assume hereafter K is connected and $d_i \in \mathbf{Z}$. It follows from the above formula that a $G^{\mathbf{C}}$-invariant Ricci-flat volume form on $G^{\mathbf{C}}/K^{\mathbf{C}}$ looks like

$$\eta \wedge \overline{\eta} \approx \prod_{i=1}^{r} \frac{|dz_i|^2}{|z_i|^{2(1+d_i)}} \prod_{k=1}^{n-r} |dw_k|^2.$$

(5.1.11) Here we compare the rank one case and the higher rank case from the viewpoint of G-action. Set

$$\widetilde{f}(e^H K^{\mathbf{C}}) = \sum_{i=1}^{r} (\overline{\alpha}_i(H))^2 \quad (H \in \underline{a}).$$

It follows from Theorem (2.4) that this defines a G-invariant Kähler potential on $G^{\mathbf{C}}/K^{\mathbf{C}}$. Set $\omega = dd^c f$; this is a complete Kähler metric on $G^{\mathbf{C}}/K^{\mathbf{C}}$. If $r = 1$ then the function $\frac{\eta\wedge\overline{\eta}}{\omega^n}$ depends only on f. So the Ricci-flat condition is reduced to an O.D.E. as in (4.4). But we cannot expect this if $r > 1$. Indeed, let $a, b \in A$ be distinct points in A outside of the origin. Suppose that $\dim Z_K(a) > \dim Z_K(A)$ and $\dim Z_K(b) = \dim Z_K(A)$. Let $c(t)$ be a geodesic (relative to ω) connecting a and b. Assume that $c(t)$ is tangent to a metric sphere at b. Let X be a nonzero element of the Lie algebra of $Z_K(a)$ which is not contained in $Z_K(A) = Z_K(b)$. The point a outside of the sphere is a fixed point of the variation of geodesics $C_s(t)$ defined by

$$c_s(t) = (\exp sX) \cdot c(t).$$

In general, the length of the variation vector field $v(t) = \frac{d}{ds}(\exp sX) \cdot c(t)$ does not assume its critical value at b. This implies that the ratio of $\eta \wedge \overline{\eta}$ and ω^n depends on the directions in X. So we cannot deform f into $\phi(f)$ so that $(dd^c\phi(f))^n = \eta \wedge \overline{\eta}$.

5.2. Construction of Background Metrics in Rank Two Case

(5.2.1) This subsection is devoted to the construction of the background metric on symmetric varieties of rank two. Let $G^{\mathbf{C}}/K^{\mathbf{C}}$ be a symmetric variety of rank two and X its canonical compactification. There exists a fibration

$$\Pi_i : D_i \to G^{\mathbf{C}}/P_i \quad (i = 1, 2)$$

with fiber F_i isomorphic to the canonical compactification of the rank one symmetric variety L_i/K_i. The parabolic subgroup P_i is generated by P and the root subgroups $U_{\pm\alpha}$ for roots α with $(\mu_i, \alpha) = 0$ for each $i = 1, 2$. So the corresponding subsystem $\pi(i) \subset S$ consists of simple roots α with $(\alpha, \mu_i) = 0$. Set

$$n_1 + 1 = \dim F_1 \quad \text{and} \quad n_2 + 1 = \dim F_2.$$

The anticanonical bundle of $G^{\mathbf{C}}/P_i$ is an ample homogeneous line bundle $L_{-\chi_i}$ with χ_i the sum of all roots in the unipotent radical of P_i, i.e., the sum of all positive roots which are not orthogonal to μ_i; since the involution θ does not fix any root in the unipotent radical of P_i, it must map their sum χ_i to $-\chi_i$, and so χ_i is a special weight. Hence χ_i is a linear combination of μ_1 and μ_2. As χ_i is a character of P_i, χ_i is orthogonal to all roots α with $(\alpha, \mu_i) = 0$. [Proof. Take a simple root α of the Levi complement of the parabolic group P_i. Since the reflection along α maps all the set of positive roots other than α to itself, we see that $w_\alpha \chi_i = \chi_i$, i.e., $(\chi_i, \alpha) = 0$. Since any root in the Levi complement of P_i is a sum of simple roots, we see that for all roots α such that $(\alpha, \mu_i) = 0$ we have $(\alpha, \chi_i) = 0$.] Therefore χ_i is a positive multiple of μ_i:

$$\chi_i = k_i \mu_i.$$

The character χ_i satisfies $(\chi_i, \check{\alpha}) > 0$ for all $\alpha \in S - \pi(i)$. As χ_i is a special weight, we can write

$$\chi_i = \sum_{j=1}^{2} 2a_{ij}\overline{\alpha}_j$$

with $a_{ij} \in \mathbf{Q}$. If we start with an irreducible Riemannian symmetric space, then we have $\mu_1, \mu_2 \in S - \pi(i)$ $(i = 1, 2)$ (so we assume that G/K is irreducible). It follows from $(\mu_i, \check{\overline{\alpha}}_j) = \delta_{ij}$ or $2\delta_{ij}$ $(i, j = 1, 2)$ that $a_{ij} > 0$. Indeed, set $\mu_i = \sum_{k=1}^{2} b_{ik}\overline{\alpha}_k$. Then $\sum_{k=1}^{2} b_{ik}(\overline{\alpha}_k, \overline{\alpha}_j)$ is δ_{ij} or $2\delta_{ij}$. Solving the equation shows $b_{ij} > 0$. Note that $\forall a_{ij} > 0$ because a_{ij} is a positive multiple of the entry of the inverse matrix of the Cartan matrix $((\overline{\alpha}_i, \overline{\alpha}_j))$. So each χ_i is written as a linear combination of $\overline{\alpha}_j$'s with positive coefficients. From the definition of d_1 and d_2, we have

$$a_{11} + a_{21} = d_1 \quad \text{and} \quad a_{12} + a_{22} = d_2.$$

Example. We have $a_{11} = a_{22} = \frac{2}{3}$, $a_{12} = a_{21} = \frac{1}{3}$ and $d_1 = d_2 = 1$ for the canonical compactification of the symmetric variety $SL_3(\mathbf{C})/SO_3(\mathbf{C})$ in Example (5.1.9).

If K is connected, $d_i \in \mathbf{Z}$ because there is a $G^{\mathbf{C}}$-invariant holomorphic n-form η with poles along D_i and $\mathcal{O}_X(D_i) = L_{-2\overline{\alpha}_i}$. Let σ_i be the canonical section of the line bundle $\mathcal{O}_X(D_i) = L_{-2\overline{\alpha}_i}$, and $\|\cdot\|_i = \|\cdot\|$ denote a Hermitian norm on this line bundle. For $i = 1, 2$ we set

$$\omega_i = \sqrt{-1}\partial\bar{\partial}\log \frac{1}{\prod_{j=1}^{2} \|\sigma_j\|^{2a_{ij}}}.$$

This is a Chern curvature form of the line bundle $L_{-\chi_i} = \mathcal{O}_X(a_{i1}D_1 + a_{i2}D_2)$ on X with $\chi_i = \sum_{j=1}^{2} 2a_{ij}\overline{\alpha}_j$. By averaging over G, we may assume that ω_i is G-invariant. Hereafter we assume G-invariance. Note that the line bundle $L_{-\chi_i}$ on X is the pull-back of k_i-times the hyperplane bundle on $\Pi_i(X) \subset \mathbf{P}(V_{\mu_i})$ via the $G^{\mathbf{C}}$-equivariant blowing down $\Pi_i : X \to \Pi_i(X)$ and that $b_1''\chi_1 + b_2''\chi_2$ is a regular special weight for any positive integers b_1'', b_2''. Hence we may choose Hermitian metrics on $\mathcal{O}_X(D_i)$ so that (in addition to G-invariance) ω_1 (resp. ω_2) is really a pull-back via Π_1 (resp. Π_2), positive semidefinite on X, positive definite on $X - D_1$ (resp. positive definite on $X - D_2$) and $b_1'\chi_1 + b_2'\chi_2$ is positive definite for any positive coefficients b_1', b_2'. Moreover [DP,Proposition 8.1] implies that
(i) the line bundle $L_{-\chi_i}$ restricted on Y is the pull-back of the anticanonical bundle $L_{-\chi_i}$ of $G^{\mathbf{C}}/P_i$ (which is ample) via the fibration $Y \to G^{\mathbf{C}}/P_i$, and
(ii) the sum $\omega_1 + \omega_2$ on Y is the curvature form of the line bundle $L_{-\chi_1-\chi_2}$ on Y, which is the anticanonical bundle of $Y = G^{\mathbf{C}}/P$.
We have chosen the Hermitian metrics of $\mathcal{O}_X(D_i)$ so that their curvature forms are G-invariant. This implies the following:

Lemma. *Let $\omega_Y = (\omega_1 + \omega_2)|_Y$. Then*
(a) ω_Y is a Kähler-Einstein metric on Y,
(b) the restriction of ω_Y to the fibers of the fibrations $\Pi_i|_Y : Y \to G^{\mathbf{C}}/P_i$ $(i = 1, 2)$ is again Kähler-Einstein.

Indeed, for any compact homogeneous Kähler manifold, the invariant Kähler form in the anticanonical class is necessarily Kähler-Einstein. We thus have (a). We shall give a systematic proof of (b) in (5.4.2).

For $a, b > 0$, we put

$$\omega(a,b) = \sqrt{-1}\partial\bar{\partial}\log \frac{1}{\|\sigma_1\|^{2a}\|\sigma_2\|^{2b}}.$$

Choose triples of positive numbers $(d_{11}, d_{12}; e_1)$ and $(d_{21}, d_{22}; e_2)$ so that the following conditions are satisfied:

$$|d_{11} - d_1|,\ |d_{12} - d_1|,\ |d_{21} - d_2|,\ |d_{22} - d_2|,\ |e_1 - \frac{1}{n}|,\ |e_2 - \frac{1}{n}| \quad \text{are small,}$$

$$\det\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \neq 0,$$

$$\begin{cases} (n-1)e_1 d_{11} + e_2 d_{12} = d_1 \\ (n-1)e_1 d_{21} + e_2 d_{22} = d_2, \end{cases}$$

and

$$e_1 d_{11} > e_2 d_{12} \quad \text{and} \quad e_1 d_{21} > e_2 d_{22}.$$

For instance, we take $e_1 = e_2 = \frac{1}{n}$. Then clearly there exist d_{ij} such that $d_{11} > d_{12}$, $d_{21} > d_{22}$, $d_{11}d_{22} - d_{12}d_{21} \neq 0$, $(n-1)d_{11} + d_{12} = nd_1$ and $(n-1)d_{21} + d_{22} = nd_2$.

Consider then (1,1)-forms

$$\begin{aligned}\Omega_i &:= \sqrt{-1}\partial\bar{\partial}\frac{1}{e_i}\left(\frac{1}{\|\sigma_1\|^{2d_{1i}}\|\sigma_2\|^{2d_{2i}}}\right)^{e_i} \\ &= \left(\frac{1}{\|\sigma_1\|^{2d_{1i}}\|\sigma_2\|^{2d_{2i}}}\right)^{e_i}\left\{\omega(d_{1i}, d_{2i}) + e_i\sqrt{-1}\partial\left(d_{1i}\log\frac{1}{\|\sigma_1\|^2} + d_{2i}\log\frac{1}{\|\sigma_2\|^2}\right)\right. \\ &\qquad \left.\wedge\,\bar{\partial}\left(d_{1i}\log\frac{1}{\|\sigma_1\|^2} + d_{2i}\log\frac{1}{\|\sigma_2\|^2}\right)\right\}\end{aligned}$$

for $i = 1, 2$. It follows that the sum of Ω_i for $i = 1, 2$:

$$\Omega_{12} := \Omega_1 + \Omega_2$$

is positive definite on $X - D$. By taking the average over G we may assume that the potential $\|\sigma_1\|^{2d_{i1}}\|\sigma_2\|^{2d_{i2}}$ is G-invariant, and so Ω_i are G-invariant. It now follows from Lemma (3.3) that Ω_{12} is a complete Kähler metric on $X - D_1 - D_2$.

To construct a background metric, in addition to Ω_{12}, we need the following (1,1)-form:

$$\begin{aligned}\Omega &= \sqrt{-1}\partial\bar{\partial}\left(\frac{1}{\|\sigma_1\|^{2d_1/n}\|\sigma_2\|^{2d_2/n}}\right) \\ &= \frac{n}{\|\sigma_1\|^{2d_1/n}\|\sigma_2\|^{2d_2/n}}\left\{\omega_1 + \omega_2 + \frac{1}{n}\sqrt{-1}\partial\left(d_1\log\frac{1}{\|\sigma_1\|^2} + d_2\log\frac{1}{\|\sigma_2\|^2}\right)\right. \\ &\qquad \left.\wedge\,\bar{\partial}\left(d_1\log\frac{1}{\|\sigma_1\|^2} + d_2\log\frac{1}{\|\sigma_2\|^2}\right)\right\}.\end{aligned}$$

This also defines a complete Kähler metric on $X - D_1 - D_2$.

Now we are ready to construct a background metric on $G^{\mathbf{C}}/K^{\mathbf{C}}$. We take tubular neighborhoods $B_{12}(\varepsilon)$ (resp. $B_1(\varepsilon)$ and $B_2(\varepsilon)$) of Y (resp. D_1 and D_2) in X of radial ε (with respect to some smooth metric on X), where ε is a small positive number to be determined later. Take a smooth nonnegative functions ρ_{12}, ρ_1 and ρ_2 defined on the union of the above tubular neighborhoods with the following properties:

(i) $\rho_{12} + \rho_1 + \rho_2 = 1$ in some neighborhood of $D_1 \cup D_2$,

(ii) $\rho_{12} \equiv 1$ in some neighborhood of Y,

(iii) $\rho_{12} \equiv 0$ outside of $B_{12}(\varepsilon)$,

(iv) $\rho_i \equiv 0$ outside of $B_i(\varepsilon)$ $(i = 1, 2)$.

