On the Duality of Quantum Liouville Field Theory

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Abstract

It has been found empirically that the Virasoro centre and 3-point functions of quantum Liouville field theory with potential $e^{2b\phi(x)}$ and external primary fields $\exp(\alpha\phi(x))$ are invariant with respect to the duality transformations $\hbar\alpha \to q - \hbar\alpha$ where $q = b^{-1} + \hbar b$. The steps leading to this result (via the Virasoro algebra and 3-point functions) are reviewed in the path-integral formalism. The duality can be traced to the fact that the quantum relationship between the α and the conformal weights Δ_α is two-to-one. It is shown that, as a consequence of this relationship, the quantum Liouville potential may actually contain two exponentials (with related parameters), and for such potentials the duality appears in a more natural way. Some open questions, such as, whether, in an SL(2,R) WZW→ Liouville reduction, the duality comes from the WZW theory or the reduction, are discussed.

1. Introduction

The interest in two-dimensional Liouville theory [1] lies in the fact that it is a non-trivial conformally invariant field theory involving only a single (scalar) field, and that it appears in many contexts, most notably in string theory [2][3]. However, since the exponential potential has an infimum but no minimum, the quantum version of the theory presents some problems. For example, in contrast to the Wess-Zumino-Witten (WZW) four-point function, the Liouville four-point function cannot be expressed in terms of elementary analytic functions (the analogue of the Knitznik-Zamolodchikov equation is second order). Even for the 3-point function, for which the space-time dependence is fixed by conformal invariance, the coefficients are difficult to obtain. On the other hand, what is known about the 3-point functions is quite interesting and exhibits a duality of the theory that is absent at the classical level [4][5]. In this talk I wish to review the quantized Liouville theory from the path-integral point of view and to discuss this duality, first describing what it is, then proposing a manner in which it can be incorporated in the functional integral and finally raising some questions about its origins and generality.

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2. The Liouville Path-Integral

The classical Liouville theory is defined by the Action

$$S(\phi) = \int d^2x \Big[\frac{1}{2}(\partial\phi)^2 + \mu e^{2b\phi}\Big] \tag{2.1}$$

where μ and b are constants and $\phi(x)$ is a scalar field, and in 2 dimensions this Action is conformally invariant. If the theory is embedded in a curved space by generalizing the Action to

$$S(\phi) = \int d^2x \Big[\frac{1}{2}\sqrt{g}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + \mu\sqrt{g}e^{2b\phi} + q\sqrt{g}R\phi\Big] \tag{2.2}$$

where R is the Riemann curvature and q a constant, the conformal invariance converts to Weyl invariance i.e. invariance with respect to the transformation $g_{\mu\nu}(x) \to \lambda(x)g_{\mu\nu}(x)$ for any scale-factor $\lambda(x)$, provided that the constant q is

$$q = b^{-1} \quad \text{and} \quad q = b^{-1} + \hbar b \tag{2.3}$$

in the classical and quantum theories respectively. The quantum duality referred to above makes its appearance already at this stage in the fact that q is invariant under the transformation $b \leftrightarrow (\hbar b)^{-1}$.

By varying $g^{\mu\nu}$ in (2.2) and then taking the flat space limit one sees that the energy momentum tensor for the flat-space theory is

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 + q(\eta_{\mu\nu}\partial^2\phi - \partial_\mu\partial_\nu\phi) \tag{2.4}$$

where η is the Minkowski metric. This energy momentum tensor is traceless on the mass-shell and leads to Virasoro operators of the form

$$T_{++} = (\partial_+\phi)^2 - q\partial_+^2\phi \quad \text{and} \quad T_{--} = (\partial_-\phi)^2 - q\partial_-^2\phi \tag{2.5}$$

For these Virasoros the centres are

$$c = \hbar + q^2 \tag{2.6}$$

and the fields $e^{\alpha\phi}$ are primary fields of conformal weights $(\Delta_\alpha, \Delta_\alpha)$ where

$$\Delta_\alpha = \alpha(q - \hbar\alpha) \tag{2.7}$$

The duality makes its second appearance at this stage in the fact that the conformal weights $\hbar\Delta_\alpha$ are invariant with respect to the transformation $\hbar\alpha \leftrightarrow (q - \hbar\alpha)$. This second form of duality is more general than the first, which is actually the special case for which $\Delta = 1$. An immediate consequence of the duality is that in quantum conformal field theories there are two primary fields of the form $e^{2\alpha\phi}$ for each conformal weight Δ_α. In other words, there are two weights

$$\alpha_\pm = \frac{q \pm \sqrt{q^2 - 4\hbar\Delta_\alpha}}{2\hbar} \tag{2.8}$$

for every Δ_α. Note that in the limit $\hbar \to 0$ we have $\alpha_+ \to \infty$ and $\alpha_- \to \alpha$ so that only one root remains finite.

