NON-KÄHLER COMPACT COMPLEX MANIFOLDS ASSOCIATED TO NUMBER FIELDS

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Abstract. For algebraic number fields K with $s > 0$ real and $2t > 0$ complex embeddings and "admissible" subgroups U of the group of integer units of K we construct and investigate $(s+t)$-dimensional compact complex manifolds $X(K, U)$. We show among other things that such manifolds are non-Kähler but admit locally conformally Kähler metrics when $t = 1$. In particular we disprove a conjecture of I. Vaisman.

1. Notations, construction and first properties

Consider an algebraic number field K, that is a finite extension field of the field of rational numbers $\mathbb{Q}$. Let $n := (K : \mathbb{Q})$ be its degree. The field K admits precisely $n = s + 2t$ distinct embeddings $\sigma_1, ...\sigma_n$ into $\mathbb{C}$, where we suppose that $\sigma_1, ...\sigma_s$ are the real embeddings, $\sigma_{s+1}, ..., \sigma_n$ are the complex ones and that $\sigma_{s+i} = \bar{\sigma}_{s+i+t}$ for $1 \leq i \leq t$. We shall suppose throughout the paper that both s and t are strictly positive. Furthermore, let $\mathcal{O}_K$ denote the ring of algebraic integers of K. This is a free $\mathbb{Z}$-module of rank n. In fact our construction works also for arbitrary orders $\mathcal{O}$ of K, i.e. for subrings $\mathcal{O}$ of $\mathcal{O}_K$ which have rank n as $\mathbb{Z}$-modules.

Set now $m := s + t$ and consider the "geometric representation" of K:

$$\sigma : K \to \mathbb{C}^m, \quad \sigma(a) := (\sigma_1(a), ..., \sigma_m(a)).$$

It is known that the image $\sigma(\mathcal{O}_K)$ of $\mathcal{O}_K$ through σ is a lattice of rank n in $\mathbb{C}^m$, cf. [1], 2.3.1., p. 95ff. We thus get a properly discontinuous action of $\mathcal{O}_K$ on $\mathbb{C}^m$ by translations. Consider furthermore the following multiplicative action of K on $\mathbb{C}^m$: for $a \in K$ and $z \in \mathbb{C}^m$ set

$$az := (\sigma_1(a)z_1, ..., \sigma_m(a)z_m).$$

For $a \in \mathcal{O}_K$, $a\sigma(\mathcal{O}_K)$ is contained in $\sigma(\mathcal{O}_K)$. Let $\mathcal{O}_K^*$ denote the group of units in $\mathcal{O}_K$ and

$$\mathcal{O}_K^{*,+} := \{a \in \mathcal{O}_K^* \mid \sigma_i(a) > 0 \text{ for all } 1 \leq i \leq s\}.$$

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Since for $s > 0$ the only torsion elements of $\mathcal{O}_K^*$ are 1 and -1, Dirichlet's Units Theorem allows us to write $\mathcal{O}_K^* = G \cup (-G)$, where G is a free abelian (multiplicative) group of rank $m-1$. One may choose G so that it contains $\mathcal{O}_K^{*,+}$, automatically with finite index. We denote by $\mathbb{H}$ the upper complex half-plane, $\mathbb{H} := \{z \in \mathbb{C} \mid \Im m z > 0\}$. Combining the additive action of $\mathcal{O}_K$ with the induced multiplicative action of $\mathcal{O}_K^{*,+}$ we get an action of $\mathcal{O}_K^{*,+} \ltimes \mathcal{O}_K$ on $\mathbb{C}^m$ which is free and leaves $\mathbb{H}^s \times \mathbb{C}^t$ invariant. We shall now choose a subgroup U of rank s of $\mathcal{O}_K^{*,+}$ such that the action of $U \ltimes \mathcal{O}_K$ on $\mathbb{H}^s \times \mathbb{C}^t$ becomes properly discontinuous, thus yielding a smooth quotient which will be shown to be compact. In order to do this we consider the logarithmic representation of units