(v) ρ's vary monotonically with respect to the modulus of the coordinate z where $z = 0$ defines the corresponding subvariety in D_i or X (for instance, in D_i, Y is given by the equation $z = 0$).

In addition to the above conditions, we need some symmetry on ρ's. We need:

(vi) ρ's are invariant under the action of a compact group G. (This is possible because only $D_i - Y$ and Y are non-principal $G^{\mathbf{C}}$-orbits.)

Moreover we need the following. We will show in (5.2.7-8) the existence of a good retraction 0f tubular neighborhoods of Y in D_i and of D_i in X (to be more precise, one can construct a retraction which is holomorphic along Y and D_i in any jet level). By using this retraction, the necessary property of ρ is described as follows: (vii) the bump functions ρ should be first defined on the corresponding subvariety and then extended to a neighborhood of it by a good retraction.

Such a partition of unity $\{\rho\}$ certainly exists. Using ρ's as gluing functions, we consider the potential

$$P(d_{ij},e_j) := \rho_{12}\left(\frac{1}{\|\sigma_1\|^{2e_1d_{11}}\|\sigma_2\|^{2e_1d_{21}}} + \frac{1}{\|\sigma_1\|^{2e_2d_{12}}\|\sigma_2\|^{2e_2d_{22}}}\right) + (\rho_1+\rho_2)\frac{1}{\|\sigma_1\|^{2d_1/n}\|\sigma_2\|^{2d_2/n}}$$

defined in the ε-neighborhood of $D_1 \cup D_2$. Let us consider its Levi form. This coincides with Ω_{12} in the region where $\rho_{12} \equiv 1$, coincides with Ω in the region $\rho_i \equiv 1$ for some $i = 1,2$. In a region where ρ's are not constant, we must examine the effect of the differentiation of ρ's. These are either of the form

$$\frac{1}{\|\sigma_1\|^{2a}\|\sigma_2\|^{2b}}\partial\bar\partial\rho$$

or of the form

$$\frac{1}{\|\sigma_1\|^{2a}\|\sigma_2\|^{2b}}\left(\partial\log\frac{1}{\|\sigma_1\|^{2a}\|\sigma_2\|^{2b}}\wedge\bar\partial\rho\right) \quad \text{or its complex conjugate.}$$

Now $\partial\rho$ and $\partial\bar\partial\rho$ have their supports near Y. We claim that, if ε is sufficiently small, the above forms are absorbed in the sum of forms which does not contain any differential of ρ. Indeed, if $\varepsilon < 1$, along F_i, we can compare $\frac{1}{n}|dz|^2/|z|^2$ and $(d\rho)^2$ directly ($F_i \cap Y$ is defined by $z=0$). Now ρ varies monotonically from 1 to 0 in the interval $\varepsilon' < |z| < \varepsilon$ and $\frac{1}{\sqrt{n}}\log|z|$ varies from $\frac{1}{\sqrt{n}}\log\varepsilon'$ to $\frac{1}{\sqrt{n}}\log\varepsilon$ on the same interval. Clearly we have $\frac{1}{\sqrt{n}}\log(\varepsilon/\varepsilon') > 1$ if ε' is sufficiently small. It follows that we can choose ρ so that $|d\rho| < \frac{1}{\sqrt{n}}\frac{|dz|}{|z|}$. Similar argument is possible for $\partial\bar\partial\rho$. Thus all terms containing differentials of ρ are absorbed in the sum of terms containing none of these. We thus have a Kähler metric

$$\Omega(d_{ij},e_j) := \sqrt{-1}\partial\bar\partial P(d_{ij},d_j)$$

in a neighborhood of $D_1 \cup D_2$, which is complete toward $D_1 \cup D_2$. Because $P(d_{ij},d_j)$ grows like an exponential of usual Fubini-Study potential and $X - D$ is affine algebraic, the function $P(d_{ij},d_j)$ extends to a strictly plurisubharmonic function on the whole $X - D$. We will write this also as $P(d_{ij},d_j)$. Then we may assume the above Kähler metric is defined on the whole $X - D$. This is our background metric. The above construction may be called the *bifurcation of Kähler potentials*, because the Kähler potential

$$\frac{1}{\|\sigma_1\|^{2d_1/n}\|\sigma_2\|^{2d_2/n}}$$

bifurcates into the sum of two potentials

$$\frac{1}{\|\sigma_1\|^{2e_1 d_{11}}\|\sigma_2\|^{2e_1 d_{21}}} + \frac{1}{\|\sigma_1\|^{2e_2 d_{12}}\|\sigma_2\|^{2e_2 d_{22}}}$$

near Y to produce two linearly independent directions

$$|\partial(d_{1i}\log\frac{1}{\|\sigma_1\|^2} + d_{2i}\log\frac{1}{\|\sigma_2\|^2})|^2 \qquad (i = 1, 2)$$

going to infinity.

(5.2.2) Here we define a certain class of Riemannian manifolds. We will seek Ricci-flat metrics in these classes. In the following definition, the α in $C^{k,\alpha}$ has, of course, nothing to do with the α in the formula $c_1(X) = \alpha[D]$.

Definition. (i) *A smooth Riemannian metric g on a smooth manifold M is of $C^{k,\alpha}$-bounded geometry if for each point $p \in M$ there exists a coordinate system $x = (x_1, \cdots, x_m)$ centered at p such that*

(a) x runs over a unit ball in $\mathbf{R}^m$,

(b) if we write $g = \sum g_{ij}dx_i dx_j$, then the matrix (g_{ij}) is uniformly equivalent to (δ_{ij}), i.e., bounded above and below by constant positive matrices independent of p,

(c) $C^{k,\alpha}$-norms of g_{ij} are uniformly bounded.

(ii) *A complete Riemannian manifold with a base point (M, g, o) is said to be of weakly $C^{k,\alpha}$-asymptotically flat geometry if for each point $p \in M$ with* $\operatorname{dist}(o,p) = \rho$, *there exists a harmonic coordinate system $(x_1, \cdots, x_m)$ centered at p such that*

(a) $x = (x_1, \cdots, x_m)$ runs over a unit ball B^m in $\mathbf{R}^m$,

(b) if we write $g = \sum g_{ij}dx_i dx_j$ then the matrix (g_{ij}) is uniformly equivalent to $r^2(\delta_{ij})$, where $r = \rho^{e(p)}$ with $e(p)$ a positive exponent depending on p not greater than one and which is bounded below by a positive number independent of p,

(c) In the coordinate system $(y_1, \cdots, y_m) = (rx_1, \cdots, rx_m)$, the components of the metric g have uniformly bounded $C^{k,\alpha}$-norms.

It is not too hard to check the following lemma by direct computation from definitions.

Lemma. *The complete Kähler metric $\Omega(d_{ij}, e_j)$ on $X - D$ is of $C^{k,\alpha}$-bounded geometry and is also of weakly $C^{k,\alpha}$-asymptotically flat geometry (with respect to any base point in $X - D$). Moreover we can take holomorphic coordinate system in Definition (5.2.2).*

Proof. Only the statement on the exponent $e(p)$ is not clear. It is explained in Remark (5.3.11) at the end of Section 5.3. To check other statements in Definition (5.2.2) we can work locally by the presence of a compact group action. Now the DeConcini-Procesi system in (5.1.6) is an example of the coordinate system satisfying the conditions in (ii) in Definition (5.2.2). *q.e.d.*

(5.2.3) We compute the volume form of $\Omega(d_{ij}, e_j)$ near a point of $Y = D_1 \cap D_2$. Let $z_i = 0$ be a local equation of D_i near a point in Y $(i = 1, 2)$. It follows from the conditions (5.2.1) on (d_{i1}, d_{i2}, e_i) that the volume form along Y is

$$\left(\frac{1}{\|\sigma_1\|^{2e_1d_{11}}\|\sigma_2\|^{2e_1d_{21}}}\right)^{(n-1)}\left(\frac{1}{\|\sigma_1\|^{2e_2d_{12}}\|\sigma_2\|^{2e_2d_{22}}}\right)\frac{|dz_1|^2|dz_2|^2}{|z_1|^2|z_2|^2}(\omega(d_{11},d_{21})|_Y)^{n-2}.$$

Reason: the form $\omega_1|_{F_2}$ has a degenerate direction at infinity, i.e., near $F_2 \cap Y$. Indeed, $\omega_1|_{F_2}$ degenerates in the direction in F_2 parallel to $F_1 \cap Y$.

Now we compute the Ricci-form "$-\sqrt{-1}\partial\bar{\partial}\log$" of the above volume form "along Y". We note that since the volume form $(\omega(d_{11},d_{21})|_Y)^{n-2}$ is G-invariant on $Y = G^{\mathbf{C}}/P$, its Ricci-form is a Kähler-Einstein metric on Y. This implies

$$\mathrm{Ric}(\omega(d_{11},d_{21})^{n-2}) = \omega(d_1,d_2) = \omega_1 + \omega_2.$$

Thus the resulting Ricci-form is computed as follows:

$$\begin{aligned} &-(n-1)e_1\omega(d_{11},d_{21}) - e_2\omega(d_{12},d_{22}) + \mathrm{Ric}(\omega(d_{11},d_{21})^n) \\ &= -\omega((n-1)e_1d_{11}+e_2d_{12},(n-1)e_1d_{21}+e_2d_{22}) + \omega(d_1,d_2) \\ &= -\omega(d_1,d_2)+\omega(d_1,d_2) = 0 \end{aligned}$$

along Y. We have thus proved

Lemma. *We can choose Hermitian metrics for $\mathcal{O}_X(D_i)$ along Y so that*

$$f = \frac{\Omega(d_{ij},e_j)^n}{\eta\wedge\bar{\eta}} = 0$$

holds along Y.

(5.2.4) The next thing we want to do is to extend the above Hermitian metrics on $\mathcal{O}_X(D_i)|_Y$ to those on $O_X(D_i)$ in a neighborhood of $D = D_1 \cup D_2$ in such a way that $f = 0$ holds along D. First of all let us observe the consequence of the degeneracy of ω_i along the fiber of $\Pi_i : D_i \to G^{\mathbf{C}}/P_i$. This implies

$$\omega_i = \sqrt{-1}\partial\bar{\partial}\left(\log\frac{1}{\|\sigma_1\|^{2a_{1i}}} + \log\frac{1}{\|\sigma_2\|^{2a_{2i}}}\right) = 0$$

along fibers of $\Pi : D_i \to G^{\mathbf{C}}/P_i$. Let

$$\begin{cases} d_{11} = h_1a_{11} + h_2a_{12} \\ d_{21} = h_1a_{21} + h_2a_{22} \end{cases}$$

Then along the fiber of $\Pi_1|_{D_1}$, we have $\omega(d_{11},d_{21}) = h_2\omega_2$ and along D_1, ω_1 is of rank $n-n_1-1$. Now introduce the quantity

$$\overline{a}^i_{jk} = a_{jk} - \frac{a_{ji}a_{ik}}{a_{ii}} \in \mathbf{Q}.$$

Then, along D_1 (say), the volume form of $\Omega(d_{ij},e_j)$ can be written as (up to constant multiplication) as follows:

$$(*)\quad \left(\frac{\omega_1^{n-n_1-2}}{\|\sigma_1\|^{2a_{11}}\|\sigma_2\|^{2a_{21}}}\frac{|dz_1|^2}{|z_1|^2}\right)\left(\frac{1}{\|\sigma_1\|^{2a_{12}}}\right)\left(\frac{1}{\|\sigma_2\|^{\frac{2a_{21}a_{12}}{a_{11}}}}\right)$$

$$\times\,(\text{some smooth function involving } \|\sigma_2\|)$$

$$\times\,\frac{1}{\|\sigma_2\|^{2\overline{a}_{22}^1}}\left(\omega_2+\frac{1}{n_1+1}\sqrt{-1}\partial\log\frac{1}{\|\sigma_2\|^{2\overline{a}_{22}^1}}\wedge\bar\partial\log\frac{1}{\|\sigma_2\|^{2\overline{a}_{22}^1}}\right)^{n_1+1}.$$

If we set

$$\begin{aligned}\widetilde{\omega}_2 &:= \frac{1}{\|\sigma_2\|^{\frac{2\overline{a}_{22}^1}{n_1+1}}}\left(\omega_2+\frac{1}{n_1+1}\partial\log\frac{1}{\|\sigma_2\|^{2\overline{a}_{22}^1}}\wedge\bar\partial\log\frac{1}{\|\sigma_2\|^{2\overline{a}_{22}^1}}\right)\\ &= \sqrt{-1}\partial\bar\partial(n_1+1)\left(\frac{1}{\|\sigma_2\|^{\frac{2\overline{a}_{22}^1}{n_1+1}}}\right),\end{aligned}$$

then the last factor in $(*)$ is replaced by $\widetilde{\omega}_2^{n_1+1}$. The smooth function in $(*)$ which compensates the effect of $\widetilde{\omega}_2^{n_1+1}$ may be considered as a part of the Hermitian metric $\|\cdot\|_1$ for $\mathcal{O}_X(D_1)$. Hereafter we consider this way.

As $\omega_2>0$ along F_1, $\widetilde{\omega}_2$ defines a complete Kähler metric on $F_1-F_1\cap Y$. Hence the Ricci-form of the volume form $\Omega(d_{ij},e_j)^n$ is

$$\Pi_1^*(\mathrm{Ric}(\omega_1^{n-n_1-2}))-\sqrt{-1}\partial\bar\partial\log\frac{1}{\|\sigma_1\|^{2a_{11}}\|\sigma_2\|^{2a_{21}}}+\sqrt{-1}\partial\bar\partial\log\frac{1}{\|\sigma_1\|^{2a_{12}}\|\sigma_2\|^{\frac{2a_{21}a_{12}}{a_{11}}}}+\mathrm{Ric}(\widetilde{\omega}_2^{n_1+1})$$

along D_1.

