MUMFORD–TATE GROUPS OF MIXED HODGE STRUCTURES AND THE THEOREM OF THE FIXED PART

(revised version)

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CONTENTS

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The present paper grew out of an attempt of understanding group–theoretically the consequences of Hodge theory which are explained in Deligne [4] II 4, with an eye towards applications to algebraic independence.

After some preliminaries about representations of linear algebraic groups, we define and study Mumford–Tate groups of mixed Hodge structures over noetherian subrings R of the field $\mathbb{R}$ of real numbers. Though in the sequel we restrict ourselves to the crucial case $R = \mathbb{Z}$, we refer to the appendix for a study of some pathologies which may occur in the case of other ground rings. We then turn to a more precise study of Mumford–Tate groups arising from 1–motives (see [4] III 10).

In the fourth paragraph a mild generalization of a result by Deligne about the monodromy of variation of Hodge structure is given; we also present our main object of study, that is Steenbrink–Zucker's notion of a good variation of mixed Hodge structure.

In paragraph 5, we give a group–theoretic formulation of the theorem of the fixed part proved in [12]: for almost all stalk of a given polarizable good variation of mixed Hodge structure, the connected monodromy group H_x is a normal subgroup of the derived Mumford–Tate group $\mathscr{D}G_x$. We then state straightforward consequences about monodromy groups. In the next paragraph, we study how big can H_x be in $\mathscr{D}G_x$; we end by applying these considerations to the study of algebraic independence of Abelian integrals depending on some parameters.

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Note: while this work was almost completed, J.P. Winterberger has pointed out to me a paper by G.A. Mustafin "Families of algebraic varieties and invariant cycles" Math. Izv. 27 (translation: 1986) n°2, where the author also compares H_x and G_x under a strong degeneration hypothesis; in that paper, the normality property in the "projective smooth" is stated, and attributed to P. Deligne.

1. Some facts about linear algebraic groups

Let K be a field of characteristic 0, and $V \cong K^N$ some K–vector space. We shall consider a closed algebraic subgroup G of $GL(V) = GL_N$. For non–negative integers m, n, we set $T^{m,n} = T^{m,n}(V) = V^{\otimes m} \otimes \check{V}^{\otimes n}$, where $\check{V}$ denotes the dual space of V (with the contragredient action of GL_N). By "representation of G" or "G–module", we shall always mean a finite–dimensional rational one. The following two properties are well–known [13; 3.5 § 16.1], [6; I 3.1]:

1) every representation of G is a subquotient representation of a finite direct sum of $T^{m,n}$'s,

2) G is the stabilizer of some one–dimensional L in some finite direct sum $\oplus T^{m_i,n_i}$: $G = \mathrm{Stab}\, L$.

For any representation W of G, and any character $\chi \in X_K(G)$ of G over K, we denote by W^G the fixed part of W under G and by W^χ the submodule of W on which G acts according to χ. We write $\mathrm{End}_G W$ for the endomorphisms of the G–module W, so that $\mathrm{End}_G W = (\mathrm{End}_K W)^G$, and we denote by $Z(\mathrm{End}_G W)$ its center.

Lemma 1. Assume that G is connected, and let $H \subset G$ be a closed subgroup. The following conditions are equivalent:

i) $H \lhd G$, that is, H is normal in G,

ii) for every tensor space $T^{m,n}$, and for every $\chi \in X_K(H)$, $(T^{m,n})^\chi$ is stable under G,

iii) every H–isotopical component of any representation of G is stable under G.

If moreover G is reductive, these conditions imply that $Z(\mathrm{End}_H V) \subset Z(\mathrm{End}_G V)$.

<u>Proof</u>: iii) ⇒ ii) is obvious, and we shall first prove that ii) ⇒ i), independently of the connectedness assumption on G. We know by 2) that there exists some one–dimensonal L in some $\oplus T^{m_i,n_i}$ such that $H = \mathrm{Stab}\, L$. Let W be the G–module spanned by L. The line L defines a character $\chi \in X_K(H)$; we have $L \subset W^\chi$, and $W^\chi = W \cap (\oplus T^{m_i,n_i})^\chi = W$, according to the hypothesis ii). Let φ be the natural morphism $G \longrightarrow GL(\mathrm{End}\, W)$; it is clear that $H \subset \ker \varphi$. Conversely if $g \in \ker \varphi$, g commutes with any endomorphism of W, that is, g is scalar; this implies that g stabilizes L, so that $g \in H$. Hence $H = \ker \varphi$ is a normal subgroup.

We now prove i) ⇒ iii). Let W be a G–module, and W' the G–submodule of the sum of its irreducible submodules. It suffices to prove that the H–isotypical components of W' are G–stable. Let H', G' denote the natural images of H and G respectively in $GL(W')$, so that $H' \triangleright G'$. The normality property implies that $(\mathrm{End}\, W')^{H'}$ is stable under G', inside the G'–module $\mathrm{End}\, W'$. For $w \in \mathrm{End}_{H'} W'$, let C_w be the kernel of the commutator map $[w,.]$ in $\mathrm{End}_{H'} W'$. It is easy to derive the formula $gC_w = C_{gw}$, so that $Z(\mathrm{End}_{H'} W') = \bigcap_{w \in \mathrm{End}_{H'} W'} C_w$ is again a G'–module. But $Z(\mathrm{End}_{H'} W')$ is a <u>finite–dimensional</u> semi–simple algebra over K. Moreover G' acts on $\mathrm{End}_{H'} W'$ by $g\varphi(x) = g\varphi(g^{-1}x)$, hence $g(\varphi \circ \psi) = g\varphi \circ g\psi$, and this gives rise to a morphism from G' to the <u>étale</u> group scheme $\underline{\mathrm{Aut}}_K(Z(\mathrm{End}_{H'} W'))$. By the connectedness of G', this morphism has trivial target, that is, $Z(\mathrm{End}_{H'} W')$ is a trivial G'–module. Now the H–isotypical components of W' are given by $p.W'$, where p runs among the minimal indempotents of $Z(\mathrm{End}_{H'} W')$. We just proved that p

commutes with the action of G' on W', and this implies that $p.W'$ is stable under G'.

When G is reductive, we have $V' = V$, and the above proof shows that $Z(\mathrm{End}_H V)$ is a trivial G–module, whence an obvious imbedding $Z(\mathrm{End}_H V) \subset Z(\mathrm{End}_G V)$.

□

2. Mumford–Tate groups and mixed Hodge structures

We first recall some definitions. Let R be some Noetherian subring of $\mathbb{R}$ such that $K := R \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field. Let V be a noeterian R–module. A (pure R–) Hodge structure of weigth $M \in \mathbb{Z}$ on V is a morphism $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G} \longrightarrow GL(V \otimes_R \mathbb{R})$ such that $hw(x)$ is the multiplication by x^M ; here w denotes the embedding $\mathbb{G}_{m\mathbb{R}} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ given by $\mathbb{R}^x \subset \mathbb{C}^x$. Equivalently, it is a bigraduation on $V \underset{R}{\otimes} \mathbb{C} =: V_{\mathbb{C}} = \underset{p+q=M}{\oplus} V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$, or a decreasing filtration F^p on $V_{\mathbb{C}}$ such that $F^p \oplus \overline{F^{(M-p+1)}} \xrightarrow{\sim} V_{\mathbb{C}}$ $(F^p = \sum_{p' \geq p} V^{p',M-p'})$. For instance, there is one and only one Hodge structure of weight $-2M$ on $V = (2\pi\sqrt{-1})^M R$, called "the Tate Hodge structure" and denotes by $R(M)$. A polarization of the Hodge structure (V,h) of weight M is a morphism of Hodge structures (in the obvious sense) $\psi : V \otimes V \longrightarrow R(-M)$ such that $(2\pi\sqrt{-1})^M \psi(.,h(\sqrt{-1}).)$ is a scalar product on $V_{\mathbb{R}} := V \otimes \mathbb{R}$. Elements of $T^{m,n}(V_K) := V^{\otimes m} \otimes (\mathrm{Hom}(V,R))^{\otimes n} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (endowed with the natural K–Hodge–structure of weight $(m-n)M$) which are of type $(0,0)$ are called "Hodge tensors". In fact Hodge tensors are nothing but elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap T^{m,n}(V_K)$ of weight 0 (and thus $m = n$).

A mixed Hodge structure (M.H.S) is a noetherian R–module V , together with a finite increasing filtration $W.$ of the K–space $V_K := V \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, and a finite decreasing filtration $F^{\cdot}$ of $V_{\mathbb{C}}$ such that for each n , $(\mathrm{Gr}^W_n(V_K),\mathrm{Gr}^W_n(F^{\cdot}))$ is a K–Hodge structure of weight n . We say that a M.H.S. V is of type $\epsilon \subset \mathbb{Z} \times \mathbb{Z}$ if its Hodge numbers $h^{p,q}$ are 0 for $(p,q) \notin \epsilon$, and that it is trivial if it is of type $\{(0,0)\}$. The category of mixed K–Hodge structures is an Abelian K–linear tensor category [4; th 12.10] which is rigid and has an obvious exact faithful K–linear tensor functor $\omega : (V_K,W.,F^{\cdot}) \longmapsto V_K$. For fixed mixed R–Hodge structure $(V,W.,F^{\cdot})$, let $< V >$ denote the Tannakian subcategory generated by $(V_K,W.,F^{\cdot})$, and ω_V the restriction of the fiber functor to $< V >$; in other words,

$< V >$ is the smallest full subcategory containing $(V_K, W_., F^{\cdot})$ and the trivial K–M.H.S. , and stable under $\oplus$, $\otimes$, taking subquotients. Then the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega_V)$ is representable by some closed K–algebraic subgroup $G = G(V)$ of $GL(V_K)$, and ω defines an equivalence of categories $< V > \xrightarrow{\sim} \underline{\mathrm{Rep}}_K G$, cf. [6; II 2.11] . We call G the Mumford–Tate group of (V, W, F) .

