Combinations of rational double points on the deformation of quadrilateral singularities II

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§0. Introduction

In this article we would like to continue to study the relation between hypersurface quadrilateral singularities and Dynkin graphs. In particular, we study 3 kinds of hypersurface quadrilateral singularities $J_{3,0}$, $Z_{1,0}$ and $Q_{2,0}$. Two kinds of transformations of Dynkin graphs, which we have proposed in previous articles (Urabe [5], [6], [7]), play essential roles. We give a proof of following Main Theorem, which have been announced in Part I (Urabe [7]). Every algebraic variety is assumed to be defined over the complex number field $\mathbf{C}$. As for the exact definition of Dynkin graphs, we follow that in Part I.

Let X be a class of quadrilateral singularities. Let $PC(X)$ denote the set of Dynkin graphs G with components of type A, D, or E only such that there exists a fiber Y in the semi-universal deformation family of a singularity belonging to X satisfying the following two conditions depending on G.

(1) The fiber Y has only rational double points as singularities.

(2) The combination of rational double points on Y just corresponds to the graph G.

(Note the phenomenon called "exceptional deformations", which was pointed out in Wall [9].)

Main Theorem. *Consider one of $J_{3,0}$, $Z_{1,0}$, and $Q_{2,0}$ as the class X of hypersurface quadrilateral singularities. A Dynkin graph G belongs to $PC(X)$ if and only if either following (1) or (2) holds.*

(1) G is one of the following exceptions.

(2) G can be made from one of the following essential basic Dynkin graphs by elementary or tie transformations applied 2 times (We can apply 2 different kinds of transformations once for each, or can apply 2 transformations of the same kind.), and G contains no vertex corresponding to a short root.

	The essential basic Dynkin graphs:	*The exceptions:*
The case $X = J_{3,0}$:	$E_8 + F_4$	$3A_3 + 2A_2$
The case $X = Z_{1,0}$:	$E_7 + F_4, E_8 + CB_3$	*None*
The case $X = Q_{2,0}$:	$E_6 + F_4, E_8 + F_2$	$3A_3 + A_2$

We know by results in Part I that the "if" part under the condition (2) is true. Thus in this Part II we would like to show the "only if" part first.

Let Λ_3 be the even unimodular lattice of signature (19, 3), and P be the lattice associated with the hypersurface quadrilateral singularity. (See Part I.) Let $Q(G)$ be the root lattice associated with a Dynkin graph G with components of type A, D or E only. If $G \in PC(X)$, then we have an embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$ come from an actual deformation fiber Y. The embedding satisfies Looijenga's conditions $\langle a \rangle$ and $\langle b \rangle$ and the induced embedding $Q(G) \hookrightarrow \Lambda_3/P$ is full. Also we have an elliptic K3 surface

$\Phi : Z \to C(\cong \mathbf{P}^1)$ corresponding to the embedding. Then, thanks to the results in Part I, we have only to show the following.

Proposition 0.1. *If $G \in PC(X)$, and if G is not in the exception list in Main Theorem, then, with respect to some full embedding $Q(G) \hookrightarrow \Lambda_3/P$, there exists a primitive isotropic element u in Λ_3/P in a nice position, i.e., such that either u is orthogonal to $Q(G)$, or there is a root basis $\Delta \subset Q(G)$ and a long root $\alpha \in \Delta$ such that $\beta \cdot u = 0$ for every $\beta \in \Delta$ with $\beta \neq \alpha$ and $\alpha \cdot u = 1$.*

To show this proposition we use the theory of the monodromy for elliptic surfaces and construct a certain transcendental cycle in the elliptic K3 surface $Z \to C$. We utilize the fact that the monodromy around a singular fiber of type I_0^* has a very simple form, i.e., simply multiplying -1.

Now, the elliptic surface $Z \to C$ has a singular fibre F_1 of type I_0^* in our situation by definition. Recall that by C_5 we have denoted the image of a section $C \to Z$, and C_5 and F_1 are contained in the curve IF at infinity. The curve IF at infinity has 6 (when $X = J_{3,0}$), 7 (when $X = Z_{1,0}$) or 8 (when $X = Q_{2,0}$) components. The lattice P has a basis associated with the dual graph of the components of IF. The union $\mathcal{E}$ of smooth rational curves on Z not intersecting IF coincides with the union of components not intersecting IF of singular fibers of $Z \to C$. The dual graph of $\mathcal{E}$ is G by definition.

We divide the case into three.

((1)) The surface $Z \to C$ has another singular fiber of type I^* apart from F_1.

((2)) $Z \to C$ has a singular fiber of type II^*, III^* or IV^*.

((3)) $Z \to C$ has no singular fiber of type I^*, II^*, III^* or IV^* apart from F_1.

In case ((1)) we can show that there exists a non-zero transcendental 2-cycle Ξ in Z with $\Xi^2 = 0$ which is orthogonal to the section C_5 and to all irreducible components of fibers of $Z \to C$. In particular, the orthogonal complement of $S = P \oplus Q(G)$ in Λ_3 contains an isotropic element, and thus we have a desired isotropic element in Λ_3/P. This case ((1)) is treated in section 1.

Case ((2)) is discussed in section 2. In this case we can show by the theory of the monodromy that there exists a transcendental 2-cycle Ξ on Z with $\Xi^2 = 4$ such that Ξ is orthogonal to C_5 and to all components of fibers of $Z \to C$. Therefore the orthogonal complement of S necessarily contains an element ξ with $\xi^2 = -4$. By drawing the Coxeter-Vinberg graph for K/P where K is the orthogonal complement of $\mathbf{Z}\xi$ in Λ_3, we show the existence of an isotropic element in a nice position in this case.

In case ((3)) it is difficult to construct a nice transcentental cycle applicable to all examples. Therefore we make the list of all possible Dynkin graph G in this case (In this case all components are of type A.), and we analyze them by the theory of K3 surfaces and by the theory of elliptic surfaces case by case. This case is discussed in section 3.

Lastly, we have to show that for a Dynkin graph G in the above exception list there exists the corresponding varieties Z, Y. This is shown in section 4. We apply Nikulin's lattice theory in this part (Nikulin [3]).

Here we have a remark. Assume that a Dynkin graph G with components of type A, D or E only can be made from one of the essential basic graphs by elementary or tie transformations applied twice. Then we can construct a full embedding $Q(G) \hookrightarrow \Lambda_3/P$ which has a primitive isotropic element in a nice position. This is a consequence of the theories in [5], [6], [7]. (See Theorem 1.1 in [6], Theorem 4.4 etc. in [7].) Of course, the constructed embedding may not be equivalent to the given embedding. However, we can use this remark to show Proposition 0.1 without any problem. Note moreover that under the assumption we have $G \in PC(X)$ by the "if" part of Main Theorem.

§1. An transcendental isotropic cycle

Recall that we have an isomorphism $H^2(Z, \mathbf{Z}) \xrightarrow{\sim} \Lambda_3$ preserving bilinear forms up to sign. (Note that it *reverses* the sign.) Via this isomorphism we can use the geometry on the elliptic K3 surface $\Phi : Z \to C$ to show an isotropic element in Λ_3.

We denote the fiber $\Phi^{-1}(a)$ over a point $a \in C$ by F_a for simplicity. By $\Sigma = \{c_1, c_2, \ldots, c_t\}$ we denote the set of critical values of Φ. We put $F_i = F_{c_i}$ for simplicity.

We can assume the following in our situation.

(1) There is a section $s_0 : C \to Z$ (i.e., a morphism of varieties with $\Phi(s_0(x)) = x$ for $x \in C$) whose image is denoted by C_5.

(2) For some point $c_1 \in \Sigma$ the fiber F_1 over c_1 is a singular fiber of type I_0^*.

We have the following facts. (Kodaira [1], Shioda [4])

- $C \cong \mathbf{P}^1$.
- For every smooth rational curve A in Z, the self-intersection number A^2 is equal to -2.
- Let $e(F)$ denote the Euler number of a fiber F. Then we have

$$\sum_{i=1}^{t} e(F_i) = 24. \tag{1}$$

- The set E of all sections of Φ has a structure of an abelian group when we fix an element, say s_0, as the unit element. This abelian group E is finitely generated. Let a be the rank of E and ρ be the Picard number of Z ($=$ rank $\mathrm{Pic}(Z)$).

$$\rho = 2 + a + \sum_{i=1}^{t}(m(F_i) - 1) \tag{2}$$

where $m(F)$ denotes the number of irreducible components of a singular fiber F. (See Lemma 1.4.)

- If a singular fiber F is not of type I, then $m(F) = e(F) - 1$, while if F is of type I, then $m(F) = e(F)$.
- Let t_1 denote the number of singular fibers of type I of Φ. The number of all singular fibers of Φ is t. By above (1) and (2) we have:

$$\rho = 26 + a - 2t + t_1. \tag{3}$$

- If the functional invariant $J : C \to \mathbf{P}^1$ of the elliptic surface Φ is not constant, then

$$20 - \rho + a \geq \nu(I_0^*) + \nu(II) + \nu(III) + \nu(IV), \tag{4}$$

where $\nu(T)$ denotes the number of singular fibers of type T. (Shioda [4]. For general elliptic surfaces 20 should be replaced by $b_2 - 2p_g$. In our case the second Betti number $b_2 = 22$, the geometric genus $p_g = 1$.)

Lemma 1.1. $t \geq 3$.

Proof. Since $\rho \leq \dim H^1(Z, \Omega_Z^1) = 20$, by (3) we have $2t \geq 6 + a + t_1 \geq 6$. Q.E.D.

Here recall the notion of *parallel translation* along a path. Let $r : [0,1] \to C - \Sigma$ be a path. We have the induced mapping

$$Z(r) = \bigcup_{0 \leq \tau \leq 1} F_{r(\tau)} \to [0,1].$$

Since $[0,1]$ is contractible, this family $Z(r) \to [0,1]$ is trivial, i.e., there is a homeomorphism $\chi : F_{r(0)} \times [0,1] \to Z(r)$ such that its composition with $Z(r) \to [0,1]$ coincides with the projection $F_{r(0)} \times [0,1] \to [0,1]$. For $0 \leq \tau \leq 1$ by χ_τ we denote the composition of the natural isomorphism $F_{r(0)} \cong F_{r(0)} \times \{\tau\}$ and the restriction of χ to $F_{r(0)} \times \{\tau\}$. The homeomorphism $\chi_\tau {\chi_{\tau'}}^{-1} : F_{r(\tau')} \to F_{r(\tau)}$ is induced for $\tau, \tau' \in [0,1]$. This is called the *parallel translation* from $r(\tau')$ to $r(\tau)$ along r. It depends on the homeomorphism χ, but the isotopy class of the parallel translation depends only on the homotopy class of the path in $C - \Sigma$ connecting $r(\tau')$ and $r(\tau)$. In particular, we can define an isomorphism of cohomology groups $r_* : H^*(F_{r(\tau')}, \mathbf{Z}) \to H^*(F_{r(\tau)}, \mathbf{Z})$ associated with the parallel translation, which depends only on the homotopy class of the path r. (Thus we can denote it by r_*.)

Now, let b_τ be the intersection point of $C_5 = s_0(C)$ and $F_{r(\tau)}$. The section s_0 induces a section $[0,1] \to Z(r)$ whose image of τ is b_τ. Here note that we can take χ such that $\chi(b_0, \tau) = b_\tau$ for $0 \leq \tau \leq 1$. Then the induced homeomorphism $F_{r(\tau')} \to F_{r(\tau)}$ sends $b_{\tau'}$ to b_τ.

The induced homomorphism $r_* : H^*(F_{r(0)}, \mathbf{Z}) \to H^*(F_{r(0)}, \mathbf{Z})$ for a closed path r is called the *monodromy* along r.

The fixed base point is denoted by $b \in C - \Sigma$. For $1 \leq i \leq t$ let l_i be a path connecting b and c_i contained in $C - \Sigma$ except the ending point c_i. Here we take them in such a way that l_i and l_j has no common point except the starting point b, if $i \neq j$.

By r_i we denote the closed path which starts from b, goes along l_i until a point just before c_i, then switches to a circle with a small radius with center c_i, proceeds on it in the positive direction round once, and then goes again along l_i in the opposite direction back untill the base point b. We can assume that no points in Σ is inside the circular part of r_i except c_i.

Set $H = H^1(F_b, \mathbf{Z})$. For any closed path r in $C - \Sigma$ with the starting point and the ending point b, we have the associated monodromy $r_* : H \to H$. It is a linear isomorphism preserving the intersection form $\cdot$ on H.

Choosing a basis α, β of H with $\alpha \cdot \beta = 1$, we can represent the monodromy r_* by an integral 2 by 2 matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with determinant 1. This implies that α is transformed to $x\alpha + z\beta$ and β to $y\alpha + w\beta$ when we go along the closed path r.

Here we would like to give a remark. Kodaira's paper on elliptic surfaces (Kodaira [1]) is a very important reference. However, we should note that in it he uses a basis α', β' of H such that $\alpha' \cdot \beta' = -1$ and $\beta' \cdot \alpha' = 1$, and moreover that he writes the transposed matrix of the one under standard representation to represent linear mappings. If

$\begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix}$ is Kodaira's matrix, it implies that α' is transformed to $x'\alpha' + y'\beta'$ and β' to $z'\alpha' + w'\beta'$. Thus Kodaira's matrix is represented by $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x' & -z' \\ -y' & w' \end{pmatrix}$ under our notation.

Proposition 1.2. *Assume that Φ has another singular fibre of type I^* except F_1. Then there exists an isotropic element $u \in H^2(Z, \mathbf{Z})$ satisfying the following conditions (1) and (2).*
(1) u is orthogonal to the cohomology class of the section $C_5 = s_0(C)$ and to all the cohomology classes of irreducible components of singular fibers.
(2) Let $s : C \to Z$ be an arbitrary section. If s has finite order in the abelian group E of sections with the unit element s_0, then u is also orthogonal to the cohomology class of the image $s(C)$.

Proof. We can assume that F_2 is of type I_n^*. According to Kodaira [1], the monodromy around c_1 and c_2 can be represented by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ n & -1 \end{pmatrix}$ respectively. In particular the monodromy r_{1*} equals to the multiplication of -1. The monodromy r_{2*} around c_2 has a primitive element $\gamma \in H$ such that it transforms $\gamma \mapsto -\gamma$. We can represent the class γ by an oriented simple closed curve Γ on F_b. We choose Γ such that it does not pass through the intersection point b_0 of $C_5 = s_0(C)$ and F_b. Let $\overline{r} = r_2 r_1$ be the connected closed path made by connecting r_2 after r_1. The path $\overline{r}$ can be regarded as a continuous mapping $\overline{r} : [0,1] \to C - \Sigma$ with $\overline{r}(0) = \overline{r}(1) = b$. Let Γ_τ be the image of Γ by the parallel translation along $\overline{r}$ from b to $\overline{r}(\tau)$. Γ_τ is an oriented simple closed curve on the Riemann surface $F_{\overline{r}(\tau)}$ of genus 1. $\Gamma = \Gamma_0$. We can assume that Γ_τ does not pass through the intersection point b_τ of $F_{\overline{r}(\tau)}$ and C_5. Consider the 2-chain

$$\widetilde{\Gamma} = \bigcup_{0 \le \tau \le 1} \Gamma_\tau \subset Z - \Phi^{-1}(\Sigma) - C_5.$$

The boundary satisfies $\partial\widetilde{\Gamma} = -\Gamma + \Gamma_1$. We have

$$\Gamma, \Gamma_1 \subset F_b - \{b_0\}.$$

The class γ of Γ is transformed to $-\gamma$ when we go along r_1. Next, $-\gamma$ is transformed to γ when we go along r_2. Thus the class of Γ_1 is also γ. Two curves Γ and Γ_1 are homologous in the compact Riemann surface F_b.

Here note that

$$H_1(F_b) \cong H_1(F_b - \{b_0\}).$$

Thus there exists a 2-chain Θ in the punctured Riemann surface $F_b - \{b_0\}$ such that $\partial\Theta = \Gamma - \Gamma_1$.

Consider the chain of sum $U = \widetilde{\Gamma} + \Theta$. This is a 2-cycle and defines a class $[U] \in H_2(Z, \mathbf{Z})$. Let $u \in H^2(Z, \mathbf{Z})$ be the Poincare-dual class of $[U]$. By construction U intersects neither C_5 nor any component of singular fibers, and thus u is orthogonal to the cohomology classes of them.

