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TWO RESULTS ON CENTRALISERS OF NILPOTENT ELEMENTS

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INTRODUCTION

Let X and Y be commuting nilpotent endomorphisms of a finite-dimensional vector space V over a field $\Bbbk$. In [4, Sect. 3], McNinch shows that, for all but finitely many points $(a : b) \in \mathbb{P}^1_{\Bbbk}$, both X and Y belong to the nilpotent radical of the centraliser of $aX + bY$ in $GL(V)$. (There is an additional restriction on $aX + bY$ if $\operatorname{char}\Bbbk =: p > 0$; namely, $(aX + bY)^{p-1}$ has to be zero.) From this, he deduces a similar result for commuting nilpotent elements of arbitrary semisimple Lie algebras if $\operatorname{char}\Bbbk$ is sufficiently large, see [4, Theorem 26 and Prop. 28]. However, the proof for $GL(V)$ is rather tedious. It requires lengthy manipulations with Jordan normal forms of X and Y and consideration of nilpotent elements over the field $\Bbbk(t)$.

The goal of this note is two-fold. First, we provide a very short alternative proof of McNinch's results if $\Bbbk$ is algebraically closed and $p = 0$ or sufficiently large. We use only standard properties of $\mathfrak{sl}_2$-triples and centralisers of nilpotent elements, and work with an arbitrary simple Lie algebra. Second, we characterise the nilpotent elements e such that $G{\cdot}e$ is the largest nilpotent orbit meeting the centraliser of e. Such nilpotent elements (orbits) are said to be *self-large*. In the last section, we discuss some problems related to self-large orbits.

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1. A SHORT PROOF OF MCNINCH'S RESULT

Throughout, G is a connected simple algebraic group over $\Bbbk$, where $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk = 0$, and $\mathfrak{g} = \operatorname{Lie} G$. Write $\mathfrak{g}_x$ for the centraliser of $x \in \mathfrak{g}$ and $\mathcal{N}$ for the nilpotent cone in $\mathfrak{g}$. The nilpotent radical of a Lie algebra $\mathfrak{q}$ is denoted by $\mathfrak{q}^u$.

Let us start with a reformulation of the McNinch's result. Given commuting (non-proportional) elements $x, y \in \mathcal{N}$, we consider the "commutative nilpotent" plane $\mathcal{P} = \Bbbk x + \Bbbk y \subset \mathcal{N} \subset \mathfrak{g}$. It is then claimed that, for almost all $e = ax + by \in \mathcal{P}$, x and y belong to $(\mathfrak{g}_e)^u$. Let us give a more precise meaning to the words "almost all". Since the closure of $G{\cdot}\mathcal{P}$ is irreducible, there is a unique nilpotent G-orbit, $\mathcal{O}$, such that $\mathcal{O} \cap \mathcal{P}$ is dense in $\mathcal{P}$. So we will actually require that $e \in \mathcal{O}$.

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Theorem 1.1. *Suppose $e, x \in \mathcal{N}$, $[e, x] = 0$, and the intersection of the orbit $G{\cdot}e$ with $\mathcal{P} = \Bbbk e + \Bbbk x$ is dense in $\mathcal{P}$. Then $x \in (\mathfrak{g}_e)^u$.*

Before giving a proof, we fix some notation and state an auxiliary result. Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple containing e and $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ the corresponding $\mathbb{Z}$-grading of $\mathfrak{g}$. Here $\mathfrak{g}(i)$ is the i-eigenspace of $\mathrm{ad}\, h$. In particular, $\mathfrak{g}(0) = \mathfrak{g}_h$. Then $\mathfrak{p} = \bigoplus_{i\geqslant 0} \mathfrak{g}(i) =: \mathfrak{g}_{\geqslant 0}$ is a parabolic subalgebra and $\mathfrak{p}^u = \mathfrak{g}_{\geqslant 1}$. Set $\mathfrak{g}_e(i) = \mathfrak{g}(i) \cap \mathfrak{g}_e$. As is well known, $\mathfrak{g}_e = \bigoplus_{i\geqslant 0} \mathfrak{g}_e(i)$ and $\mathfrak{g}_e(0)$ is a Levi subalgebra of $\mathfrak{g}_e$. Furthermore, $\mathfrak{g}_e(0) = \mathfrak{g}_e \cap \mathfrak{g}_f$ [1, Ch. 3]. Let $\alpha_h : \Bbbk^\times \to G$ be the one-parameter subgroup such that $\alpha_h(t){\cdot}y = t^i y$ for any $y \in \mathfrak{g}(i)$.