¿From now on we shall set $\hbar = 1$, assume that 2-space is a sphere and use conformal coordinates i.e. coordinates for which

$$g_{\mu\nu}(x) = e^{\sigma(x)}\eta_{\mu\nu} \tag{2.9}$$

In these coordinates the Action reduces to

$$S(\phi) = \int d^2x \Big[\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + \mu e^{2(b\phi+\sigma)} + qe^{2\sigma}R\phi\Big] \tag{2.10}$$

3. The N-Point Functions

In terms of the path integral we write the N-point function as

$$\prod_a < e^{2\alpha_a\tilde{\phi}(x_a)} >= \int d\tilde{\phi}(x)e^{iS(\tilde{\phi}(x))}\prod_a e^{2\alpha_a\tilde{\phi}(x_a)} \tag{3.1}$$

The exponential potential has the property that, if the zero mode part of $\tilde{\phi}$ is seperated from the rest by the decomposition

$$\tilde{\phi}(x) = \phi_o + \phi(x) \qquad \int d^2x\phi(x) = 0, \tag{3.2}$$

the integration over the zero-mode can be seperated from the rest [9] by writing (3.1) in the form

$$\prod_a < e^{2\alpha_a\tilde{\phi}(x_a)} >= \int d\phi e^{iS(\phi)}\prod_a e^{2\alpha_a\phi(x_a)}\Big(\int d\phi_o e^{-2\xi\phi_o + e^{2b\phi_o}V_b(\phi)}\Big) \tag{3.3}$$

where

$$\xi = q - \sum_a \alpha_a \qquad V_b(\phi) = \mu e^{2b\phi} \tag{3.4}$$

and we have used the fact that for the sphere $\int d^2x\sqrt{g}R = \chi = 2$, where χ is the Euler number. Furthermore, modulo some factors, the zero mode integral in (3.3) is just the integral for the Γ-function so it can immediately be evaluated to give

$$\prod_a < e^{2\alpha_a\tilde{\phi}(x_a)} >= \Gamma\Big(-\frac{\xi}{b}\Big)\int d\phi e^{iS(\phi)}\prod_a e^{2\alpha_a\phi(x_a)}\Big(V_b(\phi)\Big)^{\frac{\xi}{b}} \tag{3.5}$$

Here we run into the major difficulty with Liouville theory, which is that the functional integral is calculable only when ξ/b takes positive integer values, and these are precisely the points at which the coefficient $\Gamma(-\xi/b)$ has poles. So in practice all we can do is calculate the residues at the poles and try to extrapolate to other points. There is some consolation in the fact that the residues

$$R_N(x_a,\alpha_a,m)=\frac{(-1)^m}{m!}\int d\phi e^{iS(\phi)}\prod_a e^{2\alpha_a\phi(x_a)}\Big(V_b(\phi)\Big)^m \tag{3.6}$$

turn out to be free-field integrals which are already available in the literature [10]. More specifically, if we extract the standard anomaly+Polyakov factor $\exp(1+q^2)\int R\Delta^{-2}R$, renormalize by extracting the singularities of the Green's function in the usual way, and take the flat space limit, we obtain for the residues

$$R_N(x_a,\alpha_a,m)=\frac{(-\mu)^m}{m!}\prod_{a\neq b}|x_{ab}|^{-2\alpha_a\alpha_b}\prod_{i\neq j}^{i=m}\int d^2x_i|x_{ij}|^{-2b^2}|x_{ia}|^{-4b\alpha_a} \tag{3.7}$$

where $x_{ab}\equiv x_a-x_b$. Conformal invariance implies that the kernel of the integral in the second factor is $SL(2,C)$ invariant. Since the group $SL(2,C)$ has 3-parameters this means that, strictly speaking, the partition function and the N-point functions for $N=1,2$ do not exist (although in practice there is a way to define the 2-point function as the limit of the 3-point function renormalized by a universal infinite factor). For $N=3$ the residue takes the form

$$R_3(x_a,\alpha_a,m)=C_3(\alpha_a,m)\prod_{a\neq b}^{3}|x_{ab}|^{-2\Delta_{ab}} \tag{3.8}$$

where C_3 is a numerical coefficient and

$$\Delta_{ab}=\Delta_a+\Delta_b-\Delta_c \qquad \Delta_a=\alpha_a(q-\alpha_a) \tag{3.9}$$

