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Abstract

We study the entropy production of Gibbs measures for dynamical systems with *folding of the phase space*, and show that in certain cases it can be negative. The model is that generated by a hyperbolic non-invertible map f on a basic (possibly fractal) set Λ, preserving a Gibbs (equilibrium) measure μ_ϕ associated to a Hölder potential ϕ; non-invertibility creates new phenomena and techniques with respect to the diffeomorphism case. We prove a formula for the entropy production of μ_ϕ, involving an asymptotic logarithmic degree with respect to μ_ϕ. We also find the Jacobian of μ_ϕ with respect to an arbitrary iterate f^m. For hyperbolic toral endomorphisms, we show that *all* Gibbs states μ_ϕ have *non-positive entropy production* $e_f(\mu_\phi)$. We study then the entropy production of the *inverse Sinai-Ruelle-Bowen measure* μ^- and prove that, for a large family of maps it is *strictly negative*, while at the same time the entropy production of the respective (forward) Sinai-Ruelle-Bowen measure μ^+ is strictly positive. Cohomological conditions guaranteeing that the entropy production of μ^- is zero are also given.

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1 Entropy production. Outline of main results.

In statistical mechanics, one is concerned with the *stationary (steady) states*, which are probability measures on the phase space, invariant under time evolution. The study of such states can be done with the help of dynamical systems and ergodic theory (for instance [2], [3], [4], [18], [20], [22], etc.) The fundamental postulate of statistical mechanics (see [3]) says that a physical system in thermodynamical equilibrium is described by Gibbs measures.

Certain nonequilibrium steady states are described also by Gibbs states, but for a different problem (see [18]). In the nonequilibrium scenario, once a system is kept out of equilibrium and subjected to non-Hamiltonian forces, the energy is in general not conserved ([18]); so we couple it with a large external system called a thermostat, and record the entropy changes. It is thus justified to have a notion of *entropy production*, as a measure of the average differences in the entropy of the system over time. Entropy production of a measure-preserving system is an important notion both from the mathematical and the physical points of view.

From a mathematical point of view, Ruelle identifies in [20] (see also [19] and [18]) several types of entropy productions given by: i) a diffeomorphism f of a manifold M; ii) an endomorphism f on M, i.e a non-invertible smooth map f; here the folding of M by f will itself contribute to the entropy production; or iii) a diffusion model given by a map f restricted to a neighbourhood of a compact invariant set $X \subset M$.

Whether entropy production of a state is positive or not, is not clear a priori and can be a difficult question.

In this paper we are concerned with the case when f is a *smooth endomorphism* on a manifold M, having a compact invariant set $\Lambda \subset M$. Since hyperbolicity plays an important role in modelling time evolutions in statistical physics (for instance [18], [20], etc.), we shall assume that the endomorphism f is *hyperbolic* in the sense of [21], i.e that there exists a continuous splitting of the tangent bundle over the inverse limit $\hat{\Lambda}$ ($\hat{\Lambda}$ being the space of past trajectories of points in Λ), into stable and unstable directions. The map f is not assumed expanding on Λ, thus we do not have the machinery from the expanding setting here.

Hyperbolicity of f on Λ implies that we have stable directions and local stable manifolds of type $W^s_r(x), x \in \Lambda$, and unstable directions and local unstable manifolds of type $W^u_r(\hat{x})$ which depend on whole past trajectories $\hat{x} \in \hat{\Lambda}$. Thus through a given point x from Λ there may pass many (even uncountably many, as in [10]) local unstable manifolds corresponding to different prehistories of x in $\hat{\Lambda}$; these unstable manifolds may intersect also outside of Λ. At the same time, the local branches of inverse iterates do not contract small balls.

For systems given by Anosov diffeomorphisms, or for diffeomorphisms having a hyperbolic attractor, we have the existence of Sinai-Ruelle-Bowen (SRB) measures, which are natural invariant measures in the sense that they describe the distribution of trajectories of Lebesgue-almost all points in a neighbourhood of the attractor. As was shown by Sinai, the SRB measure of an Anosov diffeomorphism f, is in fact the Gibbs state of a Hölder potential Φ^u, where $\Phi^u = -\log|\det Df_u|$ (the *unstable potential*).

For a $\mathcal{C}^2$ diffeomorphism f, the *entropy production* of an arbitrary f-invariant probability measure μ is defined (see [20]) as $e_f(\mu) = -\int \log|\det(Df)(x)|d\mu(x)$. If μ is an SRB state, then Ruelle proved in [20] that $e_f(\mu) \geq 0$; moreover if the SRB state μ has no vanishing Lyapunov exponents and if $e_f(\mu) = 0$, then μ must be absolutely continuous with respect to the Lebesgue measure (see [20], [18]; and [8] for a characterization of invariant absolutely continuous measures). SRB measures exist also for diffeomorphisms having Axiom A attractors, as shown by Ruelle (see for instance [4]). Moreover SRB measures do exist also for Anosov endomorphisms and for endomorphisms with hyperbolic attractors, and they are equal to the equilibrium measures of the respective unstable potentials on the inverse limit spaces (see [16]).