The following observation is extracted from the proof of Proposition 1.2 in [6].

Lemma 1.2 (Premet). *If $x_0 \in \mathfrak{g}_e(0)$ is nonzero and nilpotent, then $e + x_0$ and e are not conjugate. Moreover, e lies in the closure of $G{\cdot}(e + x_0)$.*

Proof. For convenience of the reader, we recall Premet's argument. Since $x_0 \in \mathfrak{g}_e(0)$ is nilpotent, there is an $\mathfrak{sl}_2$-triple $\{x_0, h', y\}$ contained in $\mathfrak{g}_e(0)$. It follows that $\{e+x_0, h+h', f+y\}$ is also an $\mathfrak{sl}_2$-triple. Being a member of an $\mathfrak{sl}_2$-triple, h' lies in $[\mathfrak{g}(0), \mathfrak{g}(0)]$. Therefore h and h' are orthogonal with respect to the Killing form, κ, on $\mathfrak{g}$ and hence $\kappa(h+h', h+h') > \kappa(h, h)$. It follows that $h \underset{G}{\not\sim} h + h'$ and hence $e \underset{G}{\not\sim} e + x_0$ [1]. Finally, we have
$\alpha_{h+h'}(t)\alpha_h(-t){\cdot}(e + x_0) = e + t^2 x_0$, which implies that $e \in \overline{G{\cdot}(e + x_0)}$. □

Proof of Theorem 1.1. Using the above notation, write $x = x_0 + x_1 + \ldots$, where $x_i \in \mathfrak{g}(i)$. Our goal is to prove that $x_0 = 0$. Since $e \in \mathfrak{g}(2)$, we have $[e, x_i] = 0$ for all i.

Consider the commutative nilpotent planes $\mathcal{P}_t = \alpha_h(t){\cdot}\mathcal{P}$ for $t \in \Bbbk^\times$. Clearly, $\mathcal{P}_t$ is spanned by e and $\alpha_h(t){\cdot}x = x_0 + tx_1 + t^2x_2 + \ldots$. The limit $\lim_{t\to 0} \mathcal{P}_t$ exists in the Grassmannian of 2-planes in $\mathfrak{g}$ and for $x_0 \neq 0$ it is equal to $\mathcal{P}_0 := \Bbbk e + \Bbbk x_0$. We thus obtain another commutative plane, $\mathcal{P}_0$. Furthermore, $\mathcal{P}_0 \subset \mathcal{N}$ (as the limit of $\{\mathcal{P}_t\}$), hence x_0 is nilpotent.

By Lemma 1.2, $e + ax_0$ is not conjugate to e for every $a \neq 0$. Hence $G{\cdot}e \cap \mathcal{P}_0$ is not dense in $\mathcal{P}_0$. Since $\lim_{t\to 0} \mathcal{P}_t = \mathcal{P}_0$, we conclude that $G{\cdot}e \cap \mathcal{P}_t$ is not dense in $\mathcal{P}_t$ for almost all $t \in \Bbbk^\times$, and because all $\mathcal{P}_t$ are G-conjugate, this is also true for $\mathcal{P} = \mathcal{P}_1$. This contradiction shows that $x_0 = 0$, i.e., $x \in (\mathfrak{g}_e)^u$. □

Remark 1.3. a) Under the assumptions of the theorem, we proved that $x_0 = 0$. One may ask whether it is true that $x_1 = 0$ as well. In general, the answer is negative. This follows from Proposition 2.4 below.

b) The previous proof certainly works, if $\mathrm{char}\, \Bbbk$ is sufficiently large. E.g. if $\mathrm{char}\, \Bbbk > 4\mathsf{h} - 1$, where h is the Coxeter number of $\mathfrak{g}$.