Now, let r' be a closed path in $C - \Sigma$ homotope to $\overline{r}$. (Here we consider homotopy without any base point.) We can repeat the above construction using r' instead of $\overline{r}$. Let U' be the resulting 2-cycle. Then U' is homologous to U in Z. Since we can take r' with no intersection point with $\overline{r}$, we can construct U' without intersection points with U. Thus we have

$$u^2 = [U] \cdot [U'] = 0.$$

Lastly we would like to show $u \neq 0$. Note that we have the third singular fiber by Lemma 1.1.

◇ Case 1. One of the singular fibers F_i $(3 \leq i \leq t)$ is not of type I.

We can assume that F_3 is not of type I. F_3 is simply connected.

We can choose a smooth path $q : [0,1] \to C$ such that $q(0) = c_1$, $q(1) = c_3$, $q(\tau) \notin \Sigma$ for $0 < \tau < 1$, and q intersects $\overline{r} = r_1 r_0$ at only one point c_{11} in a neighbourhood of c_1. We can assume further that q and $\overline{r}$ intersect at c_{11} transversally.

Let B_1 and B_3 be sufficiently small non-empty open discs on C with center c_1 and c_3 respectively. We take B_1 such that B_1 is contained inside the circular part of the path r_1. Note that the inverse images $\Phi^{-1}(B_1)$ and $\Phi^{-1}(B_3)$ are simply connected.

Let $\gamma' \in H^1(F_{c_{11}})$ be the image of $\gamma \in H^1(F_b)$ by the parallel translation along r_1 to c_{11}. Let $\delta' \in H^1(F_{c_{11}})$ be a primitive element with $\gamma' \cdot \delta' \neq 0$. The class δ' can be realized by a simple closed curve Δ on the Riemann surface $F_{c_{11}}$. By Δ_τ we denote the image of Δ by the parallel translation to $q(\tau)$ along q . Δ_τ is a simple closed curve on the Riemann surface $F_{q(\tau)}$. Choose a sufficiently small positive real number ϵ such that $q(\epsilon) \in B_1$ and $q(1-\epsilon) \in B_3$. Set

$$\widetilde{\Delta} = \bigcup_{\epsilon \leq \tau \leq 1-\epsilon} \Delta_\tau.$$

$\widetilde{\Delta}$ is a 2-chain and the boundary satisfies the following.

$$\partial\widetilde{\Delta} = -\Delta_\epsilon + \Delta_{1-\epsilon}, \qquad \Delta_\epsilon \subset \Phi^{-1}(B_1), \qquad \Delta_{1-\epsilon} \subset \Phi^{-1}(B_3).$$

Since $\Phi^{-1}(B_1)$ and $\Phi^{-1}(B_3)$ are simply connected, we have 2-chains $\widetilde{\Delta}'$ and $\widetilde{\Delta}''$ in $\Phi^{-1}(B_1)$ and $\Phi^{-1}(B_3)$ respectively such that $\partial\widetilde{\Delta}' = \Delta_\epsilon, \partial\widetilde{\Delta}'' = -\Delta_{1-\epsilon}$. The chain of sum $V = \widetilde{\Delta} + \widetilde{\Delta}' + \widetilde{\Delta}''$ is a 2-cycle and it defines a class $[V] \in H_2(Z, \mathbf{Z})$. Cycles U and V intersect only on the fiber $F_{c_{11}}$ and by construction their intersection number satisfies

$$[U] \cdot [V] = \pm\gamma' \cdot \delta' \neq 0.$$

Thus we have $[U] \neq 0$ and $u \neq 0$.

◇ Case 2. Every singular fiber F_i for $3 \leq i \leq t$ is of type I.

Set $C' = \mathbf{P}^1$, and let $f : C' \to C$ be the branching double cover branching at c_1 and c_2. Let $\widetilde{Z}'$ be the normalization of the fiber product of Z and C' over C. This $\widetilde{Z}'$ is a branched double cover of Z. The branching locus is the union of 8 disjoiut smooth rational curves on Z. Singular fibers F_1 and F_2 contain 4 components with multiplicity 1 respectively. The union of them is the branching locus.

Let c_1' be the unique inverse image of c_1 by f, and c_2' be that of c_2. The fiber over c_1' of the induced morphism $\widetilde{Z}' \to C'$ contains 4 disjoint smooth rational curves which are inverse images of the branching locus, and they are exceptional curves of the first kind on $\widetilde{Z}'$. Similarly the fiber over c_2' contains 4 exceptional curves of the first kind. We can contract these 8 exceptional curves and let Z' denote the resulting smooth surface. We have the induced morphism $\Phi' : Z' \to C'$.

$$\begin{array}{ccccc} Z' & \longleftarrow & \widetilde{Z}' & \longrightarrow & Z \\ \downarrow{\scriptstyle \Phi'} & & \downarrow & & \downarrow{\scriptstyle \Phi} \\ C' & = & C' & \xrightarrow{f} & C \end{array}$$

The section $s_0 : C \to Z$ induces the section $s_0' : C' \to Z'$. Set $C_5' = s_0'(C')$. For simplicity by F_a' we denote the fiber $\Phi'^{-1}(a)$ over $a \in C'$. For every point $a \in C'$ the fiber F_a' does not contain an exceptional curve of the first kind. $F'_{c_1'}$ is a smooth elliptic curve and $F'_{c_2'}$ is a (singular) fiber of type I_{2n}. (A fiber of type I_0 is a smooth elliptic curve.) Let Σ' denote the set of critical values for Φ'. $f^{-1}(\{c_3, \ldots, c_t\}) \subset \Sigma' \subset f^{-1}(\{c_3, \ldots, c_t\}) \cup \{c_2'\}$.

Let b' and b'' be the inverse images of b by f. We fix one b' of two as the base point of C'. Let c_i' be the ending point of the lifting l_i' of l_i with the starting point $b'(3 \leq i \leq t)$. Let c_i'' be the ending point of the lifting l_i'' of l_i with the starting point $b''(3 \leq i \leq t)$. For $3 \leq i \leq t$ the fibers $F'_{c_i'}$ and $F'_{c_i''}$ are isomorphic to F_{c_i} and are of type I by definition.

We define paths on C'.

Let l be the lifting of r_1 with the starting point b'. The path l passes through a neighbourhood of c_1' and has the ending point b''. Let l_2' be the lifting of l_2 with the starting point b'. The ending point of l_2' is c_2'. We define the closed path r_2' to be the one which starts from b', goes first along l_2', then switches to a small circle with center c_2' just before c_2', proceeds round once on the circle in positive direction, and again along l_2' comes back to b'.

For $3 \leq i \leq t$, let r_i' be the lifting of r_i with the starting point b'. Note that the ending point of r_i' is not b'' but b' and r_i' is a closed path. It goes round c_i' just once. Let $\overline{r}_i$ be the lifting of r_i with the starting point b''. Set $r_i'' = l^{-1}\overline{r}_i l$. It is the composition of l, $\overline{r}_i$ and l in the inverse direction, has the starting point b', and goes round c_i'' just once.

By construction $H' = H^1(F'_{b'}, \mathbf{Z})$ is identified with $H = H^1(F_b, \mathbf{Z})$ via the induced isomorphism f_*.

For each one r of $2t-3$ closed paths

$$r_2', r_3', \ldots, r_t', r_3'', \ldots, r_t''$$

the monodromy transformation $r_* : H' \to H'$ is defined. Let G be the subgroup in the self-isomorphism group of H' generated by these $2t-3$ monodromy transformations. We define the sheaf $\mathcal{G}$ on C' by

$$\mathcal{G} = j_* j^* R^1 \Phi'_* \mathbf{Z}_{Z'},$$

where $j : C' - \Sigma' \hookrightarrow C'$ is the inclusion morphism and $\mathbf{Z}_{Z'}$ is the constant sheaf on Z' with values in the set of integers $\mathbf{Z}$. By definition we have

$$H^0(C', \mathcal{G}) \cong H'^G = \{\, x \in H' \mid g(x) = x \text{ for every } g \in G \,\}.$$

Lemma 1.3. *(Kodaira)* $\quad H^0(C', \mathcal{G}) = 0.$

Proof. Let N be the normal bundle of $C'_5 = s'_0(C')$ in Z'. By $\mathcal{F}$ we denote the pull-back of N by s'_0, which is a sheaf on C'. According to Kodaira [1], we have an injective homomorphism of sheaves $\mathcal{G} \to \mathcal{F}$. By [1] Theorem 12.3 we have $\deg \mathcal{F} = -\chi(\mathcal{O}_{Z'})$. Moreover by [1] Theorem 12.2 $\chi(\mathcal{O}_{Z'}) > 0$. Thus $H^0(\mathcal{F}) = 0$ and $H^0(\mathcal{G}) = 0$. Q.E.D.

Now, $\gamma \in H = H'$ is a primitive element transformed $\gamma \mapsto -\gamma$ by r_{2*}.

Since the monodromy $(r'_2)_*$ equals to $(r_{2*})^2$, $\gamma \in H'$ is invaliant by $(r'_2)_*$. By Lemma 1.3 the following (5) or (6) holds.

$$(r'_i)_*\gamma \neq \gamma \text{ for some } i \text{ with } 3 \leq i \leq t. \tag{5}$$

$$(r''_i)_*\gamma \neq \gamma \text{ for some } i \text{ with } 3 \leq i \leq t. \tag{6}$$

Assume that the case (5) takes place. If we regard $\gamma \in H$ by going down to the world of H via $f_* : H' \to H$, we have $(f_* r'_i)_*\gamma \neq \gamma$. However, since $f_* r'_i = r_i$ by definition, one knows

$$r_{i*}\gamma \neq \gamma. \tag{7}$$

Assume that the case (6) takes place. Similarly $(f_* r''_i)_*\gamma \neq \gamma$. Here by definition $f_* r''_i = r_1^{-1} r_i r_1$. The homomorphism r_{1*} has been the multiplication of -1. Thus $(f_* r''_i)_* = (r_1)_*^{-1} r_{i*} r_{1*} = r_{i*}$. Therefore the above (7) holds also in this case.

In the following we fix a number i with $3 \leq i \leq t$ satisfying (7).

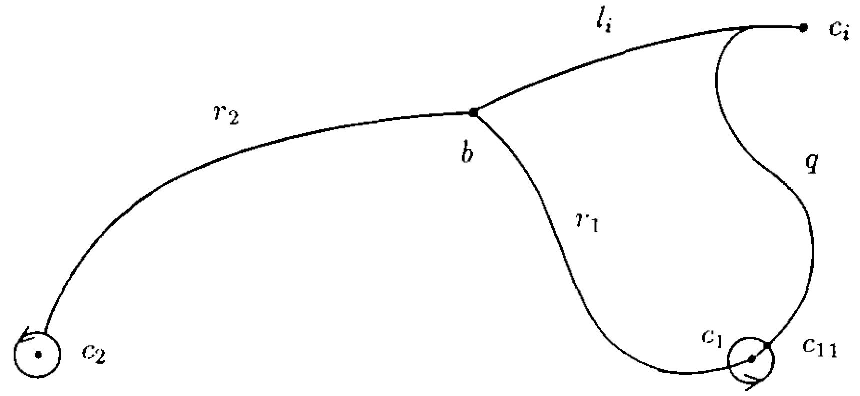
The monodromy r_{i*} has the matrix representation in the form $\begin{pmatrix} 1 & 0 \\ -b_i & 1 \end{pmatrix}$. Thus we have a unique primitive element $\delta \in H$ with $r_{i*}\delta = \delta$ up to sign. δ is a vanishing cycle of a singular point of the singular fiber F_i and it is defined associated with the path l_i with the starting point b and with the ending point c_i. By (7) $\gamma \neq \pm\delta$. Since γ is also primitive, one knows that the intersection number satisfies

$$\gamma \cdot \delta \neq 0.$$

Now, we can discuss similarly as in Case (1).

Let $q : [0,1] \to C$ be a smooth path satisfying the following conditions.

(1) The starting point c_1, the ending point c_i, i.e., $q(0) = c_1$, $q(1) = c_i$.
(2) For $0 < \tau < 1$ $q(\tau) \notin \Sigma$.
(3) The closed path $\overline{r} = r_2 r_1$ and q intersect only at one point c_{11} on the circular part of r_1 in a neighbourhood of c_1. They intersect at c_{11} transversally.
(4) The composition path $l_i l_1^{-1}$ and q are homotope in $(C - \Sigma) \cup \{\, c_1, c_i \,\}$ with the starting point c_1 and the ending point c_i keeping fixed.
(5) For some sufficiently small positive real number ϵ_1 $q(\tau)$ and $l_i(\tau)$ coincide for $1 - \epsilon_1 \leq \tau \leq 1$.



Let $\delta', \gamma' \in H^1(F_{c_{11}})$ be the image of $\delta, \gamma \in H^1(F_b)$ by the parallel translation along r_1 from b to c_{11}. We can represent the class δ' by an oriented simple closed curve Δ on the Riemann surface $F_{c_{11}}$ of genus 1. Let $\Delta_\tau \subset F_{q(\tau)}$ denote the image of Δ by the parallel translation along q from c_{11} to $q(\tau)$. Let B_i be a sufficiently small non-empty open disc on C with the center c_i, and B_1 be a sufficiently small non-empty open disc on C with the center c_1. Let ϵ be a sufficiently small positive real number with $q(\epsilon) \in B_1$, $q(1-\epsilon) \in B_i$ and $0 < \epsilon < \epsilon_1$.

Set

$$\widetilde{\Delta} = \bigcup_{\epsilon \leq \tau \leq 1-\epsilon} \Delta_\tau.$$

$\widetilde{\Delta}$ is a 2-chain satisfying

$$\partial\widetilde{\Delta} = -\Delta_\epsilon + \Delta_{1-\epsilon}, \qquad \Delta_\epsilon \subset \Phi^{-1}(B_1), \qquad \Delta_{1-\epsilon} \subset \Phi^{-1}(B_i).$$

Now, since the inverse image $\Phi^{-1}(B_1)$ is simply connected, we have a 2-chain $\widetilde{\Delta}'$ in $\Phi^{-1}(B_1)$ such that $\partial\widetilde{\Delta}' = \Delta_\epsilon$.

Next we consider $\Delta_{1-\epsilon}$. The class δ'' defined in $H^1(F_{q(1-\epsilon)})$ by $\Delta_{1-\epsilon}$ is the image of δ' by the parallel translation from c_{11} along q. Let a be the rounding number around c_1 of the closed path $l_i^{-1}q^{\#}r_1^{\#}$ where $q^{\#}$ denotes the part of q between c_{11} and c_i, and $r_1^{\#}$ denotes the part of r_1 between b and c_{11}. Since the monodromy around c_1 is the multiplication of -1, we have

$$\delta'' = (-1)^a \delta^{\#},$$

where $\delta^{\#}$ denotes the image of $\delta \in H^1(F_b)$ by the parallel translation along l_i to $q(1-\epsilon) = l_i(1-\epsilon)$. In particular, one knows that $\Delta_{1-\epsilon}$ is a vanishing cycle along l_i. Thus we have a 2-chain $\widetilde{\Delta}''$ in $\Phi^{-1}(B_i)$ with $\partial\widetilde{\Delta}'' = -\Delta_{1-\epsilon}$. The 2-chain of sum $V = \widetilde{\Delta} + \widetilde{\Delta}' + \widetilde{\Delta}''$ is a 2-cycle and defines the class $[V] \in H_2(Z, \mathbf{Z})$. The intersection number satisfies

$$[U] \cdot [V] = \pm\gamma' \cdot \delta' = \pm\gamma \cdot \delta \neq 0.$$

Thus one knows that $[U] \neq 0$ and $u \neq 0$.

Lastly we have to show (2) in Proposition 1.2.

Lemma 1.4. *Assume that an elliptic K3 surface $\Phi : Z \to C$ has a section s_0 whose image is denoted by C_5. Let $\overline{S}$ be the subgroup of the Picard group* $\mathrm{Pic}(Z)$ *of Z generated by C_5 and all irreducible components of singular fibers of Φ. By E we denote the abelian group of sections of Φ with the unit element s_0. Then there exists a natural group isomorphism*

$$\mathrm{Pic}(Z)/\overline{S} \xrightarrow{\sim} E.$$

Proof. Not difficult.

Under the condition in Proposition 1.2 (2) one knows by Lemma 1.4 that $s(C)$ can be written as a rational linear combination of the classes of C_5 and irreducible components of singular fibers. Thus (2) follows from (1) in the same proposition. This completes the proof of Proposition 1.2. Q.E.D.