For a *non-invertible* smooth map f on a Riemannian manifold M and an f-invariant probability μ on M, Ruelle defined in [20] the *entropy production* of μ by:

$$e_f(\mu) := F_f(\mu) - \int \log|\det(Df)(x)|d\mu(x), \tag{1}$$

where $F_f(\mu)$ is called the *folding entropy* of μ with respect to f. $F_f(\mu)$ is defined as the conditional entropy $H_\mu(\epsilon|f^{-1}\epsilon)$, of ϵ with respect to $f^{-1}\epsilon$, where ϵ is the single point partition.

For example we can obtain stationary measures μ, with respect to f, as weak limits (when $n \to \infty$) of averages of type

$$\frac{1}{n}\sum_{k=0}^{n-1} f^k \rho, \tag{2}$$

where ρ is an absolutely continuous probability with respect to the Lebesgue measure, having density $\bar{\rho}$. For such f-invariant limit measures μ, Ruelle showed that the entropy production is non-negative (see [20]).

There do exist in fact dynamical systems from physics presenting non-invertibility (see for example [4], [20], [18]). The study of dynamics of hyperbolic endomorphisms, and of their characteristics different from those of diffeomorphisms, appeared also in [1], [21], [10], [12], [15], etc.

From the above, it is then justified to study the entropy production of equilibrium measures of Hölder potentials for non-invertible smooth maps f on basic sets Λ on which f is hyperbolic and transitive; here by *basic set* (or locally maximal set [6]) we mean a compact f-invariant set Λ s.t $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, for a neighbourhood U of Λ. Our endomorphism f is not assumed expanding. We study equilibrium measures μ_ϕ of f, associated to arbitrary Hölder potentials ϕ on Λ. The **main results** of the paper are the following:

In **Theorem 1** we give a precise estimate for the **Jacobian** (in the sense of Parry, [14]) of the equilibrium measure μ_ϕ, with respect to an *arbitrary iterate* f^n. In this way we will be able to control the amount of measure within $\mu_\phi(A)$, coming from n iterations of f applied to local preimage branches $f_i^{-n}(A), i$. Our formula for the Jacobian of μ_ϕ is *independent of* n and will allow us to express the **folding entropy** of μ_ϕ with respect to f.

Next we will describe the folding entropy of μ_ϕ as the limit of the weighted integral, of the logarithm of the degree function of f^n with respect to μ_ϕ on Λ. In this way in **Theorem 2** we give a formula for the **entropy production** of μ_ϕ in terms of an "asymptotic logarithmic degree" (with respect to μ_ϕ) minus the integral of the Jacobian with respect to the Riemannian metric; the asymptotic logarithmic degree takes into consideration only those n-preimages (i.e preimages with respect to f^n) which behave well with respect to ϕ. In **Corollary 2** we will use the formula proved in Theorem 2 in order to calculate the folding entropy of the measure of maximal entropy.

We investigate next the case of a hyperbolic toral endomorphism on $\mathbb{T}^k$ and its Gibbs measures associated to various Hölder potentials. We prove in **Corollary 1** that in this setting, the entropy production of *any* equilibrium measure of a Hölder potential is **non-positive**.

In [11] we introduced an **inverse SRB measure** μ^- which has physical relevance since it gives the distribution of past trajectories with respect to the endomorphism f, for Lebesgue almost all points in a neighbourhood of a hyperbolic repellor (including thus the case of Anosov endomorphisms). This unique inverse SRB measure is not just the SRB measure for f^{-1}, since our map f is non-invertible in general. We proved that in fact μ^- is equal to the equilibrium measure of the stable potential (with respect to the *forward* system). Also μ^- is the only invariant probability having absolutely continuous conditional measures on local stable manifolds.

Here we will show in **Theorem 3** that for perturbations of hyperbolic toral endomorphisms, the **entropy production of μ^- is strictly negative** unless μ^- is equal to the (forward) SRB measure μ^+ of the endomorphism f, in which case both are absolutely continuous and have zero entropy production.

In **Corollary 4 a)** we show that in fact **most maps** in a neighbourhood of a hyperbolic toral endomorphism have *inverse SRB measures with negative entropy production.* And we actually construct in Corollary 4 b) a **family of perturbations** of hyperbolic toral endomorphisms, whose respective inverse SRB measures have negative entropy production.

In particular from the discussion after (2) it follows that an endomorphism with negative entropy production is not a stationary measure obtained as a weak limit of averages of iterates of absolutely continuous measures of type (2). In this way we find certain chaotic (hyperbolic) systems with folding of phase space, and Gibbs states for them having negative entropy production.