Suppose for the moment that $\|\cdot\|_2$ (a Hermitian metric on $\mathcal{O}_X(D_2)$) is determined on D_1 so that
(i) it agrees with the already existing definition $\|\cdot\|_2|_Y$,
(ii) its curvature form is G-invariant.

We consider the Ricci-form of $\Omega(d_{ij},e_j)$ along a fiber F_1 of the fibration $\Pi_1: D_1\to G^{\mathbf{C}}/P_1$. If the Hermitian metric $\|\cdot\|_1$ is determined properly, restricting $\mathrm{Ric}(\Omega(d_{ij},e_j)^n)$ to a fiber F_1 gives $\mathrm{Ric}(\widetilde{\omega}_2^{n_1+1})$ along F_1. Indeed there is a Hermitian metric $\|\cdot\|_1|_{D_1}$ such that the curvature form of the Hermitian metric $\|\cdot\|_1^{2a_{11}}\|\cdot\|_2^{2a_{21}}$ comes from the G-invariant curvature form of the anticanonical bundle of $G^{\mathbf{C}}/P_1$ (and so a Kähler-Einstein metric on it). It is easy to see that the definition of $\|\cdot\|_1|_{D_1}$ along D_1 agrees with the already existing definition of $\|\cdot\|_1|_Y$. We thus have

$$\sqrt{-1}\partial\bar\partial\log\frac{1}{\|\sigma_1\|^{2a_{11}}\|\sigma_2\|^{2a_{21}}}=0$$

with respect to the new definition of $\|\sigma_1\|$ along the fiber F_1. The contribution of $\omega_2^{n-n_1-2}$ to the Ricci-form of $\Omega(d_{ij},e_j)^n$ vanishes because this is complementary to the fiber of $Y\to G^{\mathbf{C}}/P_1$ and thus coming from the G-invariant Kähler-Einstein metric on $G^{\mathbf{C}}/P_1$. Thus, all except $\mathrm{Ric}(\widetilde{\omega}_2^{n_1+1})$ vanish along F_1. If moreover we have $\mathrm{Ric}(\widetilde{\omega}_2^{n_1+1}) = 0$ along F_1, then $f = \frac{\Omega(d_{ij}, e_j)^n}{\eta \wedge \bar{\eta}}$ becomes constant along fibers F_1. This combined with the fact that f is constant along Y shows that f is constant along D_1 (normalized to be 0). Thus it is important to get good behavior on the Ricci-form of $\widetilde{\omega}_2$ along fibers F_1. In order to determine $\|\cdot\|_2|_{F_1}$ so that $\mathrm{Ric}(\widetilde{\omega}_2^{n_1+1}) = 0$ along $F_1 - F_1 \cap Y$, we must analyze the effect on the complete Kähler metric $\widetilde{\omega}_2$ caused by a change of Hermitian metrics of $\mathcal{O}_X(D_2)|_{F_1}$. This is done in the next three paragraphs.

(5.2.5) The next proposition is a refinement of Theorem (4.3). This refined form is necessary in the induction process.

Proposition. *Let X be a Fano manifold (of dimension n) and let D a smooth hypersurface in X with $c_1(X) = \alpha[D]$ with $\alpha > 1$ and assume D admits a Kähler-Einstein metric. Suppose that for any positive integer k there exists a Hermitian metric $h = h_k$ of $\mathcal{O}_X(D)$ with the following properties:*
(a) the curvature form $\sqrt{-1}\partial\bar{\partial}\log h$ is positive definite on $X - D$ and defines a Kähler-Einstein metric on D,
(b) h is constant in the direction normal to D up to $(k+2)$-nd order normal derivatives and so the component of $\sqrt{-1}\partial\bar{\partial}\log h$ in the direction normal to D vanishes along D up to k-th order normal derivatives.
Here normal derivative means the derivative w.r.to a coordinate z such that D is defined locally by $z = 0$ (so these derivatives are smooth sections of tensor powers of the dual $N^*_{D/X}$ of the normal bundle).
Then there exists a complete asymptotically flat Kähler metric

$$\widetilde{\omega} = \sqrt{-1}\partial\bar{\partial}\left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{n}} + \sqrt{-1}\partial\bar{\partial}u$$

on $X - D$ with zero Ricci curvature and there exist a priori positive constants $C_{h,l}$ and C_n (C_n is comparable to n when n is large, i.e., the ratio $\frac{C_n}{n}$ is bounded away from 0 uniformly in n) such that

$$\|\nabla^l u\| \leq C_{h,l}\|\sigma\|^{\frac{l(\alpha-1)}{n}+C_n} \qquad \forall l \geq 0,$$

where the covariant derivative and the norm in the L.H.S. is with respect to the complete Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}\left(\frac{1}{\|\sigma\|^2}\right)^{\frac{\alpha-1}{n}}$.

Hence we have

$$|u| = O((\text{distance})^{-\varepsilon_n}), \quad |\nabla^l u| = O((\text{distance})^{-2l-\varepsilon_n})$$

with some positive constant ε_n, where the distance function is taken with respect to some fixed point in $X - D$ and with respect to the metric ω.

Proof of Proposition (5.2.5). We consider the Monge-Ampère equation

$$(\omega+\sqrt{-1}\partial\bar{\partial}u)^n=\eta\wedge\bar{\eta}.$$

The R.H.S. is the Ricci-flat volume form on $X-D$ with poles of order $s\alpha$ along D, and is written as $e^{-f}\omega^n$ where $f=\frac{\omega^n}{\eta\wedge\bar{\eta}}$. The condition (a) implies that ω is really a complete Kähler metric and the Kähler-Einstein condition in (a) together with (b) imply that $f=O(\|\sigma\|^k)$. The existence of the solution u with a priori estimates can be proved exactly with the same strategy as in [BK2] and [K1, Lemma 7]. See Lemma (5.3.2) for estimate on C_n. *q.e.d.*

(5.2.6) The following Lemma gurantees the existence of a good approximation to the Ricci-flat Kähler metric on a symmetric varietiy of rank 1.

Lemma. *Let X be the canonical compactification of a symmetric variety of rank one and D the divisor at infinity. Then there exist a neighborhood U_D of D and a holomorphic retraction $U_D\to D$.*

Proof. The canonical compactification X is the closure of the orbit of the $K^{\mathbf{C}}$-invariant vector in the projective space $\mathbf{P}$ of the representation space associated to a regular special weight. The divisor D at infinity is the orbit of the highest weight vector. Let P be the corresponding parabolic subgroup of $G^{\mathbf{C}}$ such that $D=G^{\mathbf{C}}/P$. Pick a point $p\in D$ corresponding to the highest weight vector and look at the linear isotropy representation of the maximal reductive subgroup (Levi subgroup) of P on the tangent space $T_p\mathbf{P}$. Then T_pD is an invariant subspace. Let V_p be a complementary invariant subspace determined by fixing some G-invariant Hermitian metric (e.g., the Fubini-Study metric). Then we consider the linear subspace L_p of $\mathbf{P}$ tangent to V_p at p. By using $G^{\mathbf{C}}$ action, we can construct a family of linear subspaces L_q ($q\in D$). The union $\cup_{q\in D}L_q$ defines a holomorphic retraction from a neighborhood of D onto D. We get a desired holomorphic retraction by restriction.

Another way of understanding Lemma (5.2.6) is via the holomorphic foliation on $G^{\mathbf{C}}/K^{\mathbf{C}}$ and its DeConcini-Procesi compactification given by the complexification of closed geodesics of G/K (i.e., G-translations of the maximal torus). Note that all geodesics of a symmetric space G/K of compact type of rank 1 are closed. Each leaf of this foliation is compactified in X so that the compactified leaf is transversal to D. Indeed, the divisor D parametrizes all closed geodesics in G/K and two points corresponding to a closed geodesic is just the intersection of its compactified complexification and the divisor D (cf. [PW], [LS] and [S]). The maximal reductive subgroup of P preserves the geodesic and its compactified compexification as sets and of course fixes the intersection points with D (G operates on D naturally). *q.e.d.*

(5.2.7) The followings are direct consequences of Lemma (5.2.6).

Lemma. *Let (X,D) be as in Lemma (5.2.6). Then there exists a biholomorphism ϕ between a tubular neighborhood of D in X and a tubular neighborhood of the zero section of $N_{D/X}$ such that ϕ_D is the identity of D.*

Corollary. *Let (X,D) be as in Lemma (5.2.6). Then (X,D) satisfies the conditions in Proposition (5.2.5).*

Applying Proposition (5.2.5) to $(F_1, F_1 \cap Y)$, we define the Hermitian metric $\|\cdot\|_2$ for $\mathcal{O}_X(D_2)$ so that $\widetilde{\omega}_2$ is Ricci-flat along each fiber F_1. The boundary condition is the condition (a) and (b) in Lemma (5.2.1), i.e., $\|\cdot\|_2|_Y$ is already fixed so that (a) and (b) hold along Y. We extend the hermitian metric $\|\cdot\|_2$ for $\mathcal{O}_X(D_2)$ along each $F_1 - F_1 \cap Y$ so that $\widetilde{\omega}_2|_{F_1 - F_1 \cap Y}$ is Ricci-flat. Then the estimates in Proposition (5.2.5) imply that the definition along $F_1 - F_1 \cap Y$ glues Hölder continuously (with respect to the underlying smooth structure of a compact complex manifold X) to the already existing definition of the Hermitian metric for $\mathcal{O}_X(D_2)|_Y$ ((5.2.3)). The same holds for Hermitian metric for $\mathcal{O}_X(D_1)$ by considering the Ricci-flat condition of $\widetilde{\omega}_1$ (defined in an analogous way as $\widetilde{\omega}_2$) along $F_2 \subset D_2$. Using the condition $\omega_i = 0$ along F_i we can extend the definitions of Hermitian metrics for $\mathcal{O}_X(D_i)$ along F_i $(i = 1, 2)$ smoothly outside of Y and this extension agrees Hölder continuously with the boundary condition on Y. We have thus proved

Proposition. *There exist Hermitian metrics for $\mathcal{O}_X(D_i)$ $(i = 1, 2)$ along $D_1 \cup D_2$ so that*

$$f = \frac{\Omega(d_{ij}, e_j)^n}{\eta \wedge \overline{\eta}} = 0$$

holds along $D_1 \cup D_2$ and these Hermitian metrics are compatible with those in Lemma (5.2.3) on Y.

(5.2.8) The next thing we want to do is to extend the Hermitian metrics of $\mathcal{O}_X(D_i)$ to a tubular neighborhood of D. To do this we look at the blowing down $\Pi_i : X \to \Pi_i(X) \subset \mathbf{P}(V_{\mu_i})$. This blows down D_i to $\overline{D}_i = G^{\mathbf{C}}/P_i \subset \mathbf{P}(V_{\mu_i})$. As in the proof of Lemma (5.2.6), we consider an invariant subspace complementary to the invariant subspace $T_p\overline{D}_i)$ with respect to the linear isotropy representation of the maximal reductive subgroup of the isotropy group at $p \in \overline{D}_i$ (fixing a G-invariant Hermitiann metric, e.g., the Fubini-Study metric). It follows that there exists a holomorphic retraction from a neighborhood in $\mathbf{P}(V_{\mu_i})$ of $\overline{D}_i$ onto $\overline{D}_i$. Restricting this retraction to $\Pi_i(X)$ and considering the inverse image under Π_i we get a holomorphic retraction from a tubular neighborhood of D_i onto D_i. Using these retractions, we extend the Hermitian metrics of $\mathcal{O}_X(D_i)$ defined along D_i to a neighborhood of D_i. But this is only possible in a region not too close to Y, because tubular neighborhoods of D_i intersect near Y. In such region, we have Hermitian metrics for $\mathcal{O}_X(D_i)$ $(i = 1, 2)$ in a neighborhood of $D_1 \cup D_2$ so that for any $k \in \mathbf{Z}_+$ the function

$$f = \log \frac{\Omega(d_{ij}, e_j)^n}{\eta \wedge \overline{\eta}}$$

vanishes along $D_1 \cup D_2$ in the k-th jet level, i.e., if z is a coordinate transversal to D_i ($z = 0$ is a local equation of D_i), then all derivatives $(\partial^l f/\partial z^p \bar{\partial}\bar{z}^q)_{z=0}$ $(p+q = l)$ of f in z-direction vanish for $0 \le \forall l \le k$. Moreover these Hermitian metrics are compatible with those in Proposition (5.2.7).

Now we look at a small neighborhood of Y. If we extend Hermitian metrics of $\mathcal{O}_X(D_i)$ along $D_1 \cap D_2$ by using two retractions (corresponding to D_1 and D_2), these extensions do not coincide. But the discrepancy stems only from the correction term u in Proposition (5.2.5). Now $u = 0$ along $D_1 \cap D_2$ and moreover u decays like (distance)$^{-\varepsilon_n}$ with some constant ε_n depending on n only (which is comparable to n if n is large). Because $\sqrt{-1}\partial\bar\partial u$ is involved in the metric, we have only to examine the contribution of $\partial\bar\partial u$. It follows from Proposition (5.2.5) that $\partial\bar\partial u$ decays with order like (distance)$^{-(2+\varepsilon_n)}$ with respect to the background metrics $\tilde\omega_1$ and $\tilde\omega_2$ defined on fibers F_2 and F_1. Because of the description in (5.2.1) of the asymptotic behavior of the background metric $\Omega(d_{ij}, e_j)$, we infer that the same order of decay remains true for the function $f = \log \Omega(d_{ij}, e_j)/\eta \wedge \bar\eta$:

Proposition. *There exist Hermitian metrics for $\mathcal{O}_X(D_i)$ $(i = 1, 2)$ in a neighborhood of $D_1 \cup D_2$ such that*

$$f = \log \frac{\Omega(d_{ij}, e_j)^n}{\eta \wedge \bar\eta} = O((\text{distance})^{-(2+\varepsilon_n)}).$$

As $P(d_{ij}, e_j)$ diverges to infinity at $D_1 \cup D_2$ on the affine algebraic manifold $X - D_1 - D_2$, we can extend the potential $P(d_{ij}, e_j)$ to the whole of $X - D_1 - D_2$ smoothly and obtain a strictly plurisubharmonic function on $X - D_1 - D_2$ (this is proved, for instance, using Theorem (2.4)). By taking the average over G, we may assume that the potential is G-invariant.

