MINIMAL IMMERSIONS OF PROJECTIVE SPACES INTO SPHERES

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Introduction and statement of results.

The purpose of this paper is to show positivity of the dimension of the parameter space of equivalence classes of all full isometric minimal immersions of the complex projective space $P^n(C)$ ($n \geqq 2$) or the quaternion projective space $P^2(H)$ into spheres.

Let (M,g) be a d-dimensional irreducible Riemannian symmetric space of compact type. An isometric immersion $\underline{\Phi}$ of (M,g) into the unit sphere S_1^{ℓ} in $\mathbb{R}^{\ell+1}$ is called to be minimal if for every normal deformations $\underline{\Phi}_t$ of $\underline{\Phi}$ with $\underline{\Phi}_0 = \underline{\Phi}$, the first variation of the volume $(M, \underline{\Phi}_t{}^*g_o)$ is zero at $t=0$, where g_o is the standard Riemannian metric on S_1^{ℓ} with constant curvature one. For a convenience, we call that a minimal immersion $\underline{\Phi}$ of (M,g) into $S_1^{\ell} \subset \mathbb{R}^{\ell+1}$ is full if the image $\underline{\Phi}(M)$ is not contained in a hyperplane of $\mathbb{R}^{\ell+1}$, and that two such immersions $\underline{\Phi}_1$, $\underline{\Phi}_2$ are equivalent if there exist an isometry $\underline{\rho}$ of S_1^{ℓ} such that $\underline{\Phi}_2 = \underline{\rho} \circ \underline{\Phi}_1$.

The first main problem of minimal immersions would be to determine the set $\mathfrak{A}$ of equivalence classes of all full isometric minimal immersions of M into S_1^{ℓ}. This problem was solved by do Carmo and Wallach [2], and Li [13].

We explain the standard construction of minimal immersions of a compact irreducible Riemannian symmetric space (M,g) into spheres (cf. [2],[5]): Let $\underline{\Delta}_g$ be the usual non-negative Laplace operator of (M,g) acting on the space $C^{\infty}(M)$ of all real valued C^{∞} functions on M. We denote by

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$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots ,$$

the set of all mutually distinct eigenvalues of Δ_g, and by V^k the eigenspace of Δ_g with the eigenvalue λ_k. Put $\dim(V^k) = m(k) + 1$. For each $k \geqq 1$, let $\{f_0, \ldots, f_{m(k)}\}$ be an orthonormal basis of V^k with respect to the inner product $(\varphi, \psi) = \int_M \varphi(x)\psi(x)\,d\mu$ with the canonical measure $d\mu$ of (M,g) normalized by $\int_M d\mu = m(k) + 1$. Then the mapping x_k of M into $\mathbb{R}^{m(k)+1}$ defined by

$$x_k : M \ni p \longmapsto (f_0(p), \ldots, f_{m(k)}(p)) \in \mathbb{R}^{m(k)+1}$$

gives a minimal isometric immersion of $(M, \frac{\lambda_k}{d} g)$, $d=\dim(M)$, into the unit sphere $S_1^{m(k)}$. Then the second main problem would be :

Problem (A). Is the minimal immersion x_k rigid ?

Here the <u>rigidity</u> means, if Φ is another full minimal isometric immersion of M into $S_1^{m(k)}$, then Φ is equivalent to x_k.

Now the results of do Carmo and Wallach, Li are the following :

Theorem 1 (cf. do Carmo and Wallach [2], Li [13], Ohnita [7])

1) Assume that there exists a full isometric minimal immersion Φ of (M, Cg) with a positive constant C, into a unit sphere S_1^{ℓ}. Then, for some $k \geqq 1$, $\ell \leqq m(k)$ and $C = \frac{\lambda_k}{d}$.

2) The set $\mathcal{M}$ of equivalence classes of all full isometric minimal immersions of $(M, \frac{\lambda_k}{d} g)$ into S_1^{ℓ} ($\ell \leqq m(k)$) can be smoothly parametrized by a convex body L in a vector space W_2 such that the interior points of L correspond to those $[\Phi]$ for which $\ell = m(k)$, and the boundary points of L correspond to those $[\Phi]$ for which $\ell < m(k)$.

Theorem 1 answers the first problem and Problem (A) is reduced in some sense to the following :

Problem (A'). Whether or not is $\dim(W_2)$ positive ?

In fact, do Carmo and Wallach showed :

Theorem 2 (cf. do Carmo and Wallach [2])

Assume that (M,g) is the d-dimensional unit sphere of constant curvature. Then

$$\dim(W_2) \geqq 18 \quad \text{for} \quad d \geqq 3 \text{ , and } \quad k \geqq 4.$$

Therefore the rigidity does not hold in the situation of Theorem 2. On the contrary,

Theorem 3 (cf. Calabi [12], do Carmo and Wallach [2])

In case of $M = S^2$; or S^d ($d \geqq 3$) and $k \leqq 3$, every full isometric minimal immersion Φ of $(M, \frac{\lambda_k}{d} g)$ into S_1^{ℓ} is equivalent to x_k , that is, the rigidity holds.

Theorem 4 (cf. Wallach [10] , Mashimo [5],[6])

In case of $M = P^n(C)$, $P^n(H)$, or $P^2(Cay)$, the rigidity holds in some sense for $k = 1$, i.e., $\dim(W_2) = 0$ for the immersion x_1.

In the other cases, the problems (A), (A') have been left to be open because of a technical difficulty to estimate the dimension of W_2 below. In this paper, we answer partially problems (A), (A') as follows :

Theorem B. Assume that M is the complex projective space $P^n(C) = SU(n+1)/S(U(1)\times U(n))$ with the $SU(n+1)$-invariant Riemannian metric g. Then we have

$$\dim(W_2) \geqq 91 \quad \text{for} \quad n \geqq 2 \text{ , and } \quad k \geqq 4.$$

That is, in this case, the rigidity does not hold and arbitrary two full minimal isometric immersions of $(P^n(C), \frac{\lambda_k}{2n} g)$ into $S_1^{m(k)}$ can be deformed into each other by a smooth homotopy of minimal immersions of the same type. Here $m(k)+1 = n(n+2k)\left(\frac{(n+k-1)!}{n!\ k!}\right)^2$.

Theorem C. Let $P^2(H) = Sp(3)/Sp(1)\times Sp(2)$ be the quaternion projective space of real simension 8 with the $Sp(3)$-invariant Riemannian metric g. Then we have

$$\dim(W_2) \geqq 29{,}007 \qquad \text{for} \quad k \geqq 4.$$

That is, in this case, the rigidity does not hold and arbitrary two full minimal isometric immersions of $(P^2(H), \frac{\lambda_k}{8} g)$ into $S_1^{m(k)}$ can be deformed into each other by a smooth homotopy of minimal immersions of the same type. Here $m(k)+1 = \frac{(k+4)!(k+3)!}{(k+1)!k!5!3!}(2k+5)$.

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§ 1. The standard minimal immersions.

In this section, we give the notion of the standard minimal immersions after [2], [5].

Let $M = G/K$ be a d-dimensional irreducible symmetric space of compact type, and let g be a G-invariant Riemannian metric on $M = G/K$. We denote the set of all mutually distinct eigenvalues of the Laplace-Beltrami operator Δ_g of (M,g) acting on the space $C^\infty(M)$ of all real valued C^∞ functions on M by

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots ,$$

and the eigenspace of Δ_g corresponding to the eigenvalue λ_k by V^k. Put $\dim(V^k) = m(k)+1$. We give the L^2-inner product $(\ ,\ )$ on V^k by $(f,h) = \int_M f\, h\, d\mu$, $\|f\| = (f,f)^{1/2}$, where $d\mu$ is the canonical measure of (M,g) normalized by $\int_M d\mu = m(k)+1$.

Suppose that $k \geq 1$. Let $\{f_0, f_1, \ldots, f_{m(k)}\}$ be an orthonormal basis for V^k with respect to $(\ ,\ )$ and define a mapping x_k of $\mathbb{R}^{m(k)+1}$ by

$$x_k(p) = (f_0(p), f_1(p), \ldots, f_{m(k)}(p)), \quad p \in M.$$

The action of G on M induces a natural one on V^k by $(\sigma \cdot f)(p) = f(\sigma^{-1} p)$, $\sigma \in G$, $p \in M$. The orthonormality of $\{f_i\}_{i=0}^{m(k)}$ and the homogeneity of M imply the image $x_k(M)$ is included in the unit sphere $S_1^{m(k)}$ of the Euclidean space $\mathbb{R}^{m(k)+1}$. Moreover by the G-invariance of the metric g and the assumption of the irreducibility of the linear isotropy action of K, the mapping x_k is an immersion and the induced metric $\tilde{g} = x_k{}^* g_o$ coincides with the metric g up to a positive constant C, where g_o is the standard Euclidean metric of $\mathbb{R}^{m(k)+1}$. Since $x_k : (M,g) \longrightarrow S_1^{m(k)}$ is an isometric immersion and the Laplace-Beltrami operator $\Delta_{\tilde{g}} = \frac{1}{C} \Delta_g$ of $(M,\tilde{g})$ satisfies $\Delta_{\tilde{g}} f_i = \frac{\lambda_k}{C} f_i$, $i=0,1,\ldots,m(k)$, a theorem of

Takahashi [9] implies that x_k is a minimal immersion of $(M,\tilde{g})$ into a sphere of radius $\sqrt{\frac{dC}{\lambda_k}}$. It follows that $C = \frac{\lambda_k}{d}$. The isometric minimal immersion $x_k : (M,\tilde{g}) \longrightarrow S_1^{m(k)}$ is called the k-th standard minimal immersion. Note that another orthonormal basis of V^k gives also an isometric minimal immersion of $(M,\tilde{g})$ into $S_1^{m(k)}$, which is equivalent in the sense of the introduction to the immersion x_k.

Now we choose an element f in V^k as $f(eK) \neq 0$, and put $f_0' = \int_K k\cdot f\, dk$ and $f_0 = f_0'/\|f_0'\|$, where dk is the Haar measure on K normalized by $\int_K dk = 1$. Then $k\cdot f_0 = f_0$, $k \in K$, and $f_0(eK) \neq 0$. That is, the G-module V^k is a class one representation of the pair (G,K). We can take an orthonormal basis $\{f_i\}_{i=0}^{m(k)}$ of V^k in such a way that $(f_0(eK), f_1(eK), \ldots, f_{m(k)}(eK)) = (1,0,\ldots,0)$, because there exists an isometry A of the Euclidean space $\mathbb{R}^{m(k)+1}$ such that $A(x_k(eK)) = (1,0,\ldots,0)$. Then it can be proved that

$$(1.1) \qquad x_k(\sigma K) = (f_0(\sigma K), f_1(\sigma K), \ldots, f_{m(k)}(\sigma K)) = \sigma\cdot f_0,$$

for every $\sigma \in G$, under the identification $\mathbb{R}^{m(k)+1} \ni (a_0,\ldots,a_{m(k)}) \longmapsto \sum_{i=0}^{m(k)} a_i f_i \in V^k$. Therefore the standard immersion x_k can be obtained as the orbit $x_k(\sigma K) = \sigma\cdot f_0$, $\sigma \in G$, in the class one representation V^k over $\mathbb{R}$ of (G,K).

The differential x_{k*} of x_k can be expressed in terms of the Lie algebra $\mathfrak{g}$ of G as follows : Let $\mathfrak{k}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to the Lie group K, and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. We identify $\mathfrak{p}$ with the tangent space $T_{eK}M$ by $\mathfrak{p} \ni X \longmapsto X_{eK} \in T_{eK}M$ and the tangent space $T_{\sigma\cdot f_0}V^k$ at $\sigma\cdot f_0$ with V^k itself. Then the differential $x_{k*\sigma K}$ of x_k at $\sigma K \in G/K$ is given by

$$(1.1') \qquad x_{k*\sigma K}(\tau_{\sigma *}X_{eK}) = \frac{d}{dt}\, x_k(\sigma \exp(tX)K)_{t=0} = \sigma(X\cdot f_0),$$

where $\tau_{\sigma *}$ is the differential of the translation by σ : $G/K \ni \sigma' K \longmapsto \sigma\sigma' K \in G/K$. Moreover we give an inner product $(\, , \,)$ on $\underline{p}$ from the G-invariant metric $\tilde{g} = \frac{\lambda_k}{d} g$ by

$$\tilde{g}(X_{eK}, Y_{eK}) = (X,Y), \quad X,Y \in \underline{p}.$$

Then the mapping x_k is isometric from $(M,\tilde{g})$ into V^k if and only if

$$(1.2) \qquad (\sigma X\cdot f_0, \sigma X\cdot f_0) = (X,X) \;, \quad X \in \underline{p} \;, \text{ and } \; \sigma \in G,$$

by (1.1) and the above identifications. The mapping x_k is immersion of M into V^k if and only if the mapping $\underline{p} \ni X \longmapsto X\cdot f_0 \in V^k$ is injective.