$$l : \mathcal{O}_K^* \to \mathbb{R}^m, \quad l(u) := (ln|\sigma_1(u)|, ..., ln|\sigma_s(u)|, 2ln|\sigma_{s+1}(u)|, ..., 2ln|\sigma_m(u)|),$$

cf. [1] 2.3.3.. Dirichlet's Units Theorem implies that $l(\mathcal{O}_K^{*,+})$ is a full lattice in the subspace $L := \{x \in \mathbb{R}^m | \sum_{i=1}^m x_i = 0\}$ of $\mathbb{R}^m$. Since $t > 0$, the projection $pr : L \mapsto \mathbb{R}^s$ given by the first s coordinate functions is surjective. Thus there exist subgroups U of rank s of $\mathcal{O}_K^{*,+}$ such that $pr(l(U))$ is a full lattice in $\mathbb{R}^s$. Such a subgroup will be called **admissible** for K.

Take now U admissible for K. The quotient $\mathbb{H}^s \times \mathbb{C}^t / \sigma(\mathcal{O}_K)$ is clearly diffeomorphic to a trivial torus bundle $(\mathbb{R}_{>0})^s \times (S^1)^n$ and U operates properly discontinuously on it since it induces a properly discontinuous action on the base $(\mathbb{R}_{>0})^s$ by our choice. Differentiably the quotient of this action is a fiber bundle over $(S^1)^s$ with $(S^1)^n$ as fiber. We thus get an m-dimensional compact complex affine manifold

$$X = X(K, U) := (\mathbb{H}^s \times \mathbb{C}^t)/(U \ltimes \mathcal{O}_K).$$

This paper is devoted to the description of these complex manifolds.

Remark 1. 1. *For every choice of natural numbers s and t, algebraic number fields with precisely s real and $2t$ complex embeddings exist.*

Since we don't know of any source for this observation we include here an argument we owe to Ph. Eyssidieux.

Proof. Consider the non-empty open set D of points $a = (a_1, ..., a_n) \in \mathbb{Q}^n$ such that the polynomials $P = X^n + a_1 X^{n-1} + ... + a_n$ admit exactly s real distinct roots and $2t$ complex non-real roots. The open set D will contain arbitrarily large open balls since the map $(a_1, ..., a_n) \mapsto (ba_1, b^2 a_2, ..., b^n a^n)$ leaves it invariant for any choice of rational numbers b.

Choose now a prime number p and $\tilde{P} = X^n + \tilde{a}_1 X^{n-1} + ... + \tilde{a}_n \in \mathbb{Z}[X]$ an Eisenstein polynomial with respect to p, that is $p|\tilde{a}_i$ for all i but $p^2 \nmid \tilde{a}_n$. Then the set $\tilde{a} + p^2\mathbb{Z}^n$ intersects D and consists only of Eisenstein hence irreducible polynomials. $\square$

Remark 1. 2. *For $s = 1$, $t = 1$ and $U = \mathcal{O}_K^{*,+}$, $X(K, U)$ is an Inoue-Bombieri surface S_M; cf.* [3].

Remark 1.3. *When $s = 1$ or $t = 1$ all subgroups U of rank s of $\mathcal{O}_K^{*,+}$ are admissible for K. But this need not be the case in general as the following example shows. Take two field extensions K' and K'' of $\mathbb{Q}$ with corresponding numbers of real and complex embeddings $s' = 1$, $t' = 2$, $s'' = 2$, $t'' = 1$ and K the composite of K' and K''. Then $s = 2$ but $\mathcal{O}_{K'}^{*,+}$ is not admissible for K.*

Lemma 1. 4. *Let U be a subgroup of $\mathcal{O}_K^*$ not contained in $\mathbb{Z}$. Then the following are equivalent:*