Lemma 2. a) Any tensor fixed by G in some $T^{m,n}$ is a Hodge tensor (an element of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap W_0 T^{m,n}(V_K)$) , and G is the biggest subgroup of $gL(V_K)$ which fixes Hodge tensors.

b) If $(V, W_., F^{\cdot})$ arises from pure Hodge structure (V,h) , G is the K–Zariski closure of the image of h in $GL(V_K)$ (hence G is connected), and if moreover V is polarizable, then G is inductive.

c) In general, G preserves $W_.$ and, the image of G in $GL(Gr^W V_K)$ is $G(Gr^W V_K)$; in fact $G(V)$ is an extension of $G(Gr^W V_K)$ by some unipotent group (hence it is connected); in particular if V is polarizable, $G(Gr^W V_K)$ is the quotient of G by its unipotent radical.

Remark: This definition of Mumford–Tate group is slightly different from that given in [6; I, 3.2] in the case of pure Hodge structures; if the weight is non–zero, however, this leads to an isogenous group.

Proof of the lemma: a) Any invariant tensor ℓ under G span a trivial representation L_K corresponding to a M.H.S., say L , such that $< L >$ is equivalent to $\underline{\mathrm{Vect}}_K$. Thus L is a trivial M.H.S., that is to say, ℓ is a Hodge tensor. By 1.2), we know that G is the stabilizer of some line L_K in $\oplus T^{m_i,n_i}$, which corresponds to a M.H.S. of rank one (up to isogeny), that is, to some Tate Hodge structure $L = R(N_1)$. If the weight of V is zero,

$N_1 = 0$ and $G = \mathrm{Fix}(\ell)$ for any generator of L. If the weight of V is non–zero, there exists an integer N such that the weight of $\mathrm{Det}\, W_N(V_K)$, say N_2, is non–zero. Taking if necessary $\check{V}_K$ instead of V_K, one can assume moreover that N_1 and N_2 have the same sign. Let r be the rank of the M.H.S. $W_N(V_K)$ over K, and let ℓ be a generator of the one–dimensional subspace $L_K^{\otimes|N_2|} \otimes (\overset{r}{\Lambda}\, W_N(V_K))^{\otimes 2|N_1|}$ inside $(\oplus T^{m_i,n_i})^{\otimes|N_2|} \otimes (\overset{r}{\Lambda}\, V_K)^{\otimes 2|N_1|}$. Then $G = \mathrm{Fix}(\ell)$.

b) The arguments given in [6; I, 3, 4–6] prove the statement about pure Hodge structures.

c) G preserves $W.$ because each W_k is a mixed K–Hodge substructure of V_K. In fact since $< \mathrm{Gr}^W V_K >$ is a Tannakian subcategory of $< V >$, G maps onto $G(\mathrm{Gr}^W V_K)$. Now let p be the subgroup of $GL(V_K)$ which respects the weight filtration $W.$, and N the subgroup of P which acts trivially on $\mathrm{Gr}^W(V_K)$. Then $G \subset P$ and N is unipotent. Moreover $G(\mathrm{Gr}^W V_K)$ is the image of G in P/N. Hence G is an extension of $G(\mathrm{Gr}^W V_K)$ by a (necessarily unipotent) subgroup of N. □

Remark: The description of Mumford–Tate groups by their invariant tensors implies some restrictions on the group which may occur; for example, G cannot be a Borel subgroup of $GL(V_K)$, cf. [6; I 3.2]. However, there are other restrictions on the structure of Mumford–Tate groups, as we shall see now:

Lemma 3. Let G be the Mumford–Tate group of M.H.S. over $\mathbb{R}$, say V, such that $\mathrm{Gr}^W V$ is polarizable. Then the abelianized group $G^{ab} = G/_{\mathscr{D}G}$ is a torus. The group of real points of its quotient $G^{ab}/_{G^{ab} \cap \mathbb{G}_m}$ is compact ($\mathbb{G}_m$ = homothety group).

Proof: Since all morphism in $< V >$ are strict, one has $\mathrm{Gr}^W V' \in \mathrm{Ob} < \mathrm{Gr}^W V >$ for any $V' \in \mathrm{Ob} < V >$, thus $\mathrm{Gr}^W V'$ is polarizable. Take for V' the M.H.S. corresponding to a faithful representation V'_K of the quotient U of G^{ab} by its maximal torus. We find that $G(Gr^W V') = 0$ (see lemma 2). Thus V', which is a successive extension of trivial H.S., is also a trivial H.S., and $G(V') = U = 0$. Now let $\chi \in X_{\mathbb{C}}(G) = X_{\mathbb{C}}(G^{ab})$, and let $V''_{\mathbb{R}}$ be some real plane such that $V''_{\mathbb{C}} \simeq X \oplus \overline{X}$; after twisting à la Tate, Det $V''_{\mathbb{R}}$ corresponds to a trivial real H.S. Therefore $G^{ab}/_{G^{ab} \cap \mathbb{G}_m} | \mathbb{R}$ acts trivially on Det $V''_{\mathbb{R}}$, which yields $|\chi| = 1$. All representations of $G^{ab}/_{G^{ab} \cap \mathbb{G}_m} | \mathbb{R}$, are unitary, so that this torus is compact. □

Remark. The same argument shows in the same situation that if G is nilpotent, then $G = \mathbb{G}_m \times T$ (or $G = T$ if V is pure of weight 0), where T denotes a compact torus.

3. Mumford–Tate groups of 1–motives

We recall that 1–motive over $\mathbb{C}$, denoted by $M = [\mathcal{X} \xrightarrow{u} E]$, is the following data:

i) an extension $0 \longrightarrow T \longrightarrow E \longrightarrow A \longrightarrow 0$ of an Abelian variety A by a torus T,

ii) a morphism u from a free Abelian group $\mathcal{X}$ to $E(\mathbb{C})$.

One associates to a 1–motive a mixed Hodge structure $V = V(M) = (V_{\mathbb{Z}}, W_{\cdot}, F^{\cdot})$, given by:

$$V_{\mathbb{Z}} = \{(\ell,x) \in \mathrm{Lie}\, E \times \mathcal{X} / \exp \ell = u(x)\}$$
$$W_0 = V_{\mathbb{Q}}$$
$$W_{-1} = H_1(E) \otimes \mathbb{Q} \quad (\text{thus } Gr_{-1} \simeq H_1(A) \text{ is polarizable})$$
$$W_{-2} = H_1(T) \otimes \mathbb{Q}$$
$$F^0 = \mathrm{Ker}(V_{\mathbb{Q}} \otimes \mathbb{C} \longrightarrow \mathrm{Lie}\, E)\,.$$

Morphisms of 1–motives being defined in the obvious way, this rule $M \longrightarrow V(M)$ defines a functor which is an equivalence of categories with the category of torsion–free $\mathbb{Z}$–MHS of type $\{(0,0),(0,-1),(-1,0),(-1,-1)\}$ with polarizable Gr_{-1} ([4] III 10.1.3). We denote by G the Mumford–Tate group of V, and by G_{-1} that of W_{-1}. Let E' be the connected component of identity in the Zariski closure of $u(\mathcal{X})$, and let us write
$F := (\mathrm{End}\, E') \otimes \mathbb{Q}$.

Proposition 1. Let $H \lhd G$ such that $W_0^H \subset W_{-1}$ (for instance we may take $H = G$). Let us assume that E is a split extension $(E = A \times T)$. Then $U(H) := \mathrm{Ker}(H \longrightarrow G(W_{-1}))$ is canonically isomorphic to $\mathfrak{U} := \mathrm{Hom}_F(F.u(\mathcal{X}); H_1(E') \otimes \mathbb{Q})$.

Proof (inspired by Kummer's theory of division points on Abelian varieties): let us first remark that the $\mathbb{Q}$–M.H.S. $V_{\mathbb{Q}}$ does not change if one replaces $\mathcal{X}$ by any subgroup of finite index. After such a replacement (which therefore does not affect G), one may assume that $u(\mathcal{X})$ has no torsion, and that E' is the Zariski closure of $u(\mathcal{X})$.

Given $m = (\ell,x) \in V_{\mathbb{Z}}$, the map $U(H) \longrightarrow W_{-1} : \sigma \longmapsto \sigma m - m$ depends only on $u(\mathcal{X})$ and defines therefore a G–equivariant homomorphism $U(H) \xrightarrow{\varphi} \mathrm{Hom}_{\mathbb{Z}}\,(u(\mathcal{X}); W_{-1})$. The vanishing of $\varphi(\sigma)$ implies that σ fixes W_0 , which is a faithful representation of H ; thus $\sigma = 1$, and this shows the injectivity of φ . Because of Poincaré's complete reducibility lemma applied to products of Abelian varieties and tori, the exact sequence of 1–motives $0 \longrightarrow [\mathcal{X} \rightarrow E'] \longrightarrow [\mathcal{X} \rightarrow E] \longrightarrow [0 \rightarrow E/E'] \longrightarrow 0$ splits (up to isogeny, i.e. in the category of $\mathbb{Q}$–M.H.S.).

From this follows an equality of kernels:

$$\begin{aligned} \mathrm{Ker}(H \rightarrow G(W_{-1})) &= \mathrm{Ker}(H \rightarrow G(H_1(E'))) \cap \mathrm{Ker}(H \rightarrow G(H_1(E/E')) \\ &\subseteq \mathrm{Ker}(H' \longrightarrow G(H_1(E'))) \ , \end{aligned}$$

where $H' = H \cap G(V([\mathcal{X} \xrightarrow{u} E']))$. Thus φ factorizes through $\mathrm{Hom}_{\mathbb{Z}}(u(\mathcal{X}); H_1(E') \otimes \mathbb{Q})$; also it is easily seen that elements in the image of φ are F–linear in the sense that $\varphi(U(H)) \subseteq \mho$.

Replacing E by E' and $\mathcal{X}$ by $u(\mathcal{X})$, we may now assume that u is a <u>dominant embedding</u> and identify $\mathcal{X}$ and $u(\mathcal{X})$.

Since E is a split extension, we have $F \simeq \mathrm{End}_{G_{-1}} W_{-1}$ (this is because the category of products of Abelian varieties and tori up to isogeny is equivalent to the category of polarizable $\mathbb{Q}$–Hodge structures of type $\{(-1,-1),(-1,0),(0,-1)\}$), whence $\mathrm{End}_{G_{-1}} \mho \simeq (\mathrm{End}_F\, F\mathcal{X})^{\mathrm{op}}$; also W_{-1} , whence $\mho$ (with trivial action of G_{-1} on $F\mathcal{X}$), is a semi–simple G_{-1}–module. Thus $\varphi(U(H))$ is the kernel of some G_{-1}–equivariant endomorphism ψ of $\mho$; that is to say, there exists $f \in F$ such that $(\psi\varphi(\sigma))\cdot m = \sigma fm - fm = 0$, $\forall\sigma \in U(H)$, $\forall m \in F\mathscr{X}$. If $\varphi(U(H)) \neq \mho$, then $\psi \neq 0$, therefore we can find $x \in F\chi$ such that $U(H)x = x$ and $x \neq 0$.