§ 2. Parametrization of minimal immersion.

In this section, we preserve the notations in §1. Let $(M = G/K, g)$ be an irreducible Riemannian symmetric space of compact type and let x_k be the k-th standard minimal isometric immersion of $(M,\tilde{g})$ into $S_1^{m(k)}$. Then we have :

Theorem 2.1 (cf. [2], [7], [13])

1) Assume that there exists a full isometric minimal immersion of (M,Cg) with a positive constant C, into a unit sphere S_1 . Then , for some $k \geq 1$, $\ell \leq m(k)$ and $C = \frac{\lambda_k}{d}$, where $d = \dim(M)$.

2) The set $\mathfrak{A}$ of equivalence classes of all full isometric minimal immersions of $(M,\frac{\lambda_k}{d} g)$ into S_1^{ℓ} , $\ell \leq m(k)$, can be smoothly parametrized by a convex body L in a vector space W_2 such that the interior points of L correspond to those $[\Phi]$ for which $\ell = m(k)$, and the boundary points of L correspond to those $[\Phi]$ for which $\ell < m(k)$.

The sets W_2, L in the above theorem can be constructed as follows : Let V_0, V_1 be the K-invariant subspaces of V^k defined by

$$V_0 = \mathbb{R} f_0 \quad , \text{ and } \quad V_1 = \{ X \cdot f_0 \; ; \; X \in \underline{p} \}.$$

By the G-invariance of the inner product (,) of V^k, the subspaces V_0 and V_1 are mutually orthogonal with respect to (,). Put V' the orthogonal complement of the sum $V_0 + V_1$ in the space V^k with respect to (,). Then we get the decomposition of V^k as K-modules :

$$(2.1) \qquad V^k = V_0 \oplus V_1 \oplus V' .$$

Let P_1 be the projection of V^k into V_1 under this decomposition. Let S be the set of all linear (over $\mathbb{R}$) mappings of V^k into itself which are symmetric with respect to (,). Define the G-action on S by $\underline{\sigma} \cdot A = \underline{\sigma} \, A \, \underline{\sigma}^{-1}$, $\underline{\sigma} \in G$, $A \in S$, and the G-invariant inner product (,) on S by $(A,B) = \mathrm{trace}(AB)$, $A,B \in S$.
Let S_1 be the set of all symmetric linear mappings of V_1 into itself. The set S_1 can be considered as a subset of S.
For every $u,v \in V^k$, define a linear mapping $P_{u,v}$ by $P_{u,v}(t) = (u,t)\, v$, $t \in V^k$. Then the mapping $Q_{u,v} = \frac{1}{2}(P_{u,v} + P_{v,u})$ belongs to S and the linear span of $Q_{u,u}$, $u \in V^k$, coincides with S. Moreover $Q_{u,v} \in S_1$ for $u,v \in V_1$, and the linear span of $Q_{u,u}$, $u \in V_1$, coincides with S_1. Note that

$$(2.2) \qquad (B, Q_{u,u}) = (B(u),u) \quad , \text{ for every } B \in S \text{ and } u \in V^k$$

by definition.

Now let W_1 be the linear span of the G-orbit of S_1 in S and $W_2 = \{ A \in S \; ; \; (A, W_1) = 0 \}$ its orthogonal complement.

Define the subset L of W_2 by

$$L = \{ C \in W_2 \ ; \ C + I \geqq 0 \},$$

where I is the identity mapping of V^k and $C + I \geqq 0$ means that $((C+I)(u),u) \geqq 0$ for all $u \in V^k$.

Theorem 2.1 can be proved by the same manner as Theorems 1.3 and 1.5 in [5] (cf. see Li [13]).

§ 3. Estimation of the dimension of W_2.

We preserve the notations in §2. Consider the natural isomorphism Q of the symmetric square S^2V^k of V^k onto S induced by $S^2V^k \ni u \cdot v \longmapsto Q_{u,v} \in S$. The G-action on V^k is extended naturally to S^2V^k, and the G-invariant inner product $(\ , \ )$ on V^k can be extended to the G-invariant one on S^2V^k. Since we have

$$\underline{\sigma} \cdot Q_{u,v} = \underline{\sigma} \, Q_{u,v} \, \underline{\sigma}^{-1} = Q_{\underline{\sigma} u, \underline{\sigma} v} \ , \text{ and}$$

$$(Q_{u,v}, Q_{u',v'}) = (u \cdot v, u' \cdot v'), \quad \text{for } \underline{\sigma} \in G, \ u,v,u',v' \in V^k,$$

the mapping Q is G-isomorphic and isometric. Moreover the image $Q(S^2V_1)$ of the symmetric square S^2V_1 of V_1 in (2.1) by Q coincides with S_1. Therefore the space W_1 is identified by Q with the linear span of the G-orbits of S^2V_1 in S^2V^k and W_2 is also identified with its orthogonal complement in S^2V^k.

Furthermore, in order to estimate dimension of W_2, we consider its complexification $W_2^{\mathbb{C}}$. We denote by $W^{\mathbb{C}}$ the complexification of a real vector space W. We extend the inner product $(\ , \ )$ on S^2V^k to the hermitian inner product on $(S^2V^k)^{\mathbb{C}} = S^2(V^{k\mathbb{C}})$. Then $W_1^{\mathbb{C}}$ is the linear span of the G-orbit of $S^2(V_1^{\mathbb{C}})$ in $S^2(V^{k\mathbb{C}})$ and $W_2^{\mathbb{C}}$ is its orthogonal complement in $S^2(V^{k\mathbb{C}})$. We have :

Lemma 3.1.

Let W_3 be the sum of G-submodules of $S^2(V^{k\mathbb{C}})$ over $\mathbb{C}$, not containing the K-irreducible components of $S^2(V_1^{\mathbb{C}})$. Then W_3 is included in $W_2^{\mathbb{C}}$.

Proof. It can be proved by the same manner as Lemma 5.4 in [2]. We have only to consider unitary representations instead of real orthogonal ones of compact Lie groups, making use of the Frobenius reciprocity theorem as in [1], [3]. Proof is omitted.

By Lemma 3.1, we can give an estimation of $\dim(W_2)$ by the analogous way as in [2]. In order to estimate $\dim_{\mathbb{C}}(W_3)$, note that, if the symmetric space $M = G/K$ is of <u>rank one</u>, i.e., a maximal abelian subalgebra of $\underline{g}$ contained in $\underline{p}$ is one dimensional, then every eigenspace of the Laplace-Beltrami operator is an irreducible class one representation of the pair (G,K) over $\mathbb{R}$ and its complexification is also irreducible. Therefore we can make use of a finite dimensional unitary representation theory of a compact Lie group to estimate $\dim_{\mathbb{C}}(W_3)$, which are carried out in the following sections, in case of projective spaces.

§ 4. Complex projective spaces (I).

4.1. In this section, we use the following notations:

$$G = SU(n+1),\ n \geqq 2,$$

$$K = S(U(1)\times U(n)) = \left\{ \begin{bmatrix} 1/\det\sigma & 0 \\ 0 & \sigma \end{bmatrix} ;\ \sigma \in U(n) \right\},$$

$$\underline{g} = \underline{su}(n+1) = \{ X \in M_{n+1}(\mathbb{C}) \ ;\ {}^t\overline{X} + X = 0,\ \mathrm{trace}(X) = 0 \},$$

$$\underline{k} = \left\{ \begin{bmatrix} -\mathrm{trace}(X) & 0 \\ 0 & X \end{bmatrix} ;\ X \in M_n(\mathbb{C})\ ,\ {}^t\overline{X} + X = 0 \right\},$$

$$B(X,Y) = 2(n+1)\ \mathrm{trace}(XY),\quad X,Y \in \underline{g}\ ,\ \text{the Killing form of } \underline{g}\ ,$$

$$\underline{p} = \left\{ \begin{pmatrix} 0 & -\overline{z}_1 & \cdots & -\overline{z}_n \\ z_1 & & & \\ \vdots & & 0 & \\ z_n & & & \end{pmatrix} \in M_{n+1}(\mathbb{C}) \ ;\ z_1,\dots,z_n \in \mathbb{C} \right\},$$

$$T = \left\{ \begin{pmatrix} \varepsilon_1 & & & 0 \\ & \varepsilon_2 & & \\ & & \ddots & \\ 0 & & & \varepsilon_{n+1} \end{pmatrix} \in M_{n+1}(\mathbb{C}) \ ;\ \varepsilon_i \in \mathbb{C},\ |\varepsilon_i| = 1,\ \prod_{i=1}^{n+1} \varepsilon_i = 1 \right\},$$

$$\underline{t} = \left\{ H(x_1,x_2,\dots,x_{n+1}) \ ;\ x_i \in \mathbb{R}\ ,\ \sum_{i=1}^{n+1} x_i = 0 \right\},$$

where $H(x_1,x_2,\dots,x_{n+1}) = 2\pi\sqrt{-1}\begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_{n+1} \end{pmatrix}$. Then we can identify $P^n(C)$ with the coset space G/K having the G-invariant Riemannian metric induced from the inner product $(X,Y) = -\frac{1}{n+1} B(X,Y)\ ,\ X,Y \in \underline{p}$.

Define an element λ_i in the dual space $\underline{t}^*$ of $\underline{t}$ over $\mathbb{R}$ by $\underline{t} \ni H(x_1,x_2,\dots,x_{n+1}) \longmapsto x_i,\ \ 1 \leqq i \leqq n+1$, and introduce a lexicographic order $>$ on $\underline{t}^*$ in such a way that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 > \lambda_{n+1}\ .$$

Put

$$D(G) = \left\{ \Lambda = \sum_{i=1}^{n} m_i \lambda_i \in \underline{t}^* ;\ m_i \in \mathbb{Z}\ (1 \leqq i \leqq n),\ m_1 \geqq m_2 \geqq \cdots \geqq m_n \geqq 0 \right\},$$

$$D(K) = \left\{ \underline{\Lambda} = \sum_{i=1}^{n} k_i \underline{\lambda}_i \in \underline{t}^* \ ; \ k_i \in \mathbb{Z} \ (1 \leq i \leq n),\ k_2 \geqq k_3 \geqq \cdots \geqq k_n \geqq 0 \right\}.$$

Then $D(G)$ (resp. $D(K)$) is the set of all dominant integral forms of G (resp. K) with respect to $\underline{t}$. Thus there exist a bijection between a complete set $\mathcal{D}(G)$ (resp. $\mathcal{D}(K)$) of nonequivalent irreducible modules of G (resp. K) over $\mathbb{C}$ and the set $D(G)$ (resp. $D(K)$) assigning $\underline{\Lambda} \in D(G)$ (resp. $D(K)$) to an element $V = V_{\underline{\Lambda}} \in \mathcal{D}(G)$ (resp. $\mathcal{D}(K)$) with the highest weight $\underline{\Lambda}$. Under the above situations, we have

Theorem 4.1. (the branching theorem)

Let $V = V_{\underline{\Lambda}}$ be an irreducible G-module over $\mathbb{C}$ with highest weight $\underline{\Lambda} = \sum_{i=1}^{n} m_i \underline{\lambda}_i$, $m_1 \geqq m_2 \geqq \cdots \geqq m_n \geqq 0$. Then $V = V_{\underline{\Lambda}}$ decomposes as a K-modules, into irreducible ones :

$$V_{\underline{\Lambda}} = \sum V_{k_1 \underline{\lambda}_1 + \cdots + k_n \underline{\lambda}_n},$$

where the summation runs over all the integers $k_1, \ldots, k_n$ for which there exist a non-negative integer k satisfying

$$m_1 \geqq k_2 + k \geqq m_2 \geqq k_3 + k \geqq m_3 \geqq \cdots \geqq m_{n-1} \geqq k_n + k \geqq m_n \geqq k, \text{ and}$$
$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} k_i + (n+1)k.$$

Proof. See [3].