For $N>3$ the residues take the form

$$R_N(x_a,\alpha_a,m)=|\frac{x_{23}}{x_{12}x_{31}}|^{2\xi(\xi+q)}\prod_{a\geq 2}|x_{1a}|^{4\xi\alpha_a}K(\alpha_a,r_c) \tag{3.10}$$

where the $r_c(x)$ are conformally invariant ratios of the form $r_c=x_{c2}x_{31}/x_{c1}x_{23}$ and

$$K(\alpha_a,r_c)=\prod_{c\geq 4}\int d^2x_i|x_{ij}|^{-2b^2}|x_i|^{-4b\alpha_2}|x_i-1|^{-4b\alpha_3}|x_i-r_c|^{-4b\alpha_c} \tag{3.11}$$

4. Three-Point Coefficient

The values of the coefficient $C_3(\alpha_a,-\xi/b)$ of the 3-point function at the poles $\xi=-mb$ of the Γ-function can be computed from the free-field integral

$$C_3(\alpha_a,m)=\frac{(-\mu)^m}{m!}\prod_{i\neq j}^{m}\int d^2x_i|x_{ij}|^{-2b^2}|x_{ia}|^{-4b\alpha_a} \tag{4.1}$$

and this can be integrated [10] to give

$$C_3(\alpha_a, m) = (\mu_r)^{mb} \frac{k'(0)}{k'(-mb)} \left[\frac{k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_1+\alpha_2-\alpha_3)k(\alpha_2+\alpha_3-\alpha_1)k(\alpha_3+\alpha_1-\alpha_2)} \right] \tag{4.2}$$

where $\mu_r = \left[-\mu\gamma(b^2)\right]^{1/b}$ is a renormalized version of μ and the function $k(\alpha)$, which is implicitly a function of b, is defined by the following, rather remarkable, properties

$$k(\alpha) = k(q-\alpha) \tag{4.3}$$

$$k(\alpha+b) = \gamma(b\alpha)b^{1-2b\alpha}k(\alpha) \tag{4.4}$$

and

$$k(\alpha+c) = \gamma(c\alpha)c^{1-2c\alpha}k(\alpha) \tag{4.5}$$

where

$$q = b+c \qquad c = b^{-1} \quad \text{and} \quad \gamma(x) = \Gamma(x)/\Gamma(1-x) \tag{4.6}$$

Thus the function exhibits the symmetry $\alpha \leftrightarrow q-\alpha$ explicitly and is doubly periodic modulo some simple multiplicative factors. It is actually an entire function and its only zeros are simple ones at the points

$$\alpha = mb+nc \quad \text{and} \quad (m+1)b+(n+1)c \qquad m,n = 0,1,2.... \tag{4.7}$$

Thus it has a double series of zeros. For the full function $C_3(\alpha_a, -\xi/b)$ we then have

$$C_3(\alpha_a, -\xi/b) \to \frac{(\mu_r)^{mb}}{(\xi+mb)} \frac{k'(0)}{k'(-mb)} \left[\frac{k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_1+\alpha_2-\alpha_3)k(\alpha_2+\alpha_3-\alpha_1)k(\alpha_3+\alpha_1-\alpha_2)} \right] \tag{4.8}$$

for $\xi \to -mb$ and the question is: what is the correct extrapolation $C_3(\alpha_a, -\xi/b)$ that satisfies this equation?

5. The DOZZ Ansatz

Recently there has been a suggestion by Dorn and Otto [4] and the Zamilodchikovs [5] that the simple extrapolation

$$C_3(\alpha_a, -\xi/b) = (\mu_r)^{\xi} \frac{k'(0)}{k(-\xi)} \left[\frac{k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_1+\alpha_2-\alpha_3)k(\alpha_2+\alpha_3-\alpha_1)k(\alpha_3+\alpha_1-\alpha_2)} \right] \tag{5.1}$$

is the correct one. We shall call this the DOZZ Ansatz and, apart from simplicity, there are three strong arguments in its favour:

I: Maximal Analyticity: The function $1/k(-\xi)$ has simple poles at the points given in (4.7). The replacement of $1/k'(-\xi)$ with $1/k(-\xi)$ does not introduce any new poles but its replacement by any other function would do so. Thus the DOZZ minimizes the number of new poles that are introduced.