Several interesting and important results from statistical physics point towards the profound relationship between entropy production and the time arrow/irreversibility, and also the possibility of negative entropy production on short time scales (for exp. [3], [5], [7], [18], [24], etc.) However our results are abstract mathematical ones, and we do not investigate here possible physical implications, if any.

2 Main results and proofs.

For the rest of the paper let us fix a smooth (say $\mathcal{C}^2$) non-invertible map $f: M \to M$ defined on a compact Riemannian manifold and let Λ be a fixed basic set of f, i.e there exists some neighbourhood U of Λ with $\Lambda = \bigcap\limits_{n \in \mathbb{Z}} f^n(U)$. Assume also that f is **transitive and hyperbolic** on Λ. Sometimes the set Λ may be the whole manifold as in the case of Anosov endomorphisms (for example for hyperbolic toral endomorphisms). However in general Λ may not be totally invariant, i.e we do not always have $f^{-1}(\Lambda) = \Lambda$.

Here hyperbolicity is understood in the sense of *endomorphisms* (i.e non-invertible maps) (see [21]), i.e there exists a continuous splitting of the tangent bundle into stable and unstable directions, over the inverse limit $\hat{\Lambda}$ consisting of sequences of consecutive preimages,

$$\hat{\Lambda} = \{\hat{x} = (x, x_{-1}, x_{-2}, \ldots,) \text{ with } x_{-i} \in \Lambda, f(x_{-i}) = x_{-i+1}, i \geq 1\}$$

For any element $\hat{x} = (x, x_{-1}, x_{-2}, \ldots) \in \hat{\Lambda}$ we have a stable direction E^s_x (which depends only on x) and an unstable direction $E^u_{\hat{x}}$. Consequently there exists a small $r > 0$ so that we can construct local stable and local unstable manifolds, $W^s_r(x)$ and $W^u_r(\hat{x})$ for any $\hat{x} \in \hat{\Lambda}$. We shall also denote

$$Df_s(x) := Df|_{E^s_x},\ x \in \Lambda \text{ and } Df_u(\hat{x}) := Df|_{E^u_{\hat{x}}},\ \hat{x} \in \hat{\Lambda} \tag{3}$$

The endomorphism f is assumed to have stable directions too, so it is non-expanding. More about hyperbolicity for endomorphisms can be found in [21], [13], etc. When the map is not invertible, there appear significantly different phenomena and different techniques than in the case of diffeomorphisms (as for example in [1], [18], [10], [12], etc.)

We will use in the sequel the notions of *Jacobian of an invariant measure* introduced by Parry in [14]. Let $f : M \to M$ be a smooth endomorphism on the manifold M and μ an f-invariant probability on M (whose support may be smaller than M); assume also that f is at most countable-to-one. Then as shown by Rohlin ([17], [14]), there exists a measurable partition $\xi = (A_0, A_1, \ldots)$ so that f is injective on each A_i. It was proved that the push-forward measure $((f|_{A_i})^{-1})_*\mu$ is absolutely continuous on A_i with respect to μ; so it makes sense to define (as in [14]) the respective Radon-Nykodim derivative, which will be called the **Jacobian** of μ with respect to f:

$$J_f(\mu)(x) = \frac{d\mu \circ (f|_{A_i})}{d\mu}(x),\ \mu - \text{a.e on } A_i, i \geq 0$$

Notice that from the f-invariance of μ, we have $J_f(\mu)(x) \geq 1, \mu - \text{a.e } x \in M$.

Definition 1. Given two positive quantities $Q_1(n,x), Q_2(n,x)$, we will say that they are **comparable** if there exists a positive constant C so that $\frac{1}{C} \leq \frac{Q_1(n,x)}{Q_2(n,x)} \leq C$ for all n, x.

Recall also (for example from [6]) that, given an expansive homeomorphism $f : X \to X$ on a compact metric space, having the specification property, the equilibrium measure μ_ϕ of the Hölder potential ϕ satisfies $A_\varepsilon e^{S_n\phi(x)-nP(\phi)} \leq \mu_\phi(B_n(x,\varepsilon)) \leq B_\varepsilon e^{S_n\phi(x)-nP(\phi)}$, where $B_n(x,\varepsilon) := \{y \in X, d(f^iy, f^ix) < \varepsilon, i = 0, \ldots, n-1\}$, $P(\phi)$ denotes the topological pressure of ϕ with respect to f, and where the positive constants $A_\varepsilon, B_\varepsilon$ are independent of x, n. The general homeomorphism framework above allows us to apply this result to equilibrium measures on the inverse limit $\hat{\Lambda}$. If $\pi : \hat{\Lambda} \to \Lambda, \pi(\hat{x}) := x, \hat{x} \in \hat{\Lambda}$ is the *canonical projection* and if ϕ is a Hölder potential on Λ, then μ_ϕ is the unique equilibrium measure for ϕ on Λ if and only if