Note that the irreducible modules $V_{k\underline{\lambda}_1 - k\underline{\lambda}_{n+1}}$ with highest weight $k\underline{\lambda}_1 - k\underline{\lambda}_{n+1} = 2k\underline{\lambda}_1 + k\underline{\lambda}_2 + \cdots + k\underline{\lambda}_n$, $k \geqq 0$, exhaust all class one (i.e., including the trivial representation of K) irreducible modules of the pair (G,K) over $\mathbb{C}$. The modules $V_{k\underline{\lambda}_1 - k\underline{\lambda}_{n+1}}$ are represented as follows (see for example [5]) :

Let $S^{k,k}(\mathbb{C}^{n+1})$ be the space of all complex valued C^∞ functions f on $\mathbb{C}^{n+1}$ such that $f(\lambda z) = |\lambda|^{2k} f(z)$ for every $z \in \mathbb{C}^{n+1}$, $\lambda \in \mathbb{C}$. Put $H^{k,k}(\mathbb{C}^{n+1}) = \{ f \in S^{k,k}(\mathbb{C}^{n+1}) \ ; \ \Delta_o f = o \}$, where $\Delta_o = \sum_{i=1}^{n+1} \partial^2/\partial z_i \partial \bar{z}$ the standard Laplacian of $\mathbb{C}^{n+1}$. Define an action of $U(n+1)$, also $SU(n+1)$ on $S^{k,k}(\mathbb{C}^{n+1})$ by

$$(g \cdot f)(z) = f(g^{-1} z), \quad z \in \mathbb{C}^{n+1}, \quad g \in U(n+1).$$

Then $H^{k,k}(\mathbb{C}^{n+1})$ is the $SU(n+1)$-irreducible submodule of $S^{k,k}(\mathbb{C}^{n+1})$ with heighest weight $k\lambda_1 - k\lambda_{n+1}$. Let $C^\infty(\mathbb{C}^{n+1},\mathbb{R})$ be the set of all real valued C^∞ functions on $\mathbb{C}^{n+1}$ and put $V^k = H^{k,k}(\mathbb{C}^{n+1}) \cap C^\infty(\mathbb{C}^{n+1},\mathbb{R})$. Then V^k is a class one representation over $\mathbb{R}$ of the pair (G,K) whose complexification $V^{k\mathbb{C}}$ is $V_{k\lambda_1 - k\lambda_{n+1}} = H^{k,k}(\mathbb{C}^{n+1})$, and it induces the eigenspace of the Laplace-Beltrami operator of the G-invariant Riemannian metric on G/K corresponding to the inner product $-\frac{1}{n+1} B$ with the eigenvalue $k(k+n)$.

4.2. Now by Theorem 4.1, the class one representation $V^{k\mathbb{C}}$ is decomposed into irreducible K-modules as follows :

$$(4.1) \qquad V^{k\mathbb{C}} = \sum_{p=o,1,\ldots,k} \ \sum_{q=o,1,\ldots,k} V_{p,q},$$

where $V_{p,q}$, $p,q = o,1,\ldots,k$, are the irreducible K-modules with highest weight

$$p(\lambda_1 - \lambda_{n+1}) + q(-\lambda_1 + \lambda_2) = \begin{cases} (2p-q)\lambda_1 + (p+q)\lambda_2 + p\lambda_3 + \cdots + p\lambda & (n \geq 3) \\ (2p-q)\lambda_1 + (p+q)\lambda_2 & (n = 2) \end{cases}$$

The K-module $\underline{p}^{\mathbb{C}} = \left\{ \begin{pmatrix} o & w_1 & \cdots & w_n \\ z_1 & & & \\ \vdots & & 0 & \\ z_n & & & \end{pmatrix} ; \ z_i, w_i \in \mathbb{C} \ (1 \leq i \leq n) \right\}$ is decomposed into irreducible K-modules as follows :

$$\underline{p}^{\mathbb{C}} = V_{1,o} \oplus V_{o,1}.$$

Then the components of the decomposition $V^{k\mathbb{C}} = (V_o)^{\mathbb{C}} \oplus (V_1)^{\mathbb{C}} \oplus (V')^{\mathbb{C}}$ are given as K-modules by

$$(V_o)^{\mathbb{C}} = V_{o,o} \ , \ (V_1)^{\mathbb{C}} = V_{1,o} \oplus V_{o,1} \ , \text{ and } \ (V')^{\mathbb{C}} = \sum_{(p,q)\in I} V_{p,q}$$

where $I = \{(p,q) \ ; \ p,q = o,1,\dots,k\} \setminus \{(o,o),(o,1),(1,o)\}$.

Then the K-module $S^2(V_1{}^{\mathbb{C}})$ is decomposed as follows :

$$(4.2) \quad S^2(V_1{}^{\mathbb{C}}) = V_{2(\underline{\lambda}_1 - \underline{\lambda}_{n+1})} \oplus V_{\underline{\lambda}_2 - \underline{\lambda}_{n+1}} \oplus V_{-2\underline{\lambda}_1 + 2\underline{\lambda}_2} \oplus V_{o,o}$$

Therefore we have :

Lemma 4.2.

Every G-module over $\mathbb{C}$ which contains some of the K-irreducible components (4.2) of $S^2(V_1{}^{\mathbb{C}})$ has the highest weight $\sum_{i=1}^{n} m_i \underline{\lambda}_i$, where m_i, $1 \leqq i \leqq n$, are one of the n-tuples in the following table :

(i) In case of $n \geqq 4$,

m_1	2k	2k-1	2k-2	2k+3	2k+2	2k+6
m_2	k	k+1	k+2	k+1	k+2	k+2
m_3	k	k	k	k+1	k+1	k+2
⋮						
m_{n-1}	k	k	k	k+1	k+1	k+2
m_n	k	k	k	k	k	k

(ii) in case of $n = 3$,

m_1	2k	2k-1	2k+3	2k-2	2k+2	2k+6
m_2	k	k+1	k+1	k+2	k+2	k+2
m_3	k	k	k	k	k	k

(iii) in case of $n = 2$,

m_1	2k	2k-3	2k+3	2k+6	2k-6

where, in each case, k varies over the set of all non-negative integers.

Proof. For example, we determine the G-modules containing the K-module $V_{\underline{\lambda}_2-\underline{\lambda}_{n+1}}$. The remains are proved by the same manner. The weight $\underline{\lambda}_2-\underline{\lambda}_{n+1}$ coincides with $\underline{\lambda}_1+2\underline{\lambda}_2+\underline{\lambda}_3+\dots+\underline{\lambda}_n$ $(n\geqq 3)$ or $\underline{\lambda}_1+2\underline{\lambda}_2$ $(n=2)$. By Theorem 4.1, the weight $\sum_{i=1}^{n} m_i\underline{\lambda}_i$ of the G-module should satisfy the following :

(i) in case of $n \geqq 4$,

$$m_1\geqq 2+k\geqq m_2\geqq 1+k\geqq m_3\geqq \dots \geqq m_{n-1}\geqq 1+k\geqq m_n\geqq k \text{ , and } \sum_{i=1}^{n} m_i = (n+1)(k+1),$$

(ii) in case of $n = 3$,

$$m_1\geqq 2+k\geqq m_2\geqq 1+k\geqq m_3\geqq k \text{ , and } m_1+m_2+m_3 = 4(k+1),$$

(iii) in case of $n = 2$,

$$m_1\geqq 2+k\geqq m_2\geqq k \text{ , and } m_1+m_2 = 3(k+1),$$

for a certain non-negative integer k. Thus we can determine $(m_1,\dots,m_n)$ satisfying the above conditions. Q.E.D.

4.3. We need the following lemma in order to decompose the G-module $S^2(V^{k\mathbb{C}})$ into the sum of irreducible G-modules.

Lemma 4.3.

For a G-module $(V,\underline{\rho})$ over $\mathbb{C}$ with a character $\underline{\chi}$, the character $\underline{\chi}_{(2)}$ of the symmetric square S^2V is given by

$$\underline{\chi}_{(2)}(\underline{\tau}) = \frac{1}{2}\left(\underline{\chi}(\underline{\tau})^2 + \underline{\chi}(\underline{\tau}^2)\right),\ \underline{\tau}\in G.$$

Proof. See [8] for example. For completeness, we give here its proof. For a fixed $\underline{\tau}\in G$, let $e_i\in V$ be the eigenvectors of $\underline{\rho}(\underline{\tau})$ with the eigenvalue $\underline{\lambda}_i$, i.e., $\underline{\rho}(\underline{\tau})\, e_i = \underline{\lambda}_i\, e_i$,

$i=1, \ldots, N=\dim(V)$. Then the basis $e_1^{m_1} \cdot \ldots \cdot e_N^{m_N}$ $(m_1+\ldots+m_N=k)$ of the k-th symmetric product S^kV of V satisfies

$$\underline{\rho}^{(k)}(\underline{\tau})(e_1^{m_1} \cdot \ldots \cdot e_N^{m_N}) = \underline{\lambda}_1^{m_1} \ldots \underline{\lambda}_N^{m_N}\, e_1^{m_1} \cdot \ldots \cdot e_N^{m_N},$$

where $e_i^{m_i} = e_i \cdot \ldots \cdot e_i$ (m_i times), and $\underline{\rho}^{(k)}(\underline{\tau})$ is the G action on S^kV induced from the one on V. Then the character $\underline{\chi}_{(k)}(\underline{\tau})$ of $\underline{\rho}^{(k)}(\underline{\tau})$ is given by

$$\underline{\chi}_{(k)}(\underline{\tau}) = \sum_{m_1+\ldots+m_N=k} \underline{\lambda}_1^{m_1} \cdots \underline{\lambda}_N^{m_N} .$$

Consider the following generating function of the characters :

$$P(z) = \sum_{k=0}^{\infty} z^k \underline{\chi}_{(k)}(\underline{\tau}) .$$

Then we have

$$\begin{aligned} P(z) &= \sum_{k=0}^{\infty} \sum_{m_1+\ldots+m_N=k} (z\underline{\lambda}_1)^{m_1} \ldots (z\underline{\lambda}_N)^{m_N} \\ &= \sum_{m_1,\ldots,m_N=0}^{\infty} (z\underline{\lambda}_1)^{m_1} \ldots (z\underline{\lambda}_N)^{m_N} \\ &= \prod_{i=1}^{N} (1-z\underline{\lambda}_i)^{-1} \\ &= \det(\, I - z\,\underline{\rho}(\underline{\tau})\,)^{-1} \\ &= \exp(\, \mathrm{trace}(\sum_{k=1}^{\infty} \frac{\underline{\rho}(\underline{\tau}^k)}{k} z^k)) \\ &= \exp(\sum_{k=1}^{\infty} \frac{\underline{\chi}(\underline{\tau}^k)}{k} z^k) . \end{aligned}$$

In fact, the series $P(z)$ has the convergent radius bigger than or equal to $(C|\underline{\chi}(\underline{\tau})|)^{-1}$, where the constant C satisfies $|\underline{\chi}(\underline{\tau}_1 \underline{\tau}_2)| \leqq C\,|\underline{\chi}(\underline{\tau}_1)||\underline{\chi}(\underline{\tau}_2)|$ for every $\underline{\tau}_1, \underline{\tau}_2 \in G$. Then the coefficients $P_n = P^{(n)}(0)/n!$ of P coincide with $\underline{\chi}_{(n)}(\underline{\tau})$. For example, $P_0 = 1$, $P_1 = \chi(\tau)$, $P_2 = \frac{1}{2}(\chi(\tau)^2 + \chi(\tau^2))$, ... Q.E.D.

§ 5. Complex projective spaces (II).

In this section, we investigate the irreducible decomposition of the symmetric square $S^2(V^{k\mathbb{C}})$ due to Lemma 4.3. In order to show $\dim(W_3) > 0$, we have only to show the existence of the irreducible G-submodules of $S^2(V^{k\mathbb{C}})$ which do not appear in the table in Lemma 4.2.

5.1. In this section, we use the following notations :

$$\tilde{G} = U(n+1),$$

$$\tilde{T} = \left\{ \begin{pmatrix} \varepsilon_1 & & & 0 \\ & \varepsilon_2 & & \\ & & \ddots & \\ 0 & & & \varepsilon_{n+1} \end{pmatrix} \in M_{n+1}(\mathbb{C}) \; ; \; \varepsilon_i \in \mathbb{C}, \; |\varepsilon_i| = 1 \; (1 \leqq i \leqq n+1) \right\},$$

$$\tilde{\underline{g}} = \underline{u}(n+1) = \left\{ X \in M_{n+1}(\mathbb{C}) \; ; \; {}^t\overline{X} + X = 0 \right\},$$

$$\tilde{\underline{t}} = \left\{ H(x_1, \ldots, x_{n+1}) \; ; \; x_i \in \mathbb{R} \; (1 \leqq i \leqq n+1) \right\}.$$

Define an element $\tilde{\lambda}_i$ in the dual space $\tilde{\underline{t}}^*$ of $\tilde{\underline{t}}$ over $\mathbb{R}$ by $\tilde{\underline{t}} \ni H(x_1, \ldots, x_{n+1}) \longmapsto x_i$, $1 \leqq i \leqq n+1$, and introduce a lexicographic order $>$ on $\tilde{\underline{t}}^*$ in such a way that

$$\tilde{\lambda}_1 > \tilde{\lambda}_2 > \cdots > \tilde{\lambda}_n > 0 > \tilde{\lambda}_{n+1} \, .$$

Note that λ_i is the restriction of $\tilde{\lambda}_i$ to $\underline{t}$ $(1 \leqq i \leqq n+1)$. Put

$$D(\tilde{G}) = \left\{ \tilde{\Lambda} = \sum_{i=1}^{n+1} f_i \tilde{\lambda}_i \; ; \; f_i \in \mathbb{Z}, \; f_1 \geqq f_2 \geqq \cdots \geqq f_n \geqq f_{n+1} \right\}.$$

Then $D(\tilde{G})$ coincides with the set of all dominant integral forms of $\tilde{G}$ with respect to $\tilde{\underline{t}}$ and there exists a bijection between a complete set $\mathcal{D}(\tilde{G})$ of non-equivalent irreducible modules of $\tilde{G}$ over $\mathbb{C}$ and $D(\tilde{G})$, assigning $\tilde{\Lambda} \in D(\tilde{G})$ to an element $\tilde{V} = V_{\tilde{\Lambda}} \in \mathcal{D}(\tilde{G})$ with the highest weight $\tilde{\Lambda}$. Moreover for each $\tilde{V} = V_{\tilde{\Lambda}} \in \mathcal{D}(\tilde{G})$ with $\tilde{\Lambda} \in D(\tilde{G})$, the module $V = \tilde{V}|_G$, considered as a G-module, belongs to $\mathcal{D}(G)$, its highest weight Λ is the restriction of $\tilde{\Lambda}$ to $\underline{t}$ and its character χ_Λ is the restriction of the one $\chi_{\tilde{\Lambda}}$ of $\tilde{V}$ to G.

