ON COMPLETENESS IN AFFINE DIFFERENTIAL GEOMETRY

Katsumi Nomizu

Sonderforschungsbereich
Theoretische Mathematik
Beringstraße 4
D - 5300 Bonn 1

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D - 5300 Bonn 3

MPI/SFB 84 - 33

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In affine differential geometry there are at least three notions of completeness for nondegenerate hypersurfaces M of the affine space $\mathbb{R}^{n+1}$:

(1) affine metric completeness, namely, completeness of the Levi-Civita connection of the affine metric of M (whether it is positive-definite or not);

(2) Euclidean completeness, namely, completeness of the Riemannian metric on M induced from a Euclidean metric in $\mathbb{R}^{n+1}$;

(3) completeness of the canonical equiaffine connection on M.

R. Schneider [6] has studied conditions (1) and (2) and has given an example of a surface in $\mathbb{R}^3$ which is Euclidean-complete but not affine metric-complete. E. Calabi's work in [1], [2] shows the importance of condition (1) in some global problems. In the present paper, we consider one more completeness property:

(2') Lorentzian completeness, namely, completeness of the metric (assumed nondegenerate) induced on M from a flat Lorentzian metric in $\mathbb{R}^{n+1}$.

It was shown in [4] that if M is a spacelike hypersurface in $\mathbb{R}^{n+1}$ with Lorentzian metric $\sum_{k=1}^{n} dx_k^2 - dx_{n+1}^2$ and if the induced metric on M is complete, then the metric induced on M from the Euclidean metric $\sum_{k=1}^{n+1} dx_k^2$ is also complete. In this sense, we may say that (2') implies (2) at least for a spacelike hypersurface.

We wish to propose a more systematic study of these completenes conditions, but the purpose of this paper is to give an example of a spacelike surface M in $\mathbb{R}^3$ with metric $dx^2 + dy^2 - dz^2$ (that is, the Lorentz-Minkowski space L^3) whose induced metric is complete but whose affine metric is not complete.

In order to clarify our approach to affine differential geometry we shall start with a brief introduction to the subject which emphasizes the notion of equiaffine structure. An *equiaffine structure* on a differentiable manifold is a pair (∇, θ), where ∇ is a linear connection with zero torsion and θ is a volume element which is parallel relative to ∇. This approach was first given in my talk at the Conference in Differential Geometry, Münster, June 1982.

1. Basic theory for hypersurfaces:

Let $\mathbb{R}^{n+1}$ be an (n+1)-dimensional affine space with a volume element given by the determinant: $det(e_1,\ldots e_n) = 1$, where $\{e_1,\ldots,e_n\}$ is the standard basis of the underlying vector space for $\mathbb{R}^{n+1}$. We denote by D the standard linear connection in $\mathbb{R}^{n+1}$ relative to which the volume element det is parallel.

To deal with a more general situation, let us consider an (n+1)-dimensional manifold $\tilde{M}$ with a certain equiaffine structure (D,ω), namely, a linear connection D with zero torsion and a volume element ω which is parallel relative to D. Let M be a hypersurface, namely, an n-manifold with an immersion f into $\tilde{M}$. For a local theory, we think of M as imbedded and supress f in all basic formulas we write. Let ξ be a transversal field of tangent vectors on M so that for each x in M, the tangent space $T_x(\tilde{M})$ is the direct sum of the tangent space $T_x(M)$ and the span of ξ. For tangent vector fields X and Y on M, we decompose D_XY at each point x in the form

(1) $D_XY = \nabla_XY + h(X,Y)\xi$,

where ∇_XY is the component tangent to M and $h(X,Y)\xi$ is the component in the direction of ξ. It is quite routine to check that

(2) $(X,Y) \longrightarrow \nabla_XY$

defines a linear connection on M with zero torsion and that

(3) $(X,Y) \longrightarrow h(X,Y)$

defines a bilinear symmetric form on each tangent space of M , called the second fundamental form. Note that both the connection ∇ and the form h depend on the choice of ξ. In addition to (1), we may also decompose $D_X\xi$ in the form