- *The action of U on $\mathcal{O}_K$ admits a proper non-trivial invariant submodule of lower rank.*
- *There exists some proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subset \mathcal{O}_{K'}^*$.*

Proof. Suppose M is a proper $\mathbb{Z}$-submodule of $\mathcal{O}_K$ which is invariant under U and with $0 < rank\ M = r < n$. We consider the coefficient ring of M, $\mathcal{O}_M := \{a \in K | aM \subset M\}$. We have $U \subset \mathcal{O}_M$, hence $\mathcal{O}_M$ is not contained in $\mathbb{Q}$. Let now K' be the field of fractions of $\mathcal{O}_M$. We have to show that $K' \neq K$. Let $x \in K'$ be a primitive element for $K'/\mathbb{Q}$ with $x = a/b$, $a, b \in \mathcal{O}_M$. Then the action of x on M is described by an $r \times r$ matrix with rational coefficients in terms of a basis of M. If K' and K coincided, then the characteristic polynomial of x would allow a factor of degree r over $\mathbb{Q}$. This proves the lemma in one direction. The converse is clear. □

Definition 1.5. *We shall call the manifold $X(K, U)$ of* **simple type** *if U does not satisfy the equivalent conditions of the previous lemma.*

Lemma 1. 6. *Let $\mathbb{Q} \subset K' \subset K$ be a proper intermediate extension and $U \subset \mathcal{O}_{K'}^*$ an admissible subgroup for K. Let s', $2t'$ be the numbers of distinct real and respectively complex embeddings of K'. Then $s = s'$, $t' > 0$ and U is admissible for K'.*

Proof. The restrictions to K' of two different real embeddings of K cannot coincide since $U \subset K'$ and U is admissible for K. Thus $s' \geq s$. We show now that $s \geq s'$ as well.

Let $k := (K : K')$. The restriction to K' of a real embedding of K will have to coincide with the restrictions of exactly $k - 1$ complex embeddings of K. In particular since these restrictions are real these $k - 1$ complex embeddings occur in complex conjugate pairs. So $k - 1$ is even.

Suppose now that there is a real embedding of K' which is not the restriction of any real embedding of K. Such an embedding has then to be the restriction of exactly k complex embeddings of K and k would then be even by the same reason as above. Thus $s = s'$.

By Dirichlet's Units Theorem and since U has rank s, t has to be positive. It is clear now that U is admissible for K'. □

Remark 1. 7. *If $X(K, U)$ is not of simple type with $\mathbb{Q} \subset K' \subset K$ as intermediate extension and $U \subset \mathcal{O}_{K'}$, then there exists a holomorphic foliation of $X(K, U)$ with a leaf* *isomorphic to $X(K',U)$. Just look at the foliation of $\mathbb{C}^m$ defined by the translates $V_{K'}+v$, $v\in\mathbb{C}^m$ of the complex vector subspace $V_{K'}$ of $\mathbb{C}^m$ spanned by $\sigma(\mathcal{O}_{K'})$. Its restriction to $\mathbb{H}^s\times\mathbb{C}^t$ is invariant under the action of $U\ltimes\mathcal{O}_K$ and thus descends to $X(K,U)$. It is clear that $(V_{K'}\cap(\mathbb{H}^s\times\mathbb{C}^t))/U\ltimes\mathcal{O}_{K'}$ is a leaf of this foliation which is compact since U is admissible for K'.*

Remark 1. 8. *Not all manifolds $X(K,U)$ are of simple type. Keeping the notations of Remark 1.3 take for instance K', K'' with $s'=s''=t'=t''=1$ but non isomorphic and K their composite. Then $s=1$, $t=4$ and this time $\mathcal{O}_{K'}^{*,+}$ is admissible for K. Note however that for any choice of K there are infinitely many $X(K,U)$ of simple type, since the number of intermediate extensions of K is finite.*

2. Invariants and metrics

We investigate here some properties of the varieties $X(K,U)$, where K is a number field as before and U is admissible for K.