By the character formula of Weyl [11],

$$(5.1)\qquad D(\tilde{h})\,\chi_{\tilde{\Lambda}}(\tilde{h}) = \left|\varepsilon_i^{\,l_j}\right| \qquad \text{for each } \tilde{h} = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_{n+1} \end{pmatrix} \in \tilde{T},$$

where $\left|\varepsilon_i^{\,l_j}\right|$ is the determinant of $(n+1)\times(n+1)$ matrix whose (i,j) entries are $\varepsilon_i^{\,l_j}$,

$$(5.2)\qquad l_j = f_j + n + 1 - j \quad (j=1,\ldots,n+1),$$

and $D(\tilde{h})$ is given as follows :

$$(5.3)\qquad D(\tilde{h}) = \left|\varepsilon_i^{\,n+1-j}\right| = \prod_{1\leqq i<j\leqq n+1}(\varepsilon_i - \varepsilon_j).$$

Note that the G-module $V^{k\mathbb{C}} = H^{k,k}(\mathbb{C}^{n+1})$ in 4.1 is also $\tilde{G} = U(n+1)$ irreducible module with highest weight $k\,\tilde{\lambda}_1 - k\,\tilde{\lambda}_{n+1}$.

5.2. First let us consider the irreducible decomposition of $S^2(V^{k\mathbb{C}})$ as $\tilde{G}$-modules :

$$(5.4)\qquad S^2(V^{k\mathbb{C}}) = \sum N(f_1,\ldots,f_{n+1})\, V_{f_1,\ldots,f_{n+1}},$$

where $f_1,\ldots,f_{n+1}$ vary over the set $\{(f_1,\ldots,f_{n+1})\ ;\ f_i \in \mathbb{Z},\ f_1 \geqq \cdots \geqq f_{n+1}\}$, $V_{f_1,\ldots,f_{n+1}}$ is the $\tilde{G}$-irreducible module with highest weight $\sum_{i=1}^{n+1} f_i\,\tilde{\lambda}_i$, and the number $N(f_1,\ldots,f_{n+1})$ is the multiplicity of $V_{f_1,\ldots,f_{n+1}}$ in $S^2(V^{k\mathbb{C}})$. Then since $V_{f_1,\ldots,f_{n+1}}$ is also the G-irreducible module V_{Λ} with highest weight $\Lambda = \sum_{i=1}^{n} m_i\,\lambda_i$, $m_i = f_i - f_{n+1}$ $(i=1,\ldots,n)$, we obtain the irreducible decomposition of $S^2(V^{k\mathbb{C}})$ as G-modules :

$$S^2(V^{k\mathbb{C}}) = \sum M(m_1,\ldots,m_n)\, V_{\sum_{i=1}^{n} m_i \lambda_i},$$

where $m_1,\dots,m_n$ run over the set $\{(m_1,\dots,m_n)\ ;\ m_i\in\mathbb{Z},\ m_1\geq\dots\geq m_n\geq 0\}$, and $M(m_1,\dots,m_n)=\sum_{f_1\geq\dots\geq f_{n+1},\, m_i=f_i-f_{n+1}} N(f_1,\dots,f_{n+1})$ is the the multiplicity of the G-module $V_{\sum_{i=1}^{n} m_i\lambda_i}$ in the one $S^2(V^{k\mathbb{C}})$. Then if we find an irreducible module $V_{f_1,\dots,f_{n+1}}$ of $\tilde{G}$ in (5.4) with $N(f_1,\dots,f_{n+1})>0$, then $S^2(V^{k\mathbb{C}})$ includes at least one the irreducible module $V_{\sum_{i=1}^{n} m_i\lambda_i}$ of G. Therefore we have only to consider the decomposition (5.4) of $S^2(V^{k\mathbb{C}})$ as $\tilde{G}$-modules.

Now by Lemma 4.3, the character $\chi^k{}_{(2)}$ of the $\tilde{G}$-module $S^2(V^{k\mathbb{C}})$ is given by :

$$(5.5)\qquad D_{n+1}\,\chi^k{}_{(2)}=\tfrac{1}{2}\Big\{\big|\varepsilon_i{}^{r_j}\big|^2/D_{n+1}+\big|\varepsilon_i{}^{2r_j}\big|/D'_{n+1}\Big\},$$

where $|\varepsilon_i{}^{r_j}|$ is the determinant whose (i,j)-entries are $\varepsilon_i{}^{r_j}$, $r_1=k+n$, $r_j=n+1-j$ $(j=2,\dots,n)$, $r_{n+1}=-k$, $D_{n+1}=\prod_{1\leq i<j\leq n+1}(\varepsilon_i-\varepsilon_j)$ and $D'_{n+1}=\prod_{1\leq i<j\leq n+1}(\varepsilon_i+\varepsilon_j)$. The right hand side of (5.5) can be written as

$$\prod_{i=1}^{n+1}\varepsilon_i^{-2k}\,\tilde{P}_{n+1}(\varepsilon_1,\dots,\varepsilon_{n+1}),$$

where $\tilde{P}_{n+1}(\varepsilon_1,\dots,\varepsilon_{n+1})$ is the polynomial in $(\varepsilon_1,\dots,\varepsilon_{n+1})$ given by

$$(5.6)\qquad \tilde{P}_{n+1}=\tfrac{1}{2}\Big\{\big|\varepsilon_i{}^{p_j}\big|^2/D_{n+1}+\big|\varepsilon_i{}^{2p_j}\big|/D'_{n+1}\Big\},$$

where $p_1=n+2k$, $p_j=k+n+1-j$ $(j=2,\dots,n)$ and $p_{n+1}=0$. Note that the polynomial $|\varepsilon_i{}^{p_j}|$ (resp. $|\varepsilon_i{}^{2p_j}|$) can be divided formally by the one D_{n+1} (resp. D'_{n+1}).

On the other hand, according to the decomposition (5.4), we get

$$(5.4') \qquad D_{n+1}\, \underset{\sim}{\chi}^{k}(2) = \sum_{f_1 \geqq \cdots \geqq f_{n+1}} N(f_1,\ldots,f_{n+1}) \left| \underline{\varepsilon}_i^{\,l_j} \right| ,$$

where $l_j = f_j + n+1-j$, $j=1,\ldots,n+1$. We arrange the right hand side of (5.4') as the sum of the terms $\underline{\varepsilon}_1^{\,a_1} \cdots \underline{\varepsilon}_{n+1}^{\,a_{n+1}}$ with $a_1 > \cdots > a_{n+1}$ and the terms $\underline{\varepsilon}_1^{\,b_1} \cdots \underline{\varepsilon}_{n+1}^{\,b_{n+1}}$ where there exist two integers $1 \leqq i < j \leqq n+1$ such that $b_i \leqq b_j$, that is,

$$(5.4'') \qquad D_{n+1}\, \underset{\sim}{\chi}^{k}(2) = \sum_{f_1 \geqq \cdots \geqq f_{n+1}} N(f_1,\ldots,f_{n+1})\, \underline{\varepsilon}_1^{\,l_1} \cdots \underline{\varepsilon}_{n+1}^{\,l_{n+1}} + Q(\underline{\varepsilon}_1,\ldots,\underline{\varepsilon}_{n+1}),$$

where $Q(\underline{\varepsilon}_1,\ldots,\underline{\varepsilon}_{n+1})$ is the sum of the latter type.

Now we decompose the polynomial $\widetilde{P}_{n+1}(\underline{\varepsilon}_1,\ldots,\underline{\varepsilon}_{n+1})$ in such a way that

$$(5.6') \qquad \widetilde{P}_{n+1} = \sum_{q_1 > \cdots > q_{n+1} \geqq 0} A(q_1,\ldots,q_{n+1})\, \underline{\varepsilon}_1^{\,q_1} \cdots \underline{\varepsilon}_{n+1}^{\,q_{n+1}} + R(\underline{\varepsilon}_1,\ldots,\underline{\varepsilon}_{n+1}),$$

where $R(\underline{\varepsilon}_1,\ldots,\underline{\varepsilon}_{n+1})$ is the sum of the monomials $\underline{\varepsilon}_1^{\,b_1} \cdots \underline{\varepsilon}_{n+1}^{\,b_{n+1}}$ of $\widetilde{P}_{n+1}$ where there exist two integers $1 \leqq i < j \leqq n+1$ such that $b_i \leqq b_j$. Then comparing with (5.4") and (5.6'), their first term sums coincide each other, in particular, we have

$$A(q_1,\ldots,q_{n+1}) = N(f_1,\ldots,f_{n+1}),$$

where $f_j = q_j - (n+1) - k + j$, $j=1,\ldots,n+1$. Therefore we have only to decompose $\widetilde{P}_{n+1}(\underline{\varepsilon}_1,\ldots,\underline{\varepsilon}_{n+1})$ as (5.6') and to seek the terms $\underline{\varepsilon}_1^{\,q_1} \cdots \underline{\varepsilon}_{n+1}^{\,q_{n+1}}$, $q_1 > \cdots > q_{n+1} \geqq 0$ with a non-zero coefficient

$A(q_1,\dots,q_{n+1})$. Then we obtain the G-module $V_{\sum_{j=1}^n m_j \lambda_j}$ with $m_j = q_j - q_{n+1} - (n+1) + j$, $j=1,\dots,n$, which is included in $S^2(V^{k\mathbb{C}})$.

5.3. The task of the last step in 5.2 is accomplished as follows.

(i) First, decompose $\tilde{P}_{n+1}$ as a sum of the constant term $\tilde{P}_{n+1}(\underline{\xi}_1,\dots,\underline{\xi}_n,0)$ in $\underline{\xi}_{n+1}$ and the higher order term $Q_{n+1} = Q_{n+1}(\underline{\xi}_1,\dots,\underline{\xi}_{n+1})$ in $\underline{\xi}_{n+1}$. Then the constant term $\tilde{P}_{n+1}(\underline{\xi}_1,\dots,\underline{\xi}_n,0)$ is

$$\tilde{P}_{n+1}(\underline{\xi}_1,\dots,\underline{\xi}_n,0) = \underline{\tilde{\Delta}}_n\, P_n .$$

Here $\underline{\tilde{\Delta}}_n = \prod_{i=1}^{n} \underline{\xi}_i^{\,2k+1}$ and P_n is the polynomial in $(\underline{\xi}_1,\dots,\underline{\xi}_n)$ given by

$$P_n = \frac{1}{2}\left\{ \left|\underline{\xi}_i^{\,\ell_j}\right|^2 / D_n + \left|\underline{\xi}_i^{\,2\ell_j}\right| / D'_n \right\},$$

where $\ell_1 = k+n-1$, $\ell_j = n-j$, $j=2,\dots,n$. Then we have

$$\tilde{P}_{n+1} = \underline{\tilde{\Delta}}_n\, P_n + Q_{n+1} .$$

(ii) In case of $n \geq 3$, we furthermore decompose P_n into the sum of the constant term $P_n(\underline{\xi}_1,\dots,\underline{\xi}_{n-1},0)$ in $\underline{\xi}_n$ and the higher order term $Q_n(\underline{\xi}_1,\dots,\underline{\xi}_n)$ in $\underline{\xi}_n$. The former $P_n(\underline{\xi}_1,\dots,\underline{\xi}_{n-1},0)$ is calculated as

$$P_n(\underline{\xi}_1,\dots,\underline{\xi}_{n-1},0) = \underline{\Delta}_{n-1}\, P_{n-1} .$$