Let $S \subset H^2(Z, \mathbf{Z})$ be the subgroup generated by the classes corresponding to the irreducible components of the curve IF at infinity (Urabe [7] section 1) and rational smooth curves on Z not intersecting IF. Note that such a smooth curve on Z is a component of a singular fiber. In the case of $J_{3,0}$, $Z_{1,0}$, or $Q_{2,0}$, C_5 is the unique component of IF which is an image of a section. The other components of IF than C_5 are components of singular fibers. Thus in these cases we have

$$S \subset \overline{S} \subset \mathrm{Pic}(Z) \subset H^2(Z, \mathbf{Z}),$$

where $\overline{S}$ is the group in Lemma 1.4. Under the isomorphism $H^2(Z, \mathbf{Z}) \xrightarrow{\sim} \Lambda_3$ S corresponds to $P \oplus Q(G)$, P corresponds to the subgroup in S generated by components of IF, and $Q(G)$ corresponds to the subgroup in S generated by smooth rational curves on Z not intersecting IF. Reversing the sign of bilinear forms, we have the following by Proposition 1.2.

Corollary 1.5. *Consider the case of $X = J_{3,0}, Z_{1,0}$ or $Q_{2,0}$. Let $G \in PC(X)$. If the corresponding elliptic K3 surface $Z \to C$ has a singular fiber of type I^* apart from the one of type I_0^*, then the orthogonal complement of $Q(G)$ with respect to the corresponding embedding $Q(G) \hookrightarrow \Lambda_3/P$ contains an isotropic element. In particular, then, there is an isotropic element in a nice position.*

Note that this corollary claims stronger than Proposition 0.1 under the assumption ((1)) in the introduction.

The following corollary is interesting in itself. Here recall that $IF = F_1 \cup C_5$ if $X = J_{3,0}$, $IF = F_1 \cup C_5 \cup C_6$ if $X = Z_{1,0}$, $IF = F_1 \cup C_5 \cup C_6 \cup C_7$ if $X = Q_{2,0}$, where C_6, C_7 are smooth rational curves not intersecting F_1 satisfying $C_6 \cdot C_5 = 1$, $C_7 \cdot C_6 = 1$ and $C_7 \cdot C_5 = 0$.

Corollary 1.6. *Consider the case of $X = J_{3,0}$, $Z_{1,0}$, or $Q_{2,0}$. Assume that a Dynkin graph $G \in PC(X)$ has a component of type D_k for some $k \geq 4$. Assume moreover that if $X = Z_{1,0}$, then $k \neq 6$. Then for any full embedding $Q(G) \hookrightarrow \Lambda_3/P$ the orthogonal complement of the image contains an isotropic element, and thus in particular the equivalent conditions [II](A) and [II](B) in Part I (Urabe [7]) Theorem 0.3 are satisfied.*

Proof. By assumption the corresponding K3 surface Z contains a combination $\mathcal{E}_1$ of k smooth rational curves not intersecting IF whose dual graph is the Dynkin graph of type D_k. $\mathcal{E}_1$ is contained in some singular fiber F_i of $Z \to C$ with $F_i \neq F_1$. If F_i is of type I^*, the claim follows from Corollary 1.5. Otherwise F_i is of type II^*, III^*, or IV^*, and it contains several components of the curve IF at infinity. Since the union of components of F_i disjoiut from IF is a combination of type D_k, one knows $X = Z_{1,0}$, F_i is of type III^*, and $k = 6$. Q.E.D.

§2. A transcendental cycle with the positive self-intersection number

In this section we treat the case where the corresponding elliptic K3 surface $\Phi : Z \to C$ has a singular fiber of type II^*, III^*, or IV^*. Recall that by C_5 we denote the image of the section $s_0 : C \to Z$. C_5 is a component of the curve IF at infinity. By $[\Xi]$ we denote the homology class of a cycle Ξ or the cohomology class of Ξ.

Proposition 2.1. *Assume that the elliptic K3 surface Φ with a section s_0 has a singular fiber of type II^*, III^*, or IV^* apart from the singular fiber F_1 of type I_0^*. Then there exists a cohomology class $\xi \in H^2(Z, \mathbf{Z})$ satisfying the following conditions (1)–(4).*

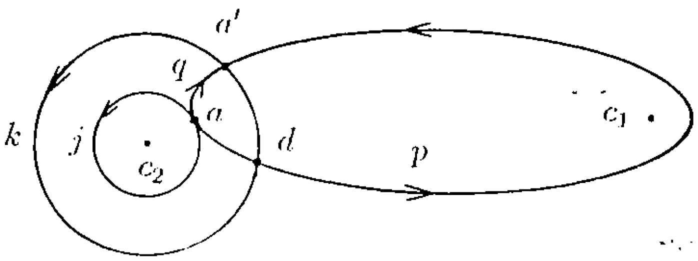
(1) $\xi^2 = +4$.

(2) The class ξ is orthogonal to the class of C_5 and to all the classes of irreducible components of singular fibers of Φ.

(3) For some two irreducible components C' and C'' of F_1 with multiplicity 1 which have no intersection with C_5, we can write $\xi + [C'] + [C''] = 2\eta$ for some $\eta \in H^2(Z, \mathbf{Z})$.

(4) Let $s : C \to Z$ be an arbitrary section. If s has finite order in the abelian group E of sections with the unit element s_0, then ξ is orthogonal to the cohomology class of the image $s(C)$.

Proof. We assume that the singular fiber F_2 over $c_2 \in \Sigma$ is of type II^*, III^*, or IV^*. Consider the paths on C as in the following figure.



The path j and k go on circles with center c_2 with a sufficiently small radius in the positive direction. j has a less radius than k. The point a and a' lie on the path j and k respectively. These are regarded as the starting point and the ending point of the respective closed path. The smooth path q has the starting point a and the ending point a', and it does not intersect j and k except at a and a'. The path p is a smooth path which has also the starting point a and the ending point a', it has no intersection with j except at a, and p and k has a unique intersection point d except the ending point a'. At d they intersect transversally. The composition path $r = q^{-1}p$ goes round the point c_1 just once in the positive direction. The inner domain surrounded by k contains no points in the set Σ of critical values of Φ except c_2, and the inner domain surrounded by r contains no points in Σ except c_1.

Let α, β be a basis of $H = H^1(F_a, \mathbf{Z})$ with $\alpha \cdot \beta = 1$ such that the associated matrix of the monodromy j_* is the following.

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \text{ (case of type } II^*) \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ (case of type } III^*)$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \text{ (case of type } IV^*)$$

Let A be an oriented simple closed curve on the Riemann surface F_a representing the cohomology class α. We can assume that A has no intersection with C_5. The closed path $j : [0, 1] \to C - \Sigma$ with $j(0) = j(1) = a$ defines the parallel translation $F_a \to F_{j(\tau)}$ for $0 \leq \tau \leq 1$. By $A_\tau \subset F_{j(\tau)}$ we denote the image of A by the parallel translation. For every τ with $0 \leq \tau \leq 1$ we can assume that A_τ has no intersection with C_5. We can define a 2-chain $\widetilde{J}$ by the following:

$$\widetilde{J} = \bigcup_{0 \leq \tau \leq 1} A_\tau \subset Z - \Phi^{-1}(\Sigma) - C_5.$$

The boundary satisfies $\partial \widetilde{J} = -A + A_1$. The cohomology class of $\partial \widetilde{J}$ in $H^1(F_a)$ is equal to $j_*\alpha - \alpha$.

Next, let $A' \subset F_{a'}$ be the image of A by the parallel translation along q. We can choose such an A' that it has no intersection with C_5. Also the closed path $k : [0, 1] \to C - \Sigma$ with $k(0) = k(1) = a'$ defines the parallel translation. By $A'_\tau \subset F_{k(\tau)}$ we denote the image of A' by the parallel translation along k. We can assume that for every τ with $0 \leq \tau \leq 1$ A'_τ does not intersect C_5. A 2-chain $\widetilde{K}$ is defined by

$$\widetilde{K} = \bigcup_{0 \leq \tau \leq 1} A'_\tau \subset Z - \Phi^{-1}(\Sigma) - C_5.$$

$\partial \widetilde{K} = -A' + A'_1$. The cohomology class of $\partial \widetilde{K}$ in $H^1(F_{a'})$ is the image of $j_*\alpha - \alpha$ by the parallel translation $q_* : H^1(F_a) \to H^1(F_{a'})$.

Now, we have an oriented simple closed curve $\Gamma \subset F_a$ representing the cohomology class $j_*\alpha - \alpha$, since $j_*\alpha - \alpha$ is primitive in H. By $\Gamma_\tau \subset F_{p(\tau)}$ we denote the image of Γ by the parallel translation along the path $p : [0, 1] \to C - \Sigma$ with $p(0) = a$, $p(1) = a'$. For every $0 \leq \tau \leq 1$ we choose such a Γ_τ that it has no intersection with C_5. Set

$$\widetilde{P} = \bigcup_{0 \leq \tau \leq 1} \Gamma_\tau.$$

The boundary satisfies $\partial \widetilde{P} = -\Gamma + \Gamma_1$.

Here the cohomology class of $-A + A_1 - \Gamma$ is zero, and the support of this cycle does not pass through the intersection point a_0 of F_a and C_5. Since $H_1(F_a) \cong H_1(F_a - \{a_0\})$, we have a 2-chain Θ in the punctured Riemann surface $F_a - \{a_0\}$ such that $\partial \Theta = A - A_1 + \Gamma$.

Consider the image $\Gamma' \subset F_a$ of $\Gamma_1 \subset F_{a'}$ by the parallel translation along the inverse of q. The homology class of Γ' coincides with the homology class of Γ applied by the monodromy r_* around c_1, and thus it is $-(j_*\alpha - \alpha)$. It follows from this that the homology class of $-A' + A_1' + \Gamma_1$ is zero in $F_{a'}$. There exists a 2-chain Θ' in $F_{a'}$ with $\partial\Theta' = A' - A_1' - \Gamma_1$ such that the support of Θ' does not pass through the intersection point a_0' of $F_{a'}$ and C_5.

The chain of sum $\Xi = \widetilde{J}+\widetilde{K}+\widetilde{P}+\Theta+\Theta'$ is a 2-cycle, and defines the homology class $[\Xi] \in H_2(Z, \mathbf{Z})$. Let $\xi \in H^2(Z, \mathbf{Z})$ be the Poincare dual class of $[\Xi]$. By construction ξ satisfies the condition (2) obviously.

In order to see the condition (3) we need several constructions. First let $\Gamma_\tau^* \subset F_{q(\tau)}$ be the image of Γ by the parallel translation along q. Adjusting the parallel translation along p, we can assume that Γ_1^* coincides with Γ_1 except that the orientation is opposite. Secondly we choose a smooth map $T : [0,1] \times [0,1] \to C - \Sigma$ such that $T(\tau, 0) = j(\tau)$, $T(\tau,1) = k(\tau)$, and $T(0,\sigma) = T(1,\sigma) = q(\sigma)$ for $\tau, \sigma \in [0,1]$. Denoting $\Phi^{-1}(T(\tau,\sigma)) = F_{\tau,\sigma}$, we have the pallarel translation $F_a = F_{0,0} \to F_{\tau,\sigma}$ associated with T. By $A_{\tau,\sigma}$ and by $\Gamma_{\tau,\sigma}$ we denote the image of A and Γ by the translation associated with T respectively. We can assume $A_\tau = A_{\tau,0}$, and $A_\tau' = A_{\tau,1}$. Setting $\widetilde{Q} = -\bigcup_{0\leq\sigma\leq 1} A_{0,\sigma} + \bigcup_{0\leq\sigma\leq 1} A_{1,\sigma}$, and $\overline{Q} = \bigcup_{0\leq\sigma\leq 1} \Gamma_\sigma^*$, we divide Ξ into three parts. Set $\Xi_1 = \widetilde{J}+\widetilde{K}+\widetilde{Q}$, $\Xi_2 = -\widetilde{Q}+\overline{Q}+\Theta+\Theta'$, and $\Xi_3 = \widetilde{P}-\overline{Q}$. Obviously $\Xi = \Xi_1+\Xi_2+\Xi_3$. However, Ξ_i $(i = 1,2,3)$ is not a cycle if we use $\mathbf{Z}$ as the coefficients.

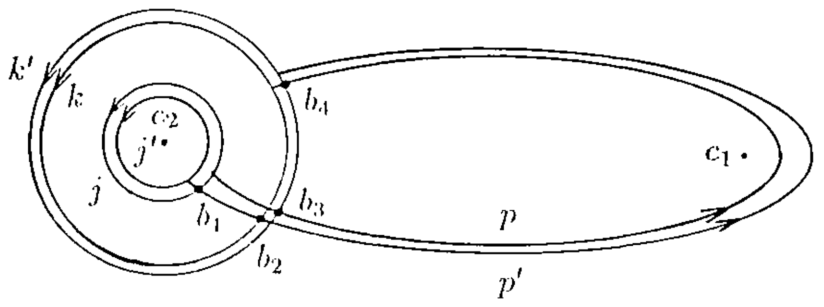
Here let us use $\mathbf{Z}/2$-coefficients. We consider homology groups over $\mathbf{Z}/2$. Then Ξ_i $(i = 1,2,3)$ are cycles. Besides, $\partial\Pi = \Xi_1$ for a 3-chain $\Pi = \bigcup_{0\leq\tau\leq 1, 0\leq\sigma\leq 1} A_{\tau,\sigma}$. Let Θ_σ be a continuous family of 2-chains such that $\partial\Theta_\sigma = A_{0,\sigma} - A_{1,\sigma} + \Gamma_\sigma^*$ as chains in $F_{q(\sigma)}$, and$\Theta_0 = \Theta$, $\Theta_1 = \Theta'$. Then we have $\partial\widetilde{\Theta} = \Xi_2$ for the 3-chain $\widetilde{\Theta} = \bigcup_{0\leq\sigma\leq 1}\Theta_\sigma$. Consequently one knows that $[\Xi] = [\Xi_3]$ in $H_2(Z, \mathbf{Z}/2)$.

Now, let $U \subset C$ be a contractible neighbourhood of the point c_1 containing the path $r = q^{-1}p$ and not containing any point in Σ except c_1. The class $[\Xi_3] \in H_2(\Phi^{-1}(U), \mathbf{Z}/2)$ is defined. Let $C_0, \ldots, C_4$ be the irreducible components of the central singular fiber F_1. We assume that C_0 has multiplicity 2, and C_4 intersects C_5. By construction the intersection $[\Xi_3]\cdot[C_i] = 0$ for $0 \leq i \leq 4$.

On the other hand, since $H_2(\Phi^{-1}(U), \mathbf{Z}/2) \cong H_2(F_1, \mathbf{Z}/2) = \sum_{i=0}^4 \mathbf{Z}/2[C_i]$, we can express $[\Xi_3]$ as the linear combination of $[C_i]$'s. Setting $u_0 = 2[C_0] + \sum_{i=1}^4 [C_i]$, we have elements ϵ, $\delta_0, \ldots, \delta_3 \in \mathbf{Z}/2$ with $[\Xi_3] = \epsilon u_0 + \sum_{i=0}^3 \delta_i[C_i]$. Then, $\epsilon = \epsilon u_0 \cdot [C_5] = [\Xi_3]\cdot[C_5] = [\Xi]\cdot[C_5] = 0$. $\delta_0 = [\Xi_3]\cdot[C_1] = 0$, and $\delta_1 + \delta_2 + \delta_3 = [\Xi_3]\cdot[C_0] = 0$. It implies that either for some two C', C'' of C_1, C_2, C_3, $[\Xi] = [\Xi_3] = [C'] + [C'']$, or $[\Xi] = [\Xi_3] = 0$. If the first case takes place, then by the universal coefficient theorem $H^2(Z, \mathbf{Z}/2) = H^2(Z, \mathbf{Z}) \otimes \mathbf{Z}/2$, we have the condition (3). If the second case takes place, we can write $\xi = 2\eta$ for some $\eta \in H^2(Z, \mathbf{Z})$. Here we assume the condition (1) $[\Xi]^2 = 4$. Then we have $\eta^2 = 1$, which contradicts that $H^2(Z, \mathbf{Z})$ is an even lattice. In the following proof of the condition (1) we do not use the condition (3). Thus we can complete the proof of condition (3).

As for the condition (4), it follows from the condition (2) as in Proposition 1.2.

The condition (1) is remaining. To compute the self-intersection number we consider the small perturbation j', k', p' of the paths j, k, p as in the following figure.



Four intersection points occur. Let b_1 be the intersection point of p' and j, b_2 be that of p' and k, and b_3, b_4 be those of k' and p. We assume that b_3 is nearer to d than b_4. Let Ξ' be the 2-cycle associated with j', k' and p' which is constructed similarly to the case for Ξ. We can check that Ξ and Ξ' are homologous. Thus the self-intersection number ξ^2 is equal to the intersection number of Ξ and Ξ'.