$$\mu_\phi = \pi_*\mu_{\phi\circ\pi},$$

where $\mu_{\phi\circ\pi}$ is the unique equilibrium measure of $\phi \circ \pi$ on the compact metric space $\hat{\Lambda}$; here the homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ is the shift map defined by $\hat{f}(x, x_{-1}, x_{-2}, \ldots) = (f(x), x, x_{-1}, \ldots)$. So we obtain for the non-invertible map f and the equilibrium measure μ_ϕ the same estimate as above:

$$A_\varepsilon e^{S_n\phi(x)-nP(\phi)} \leq \mu_\phi(B_n(x,\varepsilon)) \leq B_\varepsilon e^{S_n\phi(x)-nP(\phi)},$$

with positive constants $A_\varepsilon, B_\varepsilon$ independent of n, x, where $S_n\phi(x) := \phi(x) + \ldots + \phi(f^{n-1}(x))$ for $x \in \Lambda$ and n a positive integer.

Theorem 1. *Let f be a smooth hyperbolic endomorphism on a folded basic set Λ, which has no critical points in Λ; let also ϕ a Hölder continuous potential on Λ and denote by μ_ϕ the unique equilibrium measure of ϕ on Λ. Then for all $m \geq 1$, the Jacobian of μ_ϕ w.r.t f^m is comparable to the ratio $\frac{\sum\limits_{\zeta \in f^{-m}(f^m(x))\cap\Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}}$, i.e there exists a comparability constant $C > 0$ (independent of m, x) s.t for $\mu_\phi - a.e\ x \in \Lambda$:*

$$C^{-1} \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x))\cap\Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \leq J_{f^m}(\mu_\phi)(x) \leq C \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x))\cap\Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}}, \tag{4}$$

Proof. We know from definition that the Jacobian $J_{f^m}(\mu_\phi)$ is the Radon-Nikodym derivative of $\mu_\phi \circ f^m$ with respect to μ_ϕ on sets of injectivity for f^m. In order to estimate the Jacobian of μ_ϕ with respect to f^m, we have to compare the measure μ_ϕ on different components of the preimage set $f^{-m}(B)$, for a small borelian set B, where $m \geq 1$ is fixed. Let us consider two subsets E_1, E_2 of Λ so that $f^m(E_1) = f^m(E_2) \subset B$ and E_1, E_2 belong to two disjoint balls $B_m(y_1, \varepsilon)$, respectively $B_m(y_2, \varepsilon)$. This happens if the diameter of B is small enough, since f has no critical points in Λ and thus there exists a positive distance ε_0 between any two different preimages from $f^{-1}(y)$ for $y \in \Lambda$.

As in [6], since the borelian sets with boundaries of measure zero form a sufficient collection, we can assume that each of the sets E_1, E_2 have boundaries of μ_ϕ-measure zero. We recall that $f^m(E_1) = f^m(E_2)$. But as in [6], μ_ϕ is the limit of the sequence of measures:

$$\tilde{\mu}_n := \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \text{Fix}(f^n) \cap \Lambda} e^{S_n \phi(x)} \delta_x,$$

where $P(f, \phi, n) := \sum_{x \in \text{Fix}(f^n) \cap \Lambda} e^{S_n \phi(x)}, n \geq 1$. So we obtain

$$\tilde{\mu}_n(E_1) = \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \text{Fix}(f^n) \cap E_1} e^{S_n \phi(x)}, n \geq 1 \tag{5}$$

Let us now consider a periodic point $x \in \text{Fix}(f^n) \cap E_1$; it follows that $f^m(x) \in f^m(E_1)$, so there exists a point $y \in E_2$ such that $f^m(y) = f^m(x)$. However the point y is not necessarily periodic. Hence we will use the Specification Property ([6], [2]) on hyperbolic locally maximal sets in order to approximate y with a periodic point whose orbit follows that of y for sufficiently long time. Indeed if $\varepsilon > 0$ is fixed, there exists a constant $M_\varepsilon > 0$ such that for all $n > M_\varepsilon$, there is a point $z \in \text{Fix}(f^n) \cap \Lambda$ which ε-shadows the $(n - M_\varepsilon)$-orbit of y. In particular $z \in B_m(y_2, 2\varepsilon)$, since $E_2 \subset B_m(y_2, \varepsilon)$.