2. Self-large nilpotent elements/orbits

Recall that $e \in \mathcal{N}$ or $G{\cdot}e$ is said to be *even* if the eigenvalues of $\mathrm{ad}\, h$ are even; it is called *distinguished* if $\mathfrak{g}_e(0) = \{0\}$. It is known that "distinguished" implies "even" [1, Thm. 8.2.3].

Following Premet [6], we say that e is *almost distinguished* if $\mathfrak{g}_e(0)$ is toral (= Lie algebra of a torus). Let $\mathcal{N}(\mathfrak{g}_e)$ denote the set of nilpotent elements of $\mathfrak{g}_e$. It is easily seen that $\mathcal{N}(\mathfrak{g}_e) = \mathcal{N}(\mathfrak{g}_e(0)) \times (\mathfrak{g}_e)_{\geqslant 1} = \mathcal{N}(\mathfrak{g}_e(0)) \times (\mathfrak{g}_e)^u$. Therefore $(\mathfrak{g}_e)^u = \mathcal{N}(\mathfrak{g}_e)$ if and only if e is almost distinguished.

Definition 1. A nilpotent element e (orbit $G{\cdot}e$) is said to be *self-large* if $G{\cdot}e \cap \mathfrak{g}_e$ is dense in $\mathcal{N}(\mathfrak{g}_e)$. In other words, this means that $G{\cdot}e$ is the largest nilpotent orbit meeting $\mathfrak{g}_e$.

Our consideration of self-large orbits was motivated by attempts to better understand Premet's results on "nilpotent commuting variety" [6, Sect. 1] and generalise it to some other situations.

In this section, we give a characterisation of self-large elements. The answer is being given in terms of the $\mathbb{Z}$-grading associated with an $\mathfrak{sl}_2$-triple $\{e, h, f\}$.

Theorem 2.1. *Suppose $e \in \mathcal{N}$, and let $\mathfrak{g}_e = \bigoplus_{i\geqslant 0}\mathfrak{g}_e(i)$ be the $\mathbb{N}$-grading determined by h. Then e is self-large if and only if $\mathfrak{g}_e(0)$ is toral and $\mathfrak{g}_e(1) = 0$.*

For future use, we record the following simple assertion:

(2.1) $\qquad \mathrm{ad}\, f : \mathfrak{g}_e(1) \to \mathfrak{g}_f(-1)$ is a bijection, and the inverse map is just $\mathrm{ad}\, e$.

From this one readily deduce the following

Lemma 2.2. *For any nonzero $\xi \in \mathfrak{g}_f(-1)$ there is $\eta \in \mathfrak{g}_f(-1)$ such that $\kappa(e, [\xi, \eta]) \neq 0$. In particular, $(\xi, \eta) \mapsto \kappa(e, [\xi, \eta])$ is a non-degenerate skew-symmetric $\mathfrak{g}_e(0)$-invariant bilinear form on $\mathfrak{g}_f(-1)$.*

Lemma 2.3. *Assume that there is $z \in \mathfrak{g}_f(-1)$ such that $[z, [z, e]] \neq 0$. Then $[z, e] \in \mathfrak{g}_e(1)$ and the orbit $G{\cdot}(e + [z, e])$ is larger than $G{\cdot}e$.*

Proof. Set $v_z = [z, e]$. By Eq. (2.1), $v_z \in \mathfrak{g}_e(1)$ and also $z = [v_z, f]$. Then

$$\exp(-z)(e + v_z) = e + v_z - [z, e + v_z] + \frac{1}{2}[z, [z, e + v_z]] + \dots$$
$$= e - [z, v_z] + \frac{1}{2}[z, v_z] + \dots = e - \frac{1}{2}[z, v_z] + (\text{terms in } \mathfrak{g}_{\leqslant -1}) \ .$$

Here the element $[z, v_z]$ lies in $\mathfrak{g}(0)$ and an easy computation shows that it commutes with e. Hence it also commutes with f. Thus, we have shown that $\exp(-z)(e + v_z) \in e + \mathfrak{p}^-$, where $\mathfrak{p}^- = \mathfrak{g}_{\leqslant 0}$, and the component of degree zero lies in $\mathfrak{g}_e(0) = \mathfrak{g}_f(0)$.