(4) $D_X\xi = -S(X) + \tau(X)\xi$,

where $S(X)$ is the component tangent to M and $\tau(X)\xi$ is the component in the direction of ξ. We see that S is a (1,1) tensor and τ is a 1-form. We shall also define a volume element v on M by

(5) $\theta(X_1,\dots,X_n) = \omega(X_1,\dots,X_n,\xi)$,

for any tangent vectors $X_1,\dots,X_n$ on M. It is easy to check that

(6) $\nabla_X\theta = \tau(X)\theta$.

Our approach is the following. Asuming non-degeneracy of M (as explained below) we first show that there is a choice of ξ for which the form τ vanishes identically so that the volume element θ is parallel relative to the connection ∇. We then impose one further condition which will determine ξ,∇ and θ uniquely. The resulting pair (∇,θ) is the *canonical equiaffine structure* on M.

Now for our purpose, we begin with

__Lemma 1.__ Let $\bar\xi = Z+r\xi$ be another choice of a transversal vector field, where Z is tangent to M and $r>0$ is a differentiable function. Then we have the relationships

(i) $h = r\bar{h}$

(ii) $\nabla_X Y = \bar{\nabla}_X Y + \bar{h}(X,Y)Z$

(iii) $\bar{\tau}(X) = \tau(X) + Xr/r + h(X,Z)/r$

between ∇, h and τ for ξ and $\bar{\nabla}, \bar{h}$ and $\bar{\tau}$ for $\bar{\xi}$.

Proof: Straightforward.

From (i) it follows that the condition that h is nondegenerate is independent of the choice of ξ. In this case, we say that M is *nondegenerate*. Now we have

Lemma 2. If M is nondegenerate, then we can choose ξ so that $\tau = 0$ (and thus θ is parallel relative to ξ).

Proof: For $r = 1$, we can find Z such that $h(X,z) = -\tau(X)$ for every tangent vector X.

Remark: If Σ denotes the set of transversal vector fields for which $\tau = 0$, then the map $\xi \in \Sigma \longrightarrow \theta$ is injective. For, if $\xi, \bar{\xi} \in \Sigma$, then in the notation of Lemma 1, $\theta = \bar{\theta}$ implies $r = 1$. From $\tau = \bar{\tau} = 0$, we have $h(X,Z) = 0$ for all X, so that $Z = 0$.

Now we impose a further condition on ξ. For h corresponding to ξ, let ν be the volume element on M defined by

(6) $\nu(X_1,\ldots,X_n) = \sqrt{|\det[h(X_i,X_j)]|}$,

where $\{X_1,\ldots,X_n\}$ is any basis in the tangent space. Let us consider the condition

(C) $\nu = \theta$.

By choosing a basis $\{X_1,\ldots,X_n\}$ such that $\theta(X_1,\ldots,X_n) = 1$, let

(7) $h_{ij} = h(X_i, X_j)$ and $H = \det[h_{ij}]$.

Then $\nu(X_1, \ldots, X_n) = \sqrt{|H|}$ and hence $\nu = \sqrt{|H|}\,\theta$, Condition (C) is thus equivalent to $|H| = 1$.

__Lemma 3.__ Let $\xi, \bar{\xi} \in \Sigma$, and write H and $\bar{H}$ for the values for ξ and $\bar{\xi}$ defined in (7). If $\bar{\xi} = Z + r\xi$ as in Lemma 1, then

$$h = r\bar{h} \quad \text{and} \quad H = r^{n+2}\bar{H}$$

__Proof:__ Straightforward.

In view of Lemma 3, we see that

(8) $$\hat{h} = h/|H|^{\frac{1}{n+2}}$$

is independent of the choice of ξ. (8) is called the *affine metric* for the nondegenerate hypersurface M. If $\xi \in \Sigma$ satisfies condition (C), then $|H| = 1$ so that $\hat{h} = h$ in (8). Thus the volume element $\theta = \nu$ for ξ coincides with the volume element $\hat{\theta}$ for the affine metric $\hat{h}$. The uniqueness part in the following theorem follows from the remark just after Lemma 2.