The intersection points of Ξ and Ξ' are contained in $\bigcup_{\nu=1}^{4} F_{b_\nu}$. After computing the local intersection number in the neighbourhood of F_{b_ν}, we can take the sum.

First we consider the neighbourhood of F_{b_1}. We assume that $j(\tau_0) = b_1$ and $p'(\tau_1) = b_1$. Let B_1 be a sufficiently small meighbourhood of b_1 in C. The inverse image $\Phi^{-1}(B_1)$ can be identified with the product $B_1 \times F_{b_1}$. Let p'_B be the part in B_1 of the path p'. and j_B be the part in B_1 of j. Ξ can be identified locally with $j_B \times A_{\tau_0}$, while Ξ' can be identified with $p'_B \times \Gamma'_{\tau_1}$. Under the identification by the parallel translation along j $\tau_0 \leq \tau \leq 1$, the homology class of A_{τ_0} coincides with $j_*\alpha$, while that of Γ'_{τ_1} is $j_*\alpha - \alpha$. Thus

$$\mathrm{int}(A_{\tau_0},\ \Gamma'_{\tau_1}) = j_*\alpha \cdot (j_*\alpha - \alpha) = 1.$$

Here we denote the local intersection number of X and Y by $\mathrm{int}(X,\ Y)$. (Note that the order of two 1-cycles and the sign.) In $\Phi^{-1}(B_1)$, we have

$$\begin{aligned}\mathrm{int}(\Xi,\ \Xi') &= \mathrm{int}(j_B \times A_{\tau_0},\ p'_B \times \Gamma'_{\tau_1}) \\ &= -\mathrm{int}(j_B,\ p'_B)\,\mathrm{int}(A_{\tau_0},\ \Gamma'_{\tau_1}) \\ &= -(-1) \times 1 \\ &= +1.\end{aligned}$$

Similarly one knows that for each ν ($\nu = 1, 2, 3, 4$), Ξ and Ξ' has the local intersection number $+1$ in the neighbourhood of F_{b_ν}.

Therefore $\xi^2 = +4$. Q.E.D.

We apply Proposition 2.1 and show an nice isotropic element.

First we consider the case for $J_{3,0}$. Let $G \in PC(J_{2,0})$. Obviously under our assumption in this section G contains a component of type E. Let $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ be the corresponding lattice embedding. By Proposition 2.1 we have an element $\xi \in \Lambda_3$ orthogonal to S such that $\xi^2 = -4$. (Note that when we move from $H^2(Z,\ \mathbf{Z})$ to Λ_3, we reverse the sign of the bilinear form.) The induced embedding $Q(G) \hookrightarrow \Lambda_3/P$ is full, and the image is contained in the orthogonal complement L of $\mathbf{Z}\xi$ in Λ_3/P. L has signature $(14,1)$ and thus we can define the Coxeter-Vinberg graph for L. Since $Q(G)$ is full even in L, G is a subgraph of the Coxeter-Vinberg graph for L.

Therefore we would like to draw the Coxeter-Vinberg graph for L.

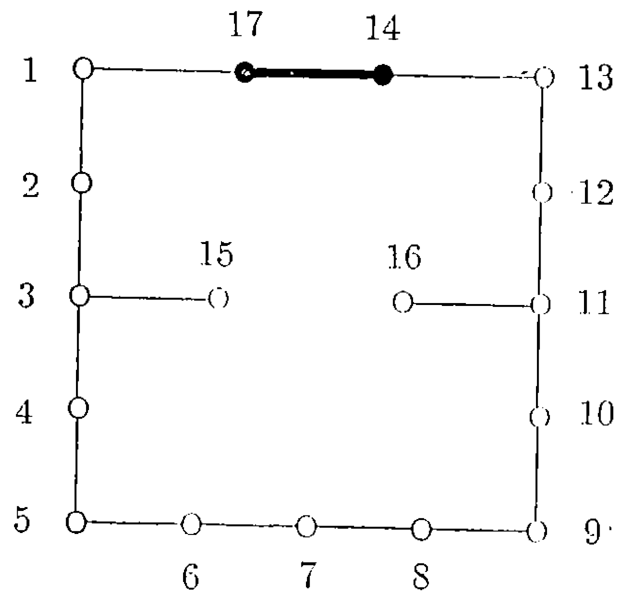
Recall that the dual module $\mathrm{Hom}(V, \mathbf{Z})$ of a $\mathbf{Z}$-module V is denoted by V^*. If V is a non-degenerate lattice, we can regard $V \subset V^* = \{x \in V \otimes \mathbf{Q} \mid \text{For every } y \in V,\ x \cdot y \in \mathbf{Z}\} \subset V \otimes \mathbf{Q}$.

Set $R = P \oplus \mathbf{Z}\xi$. The discriminant group of R is $R^*/R \cong (\mathbf{Z}/2 + \mathbf{Z}/2) \oplus \mathbf{Z}/4$. For $\overline{\alpha} = (a_1, a_2, b) \in R^*/R$, the discriminant quadratic form can be written $q_R(\overline{\alpha}) \equiv a_1^2 + a_1a_2 + a_2^2 - (b^2/4) \bmod 2\mathbf{Z}$. Thus $q_R \equiv 0 \Leftrightarrow \overline{\alpha} = (0,0,0), (1,0,2), (0,1,2)$, or $(1,1,2)$. One knows in particular that any subgroup in R^*/R consisting of elements $\overline{\alpha}$ with $q_R(\overline{\alpha}) \equiv 0$ has order≤ 2. It implies $[\widetilde{R} : R] \leq 2$ for the primitive hull $\widetilde{R}$ of R in Λ_3. On the other hand the condition (3) in Proposition 2.1 implies that $[\widetilde{R} : R]$ is a multiple of 2. Consequently one has $[\widetilde{R} : R] = 2$. After some calculation one has $\widetilde{R}^*/\widetilde{R} \cong \mathbf{Z}/4$ and $q_{\widetilde{R}}(c) \equiv 3c^2/4 \bmod 2\mathbf{Z}$.

Let M be the orthogonal complement of R in Λ_3. $M^*/M \cong \widetilde{R}^*/\widetilde{R} \cong \mathbf{Z}/4$.

Next, let K be the orthogonal complement of $\mathbf{Z}\xi$ in Λ_3. The group K^*/K has order 4. The quotient K/P can be identified with L, and we can regard M as its subgroup with finite index. Since $M \oplus P \subset K$ and since the group $(M \oplus P)^*/(M \oplus P)$ has order 16, one has $[K/P : M] = [K : M \oplus P] = 2$.

Consequently $L = K/P$ is a unimodular lattice of signature $(14,1)$. It is known that such a lattice is unique up to isomorphisms (Milnor-Husemoller [2]), and we can find its Coxeter-Vinberg graph in Vinberg [8], which is as in the following figure.



The Coxeter-Vinberg graph for L.

By γ_i we denote the fundamental root in L associated with the vertex in the above graph with the attached number i.

Proposition 2.2. *Consider the case for $J_{2,0}$. Let $G \in PC(J_{2,0})$, and G_1 be an arbitrary component of G. Assume that the corresponding elliptic K3 surface $\Phi : Z \to C$ has a singular fiber of type II^*, III^* or IV^*. We regard $Q(G)$ as a submodule of Λ_3/P*

by the corresponding embedding. Then, there exists an isotropic element u in Λ_3/P satisfying either the following condition (1) or (2).

(1) u is orthogonal to $Q(G)$.

(2) For some root basis $\Delta \subset Q(G)$ there is a long root $\alpha \in \Delta - \Delta_1$ such that $u \cdot \alpha = 1$ and $u \cdot \beta = 0$ for every $\beta \in \Delta$ with $\beta \neq \alpha$, where Δ_1 denotes the component of Δ corresponding to G_1.

Remark. If u satisfies the above condition (2), then u is automatically primitive in Λ_3/P.

Proof. We can assume that the root basis $\Delta \subset Q(G)$ is contained in the set $\{\gamma_i \mid 1 \leq i \leq 17\}$ of the fundamental roots of L. Obviously $\gamma_{14}, \gamma_{17} \notin \Delta$, since they are short roots. Moreover either γ_6, γ_7 or γ_8 does not belong to Δ, since G contains a subgraph of type E. Thus one can conclude that either $\gamma_5 \notin \Delta_1$ or $\gamma_9 \notin \Delta_1$, since the graph corresponding to Δ_1 is connected.

First we consider the case where $\gamma_5 \notin \Delta_1$. Set $u_1 = -(2\gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_{15} + 2\gamma_{17})$. We can check that u_1 is an isotropic element in $L \subset \Lambda_3/P$. Moreover,

$$u \cdot \gamma_i = 0 \text{ for } 1 \leq i \leq 17,\ i \neq 5,\ 14$$
$$u \cdot \gamma_5 = u \cdot \gamma_{14} = 1$$

Set $\alpha = \gamma_5$. If $\gamma_5 \notin \Delta$, then u_1 satisfies the above (1), while if $\gamma_5 \in \Delta$, then it satisfies (2).

The second case $\gamma_9 \notin \Delta_1$ is similar. By the symmetry of the graph it is obvious that the element $u_2 = -(\gamma_{10} + 2\gamma_{11} + 2\gamma_{12} + 2\gamma_{13} + 2\gamma_{14} + \gamma_{16})$ satisfies conditions.

Q.E.D.

We proceed to the case of $Z_{1,0}$ and $Q_{2,0}$. By X we denote either $Z_{1,0}$ or $Q_{2,0}$. Let $G \in PC(X)$. We consider the corresponding elliptic K3 surface $Z \to C$. In this case there is a unique singular fiber containing a component of IF apart from F_1. We assume that a singular fiber F_2 contains a component of IF.

We would like to reduce our case to the case of $J_{3,0}$. Now, let $\overline{IF}$ denote the union of F_1 and $C_5 = s_0(C)$. $\overline{IF}$ is a union of some components of IF, and is same as IF in the case of $J_{3,0}$. Let $\mathcal{E}$ (resp. $\overline{\mathcal{E}}$) denote the union of irreducible components not intersecting IF (resp. $\overline{IF}$) of singular fibers F_i, $2 \leq i \leq t$. One has

$$\mathcal{E} \cap (\bigcup_{i=3}^{t} F_i) = \overline{\mathcal{E}} \cap (\bigcup_{i=3}^{t} F_i) \quad \text{and} \quad (\mathcal{E} \cap F_2) \cup T \subset (\overline{\mathcal{E}} \cap F_2) \cup B = F_2.$$

Here B denotes the component of F_2 intersecting C_5, and T denotes the union of components of IF not contained in $\overline{IF}$. Note in particular that B consists of a unique component, and that it has multiplicity 1 as a component of F_2.

Assume that for some component E_1 of $\mathcal{E}$ contained in $\bigcup_{i=3}^{t} F_i$ the cohomology class $U \in H^2(Z)$ satisfies the following.

$[E_1] \cdot U = -1$, $[C_i] \cdot U = 0$ for every component C_i of $\overline{IF}$, and $[E] \cdot U = 0$ for every component E of $\overline{\mathcal{E}}$ with $E \neq E_1$.

Then U is orthogonal to the general fiber of Φ and orthogonal to all components of F_2 except possibly one component B. However, U is orthogonal also to B since the class of B can be written as an integral linear combinations of the class of the general fiber and the classes of other components of F_2. Thus one can conclude that U is orthogonal to all components of IF and to all components of $\mathcal{E}$ with a unique exception E_1.

By the same reason if U is orthogonal to all components of $\overline{IF}$ and to all components of $\overline{\mathcal{E}}$, then it is orthogonal to all components of IF and to all components of $\mathcal{E}$.

By translating this fact to the lattice theory we can get the proof.

By $\overline{P}$ we denote the sublattice of P generated by the part of the basis corresponding to the components in $\overline{IF}$. (Recall that P has a basis which has one-to-one correspondence with the components of IF.) $\operatorname{rank}\overline{P} = 6$ and $\overline{P} = P_0' \oplus H_0$ in the notation in Part I. Moreover $\overline{P}$ is isomorphic to P in the case of $J_{3,0}$. Let G (resp. $\overline{G}$) be the dual graph of $\mathcal{E}$ (resp. $\overline{\mathcal{E}}$), and G_1 (resp. $\overline{G}_1$) be the sub-dual-graph of G (resp. $\overline{G}$) corresponding to $\mathcal{E} \cap F_2$ (resp. $\overline{\mathcal{E}} \cap F_2$). $\overline{G}_1$ has a unique component and $G - G_1 = \overline{G} - \overline{G}_1 = G_0$. By the above we have $\overline{P} \oplus Q(\overline{G}_1) \supset P \oplus Q(G_1)$. The lattice $\overline{P} \oplus Q(\overline{G}_1) \oplus Q(G_0)$ has the embedding into Λ_3 comming from the geometric situation. By Proposition 2.2 one has an isotropic element $u \in \Lambda_3/\overline{P}$ satisfying either (1) or (2) in Proposition 2.2 for the pair of graphs $\overline{G}$ and $\overline{G}_1$. u is orthogonal to $Q(\overline{G}_1)$ in any case. The orthogonal complement of $\overline{P}$ in Λ_3 contains an isotropic element $\widetilde{u}$ which corresponds to u under the quotient map $\Lambda_3 \to \Lambda_3/\overline{P}$. $\widetilde{u}$ is orthogonal to $P \oplus Q(G_1)$. The image of $\widetilde{u}$ by the quotient map $\Lambda_3 \to \Lambda_3/P$ is an isotropic element in a nice position.

We have shown Proposition 0.1 for $X = J_{3,0}$, $Z_{1,0}$ or $Q_{2,0}$ under the assumption ((2)) in the introduction.

§3. Combinations of graphs of type A

Let G be a graph in $PC(X)$ with the number of vertices r. We assume moreover in this section that all singular fibers of the corresponding elliptic K3 surface $\Phi : Z \to C$ are of type I, II, III or IV except the unique exception F_1 of type I_0^*. Every component of G is of type A under this assumption.

By M_p we denote the p-Sylow subgroup of an abelian group M, and $l(M)$ denotes the minimum number of generators of M.

We begin with the case $X = J_{3,0}$. Let A_{n_i} be the dual graph of the components not intersecting C_5 of the singular fiber F_i for $2 \le i \le t$. (Here A_0 stands for the empty graph $\emptyset$.) We have $G = \sum_{i=2}^{t} A_{n_i}$.

Proposition 3.1. *We consider only the case $X = J_{3,0}$ under the assumption as above.*
(1) If $\rho = 6 + r$, then the group E of sections is finite.
(2) $\nu(G) \le 18 - r$, where $\nu(G)$ denotes the number of components of G.
(3) $r \le 13$.
(4) Set $N(p) = \{i \mid 2 \le i \le t,\ n_i + 1 \equiv 0 \pmod p\}$. If the corresponding embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$ is not primitive, then there is a subset $T \subset N(2)$ such that $\sum_{i \in T}(n_i + 1) = 12$.
(5) For any odd prime number p $N(p)$ contains at most $15 - r$ elements.

Proof. (1) Since $r = \sum_{i=2}^{t} n_i$, we have $a + \sum_{i=2}^{t}(m(F_i) - n_i - 1) = 0$ by the equality (2) at the beginning of section 1. By definition $m(F_i) > n_i$, and thus $a = 0$, which implies E is finite.

(2) Without loss of generality we can assume $\rho = 6 + r$ by Theorem 1.2 in Part I [7]. In the equality (3) at the beginning of section 1, we can substitute $a = 0$, $t - t_1 = 1 + \nu(II) + \nu(III) + \nu(IV)$, $t_1 = \nu(G) - \nu(III) - \nu(IV) + \nu(I_1)$. Thus $\nu(G) = 18 - r - \{2\nu(II) + \nu(III) + \nu(IV) + \nu(I_1)\}$.
(3) If $r > 13$, then $r = 14$ since $20 \geq \rho \geq 6 + r$. Assuming $r = 14$, we will deduce a contradiction.

First then we have $\rho = 20 = 6 + r$. Thus $a = 0$ by (1). If the functional invariant J is not constant, then we have

$$0 = 20 - \rho + a \geq \nu(I_0^*) + \nu(II) + \nu(III) + \nu(IV) \geq \nu(I_0^*) = 1,$$

by the inequality (4) in section 1, which is a contradiction. Thus J is constant and all singular fibers are of type II, III, or IV. This implies that all components of G are of type A_1 or A_2, and thus $\nu(G) \geq 7$ since G has 14 vertices. On the other hand by (2) we have $\nu(G) \leq 4$, which is a contradiction.
(4) Let S be the subgroup of $\mathrm{Pic}(Z)$ generated by the class of C_5 and the classes of all components of singular fibers. Since in our case under the isomorphism $H^2(Z, \mathbf{Z}) \xrightarrow{\sim} \Lambda_3$ S corresponds exactly to $P \oplus Q(G)$, the assumption in (4) implies $\widetilde{S}/S \neq 0$ for the primitive hull $\widetilde{S}$ of S in $H^2(Z, \mathbf{Z})$.