Set $N = \exp(\mathfrak{g}_{\leqslant -2})$. It is a unipotent group and $e + \mathfrak{p}^-$ is an N-stable subvariety of $\mathfrak{g}$. There is an isomorphism of N-varieties

$$e + \mathfrak{p}^- \simeq N \times (e + \mathfrak{g}_f) \ ,$$

where the N-action on $e+\mathfrak{g}_f$ is trivial, and N acts on itself by left translations. In other words, for every $y\in\mathfrak{p}^-$, the N-orbit of $e+y$ is isomorphic to N and contains a unique element from $e+\mathfrak{g}_f$. For regular nilpotent elements, this is implicit in [3, Sect. 4]. A general proof is given by Katsylo [2, § 5]. Let $\psi(e+y)$ denote the unique point in $N{\cdot}(e+y)\cap(e+\mathfrak{g}_f)$. It is important that the N-action does not affect the zero component of y, y_0, whenever $y_0\in\mathfrak{g}_e(0)$. It follows that

$$\psi\bigl(\exp(-z)(e+v_z)\bigr)=e-\frac{1}{2}[z,v_z]+\bigl(\text{terms in }(\mathfrak{g}_f)_{\leqslant -1}\bigr) \tag{2.2}$$

The affine subspace $e+\mathfrak{g}_f$ is the *transverse* (or Slodowy) slice to $G{\cdot}e$ at e. It follows from [7, 7.4] that $G{\cdot}e\cap(e+\mathfrak{g}_f)=\{e\}$. If $[z,v_z]\ne 0$, then Eq. (2.2) shows that $G{\cdot}(e+v_z)\cap(e+\mathfrak{g}_f)$ contains a point different from e, which implies that $e+v_z\not\in\overline{G{\cdot}e}$. Since $G{\cdot}(e+v_z)\supset e+\Bbbk^\times v_z$ (cf. Proof of Lemma 1.2), we actually have $e\in\overline{G{\cdot}(e+v_z)}$. □

Proof of Theorem 2.1. (a) The sufficiency is easy. If $\mathfrak{g}_e(0)$ is toral and $\mathfrak{g}_e(1)=0$, then $\mathcal{N}(e)=(\mathfrak{g}_e)^u\subset\mathfrak{g}_{\geqslant 2}$. Since $P{\cdot}e$ is dense in $\mathfrak{g}_{\geqslant 2}$, the assertion follows.

(b) Let us prove the necessity. If $\mathfrak{g}_e(0)$ is not toral, then there is a nilpotent element $x_0\in\mathfrak{g}_e(0)$. Then $\tilde e=e+x_0\in\mathcal{N}(e)$ and $\tilde e\not\in\overline{G{\cdot}e}$, see Lemma 1.2.

In the rest of the proof we assume that $\mathfrak{g}_e(0)$ is toral. If $\mathfrak{g}_e(1)\ne 0$, then our goal is to find an element $v\in\mathfrak{g}_e(1)$ such that $e+v$ lies in a larger orbit. By Lemma 2.3, it suffices to find $z\in\mathfrak{g}_f(-1)$ such that $[z,[z,e]]\ne 0$.

Claim 1. The space of $\mathfrak{h}$-fixed vectors in $\mathfrak{g}_f(-1)$ is trivial.

For, consider the semisimple Lie algebra $\mathfrak{s}=[\mathfrak{l},\mathfrak{l}]$, where $\mathfrak{l}=\mathfrak{g}^{\mathfrak{h}}$. Then $e,h,f\in\mathfrak{s}$ and e is distinguished as element of $\mathfrak{s}$. In particular, e is even in $\mathfrak{s}$. Since $\mathfrak{l}=\mathfrak{h}\oplus\mathfrak{s}$ and $\mathfrak{h}\subset\mathfrak{g}(0)$, we have $0=\mathfrak{s}(-1)=\mathfrak{l}(-1)=\mathfrak{g}(-1)^{\mathfrak{h}}$.