Let now $V \subset B_m(y_2, \varepsilon)$ be an arbitrary neighbourhood of the set E_2. Let us take two points $x, x' \in \text{Fix}(f^n) \cap E_1$ and assume the same periodic point $z \in V \cap \text{Fix}(f^n)$ corresponds to both of them through the previous shadowing procedure. Thus the $(n - M_\varepsilon - m)$-orbit of $f^m(z)$ ε-shadows the $(n - M_\varepsilon - m)$-orbit of $f^m(x)$ and also the $(n - M_\varepsilon - m)$-orbit of $f^m(x')$. Thus the $(n - M_\varepsilon - m)$-orbit of $f^m(x)$ 2ε-shadows the $(n - M_\varepsilon - m)$-orbit of $f^m(x')$. But recall that we took $x, x' \in E_1 \subset B_m(y_1, \varepsilon)$, so $x' \in B_m(x, 2\varepsilon)$ and hence from above, $x' \in B_{n-M_\varepsilon}(x, 2\varepsilon)$. We will partition now the set $B_{n-M_\varepsilon}(x, 2\varepsilon)$ in at most N_ε smaller Bowen balls of type $B_n(\zeta, 2\varepsilon)$. In each of these $(n, 2\varepsilon)$-Bowen balls we may have at most one fixed point for f^n. Indeed, fixed points for f^n are solutions to the equation $f^n \xi = \xi$ and Df^n does not have unitary eigenvalues. Then if $d(f^i \xi, f^i \zeta) < 2\varepsilon, i = 0, \ldots, n-1$ and if ε is small enough, we can apply the Inverse Function Theorem at each step, and thus there exists only one fixed point for f^n in the Bowen ball $B_n(\zeta, 2\varepsilon)$. So there may exist at most N_ε periodic points in Λ from $\text{Fix}(f^n) \cap E_1$ having the same point $z \in V \cap \text{Fix}(f^n)$ associated to them by the above shadowing correspondence.

Let us notice also that if $x, x' \in \text{Fix}(f^n) \cap E_1$ have the same point $z \in V$ attached to them, then as seen before, $x' \in B_{n-M_\varepsilon}(x, 2\varepsilon)$ and then, from the Hölder continuity of ϕ,

$$|S_n\phi(x) - S_n\phi(x')| \le \tilde{C}_\varepsilon,$$

for some positive constant $\tilde{C}_\varepsilon$ depending on ϕ (but independent of n, m, x). This can be used then in the estimate for $\tilde{\mu}_n(E_1)$, from (5). Notice also that, if $z \in B_{n-M_\varepsilon}(y, \varepsilon)$, then $f^m(z) \in B_{n-M_\varepsilon-m}(f^m(x), \varepsilon)$. Thus from the Hölder continuity of ϕ and the fact that $x \in E_1 \subset B_m(y_1, \varepsilon)$, it follows that there exists a positive constant $\tilde{C}'_\varepsilon$ satisfying:

$$|S_n\phi(z) - S_n\phi(x)| \le |S_m\phi(y_1) - S_m\phi(y_2)| + \tilde{C}'_\varepsilon, \text{ for } n > n(\varepsilon, m). \tag{6}$$

Then from (6), (5), and since there are at most N_ε points $x \in \text{Fix}(f^n) \cap E_1$ having the same $z \in V \cap \text{Fix}(f^n) \cap \Lambda$ corresponding to them, we obtain that there exists a constant $C_\varepsilon > 0$ s.t:

$$\tilde{\mu}_n(E_1) \le C_\varepsilon \tilde{\mu}_n(V) \cdot \frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}, \tag{7}$$

where we recall that $E_1 \subset B_m(y_1, \varepsilon), E_2 \subset B_m(y_2, \varepsilon)$ and $f^m(E_1) = f^m(E_2)$. But $\partial E_1, \partial E_2$ were assumed of μ_ϕ-measure zero, hence:

$$\mu_\phi(E_1) \le C_\varepsilon \mu_\phi(V) \cdot \frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}$$

Recall now that V was chosen arbitrarily as a neighbourhood of E_2, and by applying the same procedure for E_1 instead of E_2 we obtain the estimates:

$$\frac{1}{C}\mu_\phi(E_2)\frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}} \le \mu_\phi(E_1) \le C\mu_\phi(E_2)\frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}, \tag{8}$$

where $C > 0$ does not depend on m, E_1, E_2.

