# CONTRUCTING UNKNOTTING OPERATIONS FROM 4-MANIFOLDS: GRIDS AS UNKNOTTING OPERATIONS, II

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Abstract. In [Ask2] we defined certain grid local moves $(d_1, r_1) \times (d_2, r_2)$ on knot diagrams and showed all grids of type $(a, 0) \times (b, 0)$ with $a \in \{2, 3, 4\}$ and $b$ arbitrary positive integer to be unknotting operations. In this paper we will show that any grid of type $(d_1, 0) \times (d_2, 0)$ with $\gcd(d_1, d_2) = 1, 2, 3, 4$ is an unknotting operation. We will contruct the proof via classical results of 4-dimensional topology.

## 1. Introduction

We remind the reader the definition of a grid move introducing at the same time notation for ease of exposition:

**Definition 1.1.** *A grid local move of type $(p, r) \times (q, s)$ will be a local move as in Figure 1 below. We will refer to the left (right) to right (left) direction as a "$\pm$ to $\mp$" move. A grid move of type $(p, 0) \times (q, 0)$ will be refered to as one of type $p \times q$. By $|p \times q|$ we will denote the union $\cup_{r,s} \#_{(p,r)\times(q,s)}$.*

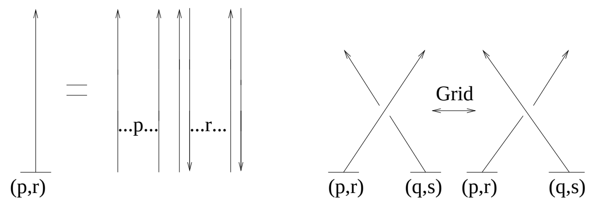


Figure 1. Left: Notation. Right: A grid move of type $(p, r) \times (q, s)$.

In [Ask1] we defined certain multi-# local moves and showed them to be unknotting operations. It immediately followed that grid local moves of type $(2, 0) \times (b, 0)$ with $b \geq 0$ are also unknotting operations. This motivated the search for more grids which unknot and lead to [Ask2] where we showed that any grid of type $(a, 0) \times (b, 0)$ with $a = 3, 4$ and $b \geq 0$ is also an unknotting operation (modulo the fact that a grid of type $4 \times 2b$ preserves the Arf invariant of a knot. It is the author's conjecture that all grids are unknotting operations and in this paper we will prove more grids to be unknotting operations in support of this conjecture. More specifically we will show:

*Key words and phrases.* 4-manifolds, unknotting operations, unknotting numbers, 4-manifolds, diffeomorphism group.

**Acknowledgements**: I would like to thank the people of Max-Planck-Institut für Mathematik for their hospitality.

**Theorem 1.1.** *Any grid local move of type $(p,0)\times(q,0)$ with $\gcd(p,q)=1,2,3,4$ is an unknotting operation (modulo the fact that grids local moves of type $4a\times 2b$ preserve the Arf invariant.*

To prove this we will use some well known classical results from 4-dimensional topology due to C. T. C. Wall and R. Kirby on the realization of the automorphism group of a 4-manifold of type $X^4\#S^2\times S^2$ by diffeomorphisms. Wall proved that the entire group is realizable and Kirby shows how to "see" these diffeomorphism in terms of handle slides. As with every unknotting operation $\lambda$ we have an associated metric $d_\lambda$ on the space $\mathcal{K}$ of all knots and knot invariants: the $\lambda$-unknotting numbers of knots $u_\lambda$ i.e. the $d_\lambda$-distance of a knot from the unknot. In this paper we are only interested in the contruction of the unknotting operations and not on estimating the invariants.

## 2. 4-dimensional topology

We first introduce some language and definitions which will facilitate further discussion.

**Definition 2.1.** *Let $M$ be a smooth 4-manifold and $x\in H_2(M)$ any homology class. By $\mathcal{K}_{(M,x)}$ we will denote the set of all knots $K$ such that there is a sphere $A$ in $M$ representing $x$ which is smooth everywhere except at a point where it is the cone on $K$.*

**Definition 2.2.** *Let $L$ be a framed link of $n=\#(L)$ components, $\overline{v}=((c_1,e_1),\dots,(c_n,e_n))$ and $L(\overline{v})$ be the link obtained from $L$ as follows. Take frame-parallel copies of each component of $L$ as follows: for the $i^{th}$ component take $c_i$ like-oriented copies and $e_i$ pairs of oppositely oriented ones. Then $\mathcal{K}_{(L,\overline{v})}(K)$ will stand for the set of all knots of the form $K\#_b L(\overline{v})$ where $\#_b$ stands for band-connected sum. $K$ will be called the base knot of $\mathcal{K}_{(L,\overline{v})}(K)$ and we will simply write $\mathcal{K}_{(L,\overline{v})}$ when $K$ is the unknot. If $L$ is the unknot with zero framing and $\overline{v}=(c,e)$ then we set $c+e=s$ and simply write $\mathcal{K}_{(O,s)}(K)$*