Here $\underline{\Delta}_{n-1} = \prod_{i=1}^{n-1} \underline{\xi}_i$ and P_{n-1} is the polynomial in $(\underline{\xi}_1,\dots,\underline{\xi}_{n-1})$ given by

$$P_{n-1} = \frac{1}{2}\left\{ \left|\underset{=}{\xi}_i^{\,\ell_j}\right|^2 / D_{n-1} + \left|\underset{=}{\xi}_i^{\,2\ell_j}\right| / D'_{n-1}\right\},$$

where $\left|\underset{=}{\xi}_i^{\,\ell_j}\right|$ is the determinant of $(n-1)\times(n-1)$ matrix whose entries are $\underset{=}{\xi}_i^{\,\ell_j}$, $1 \leq i \leq n-1$, $\ell_1 = k+n-2$, $\ell_j = n-1-j$, $j=2,\ldots,n-1$. Then we have

$$P_n = \underset{=}{\Delta}_{n-1} P_{n-1} + Q_n .$$

(iii) Go on inductively the above process. Lastly, we have

$$P_3 = \frac{1}{2}\left\{ \begin{vmatrix} \underset{=}{\xi}_1^{k+2} & \underset{=}{\xi}_1 & 1 \\ \underset{=}{\xi}_2^{k+2} & \underset{=}{\xi}_2 & 1 \\ \underset{=}{\xi}_3^{k+2} & \underset{=}{\xi}_3 & 1 \end{vmatrix}^2 / D_3 + \begin{vmatrix} \underset{=}{\xi}_1^{2(k+2)} & \underset{=}{\xi}_1^2 & 1 \\ \underset{=}{\xi}_2^{2(k+2)} & \underset{=}{\xi}_2^2 & 1 \\ \underset{=}{\xi}_3^{2(k+2)} & \underset{=}{\xi}_3^2 & 1 \end{vmatrix} / D'_3 \right\},$$

$$P_2 = \frac{1}{2}\left\{ \begin{vmatrix} \underset{=}{\xi}_1^{k+1} & 1 \\ \underset{=}{\xi}_2^{k+1} & 1 \end{vmatrix}^2 / (\underset{=}{\xi}_1 - \underset{=}{\xi}_2) + \begin{vmatrix} \underset{=}{\xi}_1^{2(k+1)} & 1 \\ \underset{=}{\xi}_2^{2(k+1)} & 1 \end{vmatrix} / (\underset{=}{\xi}_1 + \underset{=}{\xi}_2) \right\},$$

$\underset{=}{\Delta}_2 = \underset{=}{\xi}_1 \underset{=}{\xi}_2$, and

$$P_3(\underset{=}{\xi}_1, \underset{=}{\xi}_2, \underset{=}{\xi}_3) = \underset{=}{\Delta}_2 P_2 + Q_3(\underset{=}{\xi}_1, \underset{=}{\xi}_2, \underset{=}{\xi}_3) ,$$

where Q_3 is the sum of the terms of P_3 higher than the constant in $\underset{=}{\xi}_3$. Then we have , in case of $n \geq 3$,

$$(5.7) \quad \tilde{P}_{n+1} = \tilde{\underset{=}{\Delta}}_n \underset{=}{\Delta}_{n-1} \cdots \underset{=}{\Delta}_2 P_2 + \sum_{i=3}^{n} \tilde{\underset{=}{\Delta}}_n \underset{=}{\Delta}_{n-1} \cdots \underset{=}{\Delta}_i Q_i + Q_{n+1} ,$$

where

$$(5.8) \quad \tilde{\underset{=}{\Delta}}_n \underset{=}{\Delta}_{n-1} \cdots \underset{=}{\Delta}_2 = \underset{=}{\xi}_1^{2k+n-1} \prod_{j=2}^{n} \underset{=}{\xi}_j^{2k+n+1-j} ,$$

$$(5.9) \quad \tilde{\underset{=}{\Delta}}_n \underset{=}{\Delta}_{n-1} \cdots \underset{=}{\Delta}_i = \prod_{j=1}^{i} \underset{=}{\xi}_j^{2k+n+1-i} \prod_{j=i+1}^{n} \underset{=}{\xi}_j^{2k+n+1-j} ,$$

where $i = 3,\ldots, n$.

In case of $n = 2$, we have

$$(5.7') \quad \tilde{P}_3 = \tilde{\Delta}_2 \, P_2 + Q_3 \, ,$$

where

$$(5.8') \quad \tilde{\Delta}_2 = \prod_{i=1}^{2} \varepsilon_i^{\,2k+1} \, .$$

Note that the first term $\tilde{\Delta}_n \Delta_{n-1} \cdots \Delta_2 P_2$ of (5.7) is a homogeneous polynomial in $(\varepsilon_1, \ldots, \varepsilon_n)$ whose degree is $2k+n+1-i$ in the variable ε_i, $i=3,\ldots,n$, and the sum of the degrees in ε_1 and ε_2 is $6k+2n-1$. The terms $\tilde{\Delta}_n \Delta_{n-1} \cdots \Delta_i Q_i$ are homogeneous polynomials in $(\varepsilon_1, \ldots, \varepsilon_n)$ whose degrees in ε_i are greater than $2k+n+1-i$, and the degree of the last term Q_{n+1} in ε_{n+1} is greater than or equal to 1. Therefore all the monomials of $\tilde{\Delta}_n \Delta_{n-1} \cdots \Delta_2 P_2$ are different from the ones of $\sum_{i=3}^{n} \tilde{\Delta}_n \Delta_{n-1} \cdots \Delta_i P_i + Q_{n+1}$.

(iv) Now we calculate the polynomial P_2 in $(\varepsilon_1, \varepsilon_2)$: for $k \geqq 4$,

$$P_2 = \frac{1}{2}\left\{(\varepsilon_1^{\,k+1} - \varepsilon_2^{\,k+1})^2/(\varepsilon_1 - \varepsilon_2) + (\varepsilon_1^{\,2k+2} - \varepsilon_2^{\,2k+2})/(\varepsilon_1 + \varepsilon_2)\right\}$$

$$= \frac{1}{2}\left\{(\varepsilon_1^{\,k+1} - \varepsilon_2^{\,k+1}) \sum_{s=0}^{k} \varepsilon_1^{\,s} \varepsilon_2^{\,k-s} - \sum_{s=0}^{2k+1} (-1)^s \varepsilon_1^{\,s} \varepsilon_2^{\,2k+1-s}\right.$$

$$= \varepsilon_1^{\,2k+1} \varepsilon_2^{\,0} + \varepsilon_1^{\,2k-1} \varepsilon_2^{\,2} + \varepsilon_1^{\,2k-3} \varepsilon_2^{\,4} + (\text{ the lower order terms in } \varepsilon_1)$$

Thus we have, in case of $n \geqq 3$, $K \geqq 4$,

$$\tilde{\Delta}_n \Delta_{n-1} \cdots \Delta_2 P_2 = \varepsilon_1^{\,4k+n} \varepsilon_2^{\,2k+n-1} \prod_{j=3}^{n} \varepsilon_j^{\,2k+n+1-j}$$

$$+ \varepsilon_1^{\,4k+n-2} \varepsilon_2^{\,2k+n+1} \prod_{j=3}^{n} \varepsilon_j^{\,2k+n+1-j}$$

$$+ \varepsilon_1^{\,4k+n-4} \varepsilon_2^{\,2k+n+3} \prod_{j=3}^{n} \varepsilon_j^{\,2k+n+1-j}$$

$$+ (\text{ the lower order terms in } \varepsilon_1).$$

Therefore the polynomial $\tilde{P}_{n+1}$ includes the terms $\varepsilon_1^{\,q_1} \cdots \varepsilon_{n+1}^{\,q_{n+1}}$,

§ 6. Quaternion projective spaces $P^{n-1}(H) = Sp(n)/Sp(1)\times Sp(n-1)$.

6.1. In this section, we use the following terminologies :

$$G = Sp(n) = \left\{ x \in U(2n) \ ; \ {}^t x \, J_n \, x = J_n \right\}, \ n \geqq 3 ,$$

where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and I_n is the identity matrix of degree n .

$$K = Sp(1)\times Sp(n-1) = \left\{ \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & A & 0 & B \\ \hline c & 0 & d & 0 \\ 0 & C & 0 & D \end{array}\right) \ ; \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1), \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n-1) \right\},$$

$$\underline{g} = \underline{sp}(n) = \left\{ X \in \underline{u}(2n) \ ; \ {}^t X J_n + J_n X = 0 \right\}$$

$$= \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \ ; \ A, B \in M_n(\mathbb{C}), {}^t\overline{A} + A = 0, \ B = {}^t B \right\},$$

$$\underline{k} = \underline{sp}(1)\times \underline{sp}(n-1) = \left\{ \left(\begin{array}{cc|cc} x & 0 & y & 0 \\ 0 & X & 0 & Y \\ \hline -\overline{y} & 0 & \overline{x} & 0 \\ 0 & -\overline{Y} & 0 & \overline{X} \end{array}\right) ; \ \begin{array}{l} x \in \sqrt{-1}\mathbb{R}, \ y \in \mathbb{C}, X, Y \in M_{n-1}(\mathbb{C}), \\ {}^t\overline{X} + X = 0, \ {}^t Y = Y \end{array} \right\} ,$$

$B(X,Y) = (2n+2)\,\mathrm{Trace}(\ XY\ ),\ X, Y \in \underline{g}$, the Killing form of $\underline{g}$,

$$\underline{p} = \left\{ \left(\begin{array}{cc|cc} 0 & Z & 0 & W \\ -{}^t\overline{Z} & 0 & {}^t W & 0 \\ \hline 0 & -\overline{W} & 0 & \overline{Z} \\ -{}^t\overline{W} & 0 & -{}^t Z & 0 \end{array}\right) \ ; \ Z, W \in M(1, n-1, \mathbb{C}) \right\},$$ the orthocomplement of $\underline{k}$ in $\underline{g}$ relative to B ,

$$T = \left\{ \left(\begin{array}{c|c} \begin{matrix} \xi_1 & & \\ & \ddots & \\ & & \xi_n \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \xi_1^{-1} & & \\ & \ddots & \\ & & \xi_n^{-1} \end{matrix} \end{array}\right) \ ; \ \xi_i \in \mathbb{C}, \ |\xi_i| = 1 \ \ (1 \leqq i \leqq n) \right\} ,$$

$\underline{t} = \left\{ H(x_1, \ldots, x_n) \ ; \ x_i \in \mathbb{R} \ (1 \leqq i \leqq n) \right\}$, the Cartan subalgebra of $\underline{g}$ and $\underline{k}$,

where $H(x_1, \ldots, x_n) = 2\pi\sqrt{-1} \left(\begin{array}{c|c} \begin{matrix} x_1 & & \\ & \ddots & \\ & & x_n \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -x_1 & & \\ & \ddots & \\ & & -x_n \end{matrix} \end{array}\right)$. Then we can identify

$P^{n-1}(H)$ with G/K having the G-invariant Riemannian metric induced from the inner product $(X, Y) = -B(X, Y),\ X, Y \in \underline{p}$.

Define an element $\underline{\lambda}_i$ in the dual space $\underline{t}^*$ of $\underline{t}$ over $\mathbb{R}$ by $\underline{t} \ni H(x_1,\dots,x_n) \longmapsto x_i$ $(1 \leqq i \leqq n)$ and introduce a lexicographic order $>$ on $\underline{t}^*$ by

$$\underline{\lambda}_1 > \dots > \underline{\lambda}_n > o .$$

Let $\underline{\Sigma}^+(G)$ (resp. $\underline{\Sigma}^+(K)$) be the set of positive roots of the complexification $\underline{g}^{\mathbb{C}}$ (resp. $\underline{k}^{\mathbb{C}}$) of $\underline{g}$ (resp. $\underline{k}$) relative to $\underline{t}$. Then we have

$$\underline{\Sigma}^+(G) = \{\underline{\lambda}_i \pm \underline{\lambda}_j \ ; \ 1 \leqq i < j \leqq n\} \cup \{2\underline{\lambda}_i \ ; \ 1 \leqq i \leqq n\},$$

$$\underline{\Sigma}^+(K) = \{\underline{\lambda}_i \pm \underline{\lambda}_j \ ; \ 2 \leqq i < j \leqq n\} \cup \{2\underline{\lambda}_i \ ; \ 1 \leqq i \leqq n\}.$$

Put

$$D(G) = \left\{\underline{\Lambda} = \sum_{i=1}^n a_i \underline{\lambda}_i \ ; \ a_i \in \mathbb{Z} \ (1 \leqq i \leqq n), \ a_1 \geqq a_2 \geqq \dots \geqq a_n \geqq o\right\},$$

$$D(K) = \left\{\underline{\Lambda} = \sum_{i=1}^n b_i \underline{\lambda}_i \ ; \ b_i \in \mathbb{Z} (1 \leqq i \leqq n), b_1 \geqq o \text{ and } b_2 \geqq \dots \geqq b_n \geqq o\right\}.$$

Then $D(G)$ (resp. $D(K)$) is the set of all dominant integral forms of G (resp. K) with respect to $\underline{t}$. Moreover there exists a bijection between $D(G)$ (resp. $D(K)$) and a complete set $\mathcal{D}(G)$ (resp. $\mathcal{D}(K)$) of non-equivalent irreducible modules of G (resp. K) over $\mathbb{C}$ corresponding $\underline{\Lambda} \in D(G)$ (resp. $D(K)$) to an element $V = V_{\underline{\Lambda}} \in \mathcal{D}(G)$ (resp. $\mathcal{D}(K)$) with the highest weight $\underline{\Lambda}$.