Theorem. *There exists a G-invariant Kähler potential on $X - D_1 - D_2$ which is equal to $P(d_{ij}, e_j)$ in Proposition (5.2.8) in a neighborhood of $D_1 \cup D_2$.*

We will write the new Kähler potential by the same symbol $P(d_{ij}, e_j)$ and the resulting complete Kähler metric by $\Omega(d_{ij}, e_j) = \sqrt{-1}\partial\bar\partial P(d_{ij}, e_j)$. The function $f = \log(\Omega(d_{ij}, e_j)^n/\eta \wedge \bar\eta)$ decays like $O((\text{distance})^{-(2+\varepsilon_n)})$ ($\varepsilon_n > 0$, comparable to n if n is large).

(5.2.9) We describe some important properties of $\Omega(d_{ij}, e_j)$. Let f be as in Proposition (5.2.8).

Lemma. *There exists a constant $q < \dim X = n$ such that $f \in L^q(X - D_1 - D_2, \Omega(d_{ij}, e_j))$.*

Proof. It is a consequence of (5.1.10) and Theorem (5.2.8). *q.e.d.*

Proposition. *The complete Kähler metric $\Omega(d_{ij}, e_j)$ satisfies the following three properties:*
(i) The volume of metric balls of radius R grows like R^{2n}, where $n = \dim X$.
(ii) There exists an integer $h > 0$ such that the metric $\Omega(d_{ij}, e_j)$ in Theorem (5.2.8) has Ricci curvature decay faster than R^{-h} as $R \to \infty$.
(iii) The isoperimetric inequality holds with a uniform constant.

Proof. The first assertion is a consequence of the definition of the metric and the fact that ω_i degenerates along F_i. The second assertion is equivalent to the decay condition for f in Proposition (5.2.8). The third assertion is a consequence of the proof of Croke's isoperimetric inequality for complete Riemannian manifolds with nonnegative Ricci curvature and with maximal volume growth (because of rapid decay of Ricci curvature (Lemma 5.2.9), it is easy to modify the proof in [Cr] in our situation). *q.e.d.*

Corollary. *Let $(X, D_1 \cup D_2)$ be as above. Let $\Omega(d_{ij}, e_j)$ be a complete Kähler metric in Theorem (5.2.8) and let volume form and norms be with respect to this metric. Set $\gamma = \frac{n}{n-1}$. Then there exists a positive constant c such that for any compactly supported C^1-function f on $X - D_1 - D_2$, the following Sobolev inequality holds:*

$$\left(\int_{X-D_1-D_2} |f|^{2\gamma}\right)^{\frac{1}{\gamma}} \leq c \int_{X-D_1-D_2} \|df\|^2.$$

Proof. See, for instance, [Y2]. *q.e.d.*

(5.2.10) Since, by Theorem (2.4), the desired G-invariant Ricci-flat Kähler potential is determined by its values on the anisotropic torus ($\mathbf{R}^2$ in $(\mathbf{C}^*)^2$), it is desireable to compute the explicit differential equation of the Ricci-flat Kähler potential restricted to $\mathbf{R}^2$, in coordinates on $\mathbf{R}^2$ defined naturally in terms of special weights. Here we pick a simplest example $SL_3(\mathbf{C})/SO_3(\mathbf{C})$ of rank 2 and compute the explicit differential equation. As described in (5.1.9), the symmetric variety $SL_3(\mathbf{C})/SO_3(\mathbf{C})$ is identified with the set of all smooth plane conics. In matrix form, it is written as follows:

$$SL_3(\mathbf{C})/SO_3(\mathbf{C}) = \left\{ Z = \begin{pmatrix} z_1 & z_4 & z_5 \\ z_4 & z_2 & z_6 \\ z_5 & z_6 & z_3 \end{pmatrix} ; \ \det(Z) = 1 \right\}.$$

The action of $SL_3(\mathbf{C})$ is given by $Z \to gZg^t$ (g^t being the transposed matrix of g) and the above is identified with the orbit of the identity matrix. The representations corresponding to the special weights μ_1 and μ_2 in (5.1.3) are ρ_1 and ρ_2, which are the standard representation $V = \mathbf{C}^3$ and the 2-exterior representation $\bigwedge^2 V = \mathbf{C}^3$ of ρ_1, respectively. Set $r = |\rho_1(Z)|^2$ and $s = |\rho_2(Z)|^2$. Then all Weyl-invariant functions on $\mathbf{R}^2$ are functions in r and s. Explicitly, we have

$$r = |\rho_1(Z)|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2(|z_4|^2 + |z_5|^2 + |z_6|^2)$$

and

$$\begin{aligned} s &= |\rho_2(Z)|^2 \\ &= |z_1 z_2 - z_4^2|^2 + |z_1 z_3 - z_5|^2 + |z_2 z_3 - z_6^2|^2 \\ &\quad + 2(|z_1 z_6 - z_4 z_5|^2 + |z_4 z_6 - z_5 z_2|^2 + |z_4 z_3 - z_5 z_6|^2). \end{aligned}$$

The maximal torus $(\mathbf{C}^*)^2$ is given by the equation $z_4 = z_5 = z_6 = 0$. Then around the maximal torus, we can introduce λ and μ by setting $z_1 = \lambda$, $z_2 = \lambda^{-1}\mu$ and $z_3 = \mu^{-1}$ so that the system $(\lambda, \mu, z_4, z_5, z_6)$ gives a local coordinate system around the maximal torus. Let f be a G-invariant function $f = f(r, s)$. Then

$$\begin{aligned} \partial\bar{\partial} f &= f_{rr} \partial r \wedge \bar{\partial} r + f_{rs}(\partial r \wedge \bar{\partial} s + \partial s \wedge \bar{\partial} r) + f_{ss} \partial s \wedge \bar{\partial} s \\ &\quad + f_r \partial\bar{\partial} r + f_s \partial\bar{\partial} s. \end{aligned}$$

At $p = \operatorname{diag}(\lambda, \lambda^{-1}\mu, \mu^{-1})$ in the maximal torus, no z_i and dz_i and their complex conjugates ($i = 4,5,6$) appear in ∂r and ∂s. This means that along the maximal torus we can separate variables into (λ, μ) and (z_4, z_5, z_6) in the computation of $(\partial\bar{\partial} f)^5$. Along the maximal torus, we have

$$\partial\bar{\partial} r = dz_1 \wedge d\bar{z}_1 + \cdots + 2(dz_4 \wedge d\bar{z}_4 + \cdots)$$

and

$$\partial\bar{\partial} s = \partial\bar{\partial} s|_{(\mathbf{C}^*)^2} + 2(|z_3|^2 dz_4 \wedge d\bar{z}_4 + |z_2|^2 dz_5 \wedge d\bar{z}_5 + |z_1|^2 dz_6 \wedge d\bar{z}_6).$$

Hence along the maximal torus we have (up to constant multiple)

$$\begin{aligned}(\partial\bar{\partial} f)^5 &= (\partial\bar{\partial} f|_{(\mathbf{C}^*)^2})^2)^2 \\ &\quad \wedge (f_s|z_1|^2 + f_r)(f_s|z_2|^2 + f_r)(f_s|z_3|^2 + f_r) dz_4 \wedge d\bar{z}_4 \wedge \cdots .\end{aligned}$$

Now introduce a linear coordinate system (p, q) on $\mathbf{R}^2$ by setting $p = \log|\lambda|^2$ and $q = \log|\mu|^2$. Set $g(p,q) = f(r,s)$ along the maximal torus. Then we have

$$(\partial\bar{\partial} f|_{(\mathbf{C}^*)^2})^2 = (g_{pp}g_{qq} - g_{pq}^2)\frac{d\lambda \wedge d\bar{\lambda} \wedge d\mu \wedge d\bar{\mu}}{|\lambda|^2|\mu|^2}.$$

The Jacobian matrix $\left(\frac{\partial(r,s)}{\partial(p,q)}\right)$ is

$$\begin{pmatrix} e^p - e^{-p+q} & e^{-p} + e^{-q+p} \\ -e^{-q} + e^{-p+q} & e^q - e^{-q+p} \end{pmatrix}.$$

Its determinant is given by

$$\Delta = e^{p+q} - e^{-p-q} + e^{-2p+q} - e^{2p-q} + e^{p-2q} - e^{-p+2q}.$$

Then we can compute $f_s|z_1|^2 + f_r$ and so on. The result turns out to be quite simple and as follows:

$$\begin{aligned} f_s|z_1|^2 + f_r &= f_s e^p + f_r = \frac{e^{2p}(1 - e^{-p-q})(1 - e^{q-2p})g_q}{\Delta}, \\ f_s|z_2|^2 + f_r &= f_s e^{-p+q} + f_r = \frac{(1 - e^{q-2p})(e^q - e^{-q+p})(g_p + g_q)}{\Delta}, \\ f_s|z_3|^2 + f_r &= f_s e^{-q} + f_r = \frac{(e^q - e^{-q+p})(1 - e^{-p-q})g_p}{\Delta}. \end{aligned}$$

On the other hand, the holomorphic 5-form on the symmetric variety $\det(Z) = 1$ is given by the Poincaré residue:

$$\eta = \frac{dz_2 \wedge dz_3 \wedge dz_4 \wedge dz_5 \wedge dz_6}{\frac{\partial \det(Z)}{\partial z_1}},$$

which turns out to be $G^{\mathbf{C}}$-invariant. Explicitly, along the maximal torus, it is

$$\eta = \frac{d\lambda \wedge d\mu}{\lambda\mu} \wedge dz_4 \wedge dz_5 \wedge dz_6.$$

Thus, along the maximal torus $(\mathbf{C}^*)^2$, we have:

$$\chi := \frac{(\partial\bar{\partial} f)^5}{\eta \wedge \bar{\eta}} = (g_{pp}g_{qq} - g_{pq}^2)\frac{g_p g_q (g_p + g_q)(e^q - e^{-q+p})^2(e^p - e^{-q})^2(1 - e^{q-2p})^2}{\{e^{p+q} - e^{-p-q} + e^{-2p+q} - e^{2p-q} + e^{p-2q} - e^{-p+2q}\}^3}.$$

The explicit differential equation for the Ricci-flat Kähler potential is now given by $\chi = \text{constant}$.

5.3. Analysis

(5.3.1) The aim of this section is to prove the following existence theorem and the associated uniqueness theorem. For analytical background, we refer to, for instance, [Au], [CY] and [GT].

Theorem. *Let G/K be a symmetric space of compact type of rank two and let $G^{\mathbf{C}}/K^{\mathbf{C}}$ be the corresponding symmetric variety. Then there exists a G-invariant complete Ricci-flat Kähler metric on $G^{\mathbf{C}}/K^{\mathbf{C}}$ which is quasi-isometric to the background metric $\Omega(d_{ij}, e_j)$ in Theorem (5.2.8). More precisely, there exists a constant C_n comparable to n (constant times n) such that there exists a complete Ricci-flat Kähler metric of the form*

$$\Omega(d_{ij}, e_j) + \sqrt{-1}\partial\bar{\partial}u \qquad (\text{with } k \geq k_0)$$

with a priori estimates

$$\|\nabla^l_{\Omega(d_{ij},e_j)}u\|_{\Omega(d_{ij},e_j)} \leq O(R^{-C_n-l})$$

for $\forall l \in \mathbf{Z}_{\geq 0}$ and this is necessarily G-invariant if the background metric $\Omega(d_{ij}, e_j)$ is G-invariant.

The constant C_n can be estimated if one examines the constant δ in Lemma (5.3.2) carefully.

If we fix the asymptotic behavior of the metric, then the Ricci-flat metric in Theorem (5.3.1) is unique.

Corollary. *Let fix (d_{ij}, e_j) in Theorem (5.2.8). Let $\Omega(d_{ij}, e_j)$ and $\Omega(d_{ij}, e_j)'$ be two bachground metrics with the same parameter (d_{ij}, e_j). Then we have*

$$\Omega(d_{ij}, e_j) + \sqrt{-1}\partial\bar{\partial}u = \Omega(d_{ij}, e_j)' + \sqrt{-1}\partial\bar{\partial}u'$$

where u and u' are the solutions in Theorem (5.3.1) with a priori estimates of the form described in Theorem (5.3.1).

Note that this does not mean the absolute uniqueness of complete Ricci-flat Kähler metrics on $G^{\mathbf{C}}/K^{\mathbf{C}}$. This means that each time we fix the boundary condition then we have a unique complete Ricci-flat Kähler metric on $G^{\mathbf{C}}/K^{\mathbf{C}}$ with this boundary condition. As there is a freedom in choosing background metrics with different boundary conditions, there is a nontrivial moduli of complete Ricci-flat Kähler metrics on $G^{\mathbf{C}}/K^{\mathbf{C}}$.

The rest of this section is devoted to the proof of Theorem (5.3.1).

(5.3.2) We start with the existence of barrier functions as in [BK2]. Let $\omega = \Omega(d_{ij}, e_j)$ be the background metric in Theorem (5.2.8). Put

$$\rho = \rho(d_{ij}, e_j) = P(d_{ij}, e_j)^{\frac{1}{2}},$$

where d_{ij}, e_j and $P(d_{ij}, e_j)$ are as in (5.2.1). Let $\triangle$ denote the Laplacian with respect to the metric ω.

Lemma. *(i) There exists a positive constant c such that for any sufficiently small positive number δ we have*

$$\triangle \rho^{-\delta} \leq -c\rho^{-\delta-2}.$$

(ii) There exists a constant c with the following property: for any sufficiently small positive number δ and any positive number K there exists a positive number $L(\delta, K)$ such that

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}K\rho^{-\delta})^n \leq (1 - cK\delta\rho^{-2-\delta})\omega^n \\ (\omega - \sqrt{-1}\partial\bar{\partial}K\rho^{-\delta})^n \geq (1 + cK\delta\rho^{-2-\delta})\omega^n \end{cases}$$

holds if ρ is sufficiently large: $\rho > L(\delta, K)$.