We start with some preparations for the computation of the first Betti numbers of $X(K,U)$.

Remark 2. 1. *Let $a\in\mathcal{O}_K^*$ and consider its action on $Hom_{\mathbb{Z}}(\mathcal{O}_K,\mathbb{C})$ by $(af)(x):=f(a^{-1}x)$ for $f\in Hom_{\mathbb{Z}}(\mathcal{O}_K,\mathbb{C})$ and $x\in\mathcal{O}_K$. Then the restrictions to $\mathcal{O}_K$ of the embeddings $\sigma_1,...,\sigma_n$ of K give a basis of $Hom_{\mathbb{Z}}(\mathcal{O}_K,\mathbb{C})$ of eigenvectors for this action with associated eigenvalues $\sigma_1(a^{-1}),...,\sigma_n(a^{-1})$.*

Lemma 2.2. *Let $\theta=\sum_{i=1}^n a_i\sigma_i$, $a_i\in\mathbb{C}$ be a $\mathbb{Q}$-valued $\mathbb{Q}$-linear form on $\mathcal{O}_K$. Then either all coefficients a_i are non-zero or they all vanish.*

Proof. It is easy to see that there exists a primitive element α for $K/\mathbb{Q}$ in $\mathcal{O}_K$. Then $\theta(\alpha^k)\in\mathbb{Q}$ for $0\le k\le n-1$. Let $\alpha_i:=\sigma_i(\alpha)$ be the roots of the characteristic polynomial of α.

The rationality condition for θ can be written as a linear system of equations for the coefficients a_i:

$$\sum_{i=1}^n a_i\alpha_i^k=b_k,\quad 0\le k\le n-1,$$

for some rational numbers $b_0,...,b_{n-1}$. Let A be the coefficient matrix $(\alpha_i^k)_{1\le i\le n,\ 0\le k\le n-1}$ of this system. By the choice of α we have $\det A\neq 0$.

We suppose now that $\theta\neq 0$, so not all b_i-s vanish, and that one of the a_i-s is zero, say $a_n=0$. This means that the determinant of the matrix obtained from A by replacing the last column with the free vector $b=(b_0,...,b_{n-1})$ shall vanish. Hence expanding this determinant after its last column gives us the following linear dependency relation over $\mathbb{Q}$ among the coefficients of the polynomial $\Pi_{1\le i\le n-1}(X-\alpha_i)$:

$$b_{n-1}+b_{n-2}s_1+...+b_0s_{n-1}=0.$$

Here we denoted by s_i the i-th elementary symmetric function in $a_1, ..., a_{n-1}$.

Now we express inductively the elementary symmetric functions in $a_1, ..., a_{n-1}$ in terms of those in $a_1, ..., a_n$ and the powers of α_n:

$$s_1(a_1, ..., a_{n-1}) = s_1(a_1, ..., a_n) - \alpha_n,$$

$$s_2(a_1, ..., a_{n-1}) = s_2(a_1, ..., a_n) - \alpha_n s_1(a_1, ..., a_{n-1}) =$$
$$= s_2(a_1, ..., a_n) - \alpha_n s_1(a_1, ..., a_n) + \alpha_n^2, \ ...$$

This leads to a non-trivial relation over $\mathbb{Q}$ among $1, \alpha_n, ..., \alpha_n^{n-1}$ which contradicts the choice of α. $\square$

Proposition 2. 3. *For all $X = X(K, U)$ the first Betti number is $b_1(X) = s$. When X is of simple type one also has $b_2 = \binom{s}{2}$.*