On the other hand by Lemma 1.4 the quotient $\mathrm{Pic}(Z)/S$ is isomorphic to the group E of sections. Here $\widetilde{S} \subset \mathrm{Pic}(Z)$ since $\mathrm{Pic}(Z)$ is always primitive in $H^2(Z, \mathbf{Z})$, and one knows that $\widetilde{S}/S$ is isomorphic to the subgroup $\mathrm{Tor}\, E$ of E consisting of all elements with finite order. Thus we have a section $s' : C \to Z$ such that $s' \neq s_0$ and $s' \in \mathrm{Tor}\, E$. By C' we denote the image of s'. $[C'] \in \widetilde{S}$.

Now, let S_i denote the subgroup of S generated by the classes of components not intersecting IF of the singular fiber F_i. We have $S = (\mathbf{Z}[F] + \mathbf{Z}[F + C_5]) \oplus \bigoplus_{i=1}^{t} S_i$, where F denotes a general fiber of Φ. Thus we can write

$$[C'] = m[F] + [F + C_5] + \sum_{i=1}^{t} \chi_i$$

for some $m \in \mathbf{Z}$, $\chi_i \in S_i^*$.

Here we recall general facts on elliptic surfaces, which may be to be added in the beginning of section 1. (Kodaira [1], Shioda [4])

- Let $Z^\#$ be the set of points on Z at which the Jacobian matrix of Φ has rank 1. $Z^\# \to C$ has the structure of a group variety over C. In particular for every point $a \in C$ $F_a^\# = \Phi^{-1}(a) \cap Z^\#$ has the induced structure of a complex Lie group. This group structure depends only on F_a. ($F_a^\#$ is the set of simple points of the fiber $F_a = \Phi^{-1}(a)$.)
- With respect to the induced group homomorphism $E \to F_a^\#$, the induced homomorphism $\mathrm{Tor}\, E \to \mathrm{Tor}\, F_a^\#$ is injective for every point $a \in C$. Here by $\mathrm{Tor}\, M$ we denote the subgroup of an abelian group M consisting of all elements of finite order.

We consider χ_1. C' intersects a unique component of F_1 with multiplicity 1, and the component intersecting C' do not intersect C_5, since $\mathrm{Tor}\, E \to \mathrm{Tor}\, F_1^\#$ is injective and every component of F_1 contains at most one point in $\mathrm{Tor}\, F_1^\#$. It implies $\chi_1 \neq 0$, and under the isomorphism $S_1^* \cong Q(D_4)^*$ χ_1 corresponds to a fundamental weight associated with a vertex of the Dynkin graph D_4 with only one edge. Consequently one knows $\chi_1^2 = -1$. By injectivity one knows moreover that s' has order 2 in E.

Next we consider χ_i for $2 \leq i \leq t$. Assume $\chi_i \neq 0$. Since $\mathrm{Tor}\, E \to \mathrm{Tor}\, F_i^{\#}$ is injective, $F_i^{\#}$ contains a point with order 2 which is not on the component intersecting C_5. It implies F_i is of type either III or I_{2k} for some k. We have $n_i + 1 \equiv 0 \pmod 2$. The injectivity also implies that under the isomorphism $S_i^* \cong Q(A_{n_i})^*$ χ_i corresponds to the fundamental weight associated with the central vertex of the Dynkin graph A_{n_i}. In particular one has $\chi_i^2 = -(n_i+1)/4$.

We calculate m. By injectivity for $\mathrm{Tor}\, E \to \mathrm{Tor}\, F_a^{\#}$ $a \in C$, C' and C_5 have no intersection. Thus

$$0 = [C'] \cdot [C_5] = m[F] \cdot [C_5] + [F + C_5] \cdot [C_5] = m - 1.$$

We have $m = 1$.

Set $T = \{i \mid 2 \leq i \leq t, \chi_i \neq 0\}$. By the above one has $T \subset N(2)$, and

$$-2 = [C']^2 = ([F] + [F + C_5])^2 + \chi_1^2 + \sum_{i=2}^{t} \chi_i^2 = 2 - 1 - \sum_{i \in T} \frac{(n_i + 1)}{4},$$

which implies the equality in (4).

(5) Assuming that for some odd prime p $N(p)$ contains a set U with $16 - r$ elements, we deduce a contradiction.

For $M = \{i \mid 2 \leq i \leq t,\ n_i \neq 0\}$ we have $M \supset U$. First we would like to show that $M \neq U$. Indeed, if $N(p)$ contains $17 - r$ or more elements, then $M \supset N(p) \neq U$. Thus we can consider only the case where $M = N(p)$ and M has just $16 - r$ elements. Now, since $r = \sum_{i \in M} n_i$, we have $16 = r + (16 - r) = \sum_{i \in M}(n_i + 1) \equiv 0 \pmod p$, which contradicts that p is an odd prime.

Choose an element $e \in M - U$. The singular fiber F_e over $c_e \in \Sigma$ is either of type I_m for $m = n_e + 1$, or of type III, or of type IV, since $n_e > 0$.

Next we consider the homotopy theory.

Fixing a base point $b \in C - \Sigma$, we draw a path l_i connecting b and a point $c_i \in \Sigma$ for $1 \leq i \leq t$ as in the beginning part of section 1. Here by exchanging the numbering we assume moreover that when we go on a small circle with center b in the positive direction, we encounter l_i's in the order of the attached number i. Associated with l_i, we define the closed path r_i as in section 1. The homotopy classes $[r_i]$ of r_i are generators of $\pi_1(C - \Sigma, b)$ and are subject to a unique relation $[r_1][r_2]\cdots[r_t] = 1$.

Let $J : C - \Sigma_1 \to \mathbf{P}^1 - \{\infty\}$ be the functional invariant of the elliptic surface Φ where we denote $\Sigma_1 = \Sigma - \{c_1\}$. By $j : \mathcal{H} \to \mathbf{P}^1 - \{\infty\}$ we denote the j-function from the upper half plane $\mathcal{H} = \{z \in \mathbf{C} \mid \mathrm{Im}\, z > 0\}$. The multivalued function $j^{-1}J$ defines the monodromy representation $\overline{\chi} : \pi_1(C - \Sigma_1, b) \to \mathrm{PSL}(2, \mathbf{Z}) = \mathrm{SL}(2, \mathbf{Z})/\{+1, -1\}$. Let $f : \mathrm{SL}(2, \mathbf{Z}) \to \mathrm{PSL}(2, \mathbf{Z})$ denote the canonical surjective homomorphism, and Σ_2 be an arbitrary finite set with $\Sigma_1 \subset \Sigma_2 \subset C - \{b\}$.

Lemma 3.2. (Kodaira [1]) *The following two sets have one-to-one correspondence.*

(1) The set of an isomorphism class of an elliptic surface $W \to C$ over C with a section $s_0 : C \to W$ whose critical values are contained in Σ_2 and whose functional invariant coincides with J.

(2) The set of a representation $\chi : \pi_1(C - \Sigma_2, b) \to \mathrm{SL}(2, \mathbf{Z})$ such that the composition $f\chi$ coincides with the composition $\pi_1(C - \Sigma_2, b) \to \pi_1(C - \Sigma_1, b) \xrightarrow{\overline{\chi}} \mathrm{PSL}(2, \mathbf{Z})$.

The correspondence is given by associating the elliptic surface $W \to C$ with the monodromy representation on the first cohomology group $H^1(F_b, \mathbf{Z})$ of the fiber over b.

Lemma 3.3 (Kodaira [1]) *An elliptic surface $W \to D$ is a K3 surface if and only if $D \cong \mathbf{P}^1$ and the sum of the Euler numbers of singular fibers is equal to 24.*

Let $\chi_1 : \pi_1(C - \Sigma, b) \to \mathrm{SL}(2, \mathbf{Z})$ denote the representation associated with our elliptic K3 surface $Z \to C$. We have $\chi_1([r_1]) = -1$ and $\chi_1([r_1]) \cdot \chi_1([r_2]) \cdots \chi_1([r_t]) = 1$. We can construct another representation χ_2 by setting $\chi_2([r_1]) = 1$, $\chi_2([r_e]) = -\chi_1([r_e])$ and $\chi_2([r_i]) = \chi_1([r_i])$ for $1 \le i \le t$ with $i \neq 1, e$. Since -1 commutes with any element $\chi_2([r_1]) \cdot \chi_2([r_2]) \cdots \chi_2([r_t]) = 1$ and it defines a representation of $\pi_1(C - \Sigma, \mathbf{Z})$ such that $f\chi_2 = f\chi_1 = \overline{\chi}$.

Let $W \to C$ be the corresponding elliptic surface to χ_2. By Kodaira [1] the type of a singular fiber is uniquely determined by the $\mathrm{SL}(2, \mathbf{Z})$-conjugacy class of the monodromy matrix around it. Thus the fibers over c_i with $1 \le i \le t$, $i \neq 1$, e are same as those of $Z \to C$. However, the fiber over c_1 is smooth and the fiber over c_e is of type I_m^*, III^* or II^*, according as that in Z is of type I_m, III or IV. The combination $I_0^* + I_m$ has been replaced by $I_0 + I_m^*$ in the first case. Here note that for the both pairs of singular fibers the sum of the Euler numbers is $m + 6$, and they are equal. Thus by Lemma 3.3 one can conclude that W is also a K3 surface. In the second case $I_0^* + III$ has been replaced by $I_0 + III^*$. Also in this case for the both pairs the sum of the Euler numbers is 9. In the third case $I_0^* + IV$ has been replaced by $I_0 + II^*$. For the both pairs the sum of them is 10. By Lemma 3.3 W is a K3 surface even in these cases.

Next we compare Dynkin graphs. Let G_W be the dual graph associated with the set of all components of singular fibers in W not intersecting the image $s_0(C)$ of the section s_0. By construction we have $G_W = G - A_{m-1} + D_{m+4}$ in the first case. Thus G_W has $r + 5$ vertices. In the second, third case we have $G_W = G - A_1 + E_7$ and $G_W = G - A_2 + E_8$ respectively. Thus G_W has $r + 6$ vertices.

Let $S_W \subset \mathrm{Pic}(W)$ be the subgroup generated by the classes of $s_0(C)$ and all the components of singular fibers of $W \to C$. Setting $G_0 = D_{m+4}$, E_7 or E_8 according as the first, second or third case takes place, we have

$$S_W \cong H_0 \oplus Q(-G_W) \cong H_0 \oplus Q(-G_0) \oplus \bigoplus_{\substack{i=2 \\ i \neq e}}^{t} Q(-A_{n_i}),$$

where $Q(-X)$ denotes the negative definite root lattice associated with a Dynkin graph X (The bilinear form on $Q(-X)$ is (-1) times that on $Q(X)$.), and H_0 denotes a hyperbolic plane. $\mathrm{rank}\, S_W \ge r + 7$. Note that the discriminant group S_W^*/S_W has at least $(16 - r)$ of p-torsions corresponding to U.

Let $\widetilde{S}_W$ be the primitive hull of S_W in $H^2(W, \mathbf{Z})$. The quotient $\widetilde{S}_W/S_W$ is isomorphic to the group of all sections of $W \to C$ with finite order, and it is isomorphic to a subgroup in the group of the singular fiber over c_e. Since the fiber over c_e is either I_m^*, III^* or II^*, $\widetilde{S}/S$ is a 2-primary group. Thus for the p-Sylow subgroup we have $(\widetilde{S}_W/S_W)_p = 0$, since p is odd. We have $l((\widetilde{S}_W^*/\widetilde{S}_W)_p) = l((S_W^*/S_W)_p) \geq 16 - r$.

Let T_W be the orthogonal complement of S_W in $H^2(W, \mathbf{Z})$ which has rank 22. $\operatorname{rank} T_W \leq 22 - (r+7) = 15 - r$. Thus $l((T_W^*/T_W)_p) \leq l(T_W^*/T_W) \leq 15 - r$. On the other hand, since $\widetilde{S}_W^*/\widetilde{S}_W \cong T_W^*/T_W$, we have $l((T_W^*/T_W)_p) \geq 16 - r$, which is a contradiction. Q.E.D.

We have a byproduct of the proof of (4).

Lemma 3.4. *Assume that $G = \sum_{i\in I} A_{k_i} \in PC(J_{3,0})$. Then for any embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying Looijengas conditions $\langle a\rangle$, $\langle b\rangle$ the followings hold.*
(1) For the primitive hull $\widetilde{S}$ every non-zero element in the quotient $\widetilde{S}/S$ has order 2.
(2) Any non-zero element $\overline{\alpha}$ in $\widetilde{S}/S$ can be written

$$\overline{\alpha} = \overline{\chi}_0 + \sum_{i\in T} \overline{\chi}_i,$$

for some subset $T \subset I$, where $0 \neq \overline{\chi}_0 \in P^*/P \cong Q(D_4)^*/Q(D_4)$, $\overline{\chi}_i \in Q(A_{k_i})^*/Q(A_{k_i})$ has order 2, $k_i + 1 \equiv 0 \pmod 2$ for $i \in T$ and $\sum_{i\in T}(k_i+1) = 12$.
(3) If $\widetilde{S}/S$ is not cyclic, then there are subsets $T_1, T_2 \subset N(2) = \{i \in I \mid k_i + 1 \equiv 0 \pmod 2\}$ such that $\sum_{i\in T_\nu}(k_i+1) = 12$ for $\nu = 1$ and 2, and $\sum_{i\in T_1\cap T_2}(k_i+1) = 6$.

Let us consider the case $r = 13$ further. By Proposition 3.1 (2) we can consider only the graph $G = A_{k_1} + A_{k_2} + A_{k_3} + A_{k_4} + A_{k_5}$ corresponding to the division $k_1 + k_2 + k_3 + k_4 + k_5 = 13$ of 13 into a sum of 5 non-negative integers $k_1 \geq k_2 \geq k_3 \geq k_4 \geq k_5 \geq 0$. There are 57 kinds of such divisions as follows. We omit 0.

(1) 13	(2) 12+1	(3) 11+2	(4) 10+3
(5) 9+4	(6) 8+5	(7) 7+6	(8) 11+1+1
(9) 10+2+1	(10) 9+3+1	(11) 9+2+2	(12) 8+4+1
(13) 8+3+2	(14) 7+5+1	(15) 7+4+2	(16) 7+3+3
(17) 6+6+1	(18) 6+5+2	(19) 6+4+3	(20) 5+5+3
(21) 5+4+4	(22) 10+1+1+1	(23) 9+2+1+1	(24) 8+3+1+1
(25) 8+2+2+1	(26) 7+4+1+1	(27) 7+3+2+1	(28) 7+2+2+2
(29) 6+5+1+1	(30) 6+4+2+1	(31) 6+3+3+1	(32) 6+3+2+2
(33) 5+5+2+1	(34) 5+4+3+1	(35) 5+4+2+2	(36) 5+3+3+2
(37) 4+4+4+1	(38) 4+4+3+2	(39) 4+3+3+3	(40) 9+1+1+1+1
(41) 8+2+1+1+1	(42) 7+3+1+1+1	(43) 7+2+2+1+1	(44) 6+4+1+1+1
(45) 6+3+2+1+1	(46) 6+2+2+2+1	(47) 5+5+1+1+1	(48) 5+4+2+1+1
(49) 5+3+3+1+1	(50) 5+3+2+2+1	(51) 5+2+2+2+2	(52) 4+4+3+1+1
(53) 4+4+2+2+1	(54) 4+3+3+2+1	(55) 4+3+2+2+2	(56) 3+3+3+3+1
(57) 3+3+3+2+2			

Note that in each item the number of odd numbers is 1, 3, or 5.

A. For the following 24 items we can make the corresponding graph $A_{k_1}+A_{k_2}+A_{k_3}+A_{k_4}+A_{k_5}$ from the essential Dynkin graph E_8+F_4 by tie transformations repeated twice. In the following table we also indicate an example of the Dynkin graph which we can make by the first tie transformation.