Proof. This follows from direct computation similar to [BK] (cf. [K1,Lemma 7,p.164]). Here we prove (ii). A direct computation shows that $\omega + \sqrt{-1}\partial\bar{\partial}K\rho^{-\delta}$ is equivalent to

$$\rho^2\left\{\left(1 - \frac{K\delta}{2}\rho^{-2-\delta}\right)\omega_0 + \left(1 + \frac{K\delta(\delta+1)}{4}\rho^{-2-\delta}\right)\sqrt{-1}\partial\log\rho^2 \wedge \bar{\partial}\log\rho^2\right\},$$

where ω_0 is a smooth Kähler form on X comparable to $\omega_1 + \omega_2$. Here exact computation may be complicated by the effect of bifurcation, but to prove Lemma (5.3.2) approximate computation is enough. Suppose that $K\delta(n-2)/2 > cK\delta(\delta+1)/4$ (i.e., $\delta < \frac{2n-4}{c} - 1$) holds with a constant $c > 0$. Then if we choose δ so that $\delta < \frac{2(n-2)}{c} - 1$, then (ii) holds. So choosing $0 < c < 2n-5$ is enough for (ii) to be true. If we take the effect of bifurcation into account, then we must take $c <$ (const.)$(2n-5)$, where the constant is universal in n (cf. (5.2.1)). Hence we can take δ comparable to n. The argument for (i) is similar (and easier). *q.e.d.*

Note. It follows from the argument in [BK2] that if δ is comparable to n then the constant C_n in Proposition 5.2.5 is also comparable to n.

(5.3.3) Let $\omega = \sqrt{-1}\partial\bar{\partial}\Omega(d_{ij}, e_j, k)$ be the background metric as in Theorem (5.2.8) and ρ be as above. The family of the sets $\{\rho > L\}$ for large positive numbers L forms a fundamental system of neighborhoods of $D = D_1 \cup D_2$ in X. We therefore consider an exhaustion of $X - D$ by domains $\{B_m\}$ ($m \in \mathbf{Z}_{>0}$) with smooth boundary whose complement in X is equivalent to the above fundamental neighborhoods of D: Set, for instance,

$$B_m := \{\rho \leq L_m\}$$

with $\lim_{m\to\infty} L_m = \infty$. We want to solve the Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^{-f}\omega^n \ \ (= \eta \wedge \bar{\eta})$$

with decay estimates for the solution u. For fixed B_m we will solve (with estimates) the boundary value problem with Dirichlet boundary condition. Then we will let $m \to \infty$. By using the continuity method as in [BK] (with the aid of C^1 and C^2-estimates on the boundary [CKNS, pp.216-217,pp.218-221]), we see that the Dirichlet problem has a unique elliptic solution for each m (i.e., $\omega + \sqrt{-1}\partial\bar{\partial}u$ is positive definite). Let $u_m = u$ be the solution (which is shown to be smooth) for this Dirichlet problem such that $\omega + \sqrt{-1}\partial\bar{\partial}u > 0$. Then an a priori estimate for u is derived as follows. Let $\widetilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u$. Then by Lemma (5.2.9) there exists a number $q < n$ such that $f \in L^q(X - D, \omega)$, where f is as in Theorem (5.2.8). If we set $\gamma = \frac{n-1}{n}$ and $q' = \frac{q}{q-1}$ then we have $q' > \gamma$. Put $p_{-1} = \frac{q'}{q'-\gamma}$. Although $X - D$ is not a Euclidean space, we can easily modify the arguments in [CKNS, pp.216-217], because we have a global strictly plurisubharmonic exhaustion function on $X - D$. Thus the gradient estimate in [CKNS, pp.216-217] still holds in our situation. Hence we have

Lemma. *The gradient* $\|du\| := \|du\|_\omega$ *has a bound*

$$\sup_{B_m} \|du\| \leq C_1$$

independent of m.

Note. At this stage we need not know the estimate on such C_1.

(5.3.4)

Lemma. *There exists an a priori* C^0 *estimate*

$$\|u\|_\infty \leq C$$

where C *depends on* f *only. In particular if* $f \equiv 0$ *then* $u \equiv 0$.

Proof. Multiplying the equality

$$(1 - e^{-f})\omega^n = (-\sqrt{-1}\partial\bar{\partial}u) \wedge (\omega^{n-1} + \omega^{n-2}\widetilde{\omega} + \cdots + \widetilde{\omega}^{n-1})$$

by $|u|^{p-2}u$ $(p \geq p_{-1})$ and integrating by parts gives

$$\begin{aligned} &\int (1 - e^{-f})|u|^{p-2}u\omega^n \\ &= \int -|u|^{p-2}u\sqrt{-1}\partial\bar{\partial}u \wedge (\omega^{n-1} + \cdots + \widetilde{\omega}^{n-1}) \\ &= \frac{4(p-1)}{p^2}\int \sqrt{-1}\partial|u|^{\frac{p}{2}} \wedge \bar{\partial}|u|^{\frac{p}{2}}(\omega^{n-1} + \cdots + \widetilde{\omega}^{n-1}) \\ &\geq \frac{4(p-1)}{p^2}\int \|\partial|u|^{\frac{p}{2}}\|^2\omega^n. \end{aligned}$$

Here we have used the fact that u vanishes on ∂B_m and $|du|$ has an upper bound on ∂B_m. Using the Sobolev inequality (Corollary (5.2.9)), we have

$$\left(\int |u|^{p\gamma}\right)^{\frac{1}{\gamma}} \leq Cp\int |f||u|^{p-1}$$

for a constant C independent of m. Now we have an estimate for the L^q-norm of f (Lemma (5.2.9)). Applying the Hölder inequality we have

$$\left(|u|^{p\gamma}\right)^{\frac{1}{\gamma}} \leq Cp\left(\int |f|^q\right)^{\frac{1}{q}}\left(\int |u|^{q'(p-1)}\right)^{\frac{1}{q'}}.$$

Setting $p_0 = \gamma p_{-1}$, we have

$$\left(\int |u|^{p_0}\right)^{\frac{1}{p_0}} \leq C p_{-1} \left(\int |f|^q\right)^{\frac{1}{q}} \leq C'.$$

Applying the Hölder inequality again we have

$$\left(\int |u|^{p\gamma}\right)^{\frac{1}{\gamma}} \leq C p \left(\int |f|^p\right)^{\frac{1}{p}} \left(\int |u|^p\right)^{\frac{p-1}{p}}.$$

We set $p = p_i = p_0 \gamma^i$ and iterate the process (Moser's iteration technique). Thus, for each i, $\|u\|_{p_i}$ (the L^{p_i}-norm of u) is bounded by a quantity involving $\|f\|_q$, $\|f\|_\infty$ and $\|u\|_{p_0}$. Examining the behavior of this quantity as $i \to \infty$, we have an a priori C^0-estimate as desired (cf. [BK2]). *q.e.d.*

(5.3.5) We proceed to a priori decay estimates for u. Let ρ be as in (5.3.2). Let K be a sufficiently large positive number such that $-K\rho^{-\delta} < u < K\rho^{-\delta}$ on some compact set of $X - D$ (for instance, we may take a quite large compact set defined by $\rho \leq (C^{-1}K)^{\frac{1}{\delta}}$ where C is such that $\|u\|_\infty \leq C$) and $-cK\rho^{-2-\delta} < f < cK\rho^{-2-\delta}$ on $X - D$, where $c > 0$ is as in Lemma (5.3.2). Such K certainly exists and is an a priori quantity as we already have estimate for $\|u\|_\infty$ and also the decay estimate for f (Theorem (5.2.8)).

Lemma. *There is an a priori constant K such that*

$$|u| < K\rho^{-\delta}$$

holds on the region $\rho > 2K^{\frac{1}{2}}$.

Proof. Put $\omega_K = \omega + \sqrt{-1}\partial\bar{\partial}K\rho^{-\delta}$. Then Lemma (5.3.2) implies that outside a compact set defined by, say, $\rho \leq K^{\frac{1}{2+\delta}}$, we have

$$\omega_K^n \leq (1 - cK\delta\rho^{-2-\delta})\omega^n.$$

Thus

$$\begin{aligned}
& n \log\left(1 + \frac{\Delta_{\omega_K}(u - K\rho^{-\delta})}{n}\right) \\
& \geq \log \frac{(\omega_K + \sqrt{-1}\partial\bar{\partial}(u - K\rho^{-\delta}))^n}{\omega_K^n} = \log \frac{\widetilde{\omega}^n}{\omega_K^n} \\
& \geq -f - \log(1 - cK\rho^{-2-\delta}) \\
& \geq -f + cK\delta\rho^{-2-\delta} \geq 0
\end{aligned}$$

if $\rho > K^{\frac{1}{2+\delta}}$, say. Hence, on the same region, we have

$$\Delta_{\omega_K}(u - K\rho^{-\delta}) \geq 0.$$

From the choice of K we have $u - K\rho^{-\delta} < 0$ on a compact set defined by $\rho \leq 2K^{\frac{1}{2+\delta}}$, say. At the boundary we have $u - K\rho^{-\delta} < 0$. The maximum principle then implies

$$u - K\rho^{-\delta} < 0$$

outside of this compact set. Similarly we have

$$u + K\rho^{-\delta} > 0$$

outside a compact set. Thus we get an a priori decay estimate for u. *q.e.d.*

(5.3.6) To obtain a priori C^2-estimate, we need the following Bochner-type inequality due to Chern [Ch]:

Lemma ([Ch]). *Let (M, g) be a Kähler manifold, (N, g) a Hermitian manifold and $f : M \to N$ a holomorphic mapping. Set $v = \|\partial f\|^2$. Then we have*

$$\Delta_g \log v \geq \frac{\mathrm{Ric}_g(\partial f, \overline{\partial f})}{\|\partial f\|^2} - \frac{\mathrm{Bisect}_h(\partial f, \overline{\partial f}, \partial f, \overline{\partial f})}{\|\partial f\|^2}.$$

We use Lemma (5.3.6) to

$$\mathrm{id} : (B_m, \widetilde{\omega}) \to (B_m, \omega).$$

Here ω is the background metric and $\widetilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u$. Note that $\|\partial f\|^2 = \mathrm{tr}_{\widetilde{\omega}}\omega$ and $\mathrm{Ric}(\widetilde{\omega}) = 0$. Since the bisectional curvature of ω is bounded on the whole $X - D$, we have

$$\Delta \log u \geq -C_1 - C_2 \mathrm{tr}_{\widetilde{\omega}}\omega$$

where $\Delta = \Delta_{\widetilde{\omega}}$ and C_1, C_2 are positive constants depending on the geometry of ω only. Here we note that (by applying the arithmetic and geometric mean inequality)

$$\mathrm{tr}_{\widetilde{\omega}} \geq n\left(\frac{\omega^n}{\widetilde{\omega}^n}\right)^{\frac{1}{n}} = n\exp(\frac{f}{n}),$$

i.e., $\mathrm{tr}_{\widetilde{\omega}}\omega$ is bounded below by a constant independent of m. Moreover we have

$$\Delta u = n - \mathrm{tr}_{\widetilde{\omega}}\omega.$$

Choose a positive constant A so that $A - C_2 = 1$.

(5.3.7) To proceed, we need the following lemma, which is a consequence of [CKNS, Subsection 1.3, pp.218-221]:

Lemma ([CKNS]). *There exists a constant independent of m such that the second derivatives on the boundary ∂B_m have bound (with respect to the metric ω) independent of m:*

$$\sup_{\partial B_m} |D^2 u|_\omega \leq C \qquad (\exists C > 0).$$

To show this, we have only to follow the argument in [CKNS, pp.218-221]. The only difference between the situation in [CKNS] and ours is this: in [CKNS] the flat metric is used whereas we must use our background metric ω. As the metric ω is of weakly $C^{k,\alpha}$-geometry (see (5.2.2)), we can follow the arguments in [CKNS] without too much difficulty.

(5.3.8) Lemma (5.2.7) implies that there exists a lower bound for $\log \mathrm{tr}_{\tilde{\omega}}\omega$ on the boundary ∂B_m independent of m. Let B be a positive constant so that

$$B >> |\mathrm{Min}_{x\in B_m}(\log \mathrm{tr}_{\tilde{\omega}}\omega - Au) - \mathrm{Min}_{x\in\partial B_m}(\log \mathrm{tr}_{\tilde{\omega}}\omega - Au)|.$$

Such B certainly exists because we have a boundary estimate for $\log \mathrm{tr}_{\tilde{\omega}}\omega$, a lower estimate $\mathrm{tr}_{\tilde{\omega}}\omega \geq \exp(f/n)$ and a priori decay estimate for u, all of which are independent of m. With these A and B, we have

$$\Delta\{\log \mathrm{tr}_{\tilde{\omega}}\omega - Au - \frac{B}{L_m}(\rho - L_m)\} \geq -An + (A - C_2)\mathrm{tr}_{\tilde{\omega}}\omega - C_1 - \frac{B}{L_m}\mathrm{tr}_{\tilde{\omega}}\omega.$$

If m is sufficiently large, say, $\frac{B}{L_m} < \frac{1}{2}$, then the last term in the R.H.S. of the above inequality is absorbed in the second term in it. From the choice of B, the function

$$\log \mathrm{tr}_{\tilde{\omega}}\omega - Au - \frac{B}{L_m}(\rho - L_m)$$

assumes its maximum value at an interior point p_0 of B_m. At p_0 we have from the maximum principle

$$\log \mathrm{tr}_{\tilde{\omega}}\omega(p_0) \leq C$$

where C is a constant independent of m. Therefore we have

$$\mathrm{tr}_{\tilde{\omega}}\omega \leq C' \exp(\sup_{B_m} u - \inf_{B_m} u) \leq C''$$

with C' and C'' independent of m. This implies an a priori C^2-estimate without decay.