II: Reflection Invariance: The numerator in (5.1) is covariant with respect to reflection symmetry in the sense that under the reflection $\alpha_a \to q-\alpha_a$ we have

$$k(\alpha_1)k(\alpha_2)k(\alpha_3) \to \left[\lambda(\alpha_a)\right] k(\alpha_1)k(\alpha_2)k(\alpha_3) \tag{5.2}$$

where

$$\lambda(\alpha_a) = -4\alpha_a^2\gamma(2b\alpha_a)\gamma(2c\alpha_a)b^{-4q\alpha_a} \tag{5.3}$$

for any choice of $a = 1, 2, 3$. In the numerator, however, two of the factors in large bracket in (5.1) interchange, and the third one changes to $k(-\xi)$. Hence, according as $k'(-\xi)$ is replaced by $k(-\xi)$ or not, the numerator in (5.1) is *invariant* or has no particular property with respect to reflection symmetry.

III: Unitarity: If it is assumed that the $e^{\alpha\phi(x)}$ form a complete set of states $|\alpha>$ one can express the 4-point functions in terms of the 3-point functions by inserting a complete set of states according to

$$\prod_{a=1}^{4} < |e^{\alpha_a\phi(x_a)}| >= \int d\alpha \prod_{i=1}^{2} < |e^{\alpha_i\phi(x_i)}\Big(|\alpha >< \alpha|\Big) \prod_{j=1}^{2} < e^{\alpha_j\phi(x_j)}| > \tag{5.4}$$

Although the 4-point functions are unknown for most values of the parameters α there are some special values for which they are known, and Teschner [11] has shown that for these values (5.4) is true only if C_3 satisfies the DOZZ Ansatz.

6. Duality

Making the DOZZ Ansatz we see from (4.7) that although we started off from the residues at the poles $\xi = -mb$ of the zero-mode integral, we ended up with a second set of poles, namely those with b replaced by $c = b^{-1}$. In other words, although we obtained only the single sequence of poles associated with mb from the zero-mode integration, the DOZZ Ansatz produces a second set, associated with nc. This means that the $b \leftrightarrow c$ symmetry that first appeared in the Virasoro centre now appears in the coefficient of the 3-point function. Furthermore, the double sequence of poles occurs also for the residues, considered as functions of the α_a and thus the $b \leftrightarrow c$ symmetry is linked to the $\alpha \to q - \alpha$ symmetry of the conformal weights Δ_α.

Although this is a little surprising it seems to be a property of the free-field integral. However, it is a property that is not shared by the zero-mode part of the path-integral, which, it will be recalled, was carried out explicitly and yielded only a Γ-function, which has a single set of poles. Thus there is a discrepancy between the fluctuating and zero-mode part of the path-integral. This difference is compensated in an ad hoc way by the DOZZ Ansatz, through which the Γ-function with a single set of poles is extended to a function with a double set. In effect the Ansatz replaces

$$\Gamma(\xi/b + 1)\Gamma(-\xi/b) \equiv \frac{1}{\sin(\xi/b)} \quad \text{with poles at} \quad \xi = mb \qquad m \in Z \tag{6.1}$$

by

$$k^{-1}(-\xi) \quad \text{with poles at} \quad \xi = mb \quad \text{and} \quad \xi = nc \qquad m, n \in Z \tag{6.2}$$

modulo $m = 0, n \in Z_+$ and $n = 0, m \in Z_+$. Since the Ansatz is an ad hoc one made only for the 3-point function it raises the question as to whether the double set of poles has a deeper origin and a whether a prescription exists for the N-point functions with $N > 3$.

7. Generalized Proposal

A way out of this zero-mode difficulty is to incorporate the double pole prescription into the theory from the beginning [12] as follows: Instead of the Liouville theory being defined as a theory of a single scalar field with an exponential potential it can be defined as a conformal invariant theory of a single scalar field, without any a priori prejudice as to the form of the potential. In the classical theory the fact that the potential should have conformal weight $(1,1)$ limits the potential to being the usual exponential one, but in the quantum theory the fact that there are two primary fields for each conformal weight permits the potential to be a sum of these two primary fields with related coefficients i.e. permits the path-integral to take the form

$$\int d\tilde{\phi} e^{i\int d^2x\left(\frac{1}{2}(\partial\tilde{\phi})^2+\mu_b e^{2b\tilde{\phi}(x)}+\mu_c e^{2c\tilde{\phi}(x)}+qR\tilde{\phi}\right)} \quad \text{where} \quad bc=1 \tag{7.1}$$