(1)$\leftarrow B_{13}$	(2)$\leftarrow B_{13}$	(3)$\leftarrow A_{11}+A_1$
(5)$\leftarrow B_{13}$	(6)$\leftarrow A_5+E_7$	(8)$\leftarrow A_{11}+A_1$
(9)$\leftarrow A_9+A_2+A_1$	(10)$\leftarrow A_9+A_2+A_1$	(11)$\leftarrow A_9+A_2+A_1$
(12)$\leftarrow A_8+A_4$	(17)$\leftarrow A_6+E_6$	(18)$\leftarrow A_6+E_6$
(20)$\leftarrow A_5+E_7$	(21)$\leftarrow A_4+E_8$	(23)$\leftarrow A_9+A_2+A_1$
(26)$\leftarrow A_7+D_5$	(27)$\leftarrow A_7+D_5$	(30)$\leftarrow A_4+E_8$
(32)$\leftarrow A_6+E_6$	(34)$\leftarrow A_5+E_7$	(43)$\leftarrow A_7+A_2+A_1+B_2$
(47)$\leftarrow A_5+D_5+B_2$	(49)$\leftarrow A_5+D_5+B_2$	(53)$\leftarrow A_4+E_6+B_2$

B. For the three items (40), (42), (56) the item contains 5 odd numbers and $N(2)$ do not contain subsets T_1, T_2 satisfying the condition in Lemma 3.4 (3). Thus the corresponding graph G is not a member of $PC = PC(J_{3,0})$.

Indeed, let T be the orthogonal complement of $S = P \oplus Q(G)$ in Λ_3 and $\widetilde{S}$ be the primitive hull of S. $\operatorname{rank} T = 3$. Since $\widetilde{S}^*/\widetilde{S} \cong T^*/T$, $l(S^*/S) - 2l(\widetilde{S}/S) \le l(\widetilde{S}^*/\widetilde{S}) \le \operatorname{rank} T = 3$. On the other hand, $l(S^*/S) \ge l((S^*/S)_2) = l((P^*/P)_2) + l((Q(G)^*/Q(G)_2) = 2 + \#N(2) = 7$. ($\#M$ denotes the number of elements in a set M.) Consequently one has $l(\widetilde{S}/S) \ge 2$, and thus $\widetilde{S}/S$ is not cyclic. By Lemma 3.4 any embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$ do not satisfy $\langle a \rangle$, or $\langle b \rangle$. It implies $G \notin PC$ by Theorem 1.2 in Part I [7].

C. Any one of the following 12 items contains 3 odd numbers and $N(2)$ do not contain a subset T as in Proposition 3.1 (4). Thus the corresponding graph $G \notin PC$.

(14) (22) (24) (29) (31) (36) (41) (44) (45) (48) (52) (54)

Indeed, in this case $l(\widetilde{S}/S) \ge 1$ and the embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$ is not primitive. By Proposition 3.1 (4) we can conclude $G \notin PC$.

D. For the following 9 items, the corresponding $N(p) = \{i \in I \mid k_i + 1 \equiv 0 \pmod p\}$ contains 3 or more elements for $p = 3$ or 5. Thus $G \notin PC$ by Proposition 3.1 (5).

(25) (28) (33) (35) (37) (46) (50) (51) (55)

E. An item not refered to in the above **A**, **B**, **C**, **D** is one of the following 9.

(4) (7) (13) (15) (16) (19) (38) (39) (57)

For these 8 items except the last one (57) we can show that the corresponding graph G is not in PC. Note that the last one (57) $3+3+3+2+2$ correpsonds to the exception $3A_3 + 2A_2$ in Main Theorem. This item will be treated in the last section 4. For the 6 items (4), (7), (13), (15), (19), (38) there is no T satisfying the condition in Proposition 3.1 (4). Thus we can consider only the primitive embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$.

Here we explain (4) 10+3 $G = A_{10} + A_3$.

Assume that $G \in PC$. Then we have an embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying $\langle a \rangle$, $\langle b \rangle$. We can assume that it is primitive. The discriminant group $S^*/S \cong (\mathbf{Z}/2 + \mathbf{Z}/2) \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/11$. Here the first and the second direct summand $\mathbf{Z}/2 + \mathbf{Z}/2$ correspond to P^*/P. The third $\mathbf{Z}/4$ corresponds to $Q(A_3)^*/Q(A_3)$, and the fourth

$\mathbf{Z}/11$ to $Q(A_{10})^*/Q(A_{10})$. The discriminant quadratic form q_S on S^*/S can be written $q_S(a_1, a_2, b, c) \equiv a_1^2 + a_1a_2 + a_2^2 + 3b^2/4 + 10c^2/11 \bmod 2\mathbf{Z}$ for $(a_1, a_2, b, c) \in (\mathbf{Z}/2 + \mathbf{Z}/2) \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/11$. Let T denote the orthogonal complement of S. T has signature $(1, 2)$, and the discriminant D of T is positive. We have an isomorphism $T^*/T \cong S^*/S$. Via this isomorphism the discriminant quadratic form q_T of T satisfies $q_T \equiv -q_S$. One has $D = |D| = \#T^*/T = 11 \cdot 2^4$. On the other hand, we consider the lattice $T_2 = T \otimes \mathbf{Z}_2$ over 2-adic integers $\mathbf{Z}_2$. $T_2^*/T_2 \cong (T^*/T)_2 \cong (\mathbf{Z}/2 + \mathbf{Z}/2) \oplus \mathbf{Z}/4$, and the discriminant quadratic form q on T_2 satisfies $q(\overline{\alpha}) \equiv (a_1^2 + a_1a_2 + a_2^2) - 3b^2/4 \bmod 2\mathbf{Z}$ for an element $\overline{\alpha} = (a_1, a_2, b)$. This implies that T_2 is equivalent over $\mathbf{Z}_2$ to the lattice defined by the matrix $A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & -3\cdot 2^2 \end{pmatrix}$. (See Nikulin [3] Theorem 1.9.1.) Thus $D \equiv \det A = -3^2 \cdot 2^4 \bmod \mathbf{Z}_2^{*2}$. By the 2 expressions of D one knows that $11 = -\mu^2$ for some $\mu \in \mathbf{Z}_2^* = \mathbf{Z}_2 - 2\mathbf{Z}_2$, which is equivalent to $11 \equiv -1 \pmod 8$. It is a contradiction. Thus $G \notin PC$.

For (7), (13), (15), (19) and (38) the reasoning is similar. By a 2-adic method we can show $G \notin PC$ for these 5 items.

Next we discuss (16) 7+3+3 $G = A_7 + 2A_3$.

Assuming $G \in PC$, we define a lattice S, an embedding $S \hookrightarrow \Lambda_3$, the orthogonal complement T and its discriminant D similarly to the above case (4). In this case $I = \widetilde{S}/S$ is not zero, since $N(2)$ contains 3 elements. On the other hand, we have no subsets T_1, T_2 as in Lemma 3.4. Thus I is cyclic of order 2.

Now, $S^*/S \cong (\mathbf{Z}/2+\mathbf{Z}/2) \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/8$. The discriminant quadratic form q can be written $q(\overline{\alpha}) \equiv x_1^2 + x_1x_2 + x_2^2 + 3(y_1^2 + y_2^2)/4 + 7z^2/8 \bmod 2\mathbf{Z}$ for $\overline{\alpha} = (x_1, x_2, y_1, y_2, z) \in S^*/S$. Here note that $P \cong H_0 \oplus Q(D_4)$ and $Q(D_4)$ has an action of the symmetric group of degree 3 associated with the symmetry of the Dynkin graph D_4. Thus by Lemma 3.4 (2) we can assume without loss of generality that I is generated by the element $\overline{\alpha} = (1, 0, 2, 0, 4)$. One can check that the orthogonal complement $I^\perp$ of I with respect to the discriminant bilinear form b on S^*/S is generated by $\overline{\alpha}$, and $\overline{\beta}_1 = (0, 0, 0, 1, 0)$, $\overline{\beta}_2 = (0, 1, 1, 0, 0)$, $\overline{\beta}_3 = (0, 1, 0, 0, 1)$. ($b : S^*/S \times S^*/S \to \mathbf{Q}/\mathbf{Z}$ is defined by $2b(\overline{\sigma}, \overline{\tau}) = q(\overline{\sigma} + \overline{\tau}) - q(\overline{\sigma}) - q(\overline{\tau})$.) Note that $\overline{\beta}_i$ $(i = 1, 2, 3)$ are mutually orthogonal with respect to b, and $q(\overline{\beta}_1) \equiv 3/4$, $q(\overline{\beta}_2) \equiv -1/4$, $q(\overline{\beta}_3) \equiv -1/8 \pmod{2\mathbf{Z}}$. Thus the discriminant quadratic form q_1 on $\widetilde{S}^*/\widetilde{S} \cong I^\perp/I \cong \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/8$ can be written $q_1(a, b, c) \equiv 3a^2/4 - b^2/4 - c^2/8 \bmod 2\mathbf{Z}$ for $(a, b, c) \in \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/8$. One has $2^7 = D \equiv -3 \cdot 2^7 \bmod \mathbf{Z}_2^{*2}$, which is equivalent to $-3 \equiv 1 \pmod 8$. It is a contradiction. We conclude $A_7 + 2A_3 \notin PC$.

The case (39) 4+3+3+3 $G = A_4 + 3A_3$ is similar to the case (16). $S^*/S \cong (\mathbf{Z}/2+\mathbf{Z}/2) \oplus (\mathbf{Z}/4)^3 \oplus \mathbf{Z}/5$. One can assume that $I = \widetilde{S}/S$ is generated by $(1, 0, 2, 2, 2, 0)$. The orthogonal complement $I^\perp$ is generated by I and $(0, 1, 1, 0, 0, 0)$, $(0, 1, 0, 1, 0, 0)$, $(0, 1, 0, 0, 1, 0)$, $(0, 0, 0, 0, 0, 1)$. Thus the discriminant form on $\widetilde{S}^*/\widetilde{S} \cong (\mathbf{Z}/4)^3 \oplus \mathbf{Z}/5$ can be written in the form $-(a_1^2 + a_2^2 + a_3^2)/4 + 4b^2/5$. We have $5 \cdot 2^6 = D \equiv 2^6 \bmod \mathbf{Z}_2^{*2}$, which is a contradiction. $A_4 + 3A_3 \notin PC$.

Remark. For the following 11 cases the root lattice $Q = Q(G)$ associated with the corresponding Dynkin graph G satisfies the arithmetic condition "$\epsilon_p(Q) = 1$ for every prime p" which appears in Theorem 0.3 [II] in Part I.

(16), (22), (24), (36), (39), (40), (41), (42), (44), (52), (56)

Note that therefore we need not treat these items by Theorem 0.3 in Part I, if we want to show only our Main Theorem. In the above we gave an explicit proof that they are not in PC.

Let us proceed to the case $r = 12$.

By Proposition 3.1 (2) this case corresponds to the division of $12 = \sum_{i=1}^{6} k_i$ into a sum of 6 non-negative integers $k_1 \geq k_2 \geq \cdots \geq k_6 \geq 0$. There are 57 kinds of such divisions as in the following table. We omit 0.

[1] 12	[2] 11+1	[3] 10+2	[4] 9+3
[5] 8+4	[6] 7+5	[7] 6+6	[8] 10+1+1
[9] 9+2+1	[10] 8+3+1	[11] 8+2+2	[12] 7+4+1
[13] 7+3+2	[14] 6+5+1	[15] 6+4+2	[16] 6+3+3
[17] 5+5+2	[18] 5+4+3	[19] 4+4+4	[20] 9+1+1+1
[21] 8+2+1+1	[22] 7+3+1+1	[23] 7+2+2+1	[24] 6+4+1+1
[25] 6+3+2+1	[26] 6+2+2+2	[27] 5+5+1+1	[28] 5+4+2+1
[29] 5+3+3+1	[30] 5+3+2+2	[31] 4+4+3+1	[32] 4+3+3+2
[33] 3+3+3+3	[34] 8+1+1+1+1	[35] 7+2+1+1+1	[36] 6+3+1+1+1
[37] 6+2+2+1+1	[38] 5+4+1+1+1	[39] 5+3+2+1+1	[40] 5+2+2+2+1
[41] 4+4+2+1+1	[42] 4+3+3+1+1	[43] 4+3+2+2+1	[44] 4+2+2+2+2
[45] 3+3+3+2+1	[46] 3+3+2+2+2	[47] 7+1+1+1+1+1	[48] 6+2+1+1+1+1
[49] 5+3+1+1+1+1	[50] 5+2+2+1+1+1	[51] 4+4+1+1+1+1	[52] 4+3+2+1+1+1
[53] 4+2+2+2+1+1	[54] 3+3+3+1+1+1	[55] 3+3+2+2+1+1	[56] 3+2+2+2+2+1
[57] 2+2+2+2+2+2			

In order to simplify descriptions we would like to use the following proposition effectively in what follows. This proposition is a direct consequence of our theory of elementary transformations and tie transformations. (Urabe [5], [6], [7])

Proposition 3.5. *If a Dynkin graph G can be obtained from a basic Dynkin graph G_0 by elementary or tie transformations applied twice, then any subgraph G' of G can be obtained from G_0 by elementary or tie transformations applied twice.*

[**A**]. For each item $[a]$ among the above 57 items except the following 13 we can find an item (b) in the case $r = 13$ paragraph **A** such that the corresponding graph $G(b)$ to (b) contains the corresponding graph $G[a]$ to $[a]$. (Thus $G[a] \subset G(b) \in PC$.)

[16], [32], [33], [34], [36], [40], [44], [47], [48], [51], [52], [56], [57].

By Proposition 3.5 for the other items than in the above 13, we can construct the corresponding graph from $E_8 + F_4$ by elementary or tie transformations applied twice.

We can discuss only the above 13 items in what follows. It turns out that every one of the above 13 does not belong to PC.

[B]. In the case [47] the division of 12 contains 6 odd numbers, but there are no T_1, T_2 as in Lemma 3.4 (3). If the corresponding graph G is in PC, then $l(\widetilde{S}/S) \geq 2$, and by Lemma 3.4 (3) we have a contradiction. Thus $G \notin PC$.

[C]. For the case [34], [36], [48], [51], and [52], the division of 12 contains 4 odd numbers, but there is no T satisfying the condition in Proposition 3.1 (4). If $G \in PC$, then $l(\widetilde{S}/S) \geq 1$, and we have a contradiction by Proposition 3.1 (4). $G \notin PC$.

[D]. Next, we consider the case [40], [44], [56], [57]. In these cases the set $N(3)$ contains 4 or more elements. Thus by Proposition 3.1 (5) the corresponding graph is not a member of PC.

[E]. The remaining items are the following three; [16], [32], [33].

For the former 2 cases [16], [32], the division contains 2 odd numbers, and there is no T as in Proposition 3.1 (4).

We consider [16] 6+3+3 $G = A_6 + 2A_3$. If $G \in PC$, then the embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ is primitive. On the discriminant group $S^*/S \cong (\mathbf{Z}/2 + \mathbf{Z}/2) \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/7$ the discriminant quadratic form q_S of S can be written $q_S(a_1, a_2, b_1, b_2, c) \equiv a_1^2 + a_1 a_2 + a_2^2 + 3(b_1^2 + b^2)/4 + 7c^2/8 \bmod 2\mathbf{Z}$. Thus the discriminant D of the orthogonal complement T of S is $D = \# S^*/S = 7 \cdot 2^6$. On the other hand, the discriminant quadratic form of $T_2 = T \otimes \mathbf{Z}_2$ coincides with $-(q_S)_2$, where $(q_S)_2$ denotes the restriction of q_S to the 2-Sylow subgroup of S^*/S. It implies that $D \equiv 3^3 \cdot 2^6 \bmod \mathbf{Z}_2^{*2}$. Consequently we have $7 \equiv 3^3 \pmod 8$, which is a contradiction.

For the case [32] 4+3+3+2 $G = A_4 + 2A_3 + A_2$, the reasoning is similar to that in [16]. We can conclude $G \notin PC$.

Now, we consider the last case [33] 3+3+3+3 $G = 4A_3$. Since it contains 4 odd numbers, we have $l(I) \geq 1$ for $I = \widetilde{S}/S$. Since there are no T_1 and T_2 as in Lemma 3.4 (3), I is cyclic of order 2.