(5.3.9) Now we proceed to decay estimates for derivatives. We introduce a parameter $t \in [0,1]$ and consider a family of equations

$$(E_t) \qquad (\omega + \sqrt{-1}\partial\bar{\partial}u_t)^n = e^{-tf}\omega^n.$$

What we have seen so far implies that the family $\{u_t\}$ of solutions for the Dirichlet boundary value problems on B_m of the above equation have decay estimates and ordinary C^2-estimates independent of m and t. Then [GT, Theorem 17.14] implies that there exists a constant $0 < \alpha < 1$ such that $\{u_t\}$ have usual $C^{2,\alpha}$-estimate independent of m and t (note that ω has bounded geometry). The bootstrapping argument with standard interior Schauder estimates [GT, Theorem 6.2, Corollary 6.3] then yields usual $C^{k,\alpha}$-estimates for $\{u_t\}$ for all k.

To get $C^{2,\alpha}$-estimate for u with decay condition, we formally differentiate the above equation with respect to t. Putting $\widetilde{\omega}_t = \omega + \sqrt{-1}\partial\bar{\partial}u_t$, we have

$$(L_t) \qquad \Delta_{\widetilde{\omega}_t} \frac{\partial u_t}{\partial t} = -f.$$

The Dirichlet boundary value problem

$$\begin{cases} \Delta_{\widetilde{\omega}_t} h_t = f \\ h_t|_{\partial B_m} = 0 \end{cases}$$

has a unique solution h_t for each t and h_t depends on t smoothly. The equation (E_t) with $t = 0$ has a unique solution $u_0 \equiv 0$ (within functions with decay estimates). It follows that (E_t) is in fact differentiable with respect to t and (L_t) is the differentiated equation. Using Moser's iteration technique, the Sobolev inequality and the maximum principle as before (cf. [BK2]), we have a decay estimate for $\frac{\partial u_t}{\partial t}$:

$$|\frac{\partial u_t}{\partial t}| \leq O(\rho^{-\delta})$$

and usual $C^{k,\alpha}$-estimates for all k. The advantage of (L_t) is that we can get a $C^{2,\alpha}$-estimate for $\frac{\partial u_t}{\partial t}$ with decay condition for arbitrary $0 < \alpha < 1$. From Definition-Lemma (5.2.2), we have a holomorphic coordinate system $x = (x_1, \cdots, x_n)$ which runs over the unit ball in $\mathbf{R}^{2n}$ such that there exists $r > 0$ with the property that if we introduce a new coordinate system $y = (y_1, \cdots, y_n)$ by $y = rx$ then the components of ω with respect to y have uniformly bounded $C^{k,\alpha}$-norms. Here such r can be taken as $r = \rho^e$ for some $0 < e < 1$. Pick a positive number $\beta < \delta$, where $\delta > 0$ is as in Lemma (5.3.2). Set

$$\overline{\omega} = \frac{\omega}{r^2} \quad h = \frac{1}{r^2}.$$

Then (L_t) becomes

$$(L_t') \qquad \Delta_{\overline{\omega}+h\sqrt{-1}\partial\bar{\partial}u_t} \frac{\partial u_t}{\partial t} = \frac{f}{h}.$$

Let $(\overline{g}_{i\bar{j}})$ be the components of the metric $\overline{\omega}$. Then $\overline{g}_{i\bar{j}}$ has uniformly bounded $C^{k,\alpha}$-estimates relative to the coordinate system x. We apply the interior Schauder estimate [GT, Theorem 6.2, Corollary 6.3] to the linear equation (L_t'). We have

$$\|u_t\|_{C^{2,\alpha}} \leq (\text{const.})(\|u_t\|_{C^0} + \|\frac{f}{h}\|_{C^{0,\alpha}}).$$

Let us examine the constant. The constant depends on the $C^{0,\alpha}$-norm of the coefficients of the operator $\Delta_{\overline{\omega}+h\sqrt{-1}\partial\bar{\partial}u_t}$ and the value α. The coefficients of the first order terms are of degree one in the inverse matrix of $\frac{1}{r^2}\widetilde{\omega}_t$ and any other functions involved are just the Christoffel symbol of $\widetilde{\omega}_t$, all of which are uniformly bounded by an a priori constant. The coefficients in the leading term consist of entries of the inverse matrix of $\frac{1}{r^2}\widetilde{\omega}_t$, which are estimated uniformly by an a priori constant.

Thus the constant in the above estimate is an a priori quantity. On the other hand, we have

$$\|u_t\|_{C^0} \leq C\rho^{-\delta}.$$

Moreover we may assume that for some positive number ε we have

$$\|\frac{f}{h}\|_{C^{0,\alpha}} \leq C\rho^{-\varepsilon}.$$

We thus have

$$\|u_t\|_{C^{2,\alpha}} \leq C\rho^{-\delta'}$$

with some positive number δ' (we write $\delta = \delta'$ for simplicity). On the other hand, as $\partial u_t/\partial x_i = \partial u_t/\partial(r^{-1}y_i) = r\partial u_t/\partial y_i$, *e.t.c.*, we see that the Hölder semi-norm $[u_t]_{2,\alpha}$ with respect to the coordinate system x is comparable to $r^{2+\alpha}[u_t]_{2,\alpha}$ with respect to the coordinate system y. This implies that there are a priori decay estimates for $\|\nabla u_t\|_\omega$, $\|\nabla^2 u_t\|_\omega$ and α-Hölder coefficients of $\nabla^2 u_t$ of the form

$$\|\nabla^i u_t\|_\omega \leq C\rho^{-\delta}r^{-i} \leq C\rho^{-\delta-ei} \quad \text{and} \quad |[u_t]_{2,\alpha}| \leq C\rho^{-\delta}r^{-2-\alpha} \leq \rho^{-\delta-e(2+\alpha)}$$

for $i = 1, 2$, where $e = \mathrm{Min}_{p\in X-D}\{e(p)\} > 0$ (see (5.2.2)). Integrating these estimates against t gives a priori decay estimates for $\|\nabla^i u\|_\omega$ ($i = 1, 2$) and the α-Hölder coefficients of $\nabla^2 u$. Set ω (resp. $\widetilde{\omega}$) as $\sqrt{-1}\sum g_{i\bar{j}}dy_i \wedge d\bar{y}_j$ (resp. $\sqrt{-1}\sum \widetilde{g}_{i\bar{j}}dy_i \wedge d\bar{y}_j$). Differentiating the equation $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^{-f}\omega^n$ with respect to y_k gives

$$\sum \widetilde{g}^{i\bar{j}}u_{i\bar{j},k} = \sum(g^{i\bar{j}} - \widetilde{g}^{i\bar{j}})g_{i\bar{j},k} - f_k.$$

For decay estimates for higher order derivatives of u, we use the bootstrapping argument applying the interior Schauder estimates together with the rescaling argument to the above equation. We thus get

$$\|\nabla^k u\|_\omega \leq C_k\rho^{-\delta-ek}$$

for all $k \in \mathbf{Z}_{\geq 0}$. Ascoli-Arzela argument then implies that there is an infinite sequence of u_m's which converges uniformly to a smooth function $u = u_\infty$ on $X - D$. The function u_∞ satisfies the equation $(\omega + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = e^{-f}\omega^n$ with a priori decay estimate

$$\|\nabla^k u_\infty\|_\omega \leq C'_k\rho^{-\delta-ek} \quad \forall k \geq 0.$$

Moreover u_∞ satisfies the a priori estimate of the form

$$\mathrm{tr}_{\widetilde{\omega}}\omega \leq C$$

with a priori constant C. This implies that the eigenvalue of $\widetilde{\omega}$ is not too small and is estimated below by an a priori constant. This together with the Monge-Ampère equation implies an a priori estimate of the form

$$C^{-1}\omega < \widetilde{\omega} < C\omega$$

with a priori constant C. This implies that the resulting form $\widetilde{\omega}$ is positive definite and equivalent to the background metric ω. In particular $\widetilde{\omega}$ is complete. The decay estimate for u_∞ implies further that $\widetilde{\omega}$ is asymptotically equal to ω in any jet level. Therefore we have proved the existence part of Theorem (5.3.1).

(5.3.10) The solution whose existence is guaranteed by the above argument is unique. Indeed, suppose that u_1 and u_2 are two solutions with the above a priori estimates. Set $\widetilde{\omega}_i = \omega + \sqrt{-1}\partial\bar{\partial}u_i$ and $v = u_2 - u_1$. Then the difference v is a solution of the equation

$$(\widetilde{\omega}_1 + \sqrt{-1}\partial\bar{\partial}v)^n = \widetilde{\omega}_1^n$$

and hence $v \equiv 0$ (the case of $f \equiv 0$). Indeed, before going to the limit of Dirichlet problems, we have $v_m \equiv 0$ on B_m. If ω is G-invariant and all B_m are also G-invariant, then the solution $u_\infty = \lim_{m\to\infty} u_m$ is again G-invariant. Thus there exists a G-invariant complete Ricci-flat Kähler metric on $X - D$. Even if B_m is not G-invariant we can compare u_∞ and g^*u_∞ and conclude $g^*u_\infty = u_\infty$ for any $g \in G$.

(5.3.11)

Remark. The curvature of the resulting complete Ricci-flat Kähler metric does not decay quadratically if rank$(G/K)> 1$.

Explanation. As the Monge-Ampère equation has a solution with decay estimates at infinity, to examine the curvature behavior at infinity, it suffices to do so for the background metric. We look at a boundary of a small tubular neighborhood of $D = D_1 \cup D_2$ near Y. Such an object is described as follows. For simplicity we take a local two dimensional disk B transversal to Y we get two curves with an ordinary intersection at the origin. Remove from each curve a small disk centered at the origin. Then the normal S^1-bundle of each curve will have a 2-torus boundary along the circle which bounds the hole. Then glue these 2-tori by an external tube, namely by a cylinder $T \times I$ over the torus. Along the cylinder direction, the log-term involved in the background metric

$$\sqrt{-1}\partial \log \frac{1}{\|\sigma_1\|^a\|\sigma_2\|^b} \wedge \bar{\partial} \log \frac{1}{\|\sigma_1\|^a\|\sigma_2\|^b}$$

will vanish. In particular this causes the exponent $e(p)$ in the definition of weakly asymptotically flatness (see (5.2.2)) strictly less than 1. In the cylindrical direction, the curvature decays strictly slower than the inverse quadratic law (but curvature indeed decays to zero, not remains away from zero). Note that the curvature decays by inverse quadratic law for complete Ricci-flat Kähler manifolds in [BK1,2] and [TY].

Summary. In rank two case the asymptotic behavior of the resulting Ricci-flat Kähler metric is described as follows. We first consider a Kähler potential of type $P(d_{ij})$ where (d_{ij}) are parameters which describe the bifurcation of Kähler potential near Y. Then we determine Hermitian metrics along Y on $\mathcal{O}_X(D_i)|_Y$ so that $\omega_1 + \omega_2$ are Kähler-Einstein. Then we compute the volume form of $\sqrt{-1}\partial\bar{\partial}P(d_{ij})$ along D_i

and determine the Hermitian metric on $\mathcal{O}_X(D_j)|_{F_i}$ so that $\tilde{\omega}_j$ is Ricci-flat along F_i. By using the condition $\omega_i|_{F_i} = 0$ we can determine another Hermitian metric along F_i. Then by using a good retraction we extend the definitions in a tubular neighborhood of D. Thus we get a background metric whose Ricci curvature decays with any prescribed order. Then basically the same analysis as in [BK2] shows the existence of the solution to the equation

$$(\sqrt{-1}\partial\bar{\partial}P(d_{ij}) + \sqrt{-1}\partial\bar{\partial}u)^n = \eta \wedge \bar{\eta}$$

with decay estimates at infinity. Hence the background metric constructed as above yields a good approximation of the above Ricci-flat metric.