It can be verified a posteriori that the condition $bc = 1$ is both necessary and sufficient for Weyl, and therefore conformal, invariance. So a theory with path-integral (7.1) is undoubtedly a conformally invariant theory of a single scalar field. In fact, such a potential was already suggested in [13]. It turns out that, although the non-zero mode part of the path integral (7.1) takes a more complicated form, it gives exactly the same result as before provided that the constants μ_b and μ_c are related by

$$\left[-\mu_b\gamma(b^2)\right]^c = \left[-\mu_c\gamma(c^2)\right]^b \quad (= \mu_R \quad \text{say}) \tag{7.2}$$

In particular the coefficient C_3 of the 3-point function reduces to (4.2) with $\mu_r = \mu_R$. On the other hand, the zero-mode contribution to the path integral changes to

$$\frac{1}{\sin(\pi\xi/b)} = b\int du\frac{\delta(\xi-bu)}{\sin(\pi u)} \quad \to \quad \int dudv\frac{\delta(\xi-bu-cv)}{\sin(\pi u)\sin(\pi v)} = L(\xi) \tag{7.3}$$

where $L(\xi)$ is the Barnes di-logarithmic function. This function has the required double pole structure. In fact it is a natural partner of the function $k'(-\xi)/k(-\xi)$ in the sense that it has simple poles at the same points, with the same residues 1 for positive ξ but the opposite residues -1 for negative ξ. In other words it is even with respect to the symmetry $\alpha \to q-\alpha$ where $k'(\xi)/k(\xi)$ is odd. The path-integral (7.1) still has the problem that its fluctuating part can be computed only at the poles of the zero-mode integral, so that, just as before, one has to extrapolate from the residues at these poles to the full function. But, since there are now *two* sets of poles in the zero-mode integral, the conditions on the extrapolation are more stringent and the extrapolation no longer introduces new poles. Thus the extrapolation is more natural in this case. In short, the double-exponential potential works has the advantage that the pole structure is the same for the zero-mode and fluctuating pieces of the path-integral, and it remains stable under extrapolation.

8. Conclusions and Open Questions

In 2-dimensional quantum Liouville theory, with potentials $e^{2b\phi(x)}$ and primary external fields $e^{\alpha_a\phi(x_a)}$, the symmetries

$$b \leftrightarrow b^{-1} \quad \text{and} \quad \alpha \leftrightarrow q-\alpha \quad \text{where} \quad q = b+b^{-1} \tag{8.1}$$

are intrinsic and inter-related, appearing not only in the Virasoro centre but in the coefficient of the 3-point function. This symmetry can be incorporated in the path integral by using a two-exponential potential with parameters by (7.1) and (7.3).

Questions which arise are the following:

I: Given that Liouville theory may be obtained from SL(2,R) Wess-Zumino-Witten theory by Hamiltonian reduction, is the duality already present in the WZW theory or is it induced by the reduction? We hope to show in a later paper that it is actually a mixture of the two. The WZW theory has the symmetry $\alpha \to -\alpha$ and the reduction changes this to $\alpha \to q - \alpha$.

II: What is the situation for Toda theories?

III: Does a similar duality exist for quantum affine Toda theories, in particular for Sinh-Gordon theory?

References

[1] H. Poincare, J. Math. Pure App. **5** *Se 4* (1898) 157;

[2] A. Polyakov, Phys. Lett. **B103** (1981) 207;

[3] J.-L. Gervais and A. Neveu, Nuc. Phys. **B238** (1984) 125;

[4] H. Dorn and H.-J. Otto, Phys. Lett. **B291** (1992) 39; Nuc. Phys. **B429** (1994) 375;

[5] A. and Al. Zamolodchikov, Nuc. Phys. **B 477** (1996) 577; Low Dimensional Applications of Quantum Field Theory, ed. L. Baulieu, V. Kazakov, M. Picco, and P. Windey, NATO ASI Series B, Physics Volume 362, Plenum Press 1997. **B238**(1984) 396; **257**[FS14] (1985) 59;

[6] T. Curtright and C. Thorn, Phys. Rev. Lett. **48** (1982) 1309;

[7] E. Braaten, T. Curtright and C. Thorn, Phys. Lett **118** (1982) 115; Ann. Phys. **147** (1983) 365;

[8] E. D'Hoker and R. Jackiw, Phys. Rev. **D26** (1982) 3517.

[9] M. Goulian and M . Li, Phys. Rev. Lett. **66** (1991) 2051

[10] Vl. Dotsenko and V. Fateev, Nuc. Phys. **B251** (1985) 691.

[11] J. Teschner, Phys. Lett. **B363** (1995) 65.

[12] L. O'Raifeartaigh, J. M. Pawlowski and V. V. Sreedhar, Duality in Quantum Liouville Theory, hep-th/9811090

[13] S. Dotsenko, Mod. Phys. Lett. **A6** (1991) 3601.