It follows from Claim 1 and Lemma 2.2 that the weight decomposition of $\mathfrak{g}_f(-1)$ with respect to $\mathfrak{h}=\mathfrak{g}_e(0)$ can be written as

$$\mathfrak{g}_f(-1)=\bigoplus_{\gamma\in\mathcal{A}}(V_\gamma\oplus V_{-\gamma})\ ,$$

where $\mathcal{A}$ is a subset of $\mathfrak{X}(\mathfrak{h})$ such that $\mathcal{A}\cap(-\mathcal{A})=\varnothing$.

Claim 2. There are $\mu\in\mathcal{A}$ and *weight vectors* $\xi\in V_\mu$, $\eta\in V_{-\mu}$ such that $\kappa(e,[\xi,\eta])\ne 0$.

By Lemma 2.2, there are *some* $\tilde\xi,\tilde\eta\in\mathfrak{g}_f(-1)$ such that

$$\kappa(e,[\tilde\xi,\tilde\eta])\ne 0\ . \tag{2.3}$$

Let $\tilde\xi=\sum_{\gamma\in\mathcal{A}}a_\gamma\xi_\gamma$, $a_\gamma\in\Bbbk$, be the weight decomposition, and likewise for $\tilde\eta$. Substituting this to Eq. (2.3), one readily finds that for some γ, the components ξ_γ and $\eta_{-\gamma}$ satisfies the required property.

Having found such weight vectors, we take $t\in\mathfrak{h}$ such that $[t,\xi]=\xi$ and $[t,\nu]=-\nu$. Then

$$\kappa([[e,\xi+\eta],\xi+\eta],t)=2\kappa(e,[\xi,\eta])\ne 0\ ,$$

which shows that $[[e,\xi+\eta],\xi+\eta]\ne 0$. Hence $z=\xi+\eta$ is a required element. □

Notice that in order to construct a suitable element $v \in \mathfrak{g}_e(1)$, we take the sum of two different weight vectors: $v = [e, \xi] + [e, \eta]$. The reason is that a single weight vector is not suitable, as shows the following

Proposition 2.4. *Suppose $\mathfrak{h} = \mathfrak{g}_e(0)$ is toral and $v \in \mathfrak{g}_e(1)$ is an $\mathfrak{h}$-weight vector. Then $e + v \in G{\cdot}e$.*

Proof. Let $z \in \mathfrak{g}_f(-1)$ be the unique element such that $v = [z, e]$. Then $[z, v] \in \mathfrak{g}(0)$ and $[[z, v], e] = [[z, e], v] = 0$. Thus, $[z, v] \in \mathfrak{h}$ is semisimple. Let $\gamma \in \mathfrak{X}(\mathfrak{h})$ be the $\mathfrak{h}$-weight of v. Then $\gamma \neq 0$ (Claim 1), z has the same weight, and the weight of $[z, v]$ equals 2γ. If follows that $[z, v]$ is nilpotent as well. Hence $[z, v] = 0$. Therefore $\exp(z){\cdot}e = e + [z, e] = e + v$. $\square$

Example 2.5. We describe the almost distinguished orbits in all simple Lie algebras and point out the self-large ones among them.

1. For $\mathfrak{g} = \mathfrak{g}(V)$ classical, the nilpotent orbits are parametrized via partitions of $n = \dim V$. If $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_s)$ is a partition of n, then $\mathcal{O}_{\boldsymbol{\lambda}}$ stands for the corresponding orbit. For $e \in \mathcal{O}_{\boldsymbol{\lambda}}$, a description of $\mathfrak{g}_e(0)$ via $\boldsymbol{\lambda}$ is due to Springer and Steinberg, see e.g. [1, Thm. 6.1.3]. This allows us to quickly find all almost distinguished orbits.