5.4. Induction

(5.4.1) We generalize Theorem (5.3.1) to the case of arbitrary rank by induction on $r = \operatorname{rank}(G/K)$. Let G/K be an irreducible Riemannian symmetric space of compact type and $G^{\mathbf{C}}/K^{\mathbf{C}}$ its complexification. Let X be the canonical compactification of $G^{\mathbf{C}}/K^{\mathbf{C}}$ and $D = \cup_{i=1}^r D_i$ the divisor at infinity as in (5.1.6-8). Let $\underline{i} = \{i_1, i_2, \cdots, i_k\} \subset \{1, 2, \cdots, r\}$. Theorem 5.1.8 ([DP]) and Proposition (5.1.8) imply that there exists a blowing down

$$\Pi_{\underline{i}} : X \to \Pi_{\underline{i}}(X)$$

of $D_{\underline{i}} = D_{i_1 \cdots i_k} = D_{i_1} \cap \cdots \cap D_{i_k}$. On $D_{\underline{i}}$, $\Pi_{\underline{i}}$ is a $G^{\mathbf{C}}$-equivariant fibration

$$D_i \to G^{\mathbf{C}}/P_i$$

where $P_{\underline{i}}$ is the parabolic subgroup of $G^{\mathbf{C}}$ generated by B and root subgroups $U_{\pm\alpha}$ for roots α with $(\alpha, \mu_{\underline{i}}) = 0$, where $\mu_{\underline{i}} = \mu_{i_1 \cdots i_k} = \mu_{i_1} + \cdots + \mu_{i_k}$. The fiber $F_{\underline{i}} = F_{i_1 \cdots i_k}$ of $\Pi_{\underline{i}}$ is the canonical compactification of the symmetric variety $L_{i_1 \cdots i_k}/K_{i_1 \cdots i_k}$ of rank $r - k$. Set

$$n_{\underline{i}} = n_{i_1 \cdots i_k} = \dim F_{\underline{i}}.$$

The anticanonical line bundle of $G^{\mathbf{C}}/P_{\underline{i}}$ is an ample homogeneous line bundle $L_{-\chi_{\underline{i}}}$ with $\chi_{\underline{i}} = \chi_{i_1 \cdots i_k}$ the sum of all simple roots which are not orthogonal to $\mu_{\underline{i}}$. Note that $\chi_{\underline{i}}$ is a special weight. Since $\chi_{\underline{i}}$ is a character of $P_{\underline{i}}$, $\chi_{\underline{i}}$ is orthogonal to all roots α with $(\alpha, \mu_{\underline{i}}) = 0$. This implies that $\chi_{\underline{i}}$ is a sum of nonnegative multiples of μ_{i_p} $(i_p \in \underline{i})$. As in (5.1.10) we define

$$\mu = \sum_{i=1}^r d_i(\alpha_i - \alpha_i^\theta) = 2\sum_{i=1}^r \overline{\alpha}_i.$$

Now we consider the case that $\underline{i}$ consists of one element i. As χ_i is a special weight, we can write

$$\chi_i = \sum_{j=1}^r 2a_{ij}\overline{\alpha}_j$$

with $a_{ij} \in \mathbf{Q}$. Because each entry of (a_{ij}) is a positive multiple of an entry of the inverse matrix of the Cartan matrix $((\overline{\alpha}_i, \overline{\alpha}_j))$ of an irreducible symmetric space G/K, we have $\forall a_{ij} > 0$. From the definition of d_i, we have

$$\sum_{i=1}^{r} a_{ij} = d_j.$$

(5.4.2) Let $\underline{i} = \{i_1, \cdots, i_k\}$ be as in (5.4.1) and $\underline{j} = \{j_1, \cdots, j_l\}$ the complement of $\underline{i}$ in $\{1, 2, \cdots, r\}$ $(k + l = r)$. We derive the relative anticanonical bundle formula for the fibration $\Pi_{\underline{i}} : D_{\underline{i}} \to G^{\mathbf{C}}/P_{\underline{i}}$. The anticanonical bundle of $D_{\underline{i}} = D_{i_1} \cap \cdots \cap D_{i_k}$ is given by (following the rule described in (5.1.7))

$$\mu + \sum_{p \in \underline{j}} (\alpha_p - \alpha_p^{\theta}) = \mu + 2 \sum_{p \in \underline{j}} \overline{\alpha}_p = 2 \sum_{p \in \underline{j}} (1 + d_p) \overline{\alpha}_p + 2 \sum_{q \in \underline{i}} d_q \overline{\alpha}_q.$$

The anticanonical bundle of $G^{\mathbf{C}}/P_{\underline{i}}$ is given by the weight

$$\chi_{\underline{i}} = \sum_{l \in \underline{i}} \chi_l = \sum_{l \in \underline{i}} \sum_{p=1}^{r} 2 a_{lp} \overline{\alpha}_p.$$

Set

$$d_l^{\underline{i}} := \sum_{1 \le k \le r, k \in \underline{j}} a_{kl}.$$

Then the relative anticanonical bundle of the fibration $\Pi_{\underline{i}} : D_{\underline{i}} \to G^{\mathbf{C}}/P_{\underline{i}}$ is given by the weight

$$2 \sum_{p \in \underline{j}} (1 + d_p^{\underline{i}}) \overline{\alpha}_p + 2 \sum_{q \in \underline{i}} d_q^{\underline{i}} \overline{\alpha}_q.$$

Hence the relative anticanonical bundle of the $G^{\mathbf{C}}$-equivariant fibration $\Pi_{\underline{i}}|_Y : Y = G^{\mathbf{C}}/P \to G^{\mathbf{C}}/P_{\underline{i}}$ is given by

$$2 \sum_{p=1}^{r} d_p^{\underline{i}} \overline{\alpha}_p.$$

On the other hand, the anticanonical bundle of $G^{\mathbf{C}}/P_{\underline{j}}$ is given by

$$\begin{aligned} \chi_{\underline{j}} &= \sum_{l \in \underline{j}} \sum_{p=1}^{r} 2 a_{lp} \overline{\alpha}_p \\ &= \sum_{p=1}^{r} (\sum_{l \in \underline{j}} 2 a_{lp}) \overline{\alpha}_p \\ &= \sum_{p=1}^{r} d_p^{\underline{i}} \overline{\alpha}_p, \end{aligned}$$

which is equal to the weight corresponding to the relative anticanonical bundle of the $G^{\mathbf{C}}$-equivariant fibration $\Pi_{\underline{i}}|_Y : Y \to G^{\mathbf{C}}/P_{\underline{i}}$. In particular, we can prove the assertion (b) in (5.2.1). Indeed, the restriction of ω_Y to the fiber of the fibration $\Pi_i|_Y : Y \to G^{\mathbf{C}}/P_i$ $(i = 1,2)$ belongs to the anticanonical class of of the fiber. This, which is invariant under a transitive group action, is necessarily Kähler-Einstein. In particular we have proved the assertion (b) in (5.2.1).

Now the bundle $\Pi_{\underline{i}}^* L_{-\chi_{\underline{i}}}$ is trivial along the fiber $F_{\underline{i}}$. This implies $\sum_{l\in\underline{i}}\sum_{p=1}^r a_{lp}\overline{\alpha}_p = 0$ along $F_{\underline{i}}$ and so

$$\sum_{l,p\in\underline{i}} a_{lp}\overline{\alpha}_p = -\sum_{l\in\underline{i},q\in\underline{j}} a_{lq}\overline{\alpha}_q$$

along $F_{\underline{i}}$. Hence if we set $(a^{pq})_{p,q\in\underline{i}}$ to be the inverse matrix of $(a_{pq})_{p,q\in\underline{i}}$ and put

$$\begin{cases} a^{\underline{i}}_{kp} := a_{kp} - \sum_{s,t\in\underline{i}} a_{ks}a^{st}a_{tp} \\ \overline{d}^{\underline{i}}_p := \sum_{k\in\underline{j}} \overline{a}^{\underline{i}}_{kp}, \end{cases}$$

then we have

$$2\sum_{p\in\underline{j}}(1+d^{\underline{i}}_p)\overline{\alpha}_p + 2\sum_{q\in\underline{i}} d^{\underline{i}}_q\overline{\alpha}_q = 2\sum_{p\in\underline{j}}(1+\overline{d}^{\underline{i}}_p)\overline{\alpha}_p$$

along $F_{\underline{i}}$. This is the anticanonical bundle formula of the fiber $F_{\underline{i}}$.

In general, we have the following

Proposition. *Let G be complex semisimple and $Q \subset P$ two parabolics of G. Set $X = G/Q$ and $Y = P/Q$. Then*

$$c_1(K_X^{-1})|_Y = c_1(K_Y^{-1}).$$

A detailed proof will appear elesewhere.

(5.4.3) We here describe a combinatorial feature in the description of background metric. Let X be the canonical compactification of $G^{\mathbf{C}}/K^{\mathbf{C}}$ of rank r and let $n = \dim X$. We set

$$\omega_{\underline{i}} = \omega(a_{lp}; p = 1,\cdots,r; l \in \underline{i}) = \sqrt{-1}\partial\overline{\partial}\log\frac{1}{\prod_{l\in\underline{i}}\prod_{p=1}^r \|\sigma_p\|^{2a_{lp}}}.$$

This is a curvature form of the line bundle $L_{-\chi_{\underline{i}}} = \mathcal{O}_X(\sum_{l\in\underline{i}}\sum_{p=1}^r a_{lp}D_p)$ and is a sum of nonnegative multiples of the pull-back of G-invariant Kähler metrics under the blow down of $D_{\underline{i}}$

$$\Pi_{i_p} : X \to \Pi_{i_p}(X) \subset \mathbf{P}(V_{\mu_{i_p}}) \quad (i_p \in \underline{i})$$

As in (5.2.1) we consider bifurcations of Kähler potentials and glue them by partition of unity. To describe this procedure in general, it is very helpful to introduce a standard $(r-1)$-simplex with vertices $[1],[2],\cdots,[r]$. On each vertex we put a Kähler potential

$$P([i]) = P_{d_1,\cdots,d_r} = \frac{1}{\prod_{i=1}^r \|\sigma_i\|^{2d_i/n}} \quad (\forall i).$$

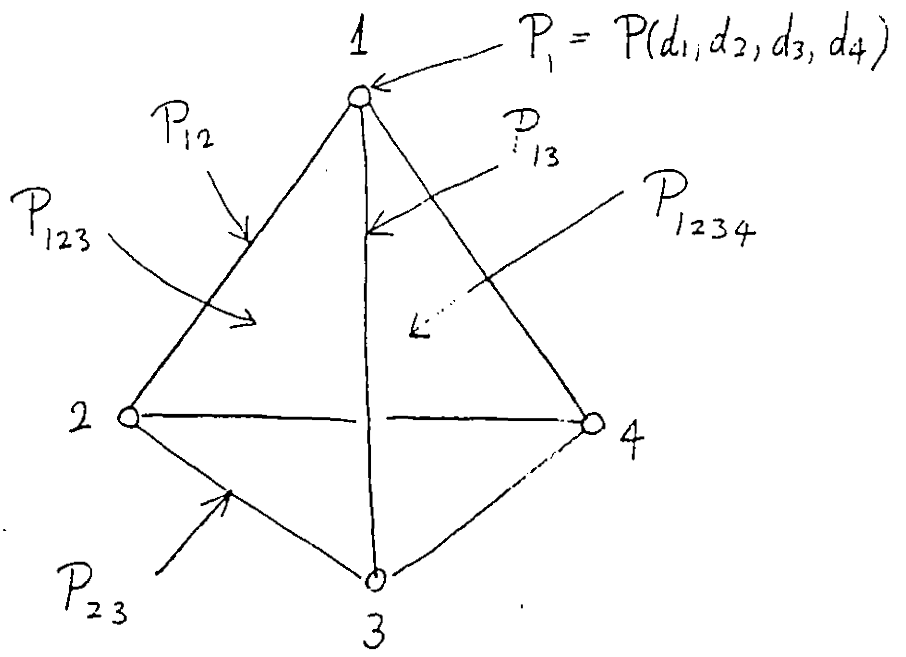

FIGURE 2. Standard simplex and bifurcation of Kähler potentials

For each $(k-1)$-simplex $[\underline{i}] = [i_1 i_2 \cdots i_k]$ we consider k (k)-ples of positive numbers $(d_{i_1 i_p}, d_{i_2 i_p}, \cdots, d_{i_k i_p})$ $(p = 1, \cdots, k)$ so that the following conditions are satisfied:

$$(*) \qquad \begin{cases} |d_{i_q i_p} - \dfrac{d_{i_q}}{n}| \quad (p = 1, \cdots, k) \quad \text{are small} \\ \det(d_{i_q i_p})_{q,p=1,\cdots,k} \neq 0 \\ (n-k+1) d_{i_q i_1} + \displaystyle\sum_{i_p \in \underline{i}, i_p \neq i_1} d_{i_q i_p} = d_{i_q} \quad (\forall i_q \in \underline{i}) \\ d_{i_q i_1} > d_{i_q i_p} \ (\forall i_p \in \underline{i} \text{ such that } i_p \neq i_1). \end{cases}$$

Associated to each $(k-1)$-simplex $[\underline{i}] = [i_1 \cdots i_k]$ is a Kähler potential

$$\begin{aligned} P([\underline{i}]) &= \sum_{p=1}^{k} P(d_{i_1 i_p}, d_{i_2 i_p}, \cdots, d_{i_k i_p}) \\ &:= \sum_{p=1}^{k} \frac{1}{\prod_{q=1}^{k} \|\sigma_{i_q}\|^{2 d_{i_q i_p}} \prod_{i \in \underline{j}} \|\sigma_i\|^{2 d_i / n}}, \end{aligned}$$

where $\underline{j}$ is the complement of $\underline{i}$. Next we put bump functions $\rho_{\underline{i}}$ on each $(k-1)$-simplex $[\underline{i}] = [i_1 \cdots i_k]$ so that the followings hold:
(i) all $\rho_{\underline{i}}$ are defined in a neighborhood of $D = \cup_{i=1}^{r} D_i$,
(ii) $\rho_{\underline{i}} \equiv 1$ on $D_{\underline{i}}$ outside a small neighborhood of $\cup_{k \notin \underline{i}} D_k \cap D_{\underline{i}}$,
(iii) $\sum_{\text{all sub-simplexes}} \rho_{\underline{i}} \equiv 1$ in a neighborhood of D.
(iv) ρ_i vary monotonically with respect to the modulus of the coordinate z_{i_p} in $D_{\underline{i} - i_p}$

where $D_{\underline{i}}$ is given by the equation $z_{i_p} = 0$ in $D_{\underline{i}}$,
(v) $\rho_{\underline{i}}$ are invariant under the action of G,
(vi) ρ_i are determined first on the corresponding subvariety $D_{\underline{i}}$ and then extended by good retractions.

Using $\rho_{\underline{i}}$'s as gluing functions, we consider the potential on $X - D$

$$P(d_{ij}) := \sum_{\text{all sub-simplexes}} \rho_{\underline{i}} P([\underline{i}]).$$

Just as in (5.2.1) we have a complete Kähler metric

$$\Omega(d_{ij}) := \sqrt{-1}\partial\bar{\partial}P(d_{ij})$$

defined on $X - D = G^{\mathbf{C}}/K^{\mathbf{C}}$. We may assume that $\Omega(d_{ij}, e_j)$ is G-invariant. We set

$$f := \log \frac{\Omega(d_{ij})^n}{\eta \wedge \bar{\eta}}.$$

It follows from the construction of $\Omega(d_{ij})$ that f is bounded on X. As in (5.2.1), we may assume that for any complementary $\underline{i}$ and $\underline{j}$:
(a) $\omega_{\underline{i}} + \omega_{\underline{j}}$ is a Kähler-Einstein metric on Y,
(b) the restriction of $\omega_{\underline{i}}$ to the fibers of the fibration $\Pi_{\underline{j}}|_Y : Y \to G^{\mathbf{C}}/P_{\underline{j}}$ is Kähler-Einstein (cf.(5.4.2)).