Then we have :

Theorem 6.1. (Lepowsky [4])

Let $\underline{\lambda} = \sum_{i=1}^n a_i \underline{\lambda}_i \in D(G)$, $\underline{\mu} = \sum_{i=1}^n b_i \underline{\lambda}_i \in D(K)$. Then the multiplicity $m(\underline{\lambda},\underline{\mu})$ of the K-module $V_{\underline{\mu}}$ in the G-module $V_{\underline{\lambda}}$ is given as follows :

Define

$$A_1 = a_1 - \max(a_2,b_2),$$

$$A_i = \min(a_i,b_i) - \max(a_{i+1},b_{i+1}) , \quad 2 \leq i \leq n-1 ,$$

$$A_n = \min(a_n,b_n) \geq 0 .$$

Then $m(\underline{\lambda},\underline{\mu}) = 0$ unless $b_1 + \sum_{i=1}^{n} A_i \in 2\mathbb{Z}$ and $A_1, A_2, \ldots, A_{n-1} \geq 0$.

Under these conditions,

$$m(\underline{\lambda},\underline{\mu}) = \sum_{L} (-1)^{|L|} \binom{n-2-|L| + \frac{1}{2}(-b_1+\sum_{i=1}^{n} A_i) - \sum_{i\in L} A_i}{n-2},$$

where L runs over all the subsets of $\{1,2,\ldots,n\}$ (also the empty set), $|L|$ denotes the number of elements in L, and $\binom{x}{y}$ denotes the binomial coefficient, which is defined to be zero if $x<y$.

It turns out by Theorem 6.1 that $V^{k\mathbb{C}} = V_{k\underline{\lambda}_1+k\underline{\lambda}_2}$, $k \geq 0$, are the class one modules of the pair (G,K) over $\mathbb{C}$.

The complexification $\underline{p}^{\mathbb{C}}$ of $\underline{p}$ is the irreducible module of K with highest weight $\underline{\lambda}_1 + \underline{\lambda}_2$. Then the symmetric square $S^2(\underline{p}^{\mathbb{C}})$ of $\underline{p}^{\mathbb{C}}$, which is $S^2(V_1{}^{\mathbb{C}})$ in §3, is decomposed as a K-module into as follows :

$$(6.1) \qquad S^2(\underline{p}^{\mathbb{C}}) = V_{2\underline{\lambda}_1+2\underline{\lambda}_2} \oplus V_{\underline{\lambda}_2+\underline{\lambda}_3} \oplus V_0 .$$

Then by Theorem 6.1, we have :

Lemma 6.2.

(I) Let $n = 3$. Then every G-module over $\mathbb{C}$ which includes certain of the K-irreducible components (6.1) of $S^2(\underline{p}^{\mathbb{C}})$ has the highest weight $\sum_{i=1}^{3} a_i\underline{\lambda}_i$, where the triple (a_1,a_2,a_3) is one of them in the following table :

a_1	k+4	k+2	k+3	k+2	k+1	k	k+4	k+2	k	3
a_2	k	k	k	k	k	k	k	k	k	2
a_3	2	2	1	1	1	1	o	o	o	o
	$(k \geqq 2)$		$(k \geqq 1)$				$(k \geqq o)$			

(II) In case of $n \geqq 4$, if $a_1 \geqq a_2 \geqq \ldots \geqq a_n \geqq o$ satisfy one of the following conditions :

(i) $a_3 \geqq 3$, (ii) $a_4 \geqq 2$, or (iii) $a_i \geqq 1$, for some $5 \leqq i \leqq n$,

then the G-module $V_{\underline{\Lambda}}$ with the highest weight $\underline{\Lambda} = \sum_{i=1}^{n} a_i \underline{\lambda}_i$ includes no the K-irreducible components of $S^2(\underline{p}^{\mathbb{C}})$.

Proof. We give only a proof of (II). Case (I) can be proved by the same manner as case (II).

By (6.1), we have only to consider the K-modules $V_{\underline{\Lambda}}$ with highest weight $\underline{\Lambda} = \sum_{i=1}^{n} b_i \underline{\lambda}_i$ as follows :

(1) $(b_1, b_2, \ldots, b_n) = (2,2,o,\ldots,o)$,

(2) $(b_1, b_2, \ldots, b_n) = (o,1,1,o,\ldots,o)$,

(3) $(b_1, b_2, \ldots, b_n) = (o,o,\ldots,o)$.

In each case, the numbers A_i , $1 \leqq i \leqq n-1$, as in Theorem 6.1 are given as follows : For (1), $A_1 = a_1 - \max(a_2, 2)$, $A_2 = \min(a_2, 2) - a_3$, $A_i = -a_{i+1}$, $3 \leqq i \leqq n-1$. For (2), $A_1 = a_1 - \max(a_2, 1)$, $A_2 = \min(a_2, 1) - \max(a_3, 1)$, $A_3 = \min(a_3, 1) - a_4$, $A_i = -a_{i+1}$, $4 \leqq i \leqq n-1$.
For (3), $A_1 = a_1 - a_2$, $A_i = -a_{i+1}$, $2 \leqq i \leqq n-1$.

If either the conditions (i), (ii) or (iii) hold, then for every case (1)$\sim$(3), one of the A_i's , $1 \leqq i \leqq n-1$, is negative. Thus Theorem 6.1 implies (II). Q.E.D.

By the character formula [11], the character $\underline{\chi}_{\underline{\Lambda}}$ of the irreducible module $V_{\underline{\Lambda}}$ with highest weight $\underline{\Lambda} = \sum_{i=1}^{n} a_i \underline{\lambda}_i$ is given by

$$(6.2) \quad D_n(\underline{\xi})\, \underline{\chi}_{\underline{\Lambda}}(\underline{\xi}) = \left| \underline{\xi}_i^{\ell_j} - \underline{\xi}_i^{-\ell_j} \right|, \text{ for each } \underline{\xi} = \left(\begin{array}{c|c} \begin{matrix} \xi_1 & & \\ & \ddots & \\ & & \xi_n \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \xi_1^{-1} & & \\ & \ddots & \\ & & \xi_n^{-1} \end{matrix} \end{array} \right),$$

where $\left| \underline{\xi}_i^{\ell_j} - \underline{\xi}_i^{-\ell_j} \right|$ is the determinant of $n \times n$-matrix whose (i,j) entries are $\underline{\xi}_i^{\ell_j} - \underline{\xi}_i^{-\ell_j}$,

$$(6.3) \quad \ell_j = a_j + n + 1 - j, \quad 1 \leqq j \leqq n, \quad \text{and}$$

$$(6.4) \quad D_n(\underline{\xi}) = \left| \underline{\xi}_i^{n+1-j} - \underline{\xi}_i^{-(n+1-j)} \right|$$

$$= \prod_{i=1}^{n} (\underline{\xi}_i - \underline{\xi}_i^{-1}) \cdot \prod_{1 \leqq i < j \leqq n} (\underline{\xi}_i - \underline{\xi}_j - \underline{\xi}_j^{-1} + \underline{\xi}_i^{-1}) .$$

6.2. In the following, we assume $n = 3$.

By Lemma 4.3, the character $\underline{\chi}^k_{(2)}$ of the symmetric square $S^2(V^{k\mathbb{C}})$ of the class one module $V^{k\mathbb{C}} = V_{k\underline{\lambda}_1 + k\underline{\lambda}_2}$ of the pair (G,K) is given by

$$(6.5) \quad D_3(\underline{\xi})\, \underline{\chi}^k_{(2)}(\underline{\xi}) = \frac{1}{2}\left\{ \frac{P_3(\underline{\xi})^2}{D_3(\underline{\xi})} + \frac{D_3(\underline{\xi})\, P_3(\underline{\xi}^2)}{D_3(\underline{\xi}^2)} \right\},$$

for $\underline{\xi} = \left(\begin{array}{c|c} \begin{matrix} \xi_1 & & \\ & \xi_2 & \\ & & \xi_3 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \xi_1^{-1} & & \\ & \xi_2^{-1} & \\ & & \xi_3^{-1} \end{matrix} \end{array} \right)$, where

$$(6.6) \quad P_3(\underline{\xi}) = \begin{vmatrix} \underline{\xi}_1^{k+3} - \underline{\xi}_1^{-(k+3)} & \underline{\xi}_1^{k+2} - \underline{\xi}_1^{-(k+2)} & \underline{\xi}_1 - \underline{\xi}_1^{-1} \\ \underline{\xi}_2^{k+3} - \underline{\xi}_2^{-(k+3)} & \underline{\xi}_2^{k+2} - \underline{\xi}_2^{-(k+2)} & \underline{\xi}_2 - \underline{\xi}_2^{-1} \\ \underline{\xi}_3^{k+3} - \underline{\xi}_3^{-(k+3)} & \underline{\xi}_3^{k+2} - \underline{\xi}_3^{-(k+2)} & \underline{\xi}_3 - \underline{\xi}_3^{-1} \end{vmatrix} .$$

Assume that

$$S^2(V^{k\mathbb{C}}) = \sum_{a_1 \geqq a_2 \geqq a_3 \geqq 0} N(a_1, a_2, a_3)\, V_{a_1\underline{\lambda}_1 + a_2\underline{\lambda}_2 + a_3\underline{\lambda}_3} .$$

Then we have the identity :

$$(6.7)\qquad D_3(\underset{\sim}{\varepsilon})\,\underset{\sim}{\chi}^k{}_{(2)}(\underset{\sim}{\varepsilon}) = \sum_{a_1 \geqq a_2 \geqq a_3 \geqq 0} N(a_1,a_2,a_3)\left|\varepsilon_i^{\ell_j} - \varepsilon_i^{-\ell_j}\right| ,$$

where $\ell_j = a_j + 4 - j$, $j=1,2,3$. And then the right hand side of (6.7) can be decomposed of the form :

$$-\sum_{a_1 \geqq a_2 \geqq a_3 \geqq 0} N(a_1,a_2,a_3)\,\varepsilon_1^{-\ell_3}\,\varepsilon_2^{-\ell_2}\,\varepsilon_3^{-\ell_1} + Q(\varepsilon_1,\varepsilon_2,\varepsilon_3) ,$$

where $Q(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ is the sum of the monomials $\varepsilon_1^{q_1}\varepsilon_2^{q_2}\varepsilon_3^{q_3}$, satisfying one of the following conditions :

$$(6.8)\qquad \text{(i) } 0 \leqq q_1 \text{ , (ii) } q_1 \leqq q_2 \text{ , or (iii) } q_2 \leqq q_3 .$$

So let us decompose $D_3\,\underset{\sim}{\chi}^k{}_{(2)}$ into the following :

$$(6.9)\qquad D_3\,\underset{\sim}{\chi}^k{}_{(2)} = -\sum_{0>q_1>q_2>q_3} A(q_1,q_2,q_3)\,\varepsilon_1^{q_1}\varepsilon_2^{q_2}\varepsilon_3^{q_3} +$$

$$+ R(\varepsilon_1,\varepsilon_2,\varepsilon_3) ,$$

where $R(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ is the sum of the monomials $\varepsilon_1^{q_1}\varepsilon_2^{q_2}\varepsilon_3^{q_3}$, satisfying one of the conditions (6.8) . Then we have

$$A(q_1,q_2,q_3) = N(a_1,a_2,a_3) \text{ , } q_1=-(a_3+1),\ q_2=-(a_2+1),\ q_3=-(a_1+3).$$

Therefore we have only to seek the monomials $A(q_1,q_2,q_3)\,\varepsilon_1^{q_1}\varepsilon_2^{q_2}\varepsilon_3^{q_3}$ with $A(q_1,q_2,q_3) \neq 0$, $0 > q_1 > q_2 > q_3$ of $D_3(\varepsilon)\,\underset{\sim}{\chi}^k{}_{(2)}(\underset{\sim}{\varepsilon})$. Then the module $S^2(V^{k\mathbb{C}})$ includes the one $V_{-(q_3+3)\underset{\sim}{\lambda}_1-(q_2+2)\underset{\sim}{\lambda}_2-(q_1+1)\underset{\sim}{\lambda}_3}$ with multiplicity $A(q_1,q_2,q_3)$.