(a) $\mathfrak{g} = \mathfrak{sl}(V)$. Here $\boldsymbol{\lambda}$ is an arbitrary partition and $\mathcal{O}_{\boldsymbol{\lambda}}$ is almost distinguished if and only if all parts of $\boldsymbol{\lambda}$ are distinct. Furthermore, $\mathfrak{g}_e(1) \neq 0$ if and only if $\lambda_i = \lambda_{i+1} + 1$ for some $i < s$ [5, Prop. 3.4]. Thus, the self-large orbits are those satisfying the property $\lambda_i - \lambda_{i+1} \geqslant 2$ for each $i < s$.

(b) $\mathfrak{g} = \mathfrak{so}(V)$. Here each even part of $\boldsymbol{\lambda}$ must occur an even number of times. The orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ is almost distinguished if and only if $\boldsymbol{\lambda}$ has no even parts and each odd part occurs at most twice. Such orbits are even, hence self-large.

(c) $\mathfrak{g} = \mathfrak{sp}(V)$. Here each odd part of $\boldsymbol{\lambda}$ must occur an even number of times. The orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ is almost distinguished if and only if $\boldsymbol{\lambda}$ has no odd parts and each even part occurs at most twice. Such orbits are even, hence self-large.

2. For $\mathfrak{g}$ exceptional, we only indicate the almost distinguished orbits with non-trivial toral part $\mathfrak{g}_e(0)$. Such orbits exist only in type **E**, see Table 1.

Table 1: Almost distinguished orbits in $\mathbf{E}_n$ with non-trivial $\mathfrak{g}_e(0)$

$\mathfrak{g}$	label	diagram	$\mathfrak{g}_e(0)$	$\dim \mathfrak{g}_e(1)$	$\dim \mathfrak{g}_e$
$\boxed{\mathbf{E}_8}$	$\mathbf{D}_7(a_1)$	$\begin{smallmatrix} 2-0-0-2-0-0-2 \\ 0 \end{smallmatrix}$	$\mathfrak{t}_1$	0	26
	$\mathbf{E}_6(a_1)+\mathbf{A}_1$	$\begin{smallmatrix} 2-0-1-0-1-0-1 \\ 0 \end{smallmatrix}$	$\mathfrak{t}_1$	2	30
	$\mathbf{D}_7(a_2)$	$\begin{smallmatrix} 1-0-1-0-1-0-1 \\ 0 \end{smallmatrix}$	$\mathfrak{t}_1$	2	32

Almost distinguished orbits in $\mathbf{E}_n$, cont.

	$\mathbf{D}_5+\mathbf{A}_2$	$\begin{array}{c}2\text{–}0\text{–}0\text{–}2\text{–}0\text{–}0\text{–}0\\ 0\end{array}$	$\mathfrak{t}_1$	0	34
$\boxed{\mathbf{E}_7}$	$\mathbf{E}_6(a_1)$	$\begin{array}{c}0\text{–}2\text{–}0\text{–}2\text{–}0\text{–}2\\ 0\end{array}$	$\mathfrak{t}_1$	0	15
	$\mathbf{A}_4+\mathbf{A}_1$	$\begin{array}{c}0\text{–}1\text{–}0\text{–}1\text{–}0\text{–}1\\ 0\end{array}$	$\mathfrak{t}_2$	4	29
$\boxed{\mathbf{E}_6}$	$\mathbf{D}_5$	$\begin{array}{c}2\text{–}0\text{–}2\text{–}0\text{–}2\\ 2\end{array}$	$\mathfrak{t}_1$	0	10
	$\mathbf{D}_5(a_1)$	$\begin{array}{c}1\text{–}1\text{–}0\text{–}1\text{–}1\\ 2\end{array}$	$\mathfrak{t}_1$	2	14
	$\mathbf{A}_4+\mathbf{A}_1$	$\begin{array}{c}1\text{–}1\text{–}0\text{–}1\text{–}1\\ 1\end{array}$	$\mathfrak{t}_1$	2	16
	$\mathbf{D}_4(a_1)$	$\begin{array}{c}0\text{–}0\text{–}2\text{–}0\text{–}0\\ 0\end{array}$	$\mathfrak{t}_2$	0	20

Remark. It turns out, a posteriori, that for $\mathfrak{g}\ne\mathfrak{sl}_n$, every self-large orbit is even.