For an unknotting operation $\lambda$ we have associated unknotting sets $\mathcal{K}_\lambda(\theta,\alpha)$ defined as follows: $K\in\mathcal{K}_\lambda(\theta,\alpha)$ if it can be unknotted by $\theta$ $\lambda$-moves and $\alpha$ inverse $\lambda$-moves. Although we will not need its full power we have the theorem below:

**Theorem 2.1.** *Let $L$ be a framed link as above representing a 4-manifold $M$. Label the components of $L$ by $K_i$ for $i=1,\dots,n$ and suppose that $x=\sum_{i=1}^n c_i[K_i]\in H_2(M)$. Then:*

$$K\in\mathcal{K}_{(M,x)}\Longleftrightarrow \exists {e_i}'s,s:\quad \left(\bigcup_S \mathcal{K}_{(L,\overline{v})}(S)\right)\cap\mathcal{K}_{(O,s)}(K)\neq\emptyset$$

*where the union is taken over all ribbon knots $S$. Furthermore let $L$ consist of $\theta+\alpha$ Hopf links $H_i=\{K_{i1},K_{i2}\}$ for $i=1,\dots,\theta+\alpha$ with zero framings i.e. $M=\#_{\theta+\alpha}S^2\times S^2$. Label the components of $L$ as $K_i$'s in lexicographic order and suppose $x=\oplus_\theta(p,q)\oplus_\alpha(-p,q)\in H_2(M)$ stands for a homology class in the standard basis. Let $c_i=q$ when $i$ is even, $=p$ when $i$ is odd and less than $2\theta$ and $=-p$ otherwise. Then:*

$$K\in\mathcal{K}_{\#_{|p\times q|}}(\theta,\alpha)\Longleftrightarrow \exists {e_i}'s:\quad K\in\mathcal{K}_{(L,\overline{v})}\Longrightarrow K\in\mathcal{K}_{(M,x)}$$

**Sketch of Proof**: The first part (see also [Ka] p. 172) consists of Morse theoretic considerations. The right hand side of the intersection which is stated to be non-empty is related to the Ribbon-Slice problem. If "Slice $\Rightarrow$ Ribbon" it can be replaced by $\{K\}$. The nonempty intersection is nothing but the level between minimums and maximums. There are no saddle points.

The second part follows by easy geometric considerations as in [Ask1] $\square$

Let $L = A \oplus H$ be an even integral unimodular quadratic form where $H$ is the $2\times 2$ hyberbolic form. Let $a_i, a$ be a basis for $A$ and $u, v$ be the standard hyperbolic basis for $H$ i.e. $u^2 = v^2 = 0$ and $u \cdot v = 1$. Define isomotries $I_{a,u}$ $I_{a,v}$ as follows:

$$I_{a,v} : a_i \to a_i - (a_i \cdot a_i)v, \quad a \to a - (a \cdot a)v, \quad u \to u + a - 1/2(a \cdot a)v, \quad v \to v$$

and $I_{a,u}$ as above with the roles of $u, v$ reversed. If $M^4$ is an even 4-manifold then the intersection form of $M^4 \# S^2 \times S^2$ splits as above. Wall then shows that the isometries $I_{a,u}$ and $I_{a,v}$ above generate the automorphism group of $L$ and that when $L$ is the intersection form of a stabilized 4-manifold as above they are realizable by diffeomorphisms (see [W1], [W2], [W3]). In [Ki] (pp. 60-63) Kirby shows how to see these diffeomorphisms in terms of handle slides.

In general given a framed link presentation of a 4-manifold a sequence of handle slides defines a diffeomorphism between the 4-manifolds represented by the first and the last framed links. If the last framed link is the same as the first we have then created a self-diffeomorphism of the 4-manifold at hand. To see the effect of the diffeomorphism on homology i.e. to see the automorphism of the intersection form this diffeomorphism realizes one has to keep track of basis change during the handle slides. To realize the isometries $I_{a,u}$ one does the following. Begin with a handle presentation of $M^4$ in which there there is one component representing $a$. Slide the handle which corresponds to $v$ over the handle which corresponds to $a$. Then use the (geometrically dual to $v$) handle $u$ to break $u$ away from the rest of the link and to unknot it as well as reduce its framing back to zero. Now you are back to the same framed link as you began with and you have a diffeomorphism which realizes the desired automorphism.

## 3. The Construction and The Proof

In this section we give the construction of the promised unknotting operations and hence the proof of Theorem 1.1. The idea is basically to start with moves which are unknotting operations of a certain dimension, insert into the picture an innocent"inconsequential paramater" which "enlarges" the grid in an unsuspected way, fit this into the picture of a suitable framed link and then apply self diffeomorphism keeping track of our knot along the way. The final book keeping reveals new grids which are then devided into the desired ones.