Proof. The cohomology groups of X with coefficients in $\mathbb{Q}$ are isomorphic to those of its fundamental group. We thus have to compute $H^1(U \ltimes \mathcal{O}_K; \mathbb{Q})$ and $H^2(U \ltimes \mathcal{O}_K; \mathbb{Q})$. The Lyndon-Hochschild-Serre spectral sequence associated to the short exact sequence

$$0 \to \mathcal{O}_K \to U \ltimes \mathcal{O}_K \to U \to 0$$

gives

$$E_2^{pq} = H^p(U; H^q(\mathcal{O}_K; \mathbb{Q})) \Longrightarrow H^{p+q}(U \ltimes \mathcal{O}_K; \mathbb{Q})$$

and an exact sequence of low degree terms:

$$0 \to H^1(U; \mathbb{Q}^{\mathcal{O}_K}) \to H^1(U \ltimes \mathcal{O}_K; \mathbb{Q}) \to H^1(\mathcal{O}_K; \mathbb{Q})^U \to H^2(U; \mathbb{Q}^{\mathcal{O}_K}) \to H^2(U \ltimes \mathcal{O}_K; \mathbb{Q});$$

cf. [4], 6.8. Here $\mathbb{Q}$ is seen as a trivial $U \ltimes \mathcal{O}_K$-module. Then $H^1(\mathcal{O}_K; \mathbb{Q}) \cong Hom(\mathcal{O}_K; \mathbb{Q})$ is a non-trivial U-module via:

$$(uf)(x) := f(u^{-1}x), \text{for all } u \in U, \quad f \in Hom(\mathcal{O}_K, \mathbb{Q}), \quad x \in \mathcal{O}_K;$$

cf. [4] 6.8.1. Thus $H^1(\mathcal{O}_K; \mathbb{Q})^U := \{f \in Hom(\mathcal{O}_K, \mathbb{Q}) \mid uf = f, \text{ for all } u \in U\}$ and this last space is trivial by Remark 2.1. Thus $H^1(U \ltimes \mathcal{O}_K; \mathbb{Q}) \cong H^1(U; \mathbb{Q}^{\mathcal{O}_K}) \cong H^1(U; \mathbb{Q}) \cong H^1(\mathbb{Z}^s; \mathbb{Q}) \cong \mathbb{Q}^s$.

Moreover, the map $H^2(U; \mathbb{Q}^{\mathcal{O}_K}) \to H^2(U \ltimes \mathcal{O}_K; \mathbb{Q})$ is injective. We only need to prove that it is surjective as well when X is of simple type. To see this it is enough to check that in this case the terms $E_2^{0,2}$ and $E_2^{1,1}$ of the spectral sequence vanish.

Consider first $E_2^{0,2} = H^0(U; H^2(\mathcal{O}_K; \mathbb{Q})) = H^2(\mathcal{O}_K; \mathbb{Q})^U \cong Alt^2(\mathcal{O}_K; \mathbb{Q})^U$. This is the space of alternating 2-forms on $\mathcal{O}_K$ which are fixed by U. Let $\gamma = \sum_{1 \leq i < j \leq n} a_{ij} \sigma_i \wedge \sigma_j \in Alt^2(\mathcal{O}_K; \mathbb{Q})^U$ with $a_{ij} \in \mathbb{C}$. The fact that γ is invariant under the action of some $u \in U$ means that $\sigma_i(u)\sigma_j(u) = 1$ whenever $a_{ij} \neq 0$; cf. Remark 2.1. From this we get $a_{ij} = 0$ for all $1 \leq i < j \leq s$ since U is admissible for K. The relation $\sigma_i(u)\sigma_j(u) = 1$ for $u \in U$ and the fact that X is of simple type imply moreover that $a_{ij} = 0$ whenever $1 \leq i \leq s$ and that for each $i > s$ there exists at most one $j = j(i) > i$ with $a_{ij} \neq 0$. (Otherwise we would get two equal embeddings $\sigma_j = \sigma_{j'}$.) Thus $\gamma = \sum_{s < i < n} a_{ij(i)} \sigma_i \wedge \sigma_{j(i)}$.