We set $\mathscr{X}_x = \mathbb{Z}x$, $M_x = [\mathscr{X}_x \hookrightarrow E]$, and we denote by a subscript x the objects G_x , V_x etc. ... associated to this 1–motive. Because $U(H)x = x$, there is a natural injection $H_x = H \cap G_x \overset{i}{\hookrightarrow} GL(W_{x,-1})$. Since E splits, $W_{x,-1} \simeq W_{-1}$ is a direct sum of polarizable pure Hodge structures, so that $H_x \lhd G_x$ is reductive. Therefore $W_{x,-1}$ is a direct summand in the H_x–module $W_{x,0}$, which means that we could choose $x \notin W_{x,-1}$ so that $H_x x = x$: indeed, H_x acts trivially (like G_x) on $W_{x,0}/W_{x,-1}$ whose type is $(0,0)$. Recall that $W_0^H \subset W_{-1}$; this implies the corresponding inclusion $W_{x,0}^{H_x} \subset W_{x,-1}$ since H commutes with the action of F . Therefore we get a contraction, and deduce that $\varphi(U(H)) = \mho$.

<u>Corollary</u>. If E splits, with non trivial Abelian part, one has a split exact sequence
$0 \longrightarrow \mho \longrightarrow G \longrightarrow G(H_1(A)) \longrightarrow 0$.

<u>Remark</u>. If one drops the assumption that E splits, $U(G)$ can be much smaller than $\mho$. In "Deficient points on extensions of abelian varieties by $\mathbb{G}_m$ " J. Number theory – (1987), O. Jacquinot and K. Ribet have constructed some examples (by means of endomorphisms of A which are antisymmetric with respect to a polarization) where $U(G) = 0$, corresponding to some self–dual 1–motives.

4. Variations of mixed Hodge structure

In the sequel we shall concentrate on the case $R = \mathbb{Z}$ (see the appendix for other ground rings). By a variation of M.H.S., we shall mean a finitely filtered object in the category of local systems of noetherian $\mathbb{Z}$–modules over a fixed connected complex manifold X,

$$(\underline{V}_{\mathbb{Z}}, W.)\,,\ W_n\,\underline{V}_{\mathbb{Z}} \subset W_{n+1}\,\underline{V}_{\mathbb{Z}}\,,$$

together with a decreasing filtration of the complex bundle $V_{\mathbb{C}}^{c}$ attached to $V_{\mathbb{C}} := \underline{V}_{\mathbb{Z}} \otimes \mathbb{C}$ by subbundles F^p, such that on each fibre $\underline{V}_{\mathbb{Z},s}$, $(W.,F^{\cdot})$ induces a M.H.S. and that the flat covariant derivative ∇ satisfies $\nabla F^p \subset F^{p-1} \otimes \Omega_X^1$. A morphism of variation of M.H.S. is a morphism of local system which respects $\underset{\cdot}{W}$ and whose complexification respects the filtration F^p pointwise. This yields an abelian category (any morphism is strictly compatible with the filtrations).

One calls such a variation $(\underline{V}_{\mathbb{Z}}, W., F^{\cdot})$ a (graded)–polarizable one if each of the local systems $\mathrm{Gr}_n^W\,\underline{V}_{\mathbb{Z}}$ carries a bilinear from with values in $\mathbb{Z}(-n)_X$ which is a morphism of local system and pointwise a polarization. Any subquotient of a polarizable variation and any object isogenious to a polarizable one are polarizable. The integral relative cohomology modules of the complement of a divisor with relatively normal crossings in a projective smooth scheme over an algebraic variety X furnish examples of polarizable variations of M.H.S. over X (see [7; 4.3] for instance). For a variaton of M.H.S., and for a point x of X, we denote by H_x the connected monodromy group, that is the connected component of identity of the smallest algebraic subgroup of $GL(V_{\mathbb{Q},x})$ containing the image of $\pi_1(X,x)$. We also denote by G_x the Mumford–Tate group of the M.H.S. carried by the stalk $V_{\mathbb{Z},x}$.

<u>Lemma 4</u> (cf. [5; 7.5]) On the complement $\overset{\circ}{X}$ of some meager subset of X, G_x is locally constant. If the variation is polarizable, then $H_x \subset G_x$ for any $x \in \overset{\circ}{X}$.

<u>Proof</u>: for a polarizable variation of pure Hodge structure, this is stated in loc. cit. We shall write down a detailed proof, though (thanks to lemma 2) there is no new complication in the mixed case. Let $\tilde{X}$ be the universal covering of $(X,0)$, for some base point $0 \in X$. The inverse image of the (polarized) variation of M.H.S. is a (polarized) variation of M.H.S. over $\tilde{X}$, whose underlying filtered local system $(\tilde{V}_{\mathbb{Z}}, \tilde{W}.)$ is constant. For $\ell \in T^{m,n}(\Gamma\tilde{V}_{\mathbb{Q}}) \cong T^{m,n}(\tilde{V}_{\mathbb{Q},0})$, we set

$$\tilde{X}(\ell) := \{x \in \tilde{X} \,/\, \ell_x \in T^{m,n}(\tilde{V}_{\mathbb{Q},x}) \text{ is a Hodge tensor}\}$$

$$= \{x \in \tilde{X} \,/\, \ell_x \in F^0 W_0 T^{m,n}(\tilde{V}_{\mathbb{C},x})\} \ .$$

Since F^0W_0 is a subbundle, $\tilde{X}(\ell)$ is an analytic subvariety of $\tilde{X}$, and its natural projection $\pi_*\tilde{X}(\ell)$ on X is an analytic subvariety too. We set

$$\overset{\circ}{X} = X \setminus \Big(\bigcup_{\substack{\ell \\ \text{such that } \pi_*\tilde{X}(\ell) \neq X}} \pi_*\tilde{X}(\ell) \Big)$$

, which is a (dense) countable intersection of dense open subsets of X. By definition of $\overset{\circ}{X}$, any $\ell \in T^{m,n}(\Gamma\tilde{V}_{\mathbb{Q}})$, whose stalk at some $x_0 \in \pi^{-1}\overset{\circ}{X}$ is a Hodge tensor, is in fact a Hodge tensor at every point of $\tilde{X}$. For $x \in \overset{\circ}{X}$, G_x is then the biggest subgroup of $GL(V_{\mathbb{Q},x})$ which fixes the various tensors in $T^{m,n}(V_{\mathbb{Q},x})$ which lift to $F^0T^{m,n}(\Gamma\tilde{V}_{\mathbb{C}})$. Therefore G_x is locally constant on $\overset{\circ}{X}$. We now assume that the variation is polarized and we shall see that $\pi_1(X,x)$ acts (through a finite group) on the spaces $HT^{m,n}_x$ of Hodge tensors in $T^{m,n}(V_{\mathbb{Q},x})$ for any $x \in \overset{\circ}{X}$; this will be sufficient to prove the lemma, since G_x can be described as $\mathrm{Fix}(\ell)$, for one element ℓ of one space $\oplus HT_x^{m_i,n_i}$. We have seen that $HT_x^{m,n}$ (for $x \in \overset{\circ}{X}$) is the subspace of $T^{m,n}(V_{\mathbb{Q},x})$ composed of tensors which lift to $F^0T^{m,n}(\Gamma\tilde{V}_{\mathbb{C}})$; in particular this subspace is locally constant. Hence $HT_x^{m,n}$ is the rational stalk at x associated to a sub–variation of M.H.S. $(\underline{V}'_{\mathbb{Z}}, W_., F'^{\cdot})$ of $(T^{m,n}(\underline{V}_{\mathbb{Z}}),\ T^{m,n}(W_.),\ T^{m,n}(F^{\cdot}))$, which is actually pure of type $(0,0)$ and which inherits a polarization. This polarization ψ on $V'_{\mathbb{R},x}$ is a scalar product, invariant under $\pi_1(X,x)$. Thus $\pi_1(X,x)$ factors through the discrete group $\operatorname{Aut} V'_{\mathbb{Z},x}$ on one hand and through the compact orthogonal group $O(V'_{\mathbb{R},x},\psi)$ on the other hand; hence the connected group H_x acts trivially on $HT_x^{m,n}$.

<u>Remark</u>: a variation of M.H.S. $\underline{V}$ is said to be <u>semi–simple</u> if for any $x \in X$, the relevant category $< \underline{V}_x >$ is semi–simple (notations of § 2). It is easily seen that a <u>polarizable</u> M.H.S. is semi–simple if and only if it is a finite direct sum of variations of pure H.S. up to isogeny. Indeed, it is easy to see that both conditions imply the reductivity of G_x for any $x \in X$. Conversely, assume that for some $x \in \overset{\circ}{X}$, G_x is reductive. Then by local constancy of G_y on $\overset{\circ}{X}$, the same is true for G_y for any $y \in \overset{\circ}{X}$.

Next consider a section σ of the inclusion $(W_m)_y \subseteq (W_{m+1})_y$ in the category $< V_y >$, and let $\gamma_{y,z}$ be a path (up to homotopy) from y to a nearby point z in $\overset{\circ}{X}$. Then because of the horizontality of the filtration $W_.$ and the local constancy of $(G_y)_{y\in\overset{\circ}{W}}$, the section $\gamma_{y,z}(\sigma)$ deduced by transporting σ along $\gamma_{y,z}$ is a section of $(W_m)_z \subseteq (W_{m+1})_z$ in the category $< V_z >$. Thus $\underline{V}|\overset{\circ}{X}$ is a direct sum of variations of pure H.S. up to isogeny, which extend to X by continuity. The semi–simplicity of $\underline{V}$ follows from this.