As in (5.2.3) we can compute the volume form of $\Omega(d_{ij})$ along Y (using degeneracy conditions of $\omega_{\underline{i}}$'s). The result is:

$$\left(\frac{1}{\prod_{i=1}^r \|\sigma_i\|^{2d_{i1}}}\right)^{(n-r+1)} \prod_{j\geq 2}\left(\frac{1}{\prod_{i=1}^r \|\sigma_i\|^{2d_{ij}}}\right) \wedge \prod_{i=1}^r \frac{|dz_i|^2}{|z_i|^2} \wedge \omega(d_{11}, d_{21}, \cdots, d_{r1})^{n-r}.$$

It follows from the conditions satisfied by (d_{ij}, e_j), the anticanonical bundle formula for Y and the G-invariance of $\omega(d_{11}, \cdots, d_{r1})$ (i.e., the assertion (b) above) that

$$\sqrt{-1}\partial\bar{\partial}f = 0$$

holds along Y. Hence we may assume that $f \equiv 0$ along Y. As in rank two case, we extend the definition of Hermitian metrics on $\mathcal{O}_X(D_i)$ in a neighborhood of D so that $f \equiv 0$ on D and more strongly f vanishes also in the normal direction of each D_i in the k-th jet level (for any fixed k). This is done inductively as follows. Let $\underline{i} = (i_1, \cdots, i_k)$ and $\underline{j}$ its complement. Define $\overline{P}([\underline{k}])$ for $\underline{k} \subset \underline{i}$ from two ingredients, namely the Kähler potential $P([\underline{k}])$ and the condition

$$\sum_{l,p\in\underline{j}} a_{lp}\overline{\alpha}_p = -\sum_{l\in\underline{j},q\in\underline{i}} a_{lq}\overline{\alpha}_q, \quad \text{i.e., } \alpha_{\underline{j}} = -A_{\underline{jj}}^{-1}A_{\underline{ji}}\alpha_{\underline{i}} \quad \text{(formally)},$$

by inserting this condition to the explicit expression of the Kähler potential $P([\underline{k}])$ formally. The computation goes as follows. We replace the monomial

$$\frac{1}{\prod_{q=1}^{|\underline{k}|}\|\sigma_{k_q}\|^{2d_{k_q k_p}}\prod_{j\in\underline{l}}\|\sigma_j\|^{2d_j/n}},$$

where $\{k_1,\cdots,k_{|\underline{k}|}\}=\underline{k}\subset\underline{i}$ and $\underline{l}$ is the complement of $\underline{k}$ in $\{1,2,\cdots,r\}$, by the following additive form

$$\sum_{q=1}^{k} d_{i_q i_p}\overline{\alpha}_{i_q}+\sum_{j\in\underline{j}}\frac{d_j}{n}\overline{\alpha}_j$$

(we have m monomials if $[\underline{k}]$ is a $(m-1)$-simplex) and then insert the condition. We write $C(\underline{i},\underline{k};k_l,i)$ for the coefficient of $\overline{\alpha}_i$ ($i\in\underline{i}$ and $k_l\in\underline{k}$). Then we have

$$C(\underline{i},\underline{k};k_1,i)>C(\underline{i},\underline{k};k_l,i)\quad\forall k_l\neq k_1,\ i\in\underline{i}.$$

Introduce then a positive number $e(\underline{i},\underline{k};i)$ by setting

$$e(\underline{i},\underline{k};i)=\frac{d_i^{\underline{j}}}{n_{\underline{j}}\{(n_{\underline{j}}-k+1)C(\underline{i},\underline{k};k_1,i)+\sum_{k_l\neq k_1}C(\underline{i},\underline{k};k_l,i)\}}$$

and introduce a notation for the power $|\underline{i}|$-vector $e(\underline{i},\underline{k})$: if $A=\frac{1}{\prod_i\|\sigma_i\|^{2a_i}}$ then

$$A^{e(\underline{i},\underline{k})}:=\frac{1}{\prod_i\|\sigma_i\|^{2a_ie(\underline{i},\underline{k};i)}}.$$

Now we follow the above procedure in the opposite direction to get a new monomial. We execute the same procedure for each monomial which is contained in $P([\underline{k}])$ and take the sum. We thus get some expression $\overline{P}([\underline{k}])$ involving only $\|\sigma_i\|$ with $i\in\underline{i}$. After carrying out this formal replacement, we consider the following "Kähler potential "on $F_{\underline{j}}$:

$$P_{\underline{i}}(d_{ij}):=\sum_{\underline{k}\subset\underline{i}}\rho_{\underline{j}\cup\underline{k}}\overline{P}([\underline{k}])^{e(\underline{i},\underline{k})}.$$

Here we have used the above notation for a $|\underline{i}|$-vector $e(\underline{i},\underline{k})$. Note that $\{\rho_{\underline{k}}\}_{\underline{j}\subset\underline{k}}$ is again a partition of unity on $F_{\underline{j}}$. From this data, for a subsimplex $\underline{i}$ of $([1]\cdots[r])$, we get a similar system $(\overline{d}_{i_1i_p},\cdots,\overline{d}_{i_ki_p})$ as before. From the definition, the condition $(*)$ is again fulfilled, for instance we have

$$(n_{\underline{j}}-k+1)\overline{d}_{i_qi_1}+\sum_{i_p\neq i_1,i_p\in\underline{i}'}\overline{d}_{i_qi_p}=\overline{d}_{i_q}^{\underline{j}}$$

for all sub-simplexes $\underline{i}'\subset\underline{i}$ with $\overline{d}_{i_q}^{\underline{j}}$ as in (5.4.2) and $\underline{j}$ the complement of $\underline{i}$.

The (1,1)-form

$$\Omega_{\underline{i}}(d_{ij}):=\sqrt{-1}\partial\overline{\partial}P_{\underline{i}}(d_{ij})$$

defined on $F_{\underline{j}}$ is a complete Kähler metric on $F_{\underline{j}}-\sum_{i\in\underline{i}}F_{\underline{j}}\cap D_i$. From the anticanonical bundle formula (see (5.4.2)) of the fiber $F_{\underline{j}}$, this metric has the same structure as $\Omega(d_{ij})$ defined on $X - D$. Indeed, it follows from definitions that the Kähler potential $P_{\underline{i}}(d_{ij})$ is constructed by considering the bifurcation of

$$\frac{1}{\prod_{i_q \in \underline{i}} \|\sigma_{i_q}\|^{2\overline{d}^{\underline{i}}_{i_q}/n_{\underline{i}}}}$$

at infinity.

(5.4.4) We construct a background metric. The basic observation here is that along $F_{\underline{i}}$ the volume form of $\Omega(d_{ij})$ is written as follows:

$$\left(\frac{\omega^{n-|\underline{i}|-n_{\underline{i}}-1}}{\prod_{i=1}^r \|\sigma_i\|^{2d_i^{\underline{j}}}} \prod_{i \in \underline{i}} \frac{|dz_i|^2}{|z_i|^2}\right)\left(\frac{1}{\prod_{i \in \underline{i}} \|\sigma_i\|^{2d_i^{\underline{i}}}}\right)\left(\frac{1}{\prod_{j \in \underline{j}} \|\sigma_j\|^{2d_j^{\underline{i}}-2\overline{d}_j^{\underline{i}}}}\right)$$
$$\times \text{ (some smooth function involving } \|\sigma_j\| \text{ for } j \in \underline{j})$$
$$\wedge (\sqrt{-1}\partial\bar{\partial}P_{\underline{j}}(d_{ij}))^{n_{\underline{i}}+1}$$

Here the expression $\omega^{n-|\underline{i}|-n_{\underline{i}}-1}$ stands for a pull-back of a G-invariant volume form on $G^{\mathbf{C}}/P_{\underline{i}}$.

Induction Step. Suppose that we can determine $\|\sigma_j\|$ ($j \in \underline{j}$) so that $\sqrt{-1}\partial\bar{\partial}P_{\underline{j}}(d_{ij})$ is a complete Ricci-flat Kähler metric on $F_{\underline{i}}$. As we are working on $F_{\underline{i}}$, we may use the relation

$$\overline{\alpha}_j = -\sum_{l \in \underline{j}, i \in \underline{i}} a^{jl} a_{li} \overline{\alpha}_i.$$

This implies that once we have determined Hermitian metrics for $\mathcal{O}_X(D_i)$ for $i \in \underline{i}$, we can determine those for $\mathcal{O}_X(D_j)$ for $j \in \underline{j}$ by the above relation. As the $\partial\bar{\partial}\log$ of the first three factors of the above volume form is cohomologically zero, we can determine Hermitian metrics for $\mathcal{O}_X(D_j)$ so that the Ricci-form is identically zero on $F_{\underline{i}}$.

Now we examine the above process. The Hermitian metrics for $\mathcal{O}_X(D_i)$ are already determined along the unique closed orbit Y. Next consider $D_{1\cdots\hat{i}\cdots r}$ where $\hat{i}$ means i removed. Set $\underline{j} = (1, \cdots, \hat{i}, \cdots, r)$ with $\underline{i} = (i)$ the complement. Direct computation shows that

$$P_{\underline{i}}(d_{ij}) = \sum_{\underline{k} \subset \underline{i}} \rho_{\underline{j} \cup \underline{k}} \overline{P}([\underline{k}])^{e(\underline{i},\underline{k})} = \overline{P}([\underline{i}])^{e(\underline{i},\underline{i})} = \frac{1}{\|\sigma_i\|^{2\overline{d}_i^{\underline{j}}/n_{\underline{i}}}}.$$

holds on $F_{\underline{j}}$. Thus

$$\Omega_{\underline{i}}(d_{ij}) = \sqrt{-1}\partial\bar{\partial}\left(\frac{1}{\|\sigma_i\|^{2d_i^{\underline{j}}}}\right)^{\frac{1}{n_{\underline{i}}}}$$

and $c_1(F_{\underline{j}}) = (1 + d_i^{\underline{j}})[F_{\underline{j}} \cap D_i]$. Hence, as in (5.2.6-7), the Hermitian metric $\|\sigma_i\|$ is extended to all $F_{\underline{j}}$ so that $\Omega_{\underline{j}}(d_{ij})$ is a Ricci-flat complete Kähler metric on

$F_{\underline{j}} - F_{\underline{j}} \cap D_i$. Next, the consequence of Section 5.2 implies that, for any $\underline{j}$ with $\underline{i} = \{p,q\}$ the complement, there exists a function $u_{\underline{j}}$ (with decay condition at infinity) on $F_{\underline{j}} - F_{\underline{j}} \cap (D_p \cup D_q)$ such that

$$P_{\underline{i}}(d_{ij}) + \sqrt{-1}\partial\bar{\partial}u_{\underline{j}}$$

is a complete Ricci-flat Kähler metric on $F_{\underline{j}} - F_{\underline{j}} \cap (D_p \cup D_q)$. This implies that the Hermitian metrics on $\mathcal{O}_X(D_p)$ and $\mathcal{O}_X(D_q)$ are extended along $F_{\underline{j}}$ so that the Kähler potential $\overline{P}_{\underline{i}}(d_{ij})$ is Ricci-flat on the affine part of $F_{\underline{j}}$. Thus $f = \log(\Omega(d_{ij})^n/\eta\wedge\overline{\eta}) \equiv 0$ along $F_{\underline{j}}$. The same argument as in (5.2.6-8) implies that the Hermitian metrics are extended nicely in a tubular neighborhood of any $F_{\underline{k}}$ ($|\underline{k}| = r-3$) such that $F_{\underline{k}}$ contains $F_{\underline{j}_1}$, $F_{\underline{j}_2}$ and $F_{\underline{j}_3}$ as the divisors at infinity, where $F_{\underline{j}_i}$ are fibers in some $D_{\underline{j}_i}$ with $|\underline{j}_i| = r-2$. Next we try to solve

$$(\Omega_{\underline{l}}(d_{ij}) + \sqrt{-1}\partial\bar{\partial}u_{\underline{k}})^n = \text{Ricci-flat volume form}$$

on $F_{\underline{k}}$, where $\underline{l}$ is the complement of $\underline{k}$. Completely the same analysis as in (5.3) can be applied to show that there exists a solution with decay estimates. And we have $f \equiv 0$ along $F_{\underline{k}}$. In this way (solving the Monge-Ampère equations along fibers and extending the Hermitian metrics by using good retractions) the inductive arguments on rank(G/K) show that there exists a background metric of the form $\Omega(d_{ij})$ with the decay condition on f (of any order). Again by analysis in (5.3) we get a solution to

$$(\Omega(d_{ij}) + \sqrt{-1}\partial\bar{\partial}u)^n = \eta\wedge\overline{\eta} = e^{-f}\Omega(d_{ij})^n$$

with decay estimates (at infinity). We thus have proved Theorem (1.1) in the introduction; more precisely we have proved

Theorem. *Let (X, D) be the canonical compactification of a symmetric variety $G^{\mathbb{C}}/K^{\mathbb{C}}$ of any rank r with $D = \cup_{i=1}^r D_i$ the divisor at infinity. Let $n = \dim X$. Determine positive rationals d_i by setting $c_1(X) = \sum_{i=1}^r (1+d_i)[D_i]$. Then there exists a G-invariant Ricci-flat complete Kähler metric of the form*

$$\sqrt{-1}\partial\bar{\partial}P(d_{ij}) + \sqrt{-1}\partial\bar{\partial}u$$

where $P(d_{ij})$ is a Kähler potential constructed basically from

$$\frac{1}{\prod_{i=1}^r \|\sigma_i\|^{2d_i/n}}$$

together with bifurcation along $D_{\underline{i}}$ for any $\underline{i} \subset \{1,2,\cdots,r\}$ described by the parameters (d_{ij}), and u satisfies a uniform estimate on $X - D$ and decay estimates (at infinity).

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