Let $\alpha \in \mathcal{O}_K$ be a primitive element for K. Then $\gamma(\alpha^k, 1) \in \mathbb{Q}$ for all $k \in \mathbb{Z}$, that is $\sum_{s<i<n} a_{ij(i)}(\sigma_i(\alpha^k) - \sigma_{j(i)}(\alpha^k)) \in \mathbb{Q}$ for all $k \in \mathbb{Z}$. But then we get a rational 1-form $\sum_{s<i<n} a_{ij(i)}(\sigma_i - \sigma_{j(i)})$ which by Lemma 2.2 has to vanish.

We now check that $E_2^{1,1} = H^1(U; H^1(\mathcal{O}_K; \mathbb{Q}))$ is trivial. Since U is free abelian we reduce ourselves by the Lyndon-Hochschild-Serre spectral sequence for $0 \to \mathbb{Z} \to \mathbb{Z}^s \to \mathbb{Z}^{s-1} \to 0$ to the computation of $H^1(\mathbb{Z}; H^1(\mathcal{O}_K; \mathbb{Q}))$ where $\mathbb{Z}$ here is the subgroup generated by some $u \in U$. Now $H^1(\mathbb{Z}; H^1(\mathcal{O}_K; \mathbb{Q})) \cong H^1(\mathcal{O}_K; \mathbb{Q})_{\mathbb{Z}} \cong H^1(\mathcal{O}_K; \mathbb{Q})/ < uf - f \mid f \in H^1(\mathcal{O}_K; \mathbb{Q}) >$; cf. [4] 6.1.4. But the action of $u - id$ is invertible by Remark 2.1 hence $H^1(\mathbb{Z}; H^1(\mathcal{O}_K; \mathbb{Q}))$ vanishes. □

Lemma 2.4. *Every holomorphic function on $\mathbb{H}^s \times \mathbb{C}^t/\sigma(\mathcal{O}_K)$ is constant.*

Proof. Take any element $v \in \mathbb{H}^s$. We shall first prove the following

Claim The image of $\{v\} \times \mathbb{C}^t$ in $(v + \mathbb{R}^s) \times \mathbb{C}^t/\sigma(\mathcal{O}_K)$ is dense in this space.

We shall just check that $0 \times \mathbb{C}^t$ has a dense image in $\mathbb{R}^s \times \mathbb{C}^t/\sigma(\mathcal{O}_K)$. For this it is enough to prove that the image of $\mathcal{O}_K$ through $\sigma' = (\sigma_1, ...\sigma_s) : \mathcal{O}_K \to \mathbb{R}^s$ is dense in $\mathbb{R}^s$.

Consider the connected component V of 0 of the topological closure of $\sigma'(\mathcal{O}_K)$ in $\mathbb{R}^s$ and the $\mathbb{Z}$-submodule $M := \sigma'^{-1}(V)$ of $\mathcal{O}_K$. If $V \neq \mathbb{R}^s$ we would have $rank\ M < n$. Take now $\alpha \in \mathcal{O}_K$ a primitive element for K. On $\mathcal{O}_K$ we have a multiplicative action of α. The submodule $\alpha\mathcal{O}_K$ of $\mathcal{O}_K$ has finite index so the induced linear action of α on $\mathbb{R}^s$ will leave V invariant. Thus M also remains invariant under the action of α. But this would imply that the characteristic polynomial of α admits a factor of degree $rank\ M$ over $\mathbb{Q}$, which is absurd.

Take now a holomorphic function f on $\mathbb{H}^s \times \mathbb{C}^t/\sigma(\mathcal{O}_K)$ and $v \in \mathbb{H}^s$. Since f is bounded on $(v+\mathbb{R}^s) \times \mathbb{C}^t/\sigma(\mathcal{O}_K) \simeq (S^1)^n$ its lift $\tilde{f}$ to $\mathbb{H}^s \times \mathbb{C}^t$ will be bounded on each $(v+\mathbb{R}^s) \times \mathbb{C}^t$ hence constant on $\{v\} \times \mathbb{C}^t$. By our Claim it follows now that $\tilde{f}$ is constant on $(v+\mathbb{R}^s) \times \mathbb{C}^t$. But then $\tilde{f}$ must be constant on $\mathbb{H}^s \times \mathbb{C}^t$ by the identity principle. □