We shall now recall a concept introduced by Steenbrink–Zucker [12] (cf. also [15]). Let us assume that X is a smooth connected algebraic variety over $\mathbb{C}$. The variation of mixed Hodge structure is considered good if it satisfies the following condition at infinity: there exists a compactification $\overline{X}$ of X, for which $\overline{X} - X$ is a divisor with normal crossings, such that

i) The Hodge filtration bundles F^p extend over $\overline{X}$ to sub–bundles $\tilde{F}^p$ of the canonical extension $\tilde{V}^c_{\mathbb{C}}$ of $V^c_{\mathbb{C}}$, such that they induce the correspondign thing for $GrW(\underline{V}_Z, W., F^{\cdot})$,

ii) for the logarithm N_j of the unipotent part of a local monodromy transformation about a component of $\overline{X} \setminus X$, the weight filtration of N_j relative to $W.$ exists.

The fact that these conditions are sufficient to imply those of [12] (3.13) is pointed out in [15] 1.5, and follows from [16] 4 and [12] A. The following classes of variations of M.H.S. are known to be good:

1) polarizable semi–simple variations of M.H.S. over algebraic bases [10], [14]

2) relative cohomology modules of the complement of a divisor with relatively normal crossings in a projective smooth X–scheme, at least when X is a curve, see [12] 5.7. Moreover, the category of good variations of M.H.S. over X is stable under standard constructions of linear algebra, $\oplus$, $\otimes$, duality ..., see [12] A.

Example: smooth 1–motives.

Recall from [4] III 10.1.10 that a smooth 1–motive $\underline{M}$ over X is the following data:

i) and extension $0 \longrightarrow \underline{T} \longrightarrow \underline{E} \longrightarrow \underline{A} \longrightarrow 0$ of a (polarizable) Abelian scheme $\underline{A}$

$$\begin{array}{c} \underline{A} \\ \downarrow f \\ X \end{array}$$

over X by a torus $\underline{T}$ over X

ii) a morphism $\underline{u} : \mathcal{X} \longrightarrow \underline{E}$ from a group scheme $\mathcal{X}$ over X to $\underline{E}$; one assumes that locally for the étale topology on X , $\mathcal{X}$ is constant and defined by a free $\mathbb{Z}$–module of finite type.

The construction $\underline{V}(M) = (\underline{V}_{\mathbb{Z}}, W., F^{\cdot})$:

$$\begin{aligned}
&\underline{V}_{\mathbb{Z}} = W_0(\underline{V}_{\mathbb{Z}}) = \underline{\text{Lie}}\ E/_X \times_{\underline{E}} \mathcal{X} \text{ defined by the exponential sequence,}\\
&W_{-1} = \text{Ker exp} = R_1\, f_*^{an}\, \mathbb{Z}\\
&W_{-2} = (X_{\mathbb{C}}(\underline{T}))^{\vee}\\
&F^0 = \text{Ker}\,(V_{\mathbb{C}}^{c} \longrightarrow \underline{\text{Lie}}\ \underline{E}/_X)\ ,
\end{aligned}$$

which is fibrewise compatible with that of § 3, yields a polarizable variation of M.H.S. over X .

Lemma 5. Assume that X is a curve. Then the variation $V(\underline{M})$ associated to the smooth 1–motive $\underline{M}$ is good.

(Sketch of) Proof: according to M. Raynaud [C.R.A.S. 262 (1966) 413–416], there exists a Néron model of $\underline{E}$ over the smooth completion $\overline{X}$ of X , such that $\underline{u}$ extends to

$$\begin{tikzcd}
\overline{\mathcal{X}} \arrow[rr, "\overline{\underline{u}}"] \arrow[dr] & & \underline{E} \arrow[dl, "f"] \\
& \overline{X} &
\end{tikzcd}$$

; note that the smooth group scheme $\underline{E}/_{\overline{X}}$ is not of finite type in general.

Replacing $\mathcal{X}$ by a subgroup–scheme of finite index, which yields an isogenous variation of M.H.S., we may assume that $\overline{\underline{u}}(\overline{\mathcal{X}})$ lies in the neutral component $\underline{E}^0$ of $\underline{E}$. Condition i) defining good variations is fulfilled with

$\tilde{\mathrm{F}}^0 = \mathrm{Ker}\,((\underline{\mathrm{Lie}}\,\underline{\mathrm{E}}^0/_{\mathrm{X}} \times_{\underline{\mathrm{E}}^0} \underline{\mathfrak{J}})^c \longrightarrow \underline{\mathrm{Lie}}\,\underline{\mathrm{E}}^0/_{\mathrm{X}})$.

In order to verify ii), we may proceed by induction since we know that both W_{-1} (by point 2) above: the geometric situation) and $W_{0/W_{-2}}$ (by duality of 1–motives and point 2)) satisfy ii).

Granting ii) for W_{-1}, it follows from theorem 2.20 of [12] (formula 2.21) that ii) for W_0 reads equivalently:

(*) $\quad N^{\ell}W_0 \cap W_{-1} \subset N^{\ell}W_{-1} + {}_{(-2)}M_{-\ell-1}$, for all $\ell > 0$; here ${}_{(-2)}M_{-\ell-1}$

is the relative weight filtration of W_{-2}, which is $W_{-\ell-1}$ since the unipotent part of the local monodromy of W_{-2} is trivial (see [12] 2.14; the point is that $\underline{T}$ is necessarily locally constant). Therefore (*) follows from property ii) for $W_0/_{W_{-2}}$.

□

5. Normality

We keep the notations of the previous paragraph. The following result is a simple consequence of the theorem of the fixed part (Griffiths–Schmid–Steenbrink–Zucker).

Theorem 1. Let $\underline{V} = (\underline{V}_{\mathbb{Z}}, W., F^{\cdot})$ be a (graded–) polarizable good variation of mixed Hodge structure over a smooth connected algebraic variety X. Then for any $x \in \overset{\circ}{X}$, the connected monodromy group H_x is a normal subgroup of the derived Mumford–Tate group $\mathscr{D}G_x$.

Proof: we first prove that $H_x \lhd G_x$, using the implication ii) $\Rightarrow$ i) in lemma 1. Since we already know by lemma 4 that $H_x \subset G_x = G_x^0$, it suffices to prove ii) for H_x, G_x. Since $\pi_1(X,x)$ acts on the free $\mathbb{Z}$–module $T^{m,n}(V_{\mathbb{Z},x})$/torsion, any action of $\pi_1(X,x)$ on a line inside $T^{m,n}(V_{\mathbb{Q},x})$ must factor through $\{\pm 1\}$ (the only possible eigenvalues). Thus the connected group H_x has only trivial rational character.
Replacing X by the finite covering defined by the maximal subgroup (of finite index) of $\pi_1(X,x)$ which factors through the connected component H_x of the monodromy group, we are reduced to prove that the largest constant sub–local system of $\underline{V}'_{\mathbb{Z}} = T^{m,n}(\underline{V}_{\mathbb{Z}})$ is a (constant) sub–variation of M.H.S.. For a finite direct sum of polarizable variation of pure H.S., this is precisely the theorem of the fixed part of Griffiths–Schmid, see [3] [10]. For a general polarizable good variation of M.H.S. in Steenbrink–Zucker' sense, this is the theorem of the fixed part of these authors, see [12] 4.19. In fact, in loc. cit., this theorem is stated for à one–dimensional basis X, but we can reduce to this case by considering curves in X, see [7] § 4.3.4.0, for the detailed argument.

So far we have proved that $H_x \rhd G_x$; to show that $H_x \rhd \mathscr{D}G_x$, it suffices to prove that $\mathcal{D}H_x = H_x$. We know that $H_x{}^{ab} \subseteq G_x{}^{ab}$ is a torus (lemma 3). Let χ a complex character of H_x . We just proved that $H_{x|\mathbb{C}} \lhd G_{x|\mathbb{C}}$, so that according to i) ⇒ ii) in lemma 1 for $K = \mathbb{C}$, $(T^{m,n}V_{\mathbb{C},x})\chi + (T^{m,n}V_{\mathbb{C},x})\bar{\chi}$ is stable under $G_{x|\mathbb{C}}$; it is even the complexification of a real space $W_{\mathbb{R}}$ stable under $G_{x|\mathbb{R}}$. Thus after suitable Tate twist, Det $W_{\mathbb{R}} \otimes \mathbb{R}(n)$ becomes a trivial $G_{x|\mathbb{R}}$–module. It follows that Det $W_{\mathbb{R}}$ is a trivial $H_{x|\mathbb{R}}$–module, which yields the equality: $|\chi| = 1$. Therefore all representations of H_x^{ab} are unitary; this means that $H^{ab}_{x|\mathbb{R}}$ is a compact torus.

Let $V' \subset \oplus T^{m_i,n_i}V_{\mathbb{Q},x}$ a faithful representation of H_x^{ab} . A subgroup of finite index of $\pi_1(X,x)$ acts on V' through $GL(V' \cap \oplus T^{m_i,n_i}V_{\mathbb{Z},x})$ which is discrete, and also through a compact torus. Because of the connectedness of H_x , it follows that V' is a trivial H_x–module, that is: $H_x = \mathcal{D}H_x$. □

As a consequence of these group–theoretic arguments, we recover:

Corollary 1 (see [4; 4.2.6–9]). The local system $\underline{V}_{\mathbb{Q}}$ underlying a polarizable variation of pure Hodge structure is semi–simple; each isotypical component carries a sub–variation of pure Hodge structure; the center of $\mathrm{End}(\underline{V}_{\mathbb{Q}})$ is purely of type $(0,0)$. For any $x \in X$, the connected monodromy group H_x is semi–simple.

Proof: since $H_x \lhd \mathcal{D}G_x$ for $x \in \overset{\circ}{X}$, and since $\mathcal{D}G_x$ is a semi–simple group (lemma 2), it follows that H_x is semi–simple; since H_x is locally constant on X , H_x is in fact semi–simple for any $x \in X$. This implies the complete reducibility of the action of $\pi_1(X,x)$ on $V_{\mathbb{Q},x}$ and the first assertion follows (the normality $H_x \lhd G_x$ would suffice here). By i) ⇒ iii) in lemma 1, applied to $H_x \lhd G_x$ for $x \in \overset{\circ}{X}$, we get on each stalk of each isotypical component of the local system $\underline{V}_{\mathbb{Q}|\overset{\circ}{X}}$ a Hodge sub–structure. By continuity, these Hodge sub–structures extend across $X \setminus \overset{\circ}{X}$ and patch together to give rise to a sub–variation of $\mathbb{Q}$–Hodge structure on the isotypical component of $\underline{V}_{\mathbb{Q}}$. The third assertion follows from lemma 1 in the same manner.