6.3. The task of 6.2 is accomplished as follows :

First, we put

$$P_3(\xi_1,\xi_2,\xi_3) = \xi_3^{-(k+3)}\,\tilde{P}_3(\xi_1,\xi_2,\xi_3)\ , \quad \text{and}$$

$$D_3(\xi_1,\xi_2,\xi_3) = \xi_3^{-3}\,\tilde{D}_3(\xi_1,\xi_2,\xi_3),$$

where $\tilde{P}_3$ and $\tilde{D}_3$ are the polynomials given by

$$\tilde{P}_3(\xi_1,\xi_2,\xi_3) = \begin{vmatrix} \xi_1^{k+3} - \xi_1^{-(k+3)} & \xi_1^{k+2} - \xi_1^{-(k+2)} & \xi_1 - \xi_1^{-1} \\ \xi_2^{k+3} - \xi_2^{-(k+3)} & \xi_2^{k+2} - \xi_2^{-(k+2)} & \xi_2 - \xi_2^{-1} \\ \xi_3^{2k+6} - 1 & \xi_3^{2k+5} - \xi_3 & \xi_3^{k+4} - \xi_3^{k+2} \end{vmatrix}$$

$$\tilde{D}_3(\xi_1,\xi_2,\xi_3) = (\xi_1-\xi_1^{-1})(\xi_2-\xi_2^{-1})(\xi_3^{2}-1)\times$$

$$\times\,(\xi_1-\xi_2-\xi_2^{-1}+\xi_1^{-1})(\xi_1\xi_3-\xi_3^{2}-1+\xi_1^{-1}\xi_3)(\xi_2\xi_3-\xi_3^{2}-1+\xi_2^{-1}\xi_3)\ .$$

Then

$$D_3\,\chi^{k}(2) = \xi_3^{-2k-3}\,\frac{1}{2}\left\{\frac{\tilde{P}_3(\xi_1,\xi_2,\xi_3)^2}{\tilde{D}_3(\xi_1,\xi_2,\xi_3)} + \frac{\tilde{D}_3(\xi_1,\xi_2,\xi_3)\tilde{P}_3(\xi_1^{2},\xi_2^{2},\xi_3^{2})}{\tilde{D}_3(\xi_1^{2},\xi_2^{2},\xi_3^{2})}\right\}$$

Here $\tilde{P}_3(\xi_1,\xi_2,\xi_3)^2$ (resp. $\tilde{D}_3(\xi_1,\xi_2,\xi_3)\tilde{P}_3(\xi_1^{2},\xi_2^{2},\xi_3^{2})$) is divided formally by $\tilde{D}_3(\xi_1,\xi_2,\xi_3)$ (resp. $\tilde{D}_3(\xi_1^{2},\xi_2^{2},\xi_3^{2})$).

Then it follows that

$$(6.10)\qquad \frac{\tilde{P}_3(\xi_1,\xi_2,\xi_3)^2}{\tilde{D}_3(\xi_1,\xi_2,\xi_3)} = \sum_{p\geq 0} a_p(\xi_1,\xi_2)\,\xi_3^{p}\ , \ \text{and}$$

$$(6.11)\qquad \frac{\tilde{D}_3(\xi_1,\xi_2,\xi_3)\ \tilde{P}_3(\xi_1^{2},\xi_2^{2},\xi_3^{2})}{\tilde{D}_3(\xi_1^{2},\xi_2^{2},\xi_3^{2})} = \sum_{p\geq 0} b_p(\xi_1,\xi_2)\,\xi_3^{p}\ ,$$

where both sums are in fact finite sums in p, and both coefficients $a_p(\xi_1,\xi_2)$, $b_p(\xi_1,\xi_2)$ are the sums of the form $A(a_1,a_2)\,\xi_1^{a_1}\xi_2^{a_2}$, a_1, a_2 , and $A(a_1,a_2)$ being integers . So decompose the constant

$\frac{1}{2}(a_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2)+b_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2))$ in $\underline{\varepsilon}_3$, into the sum of monomials $A(a_1,a_2)\ \underline{\varepsilon}_1^{a_1}\ \underline{\varepsilon}_2^{a_2}$, and seek the monomials $-A(p_1,p_2,-2k-3)\underline{\varepsilon}_1^{p_1}\underline{\varepsilon}_2^{p_2}$ with the conditions $o>p_1>p_2>-2k-3$. Then the monomial $-A(p_1,p_2,-2k-3)\underline{\varepsilon}_1^{p_1}\underline{\varepsilon}_2^{p_2}\underline{\varepsilon}_3^{-2k-3}$ does never cancel with every terms of $\frac{1}{2}\sum_{p\geqq 1}(a_p(\underline{\varepsilon}_1,\underline{\varepsilon}_2)+b_p(\underline{\varepsilon}_1,\underline{\varepsilon}_2))\ \underline{\varepsilon}_3^{-2k-3+p}$. Thus $D_3\underline{\chi}^k{}_{(2)}$ should include the monomial $-A(p_1,p_2,-2k-3)\underline{\varepsilon}_1^{p_1}\underline{\varepsilon}_2^{p_2}\underline{\varepsilon}_3^{-2k-3}$ in the decomposition (6.9). Therefore the module $S^2(V^{k\mathbb{C}})$ should include the one $V_{2k\underline{\lambda}_1-(p_2+2)\underline{\lambda}_2-(p_1+1)\underline{\lambda}_3}$ with multiplicity $A(p_1,p_2,-2k-3)$.

We have only to compute $\frac{1}{2}(a_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2)+b_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2))$. By (6.1o), and (6.11), we obtain

$$a_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2)=\frac{\widetilde{P}_3(\underline{\varepsilon}_1,\underline{\varepsilon}_2,o)^2}{\widetilde{D}_3(\underline{\varepsilon}_1,\underline{\varepsilon}_2,o)},\quad b_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2)=\frac{\widetilde{D}_3(\underline{\varepsilon}_1,\underline{\varepsilon}_2,o)\widetilde{P}_3(\underline{\varepsilon}_1^2,\underline{\varepsilon}_2^2,o)}{\widetilde{D}_3(\underline{\varepsilon}_1^2,\underline{\varepsilon}_2^2,o)},$$

where

$$\widetilde{P}_3(\underline{\varepsilon}_1,\underline{\varepsilon}_2,o)=\begin{vmatrix}\underline{\varepsilon}_1^{k+3}-\underline{\varepsilon}_1^{-(k+3)} & \underline{\varepsilon}_1^{k+2}-\underline{\varepsilon}_1^{-(k+2)} & \underline{\varepsilon}_1-\underline{\varepsilon}_1^{-1}\\ \underline{\varepsilon}_2^{k+3}-\underline{\varepsilon}_2^{-(k+3)} & \underline{\varepsilon}_2^{k+2}-\underline{\varepsilon}_2^{-(k+2)} & \underline{\varepsilon}_2-\underline{\varepsilon}_2^{-1}\\ -1 & o & o\end{vmatrix}$$

$$=(-1)\Big\{(\underline{\varepsilon}_1^{k+2}-\underline{\varepsilon}_1^{-(k+2)})(\underline{\varepsilon}_2-\underline{\varepsilon}_2^{-1})-(\underline{\varepsilon}_1-\underline{\varepsilon}_1^{-1})(\underline{\varepsilon}_2^{k+2}-\underline{\varepsilon}_2^{-(k+2)})\Big\},$$

and

$$\widetilde{D}_3(\underline{\varepsilon}_1,\underline{\varepsilon}_2,o)=(-1)(\underline{\varepsilon}_1-\underline{\varepsilon}_1^{-1})(\underline{\varepsilon}_2-\underline{\varepsilon}_2^{-1})(\underline{\varepsilon}_1-\underline{\varepsilon}_2-\underline{\varepsilon}_2^{-1}+\underline{\varepsilon}_1^{-1})$$

$$=(-1)\underline{\varepsilon}_1^{-1}(\underline{\varepsilon}_1-\underline{\varepsilon}_1^{-1})(\underline{\varepsilon}_2-\underline{\varepsilon}_2^{-1})(\underline{\varepsilon}_1-\underline{\varepsilon}_2)(\underline{\varepsilon}_1-\underline{\varepsilon}_2^{-1}).$$

Dividing formally $\widetilde{P}_3(\underline{\varepsilon}_1,\underline{\varepsilon}_2,o)^2$ (resp. $\widetilde{D}_3(\underline{\varepsilon}_1,\underline{\varepsilon}_2,o)\widetilde{P}_3(\underline{\varepsilon}_1^2,\underline{\varepsilon}_2^2,o)$) by $\widetilde{D}_3(\underline{\varepsilon}_1,\underline{\varepsilon}_2,o)$ (resp $\widetilde{D}_3(\underline{\varepsilon}_1^2,\underline{\varepsilon}_2^2,o)$), we have :

Lemma 6.3.

(i) $$a_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2) = -\sum_{s=o}^{k+2}\ \sum_{t=o}^{k+1}\ \sum_{u=o}^{k}\Big\{\underline{\varepsilon}_1^{\,2k+2-s-2t-u}\ \underline{\varepsilon}_2^{\,1-s+u} - \underline{\varepsilon}_1^{\,k+1-s-u}\ \underline{\varepsilon}_2^{\,k+2-s-2t+u} - \underline{\varepsilon}_1^{\,2k+2-s-2t-u}\ \underline{\varepsilon}_2^{\,-1+s-u} + \underline{\varepsilon}_1^{\,k+1-s-u}\ \underline{\varepsilon}_2^{\,k+s-2t-u}\Big\},$$

(ii) $$b_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2) = -\sum_{s=o}^{k}\big(\underline{\varepsilon}_1^{\,2k-2s+2} - \underline{\varepsilon}_1^{\,-2k+2s-2}\big)\sum_{u=o}^{2s+1}(-1)^u\ \underline{\varepsilon}_2^{\,2s+1-2u}$$
$$-\sum_{s=o}^{k}\underline{\varepsilon}_1^{\,2k+1-2s}\Big[\sum_{p=o}^{s}(-1)^{p+1}\underline{\varepsilon}_2^{\,2s+2-2p} + \sum_{p=o}^{s}(-1)^{p+s}\underline{\varepsilon}_2^{\,-2-2s}\Big]$$
$$-\sum_{s=o}^{k}\underline{\varepsilon}_1^{\,-1-2s}\Big[\sum_{p=o}^{k-s}(-1)^p\underline{\varepsilon}_2^{\,2k+2-2s-2p} + \sum_{p=o}^{k-s}(-1)^{k+1kp+s}\underline{\varepsilon}_2^{\,-2-2}$$

Proof. We have

$$a_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2) = (-1)\ \underline{\varepsilon}_1\ A\ B\ ,$$

where

$$A = \Big\{(\underline{\varepsilon}_1^{\,k+2} - \underline{\varepsilon}_1^{\,-(k+2)})(\underline{\varepsilon}_2 - \underline{\varepsilon}_2^{\,-1}) - (\underline{\varepsilon}_1 - \underline{\varepsilon}_1^{\,-1})(\underline{\varepsilon}_2^{\,k+2} - \underline{\varepsilon}_2^{\,-(k+2)})\Big\}/C\ ,$$
$$B = \Big\{(\underline{\varepsilon}_1^{\,k+2} - \underline{\varepsilon}_1^{\,-(k+2)})(\underline{\varepsilon}_2 - \underline{\varepsilon}_2^{\,-1}) - (\underline{\varepsilon}_1 - \underline{\varepsilon}_1^{\,-1})(\underline{\varepsilon}_2^{\,k+2} - \underline{\varepsilon}_2^{\,-(k+2)})\Big\}/D\ .$$

Here $C = (\underline{\varepsilon}_1 - \underline{\varepsilon}_1^{\,-1})(\underline{\varepsilon}_2 - \underline{\varepsilon}_2^{\,-1})$ and $D = (\underline{\varepsilon}_1 - \underline{\varepsilon}_2)(\underline{\varepsilon}_1 - \underline{\varepsilon}_2^{\,-1})$. Then

$$A = \sum_{t=o}^{k+1}(\underline{\varepsilon}_1^{\,k+1-2t} - \underline{\varepsilon}_2^{\,k+1-2t}),$$

and the numerator of B is rearranged as

$$(\underline{\varepsilon}_1^{\,k+2}\underline{\varepsilon}_2 - \underline{\varepsilon}_1^{\,-1}\underline{\varepsilon}_2^{\,-(k+2)}) + (\underline{\varepsilon}_1^{\,-1}\underline{\varepsilon}_2^{\,k+2} - \underline{\varepsilon}_1^{\,-(k+2)}\underline{\varepsilon}_2) - (\underline{\varepsilon}_1^{\,k+2}\underline{\varepsilon}_2^{\,-1} - \underline{\varepsilon}_1\underline{\varepsilon}_2^{\,-(k+2)}) - (\underline{\varepsilon}_1\underline{\varepsilon}_2^{\,k+2} - \underline{\varepsilon}_1^{\,-(k+2)}\underline{\varepsilon}_2^{\,-1})\ .$$