Proposition 2.5. *The following vector bundles on $X = X(K, U)$ are flat and admit no non-trivial global holomorphic sections:*

$$\Omega^1_X,\ \Theta_X,\ K_X^{\otimes k},\ for\ all\ k \neq 0.$$

Moreover dim $H^1(X, \mathcal{O}_X) \geq s$. In particular $\kappa(X) = -\infty$ and X is non-Kähler.

Proof. Let $z_1, ..., z_m$ be the standard complex coordinate functions on $\mathbb{H}^s \times \mathbb{C}^t$. A section ω of $K_X^{\otimes k}$ lifted to $\mathbb{H}^s \times \mathbb{C}^t$ will have the form $\tilde{\omega} = f(dz_1 \wedge ... \wedge dz_m)^{\otimes k}$. Since this section descends to $\mathbb{H}^s \times \mathbb{C}^t/\sigma(\mathcal{O}_K)$ it follows from Lemma 2.4 that f is constant on $\mathbb{H}^s \times \mathbb{C}^t$. Moreover if $f \neq 0$, the invariance of $\tilde{\omega}$ with respect to U gives $(\Pi_{i=1}^m \sigma_i(u))^k = 1$ for all $u \in U$. Multiplying this by $(\Pi_{i=1}^m \sigma_i\bar{(}u))^k = 1$ and using the fact that $(\Pi_{i=1}^n \sigma_i(u))^k = 1$ we get $(\Pi_{i=1}^s \sigma_i(u))^k = 1$ which contradicts the admissibility of U.

In the case of Ω^1_X the automorphy factors are $\sigma_i(u)$, $i = 1, ..., m$ and it is clear that none of them equals 1. An analogous argument works for Θ_X using the vector fields $\partial/\partial z_i$. The flatness of these bundles is evident.

The statement on *dim* $H^1(\mathcal{O}_X)$ follows now from Proposition 2.3 and the exact sequence of sheaves on X:

$$0 \to \mathbb{C} \to \mathcal{O} \to d\mathcal{O} \to 0.$$

□

Remark 2. 6. *The above proof also shows that the group U is uniquely determined by the complex structure of $X(K,U)$ since its elements are automorphy factors for Ω^1_X. In particular when X is of simple type its complex structure determines both K and U.*

Corollary 2. 7. *The group of holomorphic automorphisms of $X(K,U)$ is discrete. It is infinite when $t > 1$ since the elements of $\mathcal{O}^*_K/U$ induce automorphisms of $X(K,U)$.*

It is known that the Inoue-Bombieri surfaces S_M admit locally conformally Kähler metrics. This means that there is a representation $\rho : \pi_1(S_M) \to \mathbb{R}_{>0}$ and a closed strongly positive $(1,1)$-form ω on the universal cover of S_M such that $g^*\omega = \rho(g)\omega$ for all $g \in \pi_1(S_M)$; cf. [2]. We now investigate the existence of locally conformally Kähler metrics more generally on the manifolds $X(K,U)$.

Example. When $t = 1$ all manifolds $X(K,U)$ admit locally conformally Kähler metrics. Consider indeed the following potential

$$F : \mathbb{H}^s \times \mathbb{C} \to \mathbb{R},\ F(z) := \frac{1}{\Pi_{j=1}^s (i(z_j - \bar{z}_j))} + |z_m|^2.$$

Then $\omega := i\partial\bar{\partial}F$ gives the desired Kähler metric on $\mathbb{H}^s \times \mathbb{C}$.