Corollary 2 (see [4; 4.2.9b]). The radical of the connected monodromy group H_x associated to a polarizable variation of M.H.S. is unipotent.

Proof: let P_x be the subgroup of $GL(V_{\mathbb{Q},x})$ which respects the wight filtration $W.$, and N_x the subgroup of P which acts trivially on $Gr^W(V_{\mathbb{Q},x})$. Then $H_x \subset P_x$ and N_x is unipotent. Moreover the connected monodromy group, say GrH_x, of $Gr^W(\underline{V}_{\mathbb{Z}})$ at x is the image of H_x in P_x/N_x. Hence H_x is an extension of GrH_x, which (according to the previous corollary) satisfies $GrH_x = \mathscr{D}GrH_x$, by a (necessarily unipotent) subgroup of N_x. □

Remark: corollary 2 shows in particular that if G_x is solvable for some $x \in \overset{\circ}{X}$, then the variation of M.H.S. is unipotent in the sense of [15].

Remark: theorem 1 applies to the geometric situations considered in § 4 since in the course of proving it, we have made a restriction to curves.

Counterexample: we produce an example, following Steenbrink–Zucker (see [12] 3.16), to show that some extra hypothesis upon the variation of M.H.S. is necessary.

Consider a smooth 1–motive $\underline{M} = [\mathbb{Z} \xrightarrow{n \mapsto x^n} \mathbb{G}_m]$ over $X = \mathbb{G}_m$. Here the set $\overset{\circ}{X}$ is $\mathbb{C}^\times \backslash \mathbb{C}^\times_{tors}$. The corresponding good variation of M.H.S. $\underline{V}$ is an extension of $\underline{\mathbb{Z}}$ by $\underline{\mathbb{Z}(1)}$ inside $\underline{\mathbb{C}}$. We denote by ϵ_{-2} the generator $+i$ of $\mathbb{Z}(1) \simeq W_{-2}$ and by ϵ_0 any element of $\underline{V}_{\mathbb{Z}} \backslash W_{-2}$; then $< \epsilon_0, \epsilon_{-2} >$ spans $\underline{V}_{\mathbb{Z}}$. For some suitable determination of

$\log x$ (depending on the choice of ϵ_0), the section $\tilde{\epsilon}_0 := \epsilon_0 - \frac{\log x}{2i\pi}\epsilon_{-2}$ of $V^c_{\mathbb{C}}$ spans F^0 and extends to a section of $\tilde{V}^c_{\mathbb{C}}$ over $\mathbb{P}^1$. We now combine notations from § 3 and § 4. For $x \in \overset{\circ}{X}$, we have $U(H_x(\underline{M})) = U(G_x(\underline{M})) = \mathfrak{U} \simeq \mathbb{G}_a$ according to proposition 1. On the other hand $H_x(\underline{M}) = U(H_x(\underline{M}))$ according to the previous corollary.

For any entire function f, let us now consider the following perturbation $\underline{V}^f$ of $\underline{V}$: $(\underline{V}^f_{\mathbb{Z}}, W^f_{\cdot}) = (\underline{V}_{\mathbb{Z}}; W_{\cdot})$ but $(F^f)^0$ is spanned by $\tilde{\epsilon}_0 + f\,\epsilon_{-2}$. The corresponding groups $H_x(\underline{M}^f)$, $G_x(\underline{M}^f)$ admit the same description. The following assertions are easily seen to be equivalent:

a) $\underline{V}^f$ is good

b) f extends analytically at ∞

c) f is constant

d) $\underline{V}^f \simeq \underline{V}$

e) $V' := \mathrm{Hom}(\underline{V}, \underline{V}^f)$ is good.

The group $H_x(\underline{V}')$ is isomorphic to $\mathbb{G}_a$; viewed as a subgroup of $Gl_2 \times GL_2$ acting on $(V_{\mathbb{Q},x}{}^{\vee} \otimes V^f_{\mathbb{Q},x})$, its "typical" element takes the form

$$\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \times \begin{bmatrix} 1 & +a \\ 0 & 1 \end{bmatrix} .$$

The "typical" element of $G_x(\underline{V}')$ takes the form

$$\begin{bmatrix} 1/b & 0 \\ c & 1 \end{bmatrix} \times \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} ,$$

a being independent of c if (and only if) $\underline{V}^f \simeq \underline{V}$. Therefore we see in this example that $H_x(\underline{V}') \triangleleft \mathscr{D}G_x(\underline{V}')$ if and only if $\underline{V}'$ is good.

6. Maximality

Let $(\underline{V}_{\mathbb{Z}}, W., F^{\cdot})$ a polarizable good variation of mixed Hodge structures on X. Let $x \in \overset{\circ}{X}$ as in lemma 3. By the theorem, we know that $H_x \triangleleft \mathscr{D}G_x$. We now study how big can H_x be in $\mathscr{D}G_x$.

Proposition 2. Assume that for some $y \in X$, G_y is nilpotent (hence abelian, according to the remark following lemma 3). Then for any $x \in \overset{\circ}{X}$, $H_x = \mathscr{D}G_x$.

Proof: according to the remark which follows lemma 3, G_y is actually a torus. Since the assertion is invariant under taking finite coverings of X, it suffices to show that any tensor $\ell \in T^{m,n}V_{\mathbb{Q},x}$ invariant under $\pi_1(X,x)$ spans a G_x–module W_x on which the action of G_x is abelian. It follows from the "fixed part" theorem that W_x is fixed by $\pi_1(X,x)$, and the local constancy of G_x on $\overset{\circ}{X}$, together with an argument of continuity, shows that W_x extends to a constant sub–variation of M.H.S., say $(\underline{V}', W., F'^{\cdot})$, $(T^{m,n}\underline{V}_{\mathbb{Q}}, T^{m,n}h)$. In particular the action of G_x on $\underline{V}'_x = V'_x$ is the same as the action of G_y on $\underline{V}'_y$, which is abelian.

For an application to smooth one–motives, see theorem 2 below.

Remark 1. By the normality theorem, the equality $H_x = \mathscr{D}G_x$ $(x \in \overset{\circ}{X})$ holds whenever $\mathscr{D}G_x$ is $\mathbb{Q}$–simple and the variation of M.H.S. does not become constant over any finite covering of X. By way of example, we consider a non–trivial polarized family of Abelian varieties with many endomorphisms over a complex algebraic base X; by this, we mean that the generic fibre f_η of f (that makes sense since f is automatically algebraic) enjoys the following property: $(\mathrm{End}\, f_\eta) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a division ring which contains a commutative fields of degree $\dim f_\eta$ over $\mathbb{Q}$. Then the derived Mumford–Tate group of the stalk $(R_1 f_* \mathbb{Q})_x$ can be computed for any Weil generic point x of X (so that $\mathrm{End}\, f_\eta = \mathrm{End}\, f_x$) : it turns out that $\mathscr{D}G_x \cong \mathrm{Res}_{Z^+/\mathbb{Q}} G$, where Z^+ denotes the maximal totally real subfield of the center of $(\mathrm{End}\, f_\eta) \otimes_{\mathbb{Z}} \mathbb{Q}$, and G is an absolutely simple group over Z^+ (in fact $G_{|\mathbb{C}} \cong SL_2$); thus in this case $\mathscr{D}G_x$ is simple over $\mathbb{Q}$ (see also the appendix, and [9, lemma 2.3], [1, th. 2]). (1)

Remark 2. On the other hand, the equality $H_x = \mathscr{D}G_x$ $(x \in \overset{\circ}{X})$ may fail for trivial reasons, namely when some Jordan–Hölder constituent of $(V_{\mathbb{Z}}, W_., F^\cdot)$ is a locally constant V.M.H.S., with non–abelian Mumford–Tate group. However this is not the only obstruction to the maximality of H_x in general, as we shall now show. (2)

Scholie. There exists a non–isotrivial Abelian scheme $\underline{A} \longrightarrow X$ over some curve X, with simple geometric generic fibre, such that $H_x \neq \mathscr{D}G_x \simeq G_x/\mathbb{G}_m$ for any $x \in \overset{\circ}{X}$.

Proof: we use M. Borovoi's construction of a simple complex Abelian variety A of dimension 8 with Mumford–Tate group $G = \mathrm{Res}_{Z/\mathbb{Q}} SL_1(D_1 \times D_2)$, where D_1 and D_2 are quaternion algebras over some real quadratic field $\mathbf{Z}$, with the same invariants at every finite place of Z, and of type compact–non compact (resp. non–compact–compact)

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(1) Other examples of $\mathbb{Q}$–simple M–T–group are constructed in Mustafin's paper cited in the introduction.

(2) This contradicts the conjectural statement IX 3.1.6. in the author's "G–functions and Geometry" Vieweg 1989.

at ∞ .[1]. In fact, such polarized Abelian varieties (with suitable level structure) can be put into a family "of Hodge type" $\underline{A}_0 \longrightarrow X_0$, parametrized by a Shimura variety $X_c = {}_K\backslash G(\mathbb{R})/_\Gamma = {}_{K^1}\backslash G^1(\mathbb{R})/_{\Gamma^1} \times {}_{K^2}\backslash G^2(\mathbb{R})/_{\Gamma^2}$ where

$$\begin{aligned}
&G^i = \mathrm{Res}_{Z/\mathbb{Q}} SL_1(D_i) \ , \\
&\Gamma^i \text{ is a torsion–free congruence subgroup in } G^i \ , \\
&\Gamma = \Gamma^1 \, \Gamma^2 \ , \\
&K^i = \text{maximal compact subgroup in } G^i(\mathbb{R}) \ , \\
&K = K^1 \, K^2 \ .
\end{aligned}$$

Now choose $y \in \overset{\circ}{X}_0$, let y_1 denote its projection on the curve ${}_{K^1}\backslash G^1(\mathbb{R})/_{\Gamma^1}$, and let $\underline{A} \longrightarrow X := y_1 \times ({}_{K^2}\backslash G^2(\mathbb{R})/_{\Gamma^2})$ be the pull–back of $\underline{A}_0 \longrightarrow X_0$.
It is clear that $H_x \subset G^2$ for every $x \in X$. However $G_x = G_y = G^1 \times G^2$ for every $x \in \overset{\circ}{X}$.