3. Problems and examples

Results of Section 2 show that there is a hierarchy of nilpotent G-orbits:

$$\{\text{distinguished orbits}\}\subset\{\text{self-large orbits}\}\subset\{\text{almost distinguished orbits}\}\,,$$

where all inclusions are proper.

Lemma 3.1. *Suppose $e,e'\in\mathcal{N}$ are self-large and $[e,e']=0$. Then $e\underset{G}{\sim}e'$.*

Proof. Consider an $\mathfrak{sl}_2$-triple containing e and the related $\mathbb{Z}$-grading, as above. Since $e'\in\mathcal{N}(\mathfrak{g}_e)=(\mathfrak{g}_e)^u$ and $\mathfrak{g}_e(1)=0$, we have $e'\in\mathfrak{g}_{\geqslant 2}=\overline{P{\cdot}e}$. The assertion follows by the symmetry of e and e'. □

Below we discuss several related problems.

Since $\mathcal{N}(\mathfrak{g}_e)$ is irreducible, there is always a unique *maximal* nilpotent orbit meeting $\mathfrak{g}_e$. That is, we obtain the mapping $\mathcal{D}:\mathcal{N}/G\to\mathcal{N}/G$ which assigns the dense G-orbit in $G{\cdot}\mathcal{N}(\mathfrak{g}_e)$ to $G{\cdot}e$.

Problem 1. Determine explicitly $\mathcal{D}$, i.e., for every $G{\cdot}e\in\mathcal{N}/G$ describe the orbit $\mathcal{D}(G{\cdot}e)$.

For classical Lie algebras, one should expect a recipe in terms of partitions. However, this seems to be a non-trivial task. Note that if $\mathcal{O}_{min}\subset\mathcal{N}$ is the minimal nonzero orbit and $v\in\mathcal{O}_{min}$, then $\mathfrak{g}_v$ contains the nilpotent radical of a Borel subalgebra. Hence, for any $e\in\mathcal{N}$, the unique *minimal* nonzero nilpotent orbit meeting $\mathfrak{g}_e$ is always $\mathcal{O}_{min}$.

Problem 2. Describe the image of $\mathcal{D}$.

By definition, the self-large orbits are those having the property that $\mathcal{D}(\mathcal{O}) = \mathcal{O}$. In particular, they belong to $\mathrm{Im}\,\mathcal{D}$. Are there some other orbits? Equivalently, is it true that $\mathcal{D}^2 = \mathcal{D}$? At least, my direct computations of $\mathcal{D}$ for small ranks provide only self-large orbits in $\mathrm{Im}\,\mathcal{D}$.

Problem 3. Describe *all* nilpotent G-orbits meeting $\mathfrak{g}_e$.

The answer should be helpful for better understanding the structure of the nilpotent commuting variety. By Lemma 3.1, if e is self-large, then no other self-large orbits meet $\mathfrak{g}_e$.

Example 3.2. Suppose $\mathfrak{g} = \mathfrak{sl}_n$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$, and $e \in \mathcal{O}_{\boldsymbol{\lambda}}$. If e is not self-large, then it is easy to indicate larger nilpotent orbits meeting $\mathfrak{g}_e$. Namely, if $\lambda_i - \lambda_{i+1} \leqslant 1$ for some i, then one can replace two parts λ_i, λ_{i+1} with one part $\lambda_i + \lambda_{i+1}$ (with eventual rearranging the resulting parts). More generally,

$$(*) \qquad \begin{cases} \text{a substring } \dots, a^k, (a-1)^l, \dots \text{ of } \boldsymbol{\lambda} \text{ can be replaced} \\ \text{with the single part } ka + l(a-1). \end{cases}$$