__Theorem 1.__ If M is a nondegenerate hypersurface in $\tilde{M}$, we can choose a unique transversal vector field $\xi \in \Sigma$ satisfying condition (C).

__Proof:__ By Lemma 2, we choose $\xi \in \Sigma$ and compute $H = H_\xi$.

By taking $r = H^{\frac{1}{n+2}}$ we choose a tangent vector field Z such that $\bar{\xi} = Z + r\xi$ is in Σ again. Then $\bar{H} = H_{\bar{\xi}}$ is given by $H/r = H/|H|$ so that $|\bar{H}| = 1$. Thus $\bar{\xi} \in \Sigma$ satisfies condition (C).

The transversal vector field ξ established in Theorem 1 is called the *affine normal* for the nondegenerate hypersurface M. For this ξ, the second fundamental form h coincides with the affine metric $\hat{h}$, and the volume element θ coincides with the volume element $\hat{\theta}$ of the affine metric $\hat{h}$. The linear connection ∇ arising from the affine normal is called the *canonical affine connection* on M. The affine metric $\hat{h}$ is nondegenerate. The Levi-Civita connection on M for the metric $\hat{h}$ will be called the *affine metric connection*.

When $\tilde{M} = \mathbb{R}^{n+1}$ with its equiaffine structure $(D, \det)$, we obtain the canonical equiaffine structure (∇, ω) on any nondegenerate hypersurface M in $\mathbb{R}^{n+1}$. This is indeed the object of study in classical affine differential geometry.

2. An example.

We shall give an example of a spacelike surface in the Lorentz-Minkowski space L^3 whose induced metric is complete but whose affine metric is not complete. In fact, this surface is one of the surfaces constructed in [3] in the following way.

Let f be a mapping of $\mathbb{R}^2$ into L^3 with metric $dx^2 + dy^2 - dz^2$:

$$(u, \phi) \in \mathbb{R}^2 \longrightarrow f(u, \phi) = (x, y, z) \in L^3,$$

where

$$(9) \qquad x = \int_0^u \sqrt{1+e^{2t}}\, dt, \quad y = e^u \sinh \phi, \quad z = e^u \cosh \phi.$$

Then f is an imbedding of the entire (u,ϕ)-plane $\mathbb{R}^2$ into L^3 and the induced metric on $\mathbb{R}^2$

(10) $$ds^2 = du^2 + e^{2u}\, d\phi^2$$

is positive-definite. This metric is complete, since the transformation $(u,\phi) \longrightarrow (X,Y)$, where $X = \phi$ and $Y = e^{-u}$ takes it into the Poincaré metric $(dX^2 + dY^2)/Y^2$ in the upper-half plane $Y > 0$, which is known to be complete. It also follows that (10) has constant Gaussian curvature -1. We denote by M_0 this spacelike surface $f : \mathbb{R}^2 \longrightarrow L^3$.

In order to view M_0 from the affine point of view, we take a unit timelike normal vector field ξ and the corresponding second fundamental form h . It is known, in the theory of submanifolds of a Lorentzian manifold, that the Gaussian curvature K , which is -1 for our surface M_0 , is related to h by the Gauss equation

$$-K = h(X_1,X_1)\; h(X_2,X_2) - h(X_1,X_2)^2 ,$$

where $\{X_1,X_2\}$ is an orthonormal basis (relative to the metric (10)) in the tangent space. This means that from the affine point of view, the quantity H defined in (7) for ξ is equal to 1. Thus the affine metric of M_0 coincides with h . We know from [3] that

$$h(\partial/\partial u,\partial/\partial u) = e^u/\sqrt{1 + e^{2u}} , \quad h(\partial/\partial u,\partial/\partial\phi) = 0,$$

$$h(\partial/\partial\phi/\partial/\partial\phi) = e^u \sqrt{1 + e^{2u}} .$$

Thus h may be written in the form

(11) $d\sigma^2 = (e^u/\sqrt{1+e^{2u}})du^2 + (e^u\sqrt{1+e^{2u}})d\phi^2$.