Remark 3. In this example, Z is the center of the centralizer of H_x in $\mathrm{End}\, H_1(\underline{A}_x, \mathbb{Q})$, and this provides by the way a non–trivial instance where the conditions 4.4.11 of [4] II fail.

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(1) M. Borovoi, the Hodge group and the algebra of endomorphisms of an Abelian variety: Questions of Group theory and Homological Algebra (A.L. Onishchik, ed.) Yaroslav, Ges. Univ. 1981 (Russian).

7. Algebraic independence of Abelian integrals

The heuristic idea underlying this section is that "periods" describe the location of the Hodge filtration with respect to the integral lattice, so that large Mumford–Tate groups reflect randomness of periods. We illustrate this principle in the case of 1–motives (periods are then Abelian integrals).

Suppose we are given some 1–motive $\underline{M}$ over the algebraic variety X ; its generic fibre $M := \underline{M}_\eta$ is then a 1–motive over the function field $\mathbb{C}(X)$.

According to [4] III 10.1.7, there exists a universal extension $M^\natural$ of M by a vector group:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathscr{X} & = & \mathscr{X} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}^1(M,\mathbb{G}_a)^* & \longrightarrow & E^\natural & \longrightarrow & E & \longrightarrow & 0 \end{array} .$$

The De Rham cohomological realization of M is by definition $H^1_{DR}(M) := \mathrm{Co\,Lie}\, E^\natural$. Moreover, the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathscr{X},\mathbb{G}_a) \longrightarrow \mathrm{Ext}^1(M,\mathbb{G}_a) \longrightarrow \mathrm{Ext}^1(E,\mathbb{G}_a) \longrightarrow 0$$

induces an exact sequence

$$(*) \qquad 0 \longrightarrow \mathrm{Hom}(\mathscr{X},\mathbb{G}_a) \longrightarrow H^1_{DR}(M) \longrightarrow H^1_{DR}(E) \longrightarrow 0 \ ,$$

where $H^1_{DR}(E)$ is the De Rham cohomological realization of the 1–motive $[0 \longrightarrow E]$, identified with the usual first algebraic De Rham cohomology group of E .

Let K_x denote the fraction field of the local ring $\mathcal{O}_{X^{an},x}$ at some point $x \in X$.

Construction [4] III 10.1.8 then yields a canonical isomorphism:

$$\mathrm{Hom}_{\mathbb{C}(X)}(H^1_{DR}(M),K_x) = V_{\mathbb{C},x} \otimes_{\mathbb{C}} K_x \ .$$

Let ∇^{an} the flat connection over $V^c_{\mathbb{C}}$ such that $(V^c_{\mathbb{C}})^{\nabla^{an}} = \underline{V}_{\mathbb{C}}$. According to Griffiths, $Gr^W_{\cdot}\nabla^{an}$ has only regular singular points (see [3]). It follows that ∇^{an} itself has only regular singular points (because extensions of regular connections are regular), henceforth is induced by a connection ∇ over $H^1_{DR}(M)$. In fact (*) is a sequence in the category of $\mathbb{C}(X)$–vector spaces with $\mathbb{C}(X)/_{\mathbb{C}}$–connection, inducing the Gauss–Manin connection on $H^1_{DR}(E)$, and a trivial connection on $\mathrm{Hom}(\mathscr{X},\mathbb{G}_a)$.

By definition of ∇, we have

$$(**) \qquad \mathrm{Hom}_{\nabla}(H^1_{DR}(M),K_x) = V_{\mathbb{C},x} \ \text{inside} \ V_{\mathbb{C},x} \otimes K_x \ .$$

Let us translate (*) and (**) in more down–to–earth terms, assuming that $\mathscr{X} \xrightarrow{u} E$ is <u>injective</u>, and that $\mathscr{X}$ is <u>constant</u> over X. Then $\mathscr{X}$ may be considered as a group of sections of $\underline{E} \xrightarrow{f} X$, and $\underline{V}_{\mathbb{I}}$ is spanned by $< \log_{\underline{E}}\mathscr{X}, \mathrm{Ker}\exp_{\underline{E}} >$, at least if we restrict ourselves to the subset of X where u is fibrewise injective. By means of suitable bases, a fundamental solution matrix of a Picard–Fuchs differential system of order one associated to $H^1_{DR}(M)$ can be expressed in some neighbourhood of $x_0 \in X$ by:

$$Z := \left[\oint_{\gamma_j} \omega_i \ \middle| \ \int_0^{\xi_k} \omega_i \right]$$

where ω_i (resp. γ_j, resp. ξ_k) runs over some basis of $H^1_{DR}(\underline{E}/_X) \otimes \mathcal{O}_{X,x_0}$ (resp. of $(\mathbb{R}_1 f^{an}_* \mathbb{C})$, resp. of $\mathscr{X}_{x_0}$), so that the entries of Z are elements of $\mathcal{O}_{X^{an},x_0}$. On the left side, we can recognize the classical "period matrix" solution of a Picard–Fuchs differential system associated to the quotient $H^1_{DR}(E)$; such a matrix Z was already considered by Y. Manin [17].

Our next theorem deals with a smooth 1–motive of the form $[0 \longrightarrow \underline{E}]$.

Theorem 2. Assume that some fibre of $\underline{E} \xrightarrow{f} X$ splits: $E_{x_1} = T_{x_1} \times A_{x_1}$, and that A_{x_1} is of CM type.

Then the transcendence degree of the $\mathbb{C}(X)$–extension generated by all the "periods" $\oint_{\gamma_j} \omega_i$ equals the dimension of the "generic" derived Mumford–Tate group $\mathscr{D}G$.

Proof: by "generic", we mean the dimension δ of $\mathscr{D}(G_x(\underline{V}([0 \longrightarrow \underline{E}])))$ for any $x \in \overset{\circ}{X}$. The group G_{x_1} is a torus, according to the CM type hypothesis. Since the variation of M.H.S. is good (at least when restricted to curves, see the example at the end of § 4), proposition 2 applies to establish the equality $\delta = \dim H_x$. Since the connection has only regular singular points, we get furthermore that δ is the dimension of the differential Galois group associated to $H^1_{DR}(E)$. But differential Galois theory tells us that this dimension is the transcendence degree of the $\mathbb{C}(X)$–extension generated by the entries of the fundamental solution matrix Z (see [1], [2]). □

Our last theorem is concerned with a smooth 1–motive of the form $[\mathscr{X} \xrightarrow{u} \underline{A}]$, where $\underline{A} \xrightarrow{f} X$ is an Abelian scheme.

Theorem 3. Assume that, over any finite étale covering of X , the map induced by $u : \mathscr{X} \longrightarrow \underline{A}/_{\text{fixed part}}$ remains injective. Then the transcendence degree of the $\mathbb{C}(X)((\oint_{\gamma_j} \omega_i)_{ij})$–extension generated by the germs of analytic functions $\int_0^{\xi_k} \omega_i$ (ξ_k as above), equals the dimension of the generic group $\mathfrak{U}$ introduced in § 3.

Proof: using similar arguments from differential Galois theory, we can see that it is enough to show that

$$\mathfrak{U} \simeq \mathrm{Ker}(\mathrm{H}_x(\underline{\mathrm{V}}[\mathscr{X} \longrightarrow \underline{\mathrm{A}}]) \longrightarrow \mathrm{H}_x(\underline{\mathrm{V}}[0 \longrightarrow \underline{\mathrm{A}}])) := \mathrm{U}(\mathrm{H}_x) \ .$$

According to theorem 1, we have $\mathrm{H}_x \lhd \mathrm{G}_x$; thus in order to apply proposition 1, it suffices to show that $\mathrm{W}_{0,x}^{\mathrm{H}_x} \subset \mathrm{W}_{-1,x}$. At the cost of replacing X by finite étale covering, we may assume that H_x is the whole monodromy group (not only its neutral component). We identify $\mathscr{X}$ with its image in $\underline{\mathrm{A}}$ and consider it as a group of sections of f . Let $v_x \in \mathrm{W}_{0,x}^{\mathrm{H}_x}$; it extends to global section v of $\underline{\mathrm{W}}_0$; setting $\xi = \mathrm{Exp}\, v \in \mathscr{X}$, we thus have $\nabla(d/_{dx}) \int_0^{\xi(x)} \omega_x = 0$, for any section ω of $\mathrm{H}^1_{\mathrm{DR}}(\underline{\mathrm{A}}/\mathrm{X}) \otimes \mathscr{O}_{\mathrm{X},x}$ and any derivation $d/_{dx}$ of $\mathbb{C}(\mathrm{X})$. According to Manin [17], this implies that some integral multiple of ξ belongs to the fixed part of $\underline{\mathrm{A}}$. However the hypothesis we have made upon u implies in turn that ξ is torsion, so that $v \in \Gamma \underline{\mathrm{W}}_{-1}$.

Remark: this result is the geometric variant of the "Kummer theory" on Abelian varieties, which studies the extension of the field of rationality of some torsion points, generated by the division points of some non–torsion points.

Remark: the exact sequence (*) of $\mathbb{C}(\mathrm{X})$–vector spaces with connection splits if and only if $\mathrm{U}(\mathrm{H}_x) = 0$.

Appendix

Automorphisms of certain Hodge structure over number fields

So far we have been constructed only with polarized Hodge structures $(H_{\mathbb{Z}},h,\psi)$ over $\mathbb{Z}$, and we used some variants of the argument that the automorphisms of $(H_{\mathbb{Z}},h,\psi)$ form a finite group, say G : indeed G imbeds both into the discrete group $GL(H_{\mathbb{Z}})$ and into the compact orthogonal group $\mathcal{O}_{\psi} = \mathrm{Aut}(H_{\mathbb{Z}} \otimes \mathbb{R}, \psi(\cdot, h(i)\cdot))$. If $\mathbb{Z}$ is replaced by the ring of integers R of some totally real number field, the group $GL(H_R)$ is no longer discrete in general; even if one tries to use Weil's restriction of scalars from R to $\mathbb{Z}$, it could happen that the "conjugates" of $\mathcal{O}_{\psi}$ are not compact. Here we shall study those polarized Hodge structures over R which arise naturally as pieces of the cohomology of Abelian varieties with many endomorphisms, and show how the finiteness of G involves arithmetical questions.