Now the Jacobian $J_{f^m}(\mu_\phi)$ is the Radon-Nikodym derivative of $\mu_\phi \circ f^m$ with respect to μ_ϕ on sets of injectivity for f^m, hence

$$\mu_\phi(f^m(D)) = \int_D J_{f^m}(\mu_\phi)(x) d\mu_\phi(x),$$

for any borelian set D on which f^m is injective. Hence from Lebesgue derivation theorem, we have that, by putting $D = B(x, r)$ for small $r > 0$, we obtain:

$$J_{f^m}(\mu_\phi)(x) = \lim_{r \to 0} \frac{\mu_\phi(f^m(B(x,r)))}{\mu_\phi(B(x,r))}, \tag{9}$$

for μ_ϕ-almost all $x \in \Lambda$. On the other hand from the invariance of μ_ϕ, we have for any borelian set D that:

$$\mu_\phi(f^m(D)) = \mu_\phi(f^{-m}(f^m D)) \tag{10}$$

Thus if D is a small ball around x, one has to consider the m-preimages y of x, belonging to Λ. Let us notice that if $\zeta \in B_m(y, \varepsilon)$ then, from the Hölder continuity of ϕ we have that

$$|S_m\phi(\zeta) - S_m\phi(y)| \le \tilde{C}_\varepsilon,$$

where the constant $\tilde{C}_\varepsilon$ does not depend on $m > 0, y \in \Lambda$. So in the comparison inequlities of (8), we can take instead of y_1, y_2, the respective m-preimages of x from Λ.

Therefore from relationship (9), the invariance in (10) and the comparison between different pieces of the m-preimage from (8), it follows that the Jacobian of μ_ϕ with respect to f^m satisfies:

$$J_{f^m}(\mu_\phi)(x) \approx \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x))\cap\Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}}, \ \mu_\phi - \text{a.e } x \in \Lambda,$$

where the comparability constant $C > 0$ is independent of $m > 1, x \in \Lambda$.

□

Let us give now the definition of the folding entropy and the entropy production according to Ruelle, [20].

Definition 2. Let $f : M \to M$ be a smooth endomorphism and μ an f-invariant probability on M, then the **folding entropy** $F_f(\mu)$ of μ is the conditional entropy:

$$F_f(\mu) := H_\mu(\epsilon | f^{-1}\epsilon),$$

where ϵ is the partition into single points. Also define the **entropy production** of μ by:

$$e_f(\mu) := F_f(\mu) - \int \log|\det Df(x)| d\mu(x)$$

From [17] it follows that we can use the measurable single point partition ϵ in order to desintegrate the invariant measure μ into a canonical family of conditional measures μ_x supported on the finite fiber $f^{-1}(x)$ for μ-a.e x. Thus the entropy of the conditional measure of μ restricted to $f^{-1}(x)$ is $H(\mu_x) = -\Sigma_{y\in f^{-1}(x)}\mu_x(y)\log\mu_x(y)$. From [14] we have also

$$J_f(\mu)(x) = \frac{1}{\mu_{f(x)}(x)}, \ \mu - \text{a.e } x,$$

hence we obtain that

$$F_f(\mu) = \int \log J_f(\mu)(x) d\mu(x) \tag{11}$$

Let us return now to the case of a hyperbolic basic set Λ for a smooth endomorphism f and consider a Hölder potential ϕ on Λ, with its unique equilibrium measure μ_ϕ. We will give a formula for the folding entropy of the equilibrium measure μ_ϕ in terms of an "asymptotic logarithmic degree" with respect to μ_ϕ. This will take into account at step n the n-preimages of points which behave well (are generic) with respect to μ_ϕ. To this end, for an f-invariant probability (borelian) measure μ on Λ let us define, for any small $\tau > 0$, $n > 0$ integer and $x \in \Lambda$ the set

$$G_n(x, \mu, \tau) := \{y \in f^{-n}(f^n x) \cap \Lambda, \ \text{s.t} \ |\frac{S_n\phi(y)}{n} - \int \phi d\mu| < \tau\}, \tag{12}$$

where $S_n\phi(y) := \phi(y) + \ldots + \phi(f^{n-1}y), y \in \Lambda$ is the consecutive sum of ϕ on y.

Definition 3. In the above setting, denote by $d_n(x,\mu,\tau) := \text{Card}G_n(x,\mu,\tau), x \in \Lambda, n > 0, \tau > 0$. The function $d_n(\cdot,\mu,\tau)$ is measurable, nonnegative and finite on Λ.

Theorem 2. *Let $f : M \to M$ be a smooth endomorphism and Λ a basic set for f so that f is hyperbolic on Λ and does not have critical points in Λ. Let also ϕ a Hölder continuous potential on Λ and μ_ϕ the equilibrium measure associated to ϕ. Then we have the following formula for the folding entropy of μ_ϕ:*

$$F_f(\mu_\phi) = \lim_{\tau\to 0}\lim_{n\to\infty}\frac{1}{n}\int_\Lambda \log d_n(x,\mu_\phi,\tau)d\mu_\phi(x)$$