On the other hand the discriminant quadratic form of S can be written $q(\overline{\alpha}) \equiv (a_1^2 + a_1 a_2 + a_2^2) + 3\sum_{i=1}^4 b_i^2/4$ for an element $\overline{\alpha} = (a_1, a_2, b_1, b_2, b_3, b_4)$ in $S^*/S \cong (\mathbf{Z}/2 + \mathbf{Z}/2) \oplus (\mathbf{Z}/4)^4$. We can assume I is generated by $(1, 0, 2, 2, 2, 0)$. $I^\perp$ is generated by I and $(0,1,1,0,0,0)$, $(0,1,0,1,0,0)$, $(0,1,0,0,1,0)$, $(0,0,0,0,0,1)$. Thus the discriminant quadratic form on $\widetilde{S}^*/\widetilde{S} \cong I^\perp/I \cong (\mathbf{Z}/4)^4$ can be written $q_1(a_1, a_2, a_3, b) \equiv -(a_1^2 + a_2^2 + a_3^2)/4 + 3b^2/4$. Computing the discriminant of the orthogonal complement of S in two different ways, we obtain $2^8 \equiv -3 \cdot 2^8 \bmod \mathbf{Z}_2^{*2}$, which is a contradiction.

Remark. Among the 13 items mentioned in paragraph [A] for the following 11 items the arithmetic condition "$d(Q) \notin \mathbf{Q}_p^{*2}$ or $\epsilon_p(Q) = 1$" is satisfied for every prime p. Note that therefore we need not discuss these cases by Theorem 0.3 [II], if we show only our Main Theorem.

[16], [32], [33], [34], [36], [44], [47], [48], [51], [52], [56]

By the above we know that they are not members of PC.

We can complete the study in the case $J_{3,0}$.

Let us proceed to the case $Z_{1,0}$. Let $G \in PC = PC(Z_{1,0})$ be a graph with the number of vertices r. Let $\Phi : Z \to C$ be the corresponding elliptic K3 surface to G. Z carries the curve IF at infinity associated with $Z_{1,0}$. IF has 7 components. 5 of 7 are components of the singular fiber F_1 of type I_0^* of Φ. One of the remaining two is the image C_5 of the section s_0. The last remaining component C_6 is a component of a singular fiber. We assume that F_2 contains C_6. The dual graph of the set of components not intersecting IF of singular fibers is the graph G.

Here recall that we have assumed in this section that all singular fibers of Φ is of type I, II, III or IV. Note that we can assume further that the Picard number ρ of Z is equal to $r+7$. (If $\rho > r+7$, then a general deformation of Z keeping the union of the curve IF and the combination of curves corresponding to the graph G has the Picard number $r+7$, and all the singular fibers of the structure of the elliptic surface on it are of type I, II, III, or IV except the unique exception of type I_0^*.)

Lemma 3.6. *Under the above assumptions the singular fiber F_2 is of type III or I_2.*

Proof. For $2 \leq i \leq t$ let $n(F_i)$ denote the number of components of F_i not intersecting IF. By definition we have $\sum_{i=2}^{t} n(F_i) = r$. By the equality (2) in the beginning of section 1, we have

$$a + m(F_2) - n(F_2) - 2 + \sum_{i=3}^{t}(m(F_i) - n(F_i) - 1) = 0.$$

Since $m(F_2) \geq n(F_2) + 2$ and $m(F_i) \geq n(F_i) + 1$ for $3 \leq i \leq t$, we have in particular $m(F_2) = n(F_2) + 2$. It implies that a component of F_2 intersecting C_6 is unique except C_6. It is easy to see that if F_2 is of type either IV or I_n with $n \geq 3$, it has never this property. Q.E.D.

Proposition 3.7. *The following two conditions are equivalent.*

(1) There exists a K3 surface Z containing the curve IF associated with $Z_{1,0}$ such that with respect to the associated structure $\Phi : Z \to C$ of the elliptic surface every singular fiber is of type I, II, III or IV and such that the dual graph of the set $\mathcal{E}$ of smooth rational curves on Z not intersecting IF is $G \in PC(Z_{1,0})$.

(2) $G + A_1 \in PC(J_{3,0})$ and every component of G is of type A.

Proof. Assume that there is a K3 surface Z with the above mentioned properties. We can assume moreover $\rho = r + 7$. By Lemma 3.6 F_2 contains only one component C' not contained in IF. Let $\overline{IF}$ denote the union of F_1 and C_5. $\overline{IF}$ is the curve at infinity in the case of $J_{3,0}$. Obviously the set $\overline{\mathcal{E}}$ of smooth rational curves not intersecting $\overline{IF}$ coincides with $\mathcal{E} \cup \{C'\}$. Thus the dual graph of $\overline{\mathcal{E}}$ is $G + A_1$ and it belongs to $PC(J_{3,0})$. Obviously every component of G is of type A under the assumption.

Conversely assume that a Dynkin graph G with components of type A only satisfies $G + A_1 \in PC(J_{3,0})$. Let $\Phi : Z \to C$ be the associated elliptic K3 surface with $G + A_1$. Φ has a singular fiber F_1 of type I_0^* and Z has a curve C_5 which is the image of a section. The dual graph of the union of components not intersecting $\overline{IF} = F_1 \cup C_5$ of singular fibers coincides with $G + A_1$ by definition.

Note that in the case $J_{3,0}$ singular fibers of Φ with 2 or more components other than F_1 have one-to-one correspondence with components of the Dynkin graph, and the type of the singular fiber is uniquely determined by the type of the corresponding irreducible Dynkin graph except only that both type III and type I_2 correspond to A_1 and both type IV and type I_3 correspond to A_2.

Thus our $Z \to C$ has a singular fiber F_2 of type III or I_2 corresponding to the component A_1 of $G + A_1$. Every singular fiber except F_1 is of type I, II, III, or IV. Anyway F_2 has 2 components and only one C'' of two intersects C_5. The curve $IF = \overline{IF} \cup C''$ is the curve at infinity in the case $Z_{1,0}$. The dual graph of all components of singular fibers not intersecting IF coincides with G. Thus $G \in PC(Z_{1,0})$. Q.E.D.

By Proposition 3.7 and by Proposition 3.1 (3) we can assume $r \leq 12$. If $r \leq 10$, thanks to Theorem 0.3 [II] in Part I, we have nothing to verify.

Let us consider the case $r = 12$ first. Thanks to Proposition 3.7 what we have to do is to pick up items $(k_1, \ldots, k_5)$ with $\sum_{i=1}^{5} k_i = 13$ such that $k_i = 1$ for some i from the list in the paragraph **A**, and to check whether for each picked-up item the graph G defined by $G + A_1 = \sum_{i=1}^{5} A_{k_i}$ can be made from $E_7 + F_4$ or $E_8 + CB_3$ by tie transformations applied twice.

Of course, for every picked-up item the answer is affirmative. In the following we show the graph $G = \sum_{i=1}^{5} A_{k_i} - A_1$ (Note that this is different from $\sum_{i=1}^{5} A_{k_i}$.) and an example of the Dynkin graph G_1 which can be made after the first tie transformation. As the basic graph, $E_7 + F_4$ can be used for the items from (2) until (34). For (43), (47), (49), (53) $E_8 + CB_3$ can be used. There are 15 picked-up items.

G	$\leftarrow G_1$		
(2) A_{12}	$\leftarrow A_{11}$	(8) $A_{11} + A_1$	$\leftarrow A_{11}$
(9) $A_{10} + A_2$	$\leftarrow A_9 + A_2$	(10) $A_9 + A_3$	$\leftarrow A_9 + A_2$
(12) $A_8 + A_4$	$\leftarrow A_8 + F_4$	(17) $2A_6$	$\leftarrow A_6 + A_5$
(23) $A_9 + A_2 + A_1$	$\leftarrow A_9 + A_2$	(26) $A_7 + A_4 + A_1$	$\leftarrow A_7 + A_3 + A_1$
(27) $A_7 + A_3 + A_2$	$\leftarrow A_7 + A_3 + A_1$	(30) $A_6 + A_4 + A_2$	$\leftarrow E_7 + A_4$
(34) $A_5 + A_4 + A_3$	$\leftarrow E_7 + A_4$	(43) $A_7 + 2A_2 + A_1$	$\leftarrow A_7 + A_2 + A_1 + CB_1$
(47) $2A_5 + 2A_1$	$\leftarrow D_5 + A_5 + CB_1$	(49) $A_5 + 2A_3 + A_1$	$\leftarrow D_5 + A_5 + CB_1$
(53) $2A_4 + 2A_2$	$\leftarrow E_6 + A_4 + CB_1$		

Next we consider the case $r = 11$. First we can pick up items $(k_1, \ldots, k_6)$ with $\sum_{i=1}^{6} k_i = 12$ such that $k_i = 1$ for some i from items [1]–[57] in the above list. Then, it is not difficult to see that every picked-up item is either one of the 13 exceptions treated in paragraph [**A**], or the corresponding graph G is a subgraph of a graph in the above 15 just discussed. We can complete the proof by Proposition 3.5.

Remark. There are 8 items with $k_i = 1$ for some i among the 13 exceptions in [**A**]. The corresponding graph G is as follows. (Note that G does not have 12 vertices but 11 ones.)

[34] $A_8 + 3A_1$	[36] $A_6 + A_3 + 2A_1$	[40] $A_5 + 3A_2$
[47] $A_7 + 4A_1$	[48] $A_6 + A_2 + 3A_1$	[51] $2A_4 + 3A_1$
[52] $A_4 + A_3 + A_2 + 2A_1$	[56] $A_3 + 4A_2$	

We can show that $G \notin PC$ for three items [40], [48], [56]. However, for the other 5 items we can make the corresponding graph from $E_7 + F_4$ by tie transformations applied twice.

Note that this fact do not contradict Proposition 3.7, because in the case of these 5 graphs, in the corresponding elliptic K3 surface the singular fiber F_2 containing the component C_6 of IF is of type either I^*, II^*, III^* or IV^*.

We complete the case $Z_{1,0}$.

The third case is $Q_{2,0}$. We can show the following in this case.

Proposition 3.8. *The following two conditions are equivalent.*

(1) There exists a K3 surface Z containing the curve IF associated with $Q_{2,0}$ such that with respect to the associated structure $\Phi : Z \to C$ of the elliptic surface every singular fiber is of type I, II, III or IV and such that the dual graph of the set $\mathcal{E}$ of smooth rational curves not intersecting IF on Z is $G \in PC(Q_{2,0})$.

(2) $G + A_2 \in PC(J_{3,0})$ and every component of G is of type A.

By Theorem 0.3 [II] in Part I, Proposition 3.1 (3) and Proposition 3.8 we can consider only the case $r = 11$ or $r = 10$.

First we treat the case $r = 11$. By Proposition 3.8 we can consider the division $(k_1, \ldots, k_5)$ of 13 with $k_i = 2$ for some i.

Note that the case (57) 3+3+3+2+2 corresponding to the exception $3A_3 + 2A_2$ in Main Theorem satisfies $k_5 = 2$. We consider this case separatedly in the final section.

Excluding (57), what we have to do to show Main Theorem is the following; First to pick up items $(k_1, \ldots, k_5)$ with $k_i = 2$ for some i from the list in paragraph **A** in the case $J_{3,0}$ $r = 13$. Secondly to check for each picked-up item whether the graph G defined by $G + A_2 = \sum_{i=1}^{5} A_{k_i}$ can be made from $E_6 + F_4$ or $E_8 + F_2$ by tie transformations applied twice.

We can check this affirmatively for every picked-up item. The following list shows the numbering of the picked-up item, the corresponding graph G, and an example of the Dynkin graph G_1 which we can make after the first tie transformation. For every item we can use $E_6 + F_4$ at the start. The list contains 10 items.

G	$\leftarrow G_1$		
(3) A_{11}	$\leftarrow A_9 + A_1$	(9) $A_{10} + A_1$	$\leftarrow A_9 + A_1$
(11) $A_9 + A_2$	$\leftarrow A_9 + A_1$	(18) $A_6 + A_5$	$\leftarrow 2A_5$
(23) $A_9 + 2A_1$	$\leftarrow A_9 + A_1$	(27) $A_7 + A_3 + A_1$	$\leftarrow A_7 + A_1 + B_2$
(30) $A_6 + A_4 + A_1$	$\leftarrow A_6 + A_1 + F_4$	(32) $A_6 + A_3 + A_2$	$\leftarrow A_6 + 2A_2$
(43) $A_7 + A_2 + 2A_1$	$\leftarrow A_7 + A_1 + B_2$	(53) $2A_4 + A_2 + A_1$	$\leftarrow A_4 + 2A_2 + B_2$

We proceed to the case $r = 10$. In this case our problem is reduced to the analysis of the decompositions $(k_1, \ldots, k_6)$ of 12 into 6 integers. It is not difficult to show one of the following three conditions always holds for each item in the above [1]–[57].

(1) $k_i \neq 2$ for $1 \leq i \leq 6$.

(2) $k_i = 2$ for some $1 \leq i \leq 6$ and the graph G_0 defined by $G_0 + A_2 = \sum_{i=1}^{6} A_{k_i}$ is a subgraph of one of the 10 graphs G in the list just above.

(3) It is one of the 13 items discussed in paragraph [A].

By Proposition 3.5 and Proposition 3.8 we can complete the proof.

In this section we have shown Proposition 0.1 for $J_{3,0}$, $Z_{1,0}$, and $Q_{2,0}$ under the assumption ((3)) in the introduction.

§4. The exception

In this section we study the exception in Main Theorem.

First we consider $G = 3A_3 + 2A_2$ in the case $J_{3,0}$.

In this case $P = H_0 \oplus P'_0$, $P'_0 \cong Q(D_4)$ and $H_0 = \mathbf{Z}u_0 + \mathbf{Z}v_0$, $u_0^2 = v_0^2 = 0$, $u_0 \cdot v_0 = 1$. Set $S = P \oplus Q(G)$. Consider the discriminant group $S^*/S \cong (\mathbf{Z}/4)^3 \oplus (\mathbf{Z}/2)^2 \oplus (\mathbf{Z}/3)^2$. Each of three $\mathbf{Z}/4$-components corresponds to A_3, $(\mathbf{Z}/2)^2$ corresponds to P, and $(\mathbf{Z}/3)^2$ to $2A_2$. The discriminant quadratic form can be written $q(a_1, a_2, a_3, b_1, b_2, x_1, x_2) \equiv 3(a_1^2 + a_2^2 + a_3^2)/4 + b_1^2 + b_1 b_2 + b_2^2 + 2(x_1^2 + x_2^2)/3 \bmod 2\mathbf{Z}$.

Assume $G \in PC(J_{3,0})$. We have the corresponding elliptic K3 surface $\Phi : Z \to C$ and the corresponding embedding $S \hookrightarrow \Lambda_3$. Every singular fiber except one F_1 of type I_0^* is of type I, II, III, or IV, since every component of G is of type A. Thus we can apply the theories in section 3.

Since $l((S^*/S)_2) = 5 > \operatorname{rank} \Lambda_3 - \operatorname{rank} S = 3$, there is no primitive embedding $S \hookrightarrow \Lambda_3$. Let $\widetilde{S}$ be the primitive hull of S in Λ_3. Every non-zero element in the quotient $I = \widetilde{S}/S$ has order 2 by Lemma 3.4 (1). We have $I \cong \mathbf{Z}/2$, since there are no T_1, T_2 satisfying the condition in Lemma 3.4 (3). By Lemma 3.4 (2) the generator of I is either $(2,2,2,1,0,0,0)$, $(2,2,2,0,1,0,0)$ or $(2,2,2,1,1,0,0)$.

Note that these three elements are conjugate with respect to the action of the symmetric group of degree 3 on $P'_0 \cong Q(D_4)$ induced by the symmetry of the Dynkin graph D_4.

Let S_1 be the inverse image by $S^* \to S^*/S$ of the subgroup in S^*/S generated by the element $(2,2,2,1,0,0,0)$. It is an even overlattice of S with index 2.