A. Classification of Abelian varieties with many endomorphisms

Let X be a complex simple Abelian variety of dimension $g > 0$, such that $D = \mathrm{End}\, X \otimes_{\mathbb{Z}} \mathbb{Q}$ contains some commutative field E of degree g over $\mathbb{Q}$. Since X is simple, D is a division ring whose center is denoted by Z. Any polarization ψ of X defines a positive involution $*$ over D; this implies that the subfield Z^+ of Z fixed by $*$ is a totally real number field. After Albert's classification (cf. [8] 11), four cases can occur a priori:

Type I: $Z^+ = Z = E = D$; X is then called a "Hilbert–Blumenthal" Abelian variety.

Type II: $Z^+ = Z$ and for every real place ρ of Z, $D \underset{Z,\rho}{\otimes} \mathbb{R} \cong M_2(\mathbb{R})$.

According to [8] loc. cit., there exists $a \in D$, such that the reduced trace $\mathrm{Tr}_{D/Z}(a)$ vanishes, and such that the involution $*$ is given by $x^* = a[\mathrm{Tr}_{D/Z}(x) - x]a^{-1}$ for any $x \in D$. Since D is a quaternion algebra over Z, there exists $b \in D$, such that the reduced trace $\mathrm{Tr}_{D/Z}(b)$ vanishes, and which anticommutes with a. We then have $b^* = b$. So $Z(b)$ is totally real and one can assume that $E = Z(b)$.

Type III: $Z^+ = Z$ and for every place ρ of Z, $D_{Z,\rho} \otimes \mathbb{R}$ is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$. In fact this case does not occur under our assumptions on X. Indeed the representation of $\mathrm{End}_{\mathbb{H}}[H_1(X^{an},\mathbb{R}) \otimes_{Z,\rho} \mathbb{R}]$ over $H_1(X^{an},\mathbb{R}) \otimes_{Z,\rho} \mathbb{R}$ yields, after complexification, two copies of the standard representation of SO_2 ([9, lemma 2.3]). This representation thus decomposes into four sub–representations of degree one, whose endomorphism algebra has to be $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$: this is impossible.

<u>Type IV</u>: Z is a totally imaginary quadratic extension of Z^+ . Either $[Z:\mathbb{Q}] = 2g$ in which case X is said of "CM type" and we can choose $E = Z^+$, or $[Z:\mathbb{Q}] = g$ and we can assume that E is a totally imaginary quadratic extension of its subfield E^+ fixed by $*$, whence the following diagram of extensions:

$$\begin{tikzcd} & E \arrow[dl, dash] \arrow[dr, dash, "2"] & \\ Z \arrow[dr, dash, "2"'] & & E^+ \arrow[dl, dash] \\ & Z^+ & \end{tikzcd}$$

since $[D:\mathbb{Q}] \leq 2g$, $[E:Z] \leq [D:E]$ (from the commutativity of E), and $[E:\mathbb{Q}] = g$, we find that $[E:Z] \leq 2$.

Except in the CM case, E is a maximal commutative subfield of D , and in any case we shall write E^+ for the subfield of E fixed by $*$, K for the Galois closure of E^+ in $\mathbb{R}$, and R for the ring of integers of K .

B. <u>The Hodge structure</u> H_μ <u>over</u> R

Let us pick some primitive element ζ of E^+ over $\mathbb{Q}$ in the order $(\mathrm{End}\ X) \cap E^+$ of E^+. This element acts via ζ^* on the free R–module $H^1(X^{an},R)$, and its characteristic polynomial has rational integral coefficients and the same roots as the minimal polynomial of ζ; that the characteristic polynomial thus equals some power of this (separable) minimal polynomial, so that some essential R–submodule of $H^1(X^{an},R)$ decomposes into a direct sum of free R–modules H_μ, the indices running among the imbeddings of E^+ into K. Let L be the compositum in $\mathbb{C}$ of K and the image of E through some complex imbedding, so that $L = K$ except in the non–CM type IV case. Then the rank of H_μ is $2g/[E:\mathbb{Q}] = 2[L:K]$. The free R–module H_μ is naturally endowed with a polarized Hodge structure (h_μ, ψ_μ) of type $(0,1) + (1,0)$ over R, and there is an isomorphism of polarized K–Hodge structure $(H^1(X^{an},K),h,\psi) = \bigoplus_{\mu:E^+\longrightarrow K} (H_\mu \otimes_R K, h_\mu, \psi_\mu)$. Furthermore when $L \neq K$, ψ_μ comes from a L–hermitian form φ_μ on the L–vector space $H_\mu \otimes_R K$.

C. <u>Automorphisms of</u> (H_μ, h_μ, ψ_μ)

<u>Proposition 3</u>. The group G of L–linear automorphism of (H_μ, h_μ, ψ_μ) is <u>infinite</u> if and only if one of the following statements holds:

i) $K = L$, and there exists some non–totally positive element $k \in K^\times$ such that the multiple $\sqrt{k}.C$ of the Weil morphism $C = h_\mu(\sqrt{-1})$ on $H_\mu \otimes_R \mathbb{R}$ comes from an endomorphism of $H_\mu \otimes_R K$,

ii) $K \neq L$ and the direct summand $(H_\mu \otimes_R K) \otimes_L \mathbb{C}$ of $H_\mu \otimes_R \mathbb{C}$ is bihomogeneous.

We begin the proof with the case $K = L$.
Let us choose an R–basis of H_μ such that the Riemann form $\psi_\mu = <\cdot,\cdot>$ is represented by the matrix $\begin{bmatrix} 0 & e \\ -e & 0 \end{bmatrix}$ for some $e \in R^\times$, and let us consider the matrix of C in the basis (viewed as a basis of $H_\mu \otimes_R \mathbb{R}$): since $C^2 = -1$, this matrix has the shape $\begin{bmatrix} -\beta & -\gamma \\ \alpha & \beta \end{bmatrix}$, for $(\alpha,\beta,\gamma) \in \mathbb{R}^3$ satisfying the equation $a\gamma = 1 + \beta^2$. It follows that $\alpha\gamma \neq 0$. The symmetric form $<\cdot,C(\cdot)>$ is represented by $Q = \begin{bmatrix} \alpha e & \beta e \\ \beta e & \gamma e \end{bmatrix}$. Let $\theta \in G$, so that $\theta \in \mathrm{Aut}\, H_\mu \cap O(H_\mu \otimes \mathbb{R}, Q)$, and let us write $\theta_{ij} \in R$ for the coefficients of the matrix of θ. The equation ${}^t\theta Q \theta = Q$ is equivalent to the system

$$(\Sigma)\begin{cases} \alpha(\theta_{11}^2-1) + 2\beta\,\theta_{11}\theta_{21} + \gamma\theta_{21}^2 = 0 \\ \alpha\theta_{11}\theta_{12} + \beta(\theta_{12}\theta_{21} + \theta_{11}\theta_{22} - 1) + \gamma\theta_{21}\theta_{22} = 0 \\ \alpha\theta_{12}^2 + 2\beta\,\theta_{12}\theta_{22} + \gamma(\theta_{22}^2-1) = 0 \end{cases}.$$

a) Let us first deal with the case when C is defined over some totally real algebraic extension of K . Then α,β,γ are totally real algebraic numbers. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, and let α^σ, β^σ, γ^σ be conjugates (necessarily real) of α,β,γ respectively, above σ . Setting $Q^\sigma = \begin{bmatrix} \alpha^\sigma e^\sigma & \beta^\sigma e^\sigma \\ \beta^\sigma e^\sigma & \gamma^\sigma e^\sigma \end{bmatrix}$, we find ${}^t\theta^\sigma Q^\sigma \theta^\sigma = Q^\sigma$, and $\det Q^\sigma = (e^\sigma)^2 > 0$, so that θ^σ belongs to the <u>compact</u> orthogonal group $O_2(Q^\sigma)$. By restriction of scalars à la Weil from K to $\mathbb{Q}$, G imbeds into $(\mathrm{Res}_{K/\mathbb{Q}} \mathrm{Aut}(H_\mu \otimes_R K))(\mathbb{Z})$ (which is discrete) and into $\prod_\sigma O_2(Q^\sigma)$ (which is compact), so that G is finite in this case. Here we point out that the CM type is a special case: indeed the Hodge bigraduation of $H_\mu \otimes_R \mathbb{C}$ comes from the CM decomposition $H_\mu \otimes_R L' = [H^1(X^{an},\mathbb{Q}) \underset{Z,\nu}{\otimes} L'] \oplus [H^1(X^{an},\mathbb{Q}) \underset{Z,\bar\nu}{\otimes} L']$, for some complex place ν of Z over μ (here we denote by L' the compositum $K\cdot\nu(Z)$ whihc is a quadratic totally imaginary extension of K). Let us write $L' = K(h)$ with $h^2 = -g \in \mathbb{R}^-$; the matrix of C (in some basis adapted to the above decomposition) reads $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, thus C is defined over the totally real number field $K(\sqrt{g}) = K(ih)$.

b) Let us now assume that α,β,γ span a line over K ; since $\alpha\gamma \neq 0$, we write $\beta = b\alpha$, $\gamma = c\alpha$, for some $(b,c) \in K \times K^\times$. This yields $\alpha^2 = \frac{1}{a-b^2} \in K \cap \mathbb{R}^+$. Getting rid of the above possibility 2), we are reduced to the case i) of the proposition with $k = c-b^2$. Since any $\theta \in G$ commutes with $\frac{1}{\alpha}C = \begin{bmatrix} -1 & -c \\ 1 & b \end{bmatrix}$, θ has the form $\begin{bmatrix} x & -cy \\ y & x+2by \end{bmatrix}$ for x,y,cy and 2by $\in$ R . The set of all these matrices is an order R' in the field $K' = K(\sqrt{b^2-c}) = K(i\alpha)$, as is seen by identifying $\begin{bmatrix} x & -cy \\ y & x+eby \end{bmatrix}$ with $(x+by) + y\sqrt{b^2-c}$. Since θ is invertible, it is identified with some unit in R' . The equation ${}^t\theta Q\theta = Q$ then reads $X^2 + 2bxy + cy^2 = 1$, that is $(x+by) + y\sqrt{b^2-c} \in \mathrm{Ker}\, N_{K'/K}$.