This affine metric is elliptic. In order to show that it is not complete, we use the following lemma, whose proof is easy.

Lemma 4. Suppose that $d\sigma^2$ and $d\tau^2$ are two Riemannian metrics on a differentiable manifold such that $d\sigma^2 \leq d\tau^2$.

(i) If $\{x_n\}$ is a Cauchy sequence relative to $d\tau^2$, it is so relative to $d\sigma^2$

(ii) If $d\sigma^2$ is complete, so is $d\tau^2$.

To apply this lemma, let

$$d\tau^2 = \sqrt{1+e^{2u}}\,d\sigma^2 = e^u\,du^2 + e^u(1+e^{2u})d\phi^2$$

and observe that

$$d\sigma^2 \leq \sqrt{1+e^{2u}}\,d\sigma^2 = d\tau^2 .$$

We shall show that the metric $d\tau^2$ is not complete. This implies that $d\sigma^2$ is not complete.

Consider the curve C given by $u=-t$, $\phi=0$, where $0 \leq t < \infty$. The tangent vector $(du/dt, d\phi/dt) = (-1,0)$ has the length (relative to $d\tau^2$) equal to $e^{-t/2}$. So the arclength of C is

$$\int_0^\infty e^{-t/2}\,dt = 2 .$$

Obviously, the curve C has no limit point as $t \to \infty$. This proves that $d\tau^2$ is not complete.

Remark 1. The canonical affine connection ∇ of M_0 coincides with the Levi-Civita connection of the metric ds^2. Hence it is complete. Thus M_0 is also an example showing that condition (3) in the introduction does not imply condition (1).

Remark 2. It was also shown in [3] that for each $a > 0$, $a \neq 1$, there is a surface M_a in L^3 which is a non-standard imbedding of the hyperbolic plane into L^3. We can show that each of these surfaces is affine-metric complete in the following way.

The surface M_a is defined by

$$x = \int_0^u \sqrt{1 + a^2 \mathrm{sh}^2 t}\; dt, \quad y = a\ \mathrm{ch}\, u\ \mathrm{sh}\, \phi\ , \quad z = a\ \mathrm{ch}\, u\ \mathrm{ch}\, \phi\ .$$

The induced metric on M is

$$ds^2 = du^2 + a^2 \mathrm{ch}^2 u \;\; d\phi^2 \; ,$$

and the affine metric (which coincides with the second fundamental form of M_a as a spacelike surface of L^3) is given by

$$d\sigma^2 = (a\ \mathrm{ch}\, u / \sqrt{1 + a^2 \mathrm{sh}^2 u}\ )\ du^2 + \sqrt{1 + a^2 \mathrm{sh}^2 u}\;\; a\ \mathrm{ch}\, u\ d\phi^2 \; .$$

Case: $a < 1$. We have

$$1 + a^2 \mathrm{sh}^2 u < 1 + \mathrm{sh}^2 u = \mathrm{ch}^2 u \quad \text{so} \quad (a\ \mathrm{ch} u / \sqrt{1 + a^2 \mathrm{sh}^2 u}\ ) > a$$

from which we have

$$d\sigma^2 > a\ du^2 + a\ d\phi^2 \; .$$

Since the metric on the right hand side is complete, so is d^2 by Lemma 4.

Case 2: $a > 1$. We have

$$a \operatorname{ch} u / \sqrt{1 + a^2 \operatorname{sh}^2 u} > 1$$

so that

$$d\sigma^2 > du^2 + a\, d\phi^2 .$$

Again, $d\sigma^2$ is complete.

Department of Mathematics
Brown University
Providence, R. I 02912
U.S.A.

and

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
5300 Bonn 3
Federal Republic of Germany

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