**Theorem 3.1.** *If a grid local move of type $d \times dpq$, with $\gcd(p, q) = 1$, unknots a knot $K$ then a grid local move of type $dp \times dq$ also unknots it.*

**Proof**: Suppose that $K$ can be unknotted by one grid local move of type $d \times dpq$. We will then show that it can be unknotted by grid local moves of type $dp \times dq$. The general case will then be clear from the proof of this case. It follows easily that $K$ can be unknotted by one grid local move of type $(d, 0) \times (dpq, dpql)$ for any $l \geq 0$ (just do and undo a twist). It follows from the second part of Theorem 2.1 that the knot $K$ sits in the boundary of a punctured $S^2 \times S^2 \# S^2 \times S^2$ as in the figure below:

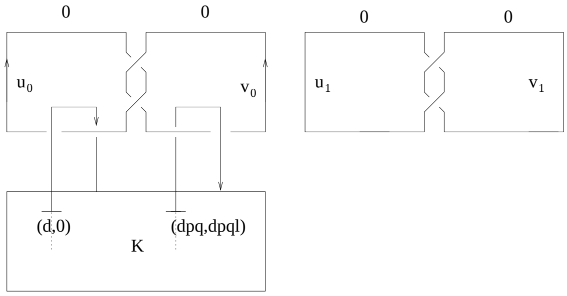


FIGURE 2. Beggining position

If we now perform any (self-diffeomorphism creating) number of handle slides amongst the Hopf links, carrying the knot $K$ along, then zero framed surgery on the first two Hopf links and zero framed surgery on the final framed links will both unknot $K$ because the results are images of each other under the restriction of the self-diffeomorphism of the punctured $S^2 \times S^2 \# S^2 \times S^2$ to the boundary three sphere.

Suppose now that $pp^{'} + qq^{'} = 1$ and set $r = pql$. Suppose also that $p^{'} \leq 0$ and $q^{'} \geq 0$. The parameter $r$ which was all innocently introduced via inconsequential twists will be the key in pulling the proof off.

Let $L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ stand for the intersection form of $S^2 \times S^2 \# S^2 \times S^2$ with standard hyperbolic basis $u_i, v_i$, $i = 0, 1$. As the reader can easily check the isometry $I^{q^{'}}_{u_0,u_1} \circ I^{p^{'}}_{v_0,u_1} \circ I^{p-1}_{u_0,v_1} \circ I^{(p-1)q}_{-u_0,v_1} \circ I_{-v_0,u_1}$ sends $u_0 + pqv_0$ to $pu_0 + qv_0$. After performing the handle slides which realize these isometries we see our knot sitting in the boundary of a punctured $S^2 \times S^2 \# S^2 \times S^2$ as in the Figure below:

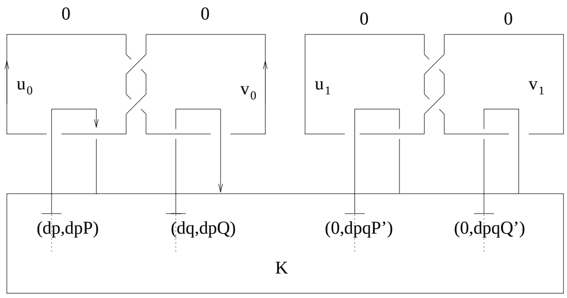


FIGURE 3. Ending Position

where $P = q^{'}(dp-1)(dq+ql)$, $Q = q^{'}[(dp-1)dq+dr]+plp^{'}$, $P^{'} = (dp-1)(d+l)$ and $Q^{'} = dq^{'}$. Hence our knot can be unknotted by two grids of type $(dp, dpP) \times (dq, dqQ)$ and $(0, dpqP^{'}) \times (0, dpqQ^{'})$. A simple inspection of these grids shows that we can break them to grids of type $dp \times dq$. $\square$

**Proof of Theorem 1.1.** It follows from [Su] that any grid of type $(1,0) \times (x,y)$ is an unknotting operation. We have proven in [Ask2] that grids of type $(d, dx)$ with $d = 2, 3, 4$ are unknotting operations (modulo the fact that a grid of type $4 \times 2x$ must preserve the Arf invariant). $\square$

**Remark 3.1.** *So far as the author knows this is the first constructive input of* 4*-manifold theory to the theory of Unknotting Operations. Notice that this technique preserves the "torsion"* $\gcd(p,q)$ *of a grid local move* $p \times q$ *so our work in* [Ask2] *is essential for the proof of Theorem 1.1. We should mention that the unknotting numbers of these unknotting operations can be estimated via classical* 4*-dimensional topology. The problem of which grids are unknotting operations is still not completely resolved. The author believes every grid to be an unknotting operation (modulo the Arf invariant issue).*

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