Proof. First let us recall formula (11) for an arbitrary f-invariant measure μ, namely

$$F_f(\mu) = \int_\Lambda \log J_f(\mu)(x)d\mu_(x)$$

From the Chain Rule for Jacobians, $J_{f^n}(\mu)(x) = J_f(\mu)(x)\dots J_f(\mu)(f^{n-1}(x))$ μ-a.e, for any $n \geq 1$. On the other hand, since μ is f-invariant, we have that

$$\int \log J_f(\mu)(x)d\mu(x) = \int \log J_f(\mu)(f(x))d\mu(x) = \int \log J_f(\mu)(f^k x)d\mu(x),$$

for all $k \geq 1$. These facts imply that for any $n \geq 1$,

$$F_f(\mu) = \frac{1}{n}\int \log J_{f^n}(\mu)(x)d\mu(x) \tag{13}$$

Therefore from Theorem 1, since the constant C is independent of n we obtain that:

$$F_f(\mu_\phi) = \lim_{n\to\infty}\frac{1}{n}\int_\Lambda \log\frac{\sum\limits_{y\in f^{-n}(f^n(x))\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}}d\mu_\phi(x) \tag{14}$$

Now since Λ is compact, each point $x \in \Lambda$ has only finitely many f-preimages in Λ, i.e there exists a positive integer d s.t $\text{Card}(f^{-1}x) \leq d, x \in \Lambda$. Since μ_ϕ is an ergodic measure (as it is an equilibrium state) and from Birkhoff Ergodic Theorem we obtain that

$$\mu_\phi(x \in \Lambda, |\frac{S_n\phi(x)}{n} - \int \phi d\mu| > \tau/2) \underset{n\to\infty}{\to} 0,$$

for any small $\tau > 0$. Thus for any $\eta > 0$ there exists a large integer $n(\eta)$ s.t for $n \geq n(\eta)$,

$$\mu_\phi(x \in \Lambda, |\frac{S_n\phi(x)}{n} - \int \phi d\mu| > \tau/2) < \eta \tag{15}$$

Let us now take a point $x \in \Lambda$ with $|\frac{S_n\phi(x)}{n} - \int\phi d\mu| < \tau$. From Definition 3 we have

$$\frac{e^{n(\int\phi d\mu_\phi-\tau)}d_n(x,\mu_\phi,\tau) + r_n(x,\mu_\phi,\tau)}{e^{n(\int\phi d\mu+\tau)}} \leq \frac{\sum\limits_{y\in f^{-n}(f^n x)\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} \leq \frac{e^{n(\int\phi d\mu_\phi+\tau)}d_n(x,\mu_\phi,\tau) + r_n(x,\mu_\phi,\tau)}{e^{n(\int\phi d\mu_\phi-\tau)}}, \tag{16}$$

where $r_n(x, \mu_\phi, \tau)$ is the remainder $\sum\limits_{y \in f^{-n}f^n(x) \setminus G_n(x,\mu_\phi,\tau)} e^{S_n\phi(y)}$. In order to simplify notation, we will also denote $r_n(x, \mu_\phi, \tau)$ by r_n when no confusion can arise.

Given n large, let us consider now a partition $(A_i^n)_{1 \le i \le K}$ of Λ (modulo μ_ϕ) so that for each $0 \le i \le K$, there exists a point $z_i \in A_i^n$ so that for any n-preimage $\xi_{ij} \in f^{-n}(z_i) \cap \Lambda, 1 \le j \le d_{n,i}$, we have $A_i^n \subset f^n(B_n(\xi_{ij}, \varepsilon)), 1 \le j \le d_{n,i}, 1 \le i \le K$. For the above partition, let us denote by A_{ij}^n the part of the n-preimage of A_i^n which belongs to the Bowen ball $B_n(\xi_{ij}, \varepsilon)$, i.e $A_{ij}^n := f^{-n}(A_i^n) \cap B_n(\xi_{ij}, \varepsilon), 1 \le j \le d_{n,i}, 1 \le i \le K$. Since the sets A_i^n were chosen disjoint, also the pieces of their preimages, namely A_{ij}^n, i, j, are mutually disjoint.

We will decompose the integral in (14) over the sets A_{ij}^n. Notice that if $y, z \in A_{ij}^n$, then since ϕ is Hölder continuous and $A_{ij}^n \subset B_n(\xi_{ij}, \varepsilon)$, it follows that we have

$$|S_n\phi(y) - S_n\phi(z)| \le C(\varepsilon), \tag{17}$$

where $C(\varepsilon)$ is a positive function with $C(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$. So we will obtain now:

$$\int_\Lambda \log \frac{\sum\limits_{y \in f^{-n}f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) = \sum_{0 \le j \le d_i, 0 \le i \le K} \int_{A_{ij}^n} \log \frac{\sum\limits_{y \in f^{-n}f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \tag{18}$$