Proposition 4.1. *Any embedding $S = H_0 \oplus P'_0 \oplus Q(3A_3 + 2A_2) \hookrightarrow \Lambda_3$ satisfying Looijenga's conditions (a) and (b) is the composition of an isomorphism $S \xrightarrow{\sim} S$ induced by an isomorphism $P'_0 \xrightarrow{\sim} P'_0$ of the direct summand and a primitive embedding $S_1 \hookrightarrow \Lambda_3$.*

Next, we compute the discriminant quadratic form q_1 of S_1. Let $\overline{\sigma}_1$, $\overline{\sigma}_2$, $\overline{\sigma}_3 \in S^*/S$ be the elements of order 4 corresponding to $(1,0,0,0,1,0,0)$, $(0,1,0,0,1,0,0)$, $(0,0,1,0,1,0,0) \in (\mathbf{Z}/4)^3 \oplus (\mathbf{Z}/2)^2 \oplus (\mathbf{Z}/3)^2$ respectively. Let $\overline{\tau}_1$, $\overline{\tau}_2 \in S^*/S$ be the elements of order 3 corresponding to $(0,0,0,0,0,1,0)$ and $(0,0,0,0,0,0,1)$ respectively. We can check that the orthogonal complement $I^\perp$ of I with respect to the discriminant bilinear form b on S^*/S is the direct sum of I and the 5 cyclic groups generated by $\overline{\sigma}_1$, $\overline{\sigma}_2$, $\overline{\sigma}_3$, $\overline{\tau}_1$, $\overline{\tau}_2$. Thus we have $S_1^*/S_1 \cong I^\perp/I \cong (\mathbf{Z}/4)^3 \oplus (\mathbf{Z}/3)^2$. Note that any two of $\overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3, \overline{\tau}_1, \overline{\tau}_2$ are orthogonal with respect to b, $q(\overline{\sigma}_\nu) \equiv -1/4 \bmod 2\mathbf{Z}$ $(\nu = 1,2,3)$, and $q(\overline{\tau}_\nu) \equiv 2/3 \bmod 2\mathbf{Z}$ $(\nu = 1,2)$. Thus the discriminant quadratic form q_1 of S_1 can be written

$$q_1(\overline{\sigma}) \equiv -\frac{1}{4}(a_1^2 + a_2^2 + a_3^2) + \frac{2}{3}(b_1^2 + b_2^2) \bmod 2\mathbf{Z},$$

for an element $\overline{\sigma} \in S_1^*/S_1$ corresponding to $(a_1, a_2, a_3, b_1, b_2) \in (\mathbf{Z}/4)^3 \oplus (\mathbf{Z}/3)^2$.

In what follows we consider $S_1^*/S_1 \cong (\mathbf{Z}/4)^3 \oplus (\mathbf{Z}/3)^2$. Let $\overline{\kappa}_1$ and $\overline{\kappa}_2$ be the elements of order 12 in S_1^*/S_1 corresponding to $(1,2,0,1,1)$ and $(2,1,0,1,-1)$ respectively. Let $\overline{\lambda} \in S_1^*/S_1$ be the element of order 4 corresponding to $(0,0,1,0,0)$. We can check that S_1^*/S_1 is the direct sum of three cyclic groups generated by $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\lambda}$. After some calculation one has

$$q_1(a_1\overline{\kappa}_1 + a_2\overline{\kappa}_2 + b\overline{\lambda}) \equiv \frac{1}{12}(a_1^2 + a_2^2) - \frac{1}{4}b^2 \bmod 2\mathbf{Z},$$

for $a_1, a_2 \in \mathbf{Z}/12$, $b \in \mathbf{Z}/4$.

Proposition 4.2. *(1) S_1 has a primitive embedding into the even unimodular lattice Λ_3 of signature $(19,3)$.*
(2) If $\eta \in S_1$, $\eta \notin S$ and $\eta \cdot u_0 = 0$, then $\eta^2 \geq 4$.

Proof. (1) Let T be the lattice of rank 3 defined by the diagonal matrix whose diagonal entries are -12, -12 and 4. T is an even lattice and we can define the discriminant quadratic form q_T on T^*/T. By the above calculation one knows that we have an isomorphism $\phi : S_1^*/S_1 \xrightarrow{\sim} T^*/T$ such that $-q_T\phi$ coincides with the discriminant quadratic form q_1 of S_1.

Consider the direct sum $U = S_1 \oplus T$. Let J be the graph of the isomorphism ϕ defined in the discriminant group $U^*/U = (S_1^*/S_1) \oplus (T^*/T)$ of U. J is a subgroup in U^*/U and the restriction to J of the discriminant quadratic form of U is zero. Thus the inverse image Λ of J by the natural surjective homomorphism $U^* \to U^*/U$ is an even lattice. Since the square of the order of J is equal to the order of U^*/U, Λ is unimodular. The signature of S_1 is equal to that of S and thus it is equal to $(18,1)$. Since the signature of T is $(1,2)$, Λ has signature $(19,3)$. The even unimodular lattice with signature $(19,3)$ is unique up to isomorphisms (Milnor-Husemoller [2]), and $\Lambda \cong \Lambda_3$.

Let $\widetilde{S}_1$ denote the primitive hull of S_1 in Λ. Since $\widetilde{S}_1/S_1 \cong (\widetilde{S}_1+T)/U = ((S_1^*/S_1)\oplus \{0\}) \cap J = \{0\}$, S_1 is primitive in Λ.
(2) Let $\omega_0 \in (P_0')^*$ be the element corresponding under the isomorphism $(P_0')^* \cong Q(D_4)^*$ to the fundamental weight associated with one of three vertices at the end of the Dynkin graph D_4. For $\nu = 1,2,3$ let $\chi_\nu \in Q(G)^*$ be the fundamental weight associated with the central vertex of the ν-th component of G of type A_3. Set $\xi = \omega_0+\chi_1+\chi_2+\chi_3$. We can assume $S_1 = S \cup (S+\xi)$.

Now, by assumption we can write $\eta = \xi + \zeta$ for some $\zeta \in S$. We have $0 = \eta \cdot u_0 = \xi \cdot u_0 + \zeta \cdot u_0 = \zeta \cdot u_0$. Thus $\zeta = mu_0 + \zeta_0$ for some $m \in \mathbf{Z}$, $\zeta_0 \in P_0' \oplus Q(G)$. Setting $\eta_0 = \xi + \zeta_0$, one has $\eta^2 = \eta_0^2$, since $\eta = \eta_0 + mu_0$ and $\eta_0 \cdot u_0 = 0$. Our problem is reduced to showing $\eta_0^2 \geq 4$.

Here recall the notion of the characteristic number. (Part I Urabe [7] section 2 Lemma 2.1 etc.) $S_0 = P_0' \oplus Q(G)$ is an even positive definite lattice, and we can define the characteristic number $\nu(\overline{x})$ for each element $\overline{x} \in S_0^*/S_0$. For an element $x \in S_0^*$ we write $\overline{x} = x \bmod S_0 \in S_0^*/S_0$. By definition $\eta_0^2 \geq \nu(\overline{\eta}_0) = \nu(\overline{\xi}) = \nu(\overline{\omega}_0 + \overline{\chi}_1 + \overline{\chi}_2 + \overline{\chi}_3) = 4$.
Q.E.D.

Corollary 4.3. *In the case of $J_{3,0}$ the lattice $P \oplus Q(3A_3+2A_2)$ has an embedding into Λ_3 satisfying Looijenga's conditions (a), (b). In particular $3A_3+2A_2 \in PC(J_{3,0})$.*

Proposition 4.4. *In case $J_{3,0}$ with respect to any full embedding $Q(3A_3+2A_2) \hookrightarrow \Lambda_3/P$ there is no isotropic element in a nice position.*

Proof. Assume that we have a primitive embedding $P \hookrightarrow \Lambda_3$, a full embedding $Q(G) \hookrightarrow \Lambda_3/P$ and a primitive isotropic element $\overline{u} \in \Lambda_3/P$ in a nice position. We will deduce a contradiction.

Let $u \in \Lambda_3$ be the element in the orthogonal complement of P whose image under the canonical surjective homomorphism $\Lambda_3 \to \Lambda_3/P$ is $\overline{u}$. Such a u exists by Proposition 3.6 in Part I. By the definition of a nice position we have a root basis $\Delta \subset Q(G)$ and a long root $\alpha \in \Delta$ such that $\beta \cdot u = 0$ for every $\beta \in \Delta$ with $\beta \neq \alpha$ and $\alpha \cdot u = 1$ or 0. We have the induced embedding $S = P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying Looijenga's $\langle a \rangle$ and $\langle b \rangle$.

Assume that $\alpha \cdot u = 0$. Then we have $\epsilon_p(Q(3A_3 + 2A_2)) = 1$ for every prime p, since the orthogonal complement of $P \oplus Q(G)$ contains an isotropic element u. However, $\epsilon_3(Q(3A_3 + 2A_2)) = (3,3)_3 = -1$, which is a contradiction. One knows $\alpha \cdot u = 1$.

Let T be the orthogonal complement of S in Λ_3. Since $P \oplus Q(G) \oplus T \subset \Lambda_3 \subset P^* \oplus Q(G)^* \oplus T^*$ and since u is orthogonal to P, we can write $u = \omega + \tau$ for $\omega \in Q(G)^*$, $\tau \in T^*$. The element $\omega \in Q(G)$ is the fundamental weight associated with α and Δ.

Here note that considering $w(u)$ instead of u for an element w of the Weyl group of $Q(G)$, we can assume moreover that the root basis Δ coincides with a previously given root basis.

We have two cases.

(a) The long root α lies on a component of type A_3.

(b) α lies on a component of type A_2.

First we consider case (a). We use the notations ξ, ω_0, χ_ν in the proof of Proposition 4.2 (2). By Proposition 4.1 there is an isomorphism $\sigma : S \to S$ keeping every element in $Q(G)$ fixed such that $\sigma(\xi) \in \Lambda_3$. We have $\mathbf{Z} \ni u \cdot \sigma(\xi) = \omega \cdot \xi = \omega \cdot \chi_\nu$. Here we assumed that α lies on the ν-th A_3-component of Δ. If α corresponds to a vertex of the Dynkin graph A_3 at the end, then $\omega \cdot \chi_\nu = 1/2$. Thus α corresponds to the central vertex of A_3 and $\omega = \chi_\nu$. In particular $\omega^2 = 1$.

Since $0 = u^2 = \omega^2 + \tau^2$, one has $\tau^2 = -1$ for $\tau \in T^*$.

Next, we consider the lattice $T^* \otimes \mathbf{Z}_2 = (T \otimes \mathbf{Z}_2)^*$ over 2-adic integers $\mathbf{Z}_2$. By Proposition 4.1 the discriminant quadratic form of $T_2 = T \otimes \mathbf{Z}_2$ coincides with $-(q_1)_2$, where $(q_1)_2$ denotes the restriction to the 2-Sylow subgroup of the discriminant quadratic form q_1 of S_1. The discriminant D of T is equal to that of T_2, and it satisfies $D = 2^6 \cdot 3^2 \equiv 2^6 \bmod \mathbf{Z}_2^{*2}$.

Let $T' = (\mathbf{Z}_2)^3$ be the $\mathbf{Z}_2$-lattice whose intersection matrix is the diagonal matrix with diagonal entries 4, 4, 4. By the calculation of q_1 before Proposition 4.2 one knows that T_2 and T' have the same rank, the same discriminant quadratic form, and the same discriminant modulo $\mathbf{Z}_2^{*2}$. By Corollary 1.9.3 in Nikulin [3] they are isomorphic as $\mathbf{Z}_2$-lattices. Thus there is an isomorphism $T_2^* \cong (\mathbf{Z}_2)^3$ such that the quadratic form is given by $x^2 = (x_1^2 + x_2^2 + x_3^2)/4$ for an element $x \in T_2^*$ corresponding to $(x_1, x_2, x_3) \in (\mathbf{Z}_2)^3$.

Assume $\tau \in T_2^*$ corresponds to $(x_1, x_2, x_3) \in (\mathbf{Z}_2)^3$. One has $x_1^2 + x_2^2 + x_3^2 = -4$. Let k_1 be the number of x_ν's ($\nu = 1, 2, 3.$) with $x_\nu \notin 2\mathbf{Z}_2$. One has $-4 \equiv k_1 \pmod 4$. Thus $k_1 = 0$. We can write $x_\nu = 2y_\nu$ with $y_\nu \in \mathbf{Z}_2$ for $\nu = 1, 2, 3$. Let k_2 be the number of y_ν's ($\nu = 1, 2, 3.$) with $y_\nu \notin 2\mathbf{Z}_2$. One has $-1 = y_1^2 + y_2^2 + y_2^2 \equiv k_2 \pmod 4$, and $k_2 = 3$. Then one has congruent relations modulo 8 $y_\nu^2 \equiv 1 \pmod 8$ ($\nu = 1, 2, 3$). We have $-1 = y_1^2 + y_2^2 + y_3^2 \equiv 3 \pmod 8$, which is a contradiction. The case (a) never takes place.

We proceed to the case (b). Since ω is a fundamental weight of $Q(A_2)$, $3\omega \in Q(G) \subset \widetilde{S}$ and $\omega^2 = 2/3$. 3 is invertible in $\mathbf{Z}_2$ and we have $\omega \in \widetilde{S} \otimes \mathbf{Z}_2$. Since $\omega \bmod \widetilde{S} \otimes \mathbf{Z}_2 = 0$ corresponds to $\tau \bmod T_2$ under the canonical isomorphism $(\widetilde{S} \otimes \mathbf{Z}_2)^* / (\widetilde{S} \otimes \mathbf{Z}_2) \cong T_2^*/T_2$, $\tau \in T_2 = T \otimes \mathbf{Z}_2$. We can identify the quadratic form on T_2 with $2^2(x_1^2 + x_2^2 + x_3^2)$ and $\tau^2 \in \mathbf{Z}_2$ is a multiple of 2^2.

On the other hand $\tau^2 = u^2 - \omega^2 = -2/3$, which is not a multiple of 2^2. We have a contradiction. Thus the case (b) never takes place, either. Q.E.D.

Corollary 4.5. *The Dynkin graph $3A_3 + 2A_2$ can never be made from $E_8 + F_4$ or B_{12} by applying elementary transformations or tie transformations twice.*

The case $G = 3A_3 + A_2$ for $X = Q_{2,0}$ follows easily from the above case. By Proposition 3.8 one has $G \in PC(Q_{2,0})$ since $G + A_2 = 3A_3 + 2A_2 \in PC(J_{3,0})$.

Next, assume that there are an embedding $P \oplus Q(G) \hookrightarrow \Lambda_3$ satisfying $\langle a \rangle$ and $\langle b \rangle$ and a primitive isotropic element $\overline{u} \in \Lambda_3/P$ in a nice position. Let $u \in \Lambda_3$ be the element orthogonal to P and mapped to $\overline{u}$ by $\Lambda_3 \to \Lambda_3/P$. u is also a primitive isotropic element.

Here note that the lattice P in our case $Q_{2,0}$ has the decomposition $P = \overline{P} \oplus T$ where $\overline{P}$ is isomorphic to P defined in the case $J_{3,0}$ and $T \cong Q(A_2)$. Thus the given embedding induces the embedding $\overline{P} \oplus Q(G + A_2) \hookrightarrow \Lambda_3$. Regarding this embedding as the one defined in the case $J_{3,0}$, we would like to show that this also satisfies $\langle a \rangle$ and $\langle b \rangle$.

Let us consider the associated elliptic surface $\Phi : Z \to C$ with $\rho = r + 8 = 19$. Let F_1 be the singular fiber of Φ contained in the curve at infinity IF. F_1 is of type I_0^*. Let F_2 be another singular fiber containing 2 components of IF. The other singular fibers than F_1 and F_2 are of type I, II, III, or IV, since G has only components of type A.

If F_2 is of type I^* then the orthogonal complement of $P \oplus Q(G)$ contains an isotropic element by Proposition 1.2. Thus $\epsilon_p(Q(G)) = (3, -d(Q(G)))_p$ for every prime p. However, $\epsilon_3(Q(G)) = (-1, 3)_3 = -1$ and $(3, -d(Q(G)))_3 = (3, -2^6 \cdot 3)_3 = (3, -3)_3 = +1$, which is a contradiction.

Assume that F_2 is of type either II^*, III^* or IV^*. Let G_1 be the dual graph of the set of components not intersecting IF in F_2. One knows G_1 is of type either E_6, A_5 or $2A_2$. Neither of them is contained in $G = 3A_3 + A_2$, which is a contradiction.

Consequently F_2 is also of type I or IV, and by the proof of Proposition 3.7 one knows that the embedding $\overline{P} \oplus Q(G + A_2) \hookrightarrow \Lambda_3$ also satisfies $\langle a \rangle$ and $\langle b \rangle$.

The image of u by the surjective homomorphism $\Lambda_3 \to \Lambda_3/\overline{P}$ is a primitive isotropic element in a nice position with respect to the embedding $Q(3A_3 + 2A_2) = Q(G + A_2) \hookrightarrow \Lambda_3/\overline{P}$, which contradicts Proposition 4.4.

Proposition 4.6. *(1) $3A_3 + A_2 \in PC(Q_{2,0})$.*
(2) In case $Q_{2,0}$ with respect to any full embedding $Q(3A_3 + A_2) \hookrightarrow \Lambda_3/P$ there is no isotropic element in a nice position.
(3) The Dynkin graph $3A_3 + A_2$ can never be made from any one of $E_6 + F_4$, $E_8 + F_2$, B_9 by applying elementary transformations or tie transformations twice.

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