Remark 2. 8. *The manifolds $X(K,U)$ with $s = 2$ and $t = 1$ give counterexamples to a conjecture of I. Vaisman, according to which a compact locally conformally Kähler manifold admitting even Betti numbers with odd index and non-zero Betti numbers with even index should already be Kähler; (see* [2]*, p. 8).*

Proof. We have the following Betti numbers for $X(K,U)$: $b_0 = b_6 = 1$, $b_1 = b_5 = 2$, $b_2 = b_4 = 1$ and $b_3 = 0$. In fact, here $X(K,U)$ is of simple type and therefore we can apply Proposition 2.3 to get b_1 and b_2. For b_3 note that the Euler characteristic equals $c_3(X(K,U)) = 0$, since Θ is flat. □

Proposition 2. 9. *When $s = 1$ and $t > 1$ there exists no locally conformally Kähler metric on $X(K,U)$.*

Proof. Let $s = 1$, $\omega = \sum_{1\le i,\ j\le m} g_{ij} dz_i \wedge d\bar{z}_j$ a closed strictly positive $(1,1)$-form on $\mathbb{H} \times \mathbb{C}^t$ and $\rho : U \ltimes \mathcal{O}_K \to \mathbb{R}_{>0}$ a representation such that $g^*\omega = \rho(g)\omega$ for all $g \in U \ltimes \mathcal{O}_K$. We shall show that $t = 1$.

It is clear that ρ factorizes through a representation of U which we denote again by ρ. Since ω descends to $(\mathbb{H} \times \mathbb{C}^t)/\sigma(\mathcal{O}_K) \simeq \mathbb{R}_{>0} \times (S^1)^n$, we may assume by averaging over $(S^1)^n$ that the coefficients g_{ij} are constant in the directions of $\sigma(\mathcal{O}_K)$. In particular they are constant on the subspaces $\{v\} \times \mathbb{C}^t$ for each $v \in \mathbb{H}$. Since $d\omega = 0$, this implies that for $i, j > 1$ the coefficients g_{ij} are constant on the whole of $\mathbb{H} \times \mathbb{C}^t$. By the compatibility of ω with ρ we thus get

$$\rho(u) = |\sigma_2(u)|^2 = |\sigma_3(u)|^2 ... = |\sigma_m(u)|^2, \quad \forall u \in U.$$

Consider now a non-trivial element u of U and its characteristic polynomial $X^n - a_1 X^{n-1} + ... + a_{2t} X - 1$. This polynomial must be irreducible, otherwise there would exist some $i > 1$ such that $\sigma_1(u) = \sigma_i(u) \ \ \forall u \in U$. But this would imply $\sigma_1(u) = 1$ which is impossible.

We have

$$\sigma_1(u) = \frac{1}{\rho(u)^t},$$

$$a_1 = \frac{1}{\rho(u)^t} + \sum_{j=2}^{m} (\sigma_j(u) + \bar{\sigma}_j(u)),$$

$$a_{2t} = \sum_{j=1}^{m} \frac{1}{\sigma_j(u)} = \rho(u)^t + \frac{\sum_{j=2}^{m} (\sigma_j(u) + \bar{\sigma}_j(u))}{\rho(u)} = \rho(u)^t + \frac{a_1}{\rho(u)} - \frac{1}{\rho(u)^{t+1}}.$$

Thus $\rho(u)$ satisfies the following equation:

$$\rho(u)^n - a_{2t}\rho(u)^{t+1} + a_1 \rho(u)^t - 1 = 0.$$

Since $\mathbb{Q}[\sigma_1(u)] \subset \mathbb{Q}[\rho(u)]$ these field extensions must be equal, hence $\rho(u)$ is a non-torsion unit in $\mathcal{O}_K$ having the same property as u, namely that its images through the complex embeddings of K have the same absolute value: $\rho(u)^{-1/t} = \sigma_1(u)^{1/t^2}$. But the same argument as before yields a new non-torsion unit $\rho(u)^{-1/t}$ which for $t > 1$ satisfies the equation $X^n - 1 = 0$. This is a contradiction! $\square$

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