One can do the same thing with *other* parts of the initial partition, if possible, but it is not allowed to apply this to newly obtained parts. However, concatenation of such steps is not sufficient for constructing $\mathcal{D}(\mathcal{O}_{\boldsymbol{\lambda}})$. For instance, take $\boldsymbol{\lambda} = (3, 1, 1)$ for $\mathfrak{sl}_5$. Then

$$(3, 1, 1) \mapsto (3, 2) \not\mapsto (5) .$$

That is, $\mathcal{O}_{(3,2)}$ meets the centraliser of $e \in \mathcal{O}_{(3,1,1)}$. However, a direct verification shows that $\mathcal{D}(\mathcal{O}_{(3,1,1)}) = \mathcal{O}_{(4,1)}$. Note that $\mathcal{O}_{(4,1)}$ is self-large, while $\mathcal{O}_{(3,2)}$ is not. Similarly, for $\mathfrak{g} = \mathfrak{sl}_7$, we have $\mathcal{D}(\mathcal{O}_{(4,2,1)}) = \mathcal{O}_{(5,2)}$.

Let us justify rule $(*)$. Taking the respective Jordan subspaces, it suffices to assume that $\boldsymbol{\lambda} = (a^k, (a-1)^l)$. Let e be a regular nilpotent element of $\mathfrak{sl}_n$ with $n = ka + l(a-1)$. Then $\mathcal{O}_{\boldsymbol{\lambda}}$ is the orbit of e^{k+l}, hence the assertion.

Example 3.3. For some classes of orbits, the description of all orbits meeting $\mathcal{N}(\mathfrak{g}_e)$ is available. If $e \in \mathfrak{g} = \mathfrak{sl}_n$ is regular nilpotent, then $e, e^2, \dots, e^{n-1}$ form a basis for $\mathfrak{g}_e$. It is easily seen that if $\mathcal{O}$ meets $\mathfrak{g}_e$, then $\mathcal{O} = SL_n{\cdot}e^k$ for some k. The partition of e^k has k nonzero parts; $n - k\left[\frac{n}{k}\right]$ parts are of size $\left[\frac{n}{k}\right] + 1$ and the remaining parts are of size $\left[\frac{n}{k}\right]$.

Similar situation occurs for $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$, where one has to take odd powers of e.

Example 3.4. For $\mathfrak{g} = \mathfrak{sl}_7$, we have $\mathrm{Im}\,\mathcal{D} = \{\mathcal{O}_{(7)}, \mathcal{O}_{(6,1)}, \mathcal{O}_{(5,3)}\}$, i.e., precisely the set of self-large orbits. The full description of $\mathcal{D}$ is given by the following data:

$\mathcal{D}^{-1}(\mathcal{O}_{(7)}) = \{\mathcal{O}_{(7)}, \mathcal{O}_{(4,3)}, \mathcal{O}_{(3,2,2)}, \mathcal{O}_{(2^3,1)}, \mathcal{O}_{(2^2,1^3)}, \mathcal{O}_{(2,1^5)}\}$;

$\mathcal{D}^{-1}(\mathcal{O}_{(6,1)}) = \{\mathcal{O}_{(6,1)}, \mathcal{O}_{(3,3,1)}, \mathcal{O}_{(3,2,1,1)}, \mathcal{O}_{(3,1^4)}\}$;

$\mathcal{D}^{-1}(\mathcal{O}_{(5,2)}) = \{\mathcal{O}_{(5,2)}, \mathcal{O}_{(5,1,1)}, \mathcal{O}_{(4,2,1)}, \mathcal{O}_{(4,1^3)}\}$.

Example 3.5. For $\mathfrak{g} = \mathfrak{so}_7$, we again have 3 self-large orbits and

$\mathcal{D}^{-1}(\mathcal{O}_{(7)}) = \{\mathcal{O}_{(7)}, \mathcal{O}_{(3,2,2)}, \mathcal{O}_{(2^2,1^3)}\}$,

$\mathcal{D}^{-1}(\mathcal{O}_{(5,1,1)}) = \{\mathcal{O}_{(5,1,1)}, \mathcal{O}_{(3,1^4)}\}$, $\quad \mathcal{D}^{-1}(\mathcal{O}_{(3,3,1)}) = \{\mathcal{O}_{(3,3,1)}\}$.

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