But $N_{K'/K}$ has maximal rank as a morphism between unit groups $(R')^{\times} \longrightarrow R^{\times}$. By assumption, K' is not totally imaginary, so that by Dirichlet's theorem $\mathrm{rk}(R')^{\times} > \mathrm{rk}\, R^{\times}$. Thus the kernel of $N_{K'/K}$ in $(R')^{\times}$ contains infinitely many elements, and so does G in this case.

c) It remains to deal with the case when α,β,γ span a K–vector space of dimension at least 2. This implies that all minors of (Σ) vanish. In particular,

$$(1) \quad (\theta_{11}^2-1)(\theta_{12}\theta_{21} + \theta_{11}\theta_{22}-1) = 2\theta_{11}^2\theta_{12}\theta_{21}$$

$$(2) \quad (\theta_{22}^2-1)(\theta_{12}\theta_{21} + \theta_{11}\theta_{22}-1) = 2\theta_{12}\theta_{22}\theta_{22}^2$$

$$(3) \quad (\theta_{22}^2-1)(\theta_{22}^2-1) = \theta_{12}^2\theta_{21}^2$$

from which it follows that $(\theta_{12}\theta_{21} + \theta_{11}\theta_{22}-1)\theta_{22}-1)\theta_{12}^2\theta_{21}^2 = 2\theta_{11}\theta_{12}^2\theta_{21}^2\theta_{22}$, so that $\theta_{12}\theta_{21} = 1 + \theta_{22}\theta_{11}$ if $\theta_{12}\theta_{21} \neq 0$. Squaring, we find (using (3) again) that $\theta_{11} = -\theta_{22}$ in this case, and from (1) we get $\theta_{12}\theta_{21} = 1-\theta_{11}^2$; that is, $\det\theta = -1$ and $\mathrm{tr}\,\theta = 0$, from which it follows that $\theta^2 = 1$. If $\theta_{12}\theta_{21} = 0$, we get from the vanishing of the other minors) that $\theta_{11}^2 = \theta_{22}^2 = 1$, and moreover that $\theta_{11}\theta_{22} = -1$ if θ_{12} and θ_{21} do not vanish simultaneously; so we are reduced to the previous case where $\theta_{11} = -\theta_{22}$, except if $\theta = \pm 1$. From this description we see that any two elements of G, distinct from ± 1, are inverse up to sign; this implies that G is finite (with at most 4 elements).

We now turn to the case $K \neq L$.

Let us choose a R–basis of H_μ such that the L–hermitian form $\varphi_\mu = \langle\cdot,\cdot\rangle$ is represented by the matrix $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$, for some $(e,f) \in (R^{\times})^2$. We identify $L \otimes_K \mathbb{R}$ with $\mathbb{C}$ by means of an element h of L such that $h^2 = -g \in K \cap \mathbb{R}'$; since L is totally imaginary (like E), g is totally positive. The Weil morphism C is linear with respect to the complex structure induced by $L \otimes_K \mathbb{R}$ on $H_\mu \otimes_R \mathbb{R}$, since it commutes with the action of L .

a) Let us first deal with the case when $(H_\mu \otimes_R K) \otimes_L \mathbb{C}$ is <u>not</u> bihomogeneous. Through the isomorphism $\mathbb{C} \cong L \otimes_K \mathbb{R}$, $(H_\mu \otimes_R K) \otimes_L \mathbb{C}$ can be identified with the complex plan $H_\mu \otimes_R \mathbb{R}$, and C denotes the two eigenvalues $\pm i$ on $H_\mu \otimes_R \mathbb{R}$. Since ψ_μ is a morphism of the Hodge structure and since C is $\mathbb{C}$–linear, C belongs to the unitary group of φ_μ . Using this property, and the equations $C^2 = -1$ and $\mathrm{tr}\, C = 0$, we get the following matrix representation for C : $\begin{bmatrix} ht & 7 \\ -\alpha & -ht \end{bmatrix}$ for $t \in \mathbb{R}$, $(\alpha,\gamma) \in \mathbb{C}^2$, and with the following equation:

$$(*) \qquad \alpha\gamma + gt^2 = 1 \ \text{ and } \ f\bar{\alpha} = e\gamma \ .$$

Let us write $\alpha = v + hw$, for $(v,w) \in \mathbb{R}^2$. Taking into account $(*)$, we find the following matrix representation for the symmetric form $\mathrm{Re}\, h/g <\cdot, C(\cdot)>$ in the <u>real</u> basis of $H_\mu \otimes_R \mathbb{R}$ attached to the chosen complex basis:

$$Q_\mu = \begin{bmatrix} -et & 0 & fw & -fv \\ 0 & -get & fv & gfw \\ fw & fv & ft & 0 \\ -fv & gfw & 0 & gft \end{bmatrix} \quad \text{for } (t,v,w) \in \mathbb{R}^3 .$$

Since Q_μ has maximal rank and index 0, the first main 1–minor is non–zero: $t \neq 0$. Let us first assume that $\alpha \neq 0$. Since $\Theta \in G$ commutes with C , we find that Θ has following matrix representation:

$$\begin{bmatrix} x & -\gamma y/\alpha \\ y & x+2hty/\alpha \end{bmatrix} = \begin{bmatrix} x & -f\bar{\alpha}y/\alpha e \\ y & x+2hty/\alpha \end{bmatrix} , \text{ for } (x,y) \in L^2 \ .$$

Furthermore the relation ${}^t\bar\theta \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \theta = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ yields the system

$$\begin{cases} x\bar{x} + f/e \cdot y\bar{y} = 1 & (1) \\ x\bar{x} + (f/e + 4gt^2/_{\alpha\bar\alpha})y\bar{y} = 1 + 2ht/_{\alpha\bar\alpha}(\alpha x\bar{y} + \bar{\alpha}\bar{x}y) & (2) \\ 2ht\, y\bar{y} = \bar{\alpha}\bar{x}y - \alpha x\bar{y} & (3)\ . \end{cases}$$

Eliminating $x\bar{x}$ between (1) and (2) and $y\bar{y}$ between (2) and (3), one obtains $\bar{x}y = 0$; reporting this equation in (1) and (3) gives $y = 0$ and $x\bar{x} = 1$. (Note that since θ is invertible, x is a unit in L).

If on the contrary $\alpha = 0$, then $\gamma = 0$ according to (∗), so that θ is diagonal and $x\bar{x} = 1$ again. In both cases, to show that G is finite, it suffices to prove that the unit in Ker $N_{L/K}$ form a finite group. Since L is a totally imaginary quadratic extension of K , the unit groups U_L and U_K have the same rank $[K:\mathbb{Q}]-1$, thus the desired statement comes from Dirichlet's theorem.

b) It remains to deal with the case ii) of the proposition. In this case C is the homothety with scale $\pm i \in L \otimes_K \mathbb{R}$ on $H_\mu \otimes_R \mathbb{R}$. The matrix of the symmetric form Re $h<\cdot, C(\cdot)>$ in the real basis of $H_\mu \otimes_R \mathbb{R}$ attached to the chosen complex basis reads:

$$Q = \sqrt{g} \begin{bmatrix} e & & & 0 \\ & ge & & \\ & & f & \\ 0 & & & gf \end{bmatrix} \quad g > 0\ ,\ e,f \in \mathbb{R}^{x}\ .$$

Since Q is definite (positive or negative) it follows from Sylvester's criterion that the product $\delta_1\delta_3$ of the first and third main minors of Q is positive: $ef > 0$.

Let K' the imaginary quadratic extension of K generated by $\sqrt{-e/f}$. We shall show that K' is not totally imaginary. Indeed, according to a result of Shimura [11, th. 5], there exists at least one place $\sigma\mu$ of K ($\sigma \in \mathrm{Gal}(K/\mathbb{Q})$) such that $H_{\sigma\mu}$ falls into case a). We apply Sylvester's criterion to the matrix

$$Q_{\sigma\mu} = \begin{bmatrix} -e^{\sigma} t & 0 & f^{\sigma} w & -f^{\sigma} v \\ 0 & -g^{\sigma} e^{\sigma} t & f^{\sigma} v & g^{\sigma} f^{\sigma} w \\ f^{\sigma} w & f^{\sigma} v & f^{\sigma} t & 0 \\ -f^{\sigma} v & g^{\sigma} f^{\sigma} w & 0 & g^{\sigma} f^{\sigma} t \end{bmatrix}$$

considered in case a) (for $H_{\sigma\mu}$ instead of H_{μ}).

The product $\delta_1 \delta_3$ is $-(e^2 f)^{\sigma} t^2 (f^{\sigma} v^2 + f^{\sigma} g^{\sigma} w^2 + g^{\sigma} e^{\sigma} t^2)$. Because of the relations (*), this can be simplified: $\delta_1 \delta_3 = -(e^{\sigma} f^{\sigma} t)^2 e^{\sigma} f^{\sigma}$. We find $e^{\sigma} f^{\sigma} < 0$, so that K' is not totally imaginary. Let $\theta \in G$ and δ its L–determinant. The relation ${}^t\bar{\theta} \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$
$\theta = \begin{bmatrix} a & -(f/e)\bar{c}\delta \\ c & \bar{a}\delta \end{bmatrix}$ for the matrix of θ, with $\delta\bar{\delta} = 1$ and $ea\bar{a} + fc\bar{c} = e$. To show that G is infinite, it suffices to consider the case where $a,c \in R$ and $\delta = 1$. Then the set of matrices $\begin{bmatrix} a & -(f/e)c \\ c & a \end{bmatrix}$ with $(a,c) \in K^2$ is a field isomorphic to K'. The subring consisting of matrices with entries in R is an order R', and the subgroup of $(R')^{\times}$ consisting of unimodular matrices satisfying $ea^2 + fc^2 = e$ is the kernel of $N_{K'/K}$ in $(R')^{\times}$. The same argument as in the first part of the proof (K = L , case b), shows that this group is infinite. This completes the proof of the proposition.

□

Along the lines of [4; II 4.4.8], proposition 3 can be used to reprove the conjecture of § 6 for families of Abelian varieties with many endomorphisms. The point is that, except in case ii), the Hodge filtration of $\underline{H}_{\mu}$ is locally constant if and only if the monodromy is finite. Indeed, the local constancy of $F^{\cdot}$ implies that the monodromy group (whose component of identity is semi–simple) imbeds into the automorphism group $\overline{G}$ which is finite except in cases i), ii) and which is a torus in case i); here $\overline{G}$ denotes the Zariski closure of the group G determined by proposition 3.

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