Thus we have

$$B = \left\{ \sum_{s=o}^{k+2} (\underline{\varepsilon}_1^{k+1-s} \underline{\varepsilon}_2^{1-s} - \underline{\varepsilon}_1^{-s} \underline{\varepsilon}_2^{k+2-s}) - \sum_{s=o}^{k} (\underline{\varepsilon}_1^{k+1-s} \underline{\varepsilon}_2^{-1-s} - \underline{\varepsilon}_1^{-2-s} \underline{\varepsilon}_2^{k+2-s}) \right\} / (\underline{\varepsilon}_1 - \underline{\varepsilon}_2)$$

$$= \sum_{s=o}^{k+2} \sum_{u=o}^{k} \underline{\varepsilon}_1^{k-s-u} (\underline{\varepsilon}_2^{1-s+u} - \underline{\varepsilon}_2^{-1+s-u}) .$$

Hence we have (i). For (ii), it follows that

$$b_o(\underline{\varepsilon}_1, \underline{\varepsilon}_2) = (-1) \underline{\varepsilon}_1 \left\{ (\underline{\varepsilon}_1^{2k+4} - \underline{\varepsilon}_1^{-2k-4})(\underline{\varepsilon}_2^{2} - \underline{\varepsilon}_2^{-2}) - (\underline{\varepsilon}_1^{2} - \underline{\varepsilon}_1^{-2})(\underline{\varepsilon}_2^{2k+4} - \underline{\varepsilon}_2^{-2k-4}) \right\} / (\underline{\varepsilon}_1 + \underline{\varepsilon}_1^{-1})(\underline{\varepsilon}_2 + \underline{\varepsilon}_2^{-1})(\underline{\varepsilon}_1 + \underline{\varepsilon}_2)(\underline{\varepsilon}_1 + \underline{\varepsilon}_2^{-1})$$

$$= (-1)\ \underline{\varepsilon}_1\ E \ / (\underline{\varepsilon}_1 + \underline{\varepsilon}_1^{-1})(\underline{\varepsilon}_2 + \underline{\varepsilon}_2^{-1})(\underline{\varepsilon}_1 + \underline{\varepsilon}_2) ,$$

where

$$E = \left\{ (\underline{\varepsilon}_1^{2k+4} - \underline{\varepsilon}_1^{-2k-4})(\underline{\varepsilon}_2^{2} - \underline{\varepsilon}_2^{-2}) - (\underline{\varepsilon}_1^{2} - \underline{\varepsilon}_1^{-2})(\underline{\varepsilon}_2^{2k+4} - \underline{\varepsilon}_2^{-2k-4}) \right\} / (\underline{\varepsilon}_1 + \underline{\varepsilon}_2^{-1})$$

$$= \left\{ (\underline{\varepsilon}_1^{2k+4} \underline{\varepsilon}_2^{2} - \underline{\varepsilon}_1^{-2} \underline{\varepsilon}_2^{-2k-4}) + (\underline{\varepsilon}_1^{-2} \underline{\varepsilon}_2^{2k+4} - \underline{\varepsilon}_1^{-2k-4} \underline{\varepsilon}_2^{2}) - (\underline{\varepsilon}_1^{2k+4} \underline{\varepsilon}_2^{-2} - \underline{\varepsilon}_1^{2} \underline{\varepsilon}_2^{-2k-4}) - (\underline{\varepsilon}_1^{2} \underline{\varepsilon}_2^{2k+4} - \underline{\varepsilon}_1^{-2k-4} \underline{\varepsilon}_2^{-2}) \right\} / (\underline{\varepsilon}_1 + \underline{\varepsilon}_2^{-1}) .$$

Then we have

$$\underline{\varepsilon}_1 E = \sum_{t=o}^{2k+5} (-1)^t (\underline{\varepsilon}_1^{2k+4-t} \underline{\varepsilon}_2^{2-t} - \underline{\varepsilon}_1^{2-t} \underline{\varepsilon}_2^{2k+4-t}) - \sum_{t=o}^{2k+1} (-1)^t (\underline{\varepsilon}_1^{2k+4-t} \underline{\varepsilon}_2^{-2-t} - \underline{\varepsilon}_1^{-2-t} \underline{\varepsilon}_2^{2k+4-t}).$$

Thus we obtain

$$F = \underline{\varepsilon}_1\ E/(\underline{\varepsilon}_1 + \underline{\varepsilon}_2) = \sum_{t=o}^{2k+5} \sum_{u=o}^{2k+1} (-1)^{t+u} (\underline{\varepsilon}_1^{2k+3-t-u} \underline{\varepsilon}_2^{2-t+u} - \varepsilon_1^{2k+3-t-u} \varepsilon_2^{-2+t-u}) .$$

We rearrange F as follows :

$$F = \sum_{s=-(2k+3)}^{2k+3} \sum_{t=a_s}^{b_s} (-1)^s \left\{ \varepsilon_1^{2k+5-s-2t} \varepsilon_2^{s} - \varepsilon_1^{2k+5-s-2t} \varepsilon_2^{-s} \right\} ,$$

where $a_o = 2$, $b_o = 2k+3$, $a_1 = 1$, $b_1 = 2k+2$, $a_{-1} = 3$, $b_{-1} = 2k+4$, $a_s = o$, $b_s = 2k+3-s$ ($s \geq 2$) and $a_{-s} = 2+s$, $b_{-s} = 2k+5$ ($s \geq 2$).

Then we have

$$F = -(\varepsilon_1^{2k+2} - \varepsilon_1^{-2k-2})(\varepsilon_2 - \varepsilon_2^{-1})$$
$$+ \sum_{s=o}^{2k+1} (-1)^s \sum_{t=o}^{2k+1-s} (\varepsilon_1^{2k+3-s-2t} - \varepsilon_1^{2k-1-s-2t})(\varepsilon_2^{s+2} - \varepsilon_2^{-s-2}) .$$

Thus

$$G = F/(\varepsilon_1 + \varepsilon_1^{-1}) = - (\sum_{u=o}^{2k+1} (-1)^u \varepsilon_1^{2k+1-2u})(\varepsilon_2 - \varepsilon_2^{-1})$$
$$+ \sum_{s=o}^{2k+1} (-1)^s \sum_{t=o}^{2k+1-s} (\varepsilon_1^{2k+2-s-2t} - \varepsilon_1^{2k-s+2t})(\varepsilon_2^{s+2} - \varepsilon_2^{-s-2})$$

Here we rearrange G as follows :

$G = H + I$,

H = the sum of terms of even order in ε_2, and

I = the sum of terms of odd order in ε_2 .

Then

$$H = \sum_{s=o}^{k} (\varepsilon_1^{2k+2-2s} - \varepsilon_1^{-2k-2+2s})(\varepsilon_2^{2s+2} - \varepsilon_2^{-2s-2}) , \text{ and}$$

$$I = - \sum_{s=o}^{k} \varepsilon_1^{2k+1-2s} \left\{ \varepsilon_2^{2s+3} + (-1)^s \varepsilon_2 - (-1)^s \varepsilon_2^{-1} - \varepsilon_2^{-2s-3} \right\}$$
$$+ \sum_{s=o}^{k} \varepsilon_1^{-1-2s} \left\{ \varepsilon_2^{2(k-s)+3} + (-1)^{k-s} \varepsilon_2 - (-1)^{k-s} \varepsilon_2^{-1} - \varepsilon_2^{-2(k-s)-3} \right\} .$$

Thus

$$H/(\underline{\varepsilon}_2+\underline{\varepsilon}_2^{-1}) = \sum_{s=0}^{k}(\underline{\varepsilon}_1^{2k+2-2s}-\underline{\varepsilon}_1^{-2k-2+2s})\sum_{u=0}^{2s+1}(-1)^u\underline{\varepsilon}_2^{2s+1-2u}\ , \text{ and}$$

$$I/(\underline{\varepsilon}_2+\underline{\varepsilon}_2^{-1}) = -\sum_{s=0}^{k}\underline{\varepsilon}_1^{2k+1-2s}\left[\sum_{p=0}^{s}(-1)^p\underline{\varepsilon}_2^{2s+2-2p}+(-1)^{s+1}\sum_{p=0}^{s}(-1)^p\underline{\varepsilon}_2^{-2-2p}\right]$$
$$+\sum_{s=0}^{k}\underline{\varepsilon}_1^{-1-2s}\left[\sum_{p=0}^{k-s}(-1)^p\underline{\varepsilon}_2^{2(k-s)+2-2p}+(-1)^{k-s+1}\sum_{p=0}^{k-s}(-1)^p\underline{\varepsilon}_2^{-2-2p}\right].$$

Therefore we obtain (ii). Q.E.D.

By Lemma 6.3, we obtain the following tables:

(i) the monomials of $-a_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2) = -\sum A(a_1,a_2)\underline{\varepsilon}_1^{a_1}\underline{\varepsilon}_2^{a_2}$:

	$-a_1$	$-a_2$	$A(a_1,a_2)$
1)	$-2k-2+s+2t+u$	$-1+s-u$	1
2)	$-k-1+s+u$	$-k-2+s+2t-u$	-1
3)	$-2k-2+s+2t+u$	$1-s+u$	-1
4)	$-k-1+s+u$	$-k-s+2t+u$	1

where $o\leqq s\leqq k+2$, $o\leqq t\leqq k+1$, and $o\leqq u\leqq k$.

(ii) The monomials of $-b_o(\underline{\varepsilon}_1,\underline{\varepsilon}_2) = -\sum B(b_1,b_2)\underline{\varepsilon}_1^{b_1}\underline{\varepsilon}_2^{b_2}$:

	$-b_1$	$-b_2$	$B(b_1,b_2)$	
5)	$-2k+2s-2$	$-2s-1+2u$	$(-1)^u$	$o\leqq u\leqq 2s+1$
6)	$2k-2s+2$	$-2s-1+2u$	$(-1)^{u+1}$	
7)	$-2k-1+2s$	$-2s-2+2p$	$(-1)^{p+1}$	$o\leqq p\leqq s$
8)	$-2k-1+2s$	$2+2p$	$(-1)^{p+s}$	
9)	$1+2s$	$-2k-2+2s+2p$	$(-1)^p$	$o\leqq p\leqq k-s$
1o)	$1+2s$	$2+2p$	$(-1)^{k+1+p+s}$	

where $o\leqq s\leqq k$.

Making use of the above tables, it turns out that

$\frac{1}{2}(a_o(\underline{\varepsilon}_1, \underline{\varepsilon}_2)+b_o(\underline{\varepsilon}_1, \underline{\varepsilon}_2))$ includes the following monomials :

(i) $-\ \underline{\varepsilon}_1^{-1}\ \underline{\varepsilon}_2^{-(2k+2)}$ ($k \geqq o$) ,

(ii) $-\ \underline{\varepsilon}_1^{-1}\ \underline{\varepsilon}_2^{-(2k-6)}$ ($k \geqq 4$) , and

(iii) $-\ \underline{\varepsilon}_1^{-4}\ \underline{\varepsilon}_2^{-(2k-3)}$ ($k \geqq 4$) .

Therefore $S^2(V^{k\mathbb{C}})$ includes the following G-irreducible modules with multiplicity one :

(i) $V_{2k\underline{\lambda}_1+2k\underline{\lambda}_2}$ ($k \geqq o$) ,

(ii) $V_{2k\underline{\lambda}_1+(2k-8)\underline{\lambda}_2}$ ($k \geqq 4$) , and

(iii) $V_{2k\underline{\lambda}_1+(2k-5)\underline{\lambda}_2+3\underline{\lambda}_3}$ ($k \geqq 4$) .

The module $V_{2k\underline{\lambda}_1+2k\underline{\lambda}_2}$ appears in the table in Lemma 6.2 , but both the latter ones $V_{2k\underline{\lambda}_1+(2k-8)\underline{\lambda}_2}$, $V_{2k\underline{\lambda}_1+(2k-5)\underline{\lambda}_2+3\underline{\lambda}_3}$ ($k \geqq 4$) do not so . Thus we obtain , if $k \geqq 4$,

$$\dim(W_3) \geqq \dim(V_{2k\underline{\lambda}_1+(2k-8)\underline{\lambda}_2}) + \dim(V_{2k\underline{\lambda}_1+(2k-5)\underline{\lambda}_2+3\underline{\lambda}_3})$$

$$\geqq 1{,}287 + 27{,}72o = 29{,}oo7 .$$

By Lemma 3.1, we obtain Theorem C.

Remark. In case of $P^2(H)$ and $k = 4$, it follows that $m(4)+1 = 1{,}274$. Then we have

$$29{,}007 \leqq \dim(W_2) \leqq \frac{1}{2}(m(4)+1)(m(4)+2) = 812{,}175.$$

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