Let us now denote by $R_n(i, \mu_\phi, \tau)$ the set of preimages ξ_{ij} with $\xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)$, and denote simply by $R_{n,i}$ the set of indices $j, 1 \le j \le d_{n,i}$ with $\xi_{ij} \in R_n(i, \mu_\phi, \tau)$ for every $1 \le i \le K$. Now in the decomposition from (18) we notice that the integral over those sets A_{ij}^n with $j \in R_{n,i}$ will not matter significantly. Indeed as $\text{Card}(f^{-1}x \cap \Lambda) \le d, x \in \Lambda$ and since $-M \le \phi(x) \le M, x \in \Lambda$ we have

$$1 \le \frac{\sum\limits_{y \in f^{-n}f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} \le d^n e^{2nM}$$

Now recall that each $A_{ij}^n \subset B_n(\xi_{ij}, \varepsilon)$ and the sets A_{ij}^n, i, j are mutually disjoint (with respect to μ_ϕ). Hence by using inequalities (15) and (17) and the fact that $\xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)$ whenever $j \in R_{n,i}$, we obtain:

$$\sum_{0 \le i \le K, j \in R_{n,i}} \frac{1}{n} \int_{A_{ij}^n} \log \frac{\sum\limits_{y \in f^{-n}f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \le \frac{1}{n} \log(d^n e^{2nM}) \cdot \eta = \eta(\log d + 2M) \tag{19}$$

But by using the comparison between different parts of the n-preimage of a small set from the proof of Theorem 1 (see (8)), we deduce that the last term of formula (18) is comparable to

$$\sum_{i,j} \mu_\phi(A_{ij}^n) \log \frac{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A_{ij}^n) + \tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A_{ij}^n)}, \tag{20}$$

where $\tilde{r}_n(z_i, \mu, \tau) := \sum\limits_{\xi_{ij} \in f^{-n}(z_i) \cap \Lambda, \ \xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)} \mu_\phi(A_{ij}^n)$, Hence from (8), (19) and (20) we obtain:

$$\begin{aligned}&\frac{1}{n}\sum_{i,j\notin R_{n,i}}\mu_\phi(A_{ij}^n)\log d_n(z_i,\mu_\phi,\tau)+\frac{1}{n}\sum_{i,j\notin R_{n,i}}\mu_\phi(A_{ij}^n)\log(1+\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A_{ij}^n)})-\delta(\tau)-\eta C'\le\\ &\le\int_\Lambda\frac{1}{n}\log\frac{\sum\limits_{y\in f^{-n}f^nx\cap\Lambda}e^{S_n\phi(y)}}{e^{S_n\phi(x)}}d\mu_\phi(x)\le\\ &\le\frac{1}{n}\sum_{i,j\notin R_{n,i}}\mu_\phi(A_{ij}^n)\log d_n(z_i,\mu_\phi,\tau)+\frac{1}{n}\sum_{i,j\notin R_{n,i}}\mu_\phi(A_{ij}^n)\log(1+\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A_{ij}^n)})+\delta(\tau)+\eta C',\end{aligned} \tag{21}$$

with $C'=\log d+2M$ being the constant found in (19), and where the positive constant $\delta(\tau)$ comes from the uniformly bounded variation of $\frac{1}{n}S_n\phi(x)$ when x is in A_{ij}^n and when $1\le i\le K, j\notin R_{n,i}$ vary; clearly we have $\delta(\tau)\underset{\tau\to 0}{\to}0$.

Now we know that in general $\log(1+x)\le x$, for $x>0$. Thus $\log(1+\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A_{ij}^n)})\le\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A_{ij}^n)}$, i,j and hence in (21) we have, for n large enough that:

$$\begin{aligned}&\sum_{i,j\notin R_{n,i}}\mu_\phi(A_{ij}^n)\log(1+\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A_{ij}^n)})\le\sum_{i,j\notin R_{n,i}}\mu_\phi(A_{ij}^n)\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A_{ij}^n)}=\\ &=\sum_{1\le i\le K}\tilde r_n(z_i,\mu_\phi,\tau)\le\eta,\end{aligned} \tag{22}$$

where we used that by definition, there are $d_n(z_i,\mu_\phi,\tau)$ indices j in $\{1,\dots,d_{n,i}\}\setminus R_{n,i}$ for any $1\le i\le K$. Therefore from the last displayed inequality and from (21) we obtain, for $n\ge n(\eta)$, that:

$$\left|\frac{1}{n}\int_\Lambda\log\frac{\sum\limits_{y\in f^{-n}f^nx\cap\Lambda}e^{S_n\phi(y)}}{e^{S_n\phi(x)}}d\mu_\phi(x)-\frac{1}{n}\int_\Lambda\log d_n(z,\mu_\phi,\tau)d\mu_\phi(z)\right|\le\delta(\tau)+\eta, \tag{23}$$

where $\delta(\tau)\underset{\tau\to 0}{\to}0$. Then by taking $n\to\infty$ and $\tau\to 0$, we will obtain the conclusion of the Theorem from (14) and (23), namely that

$$F_f(\mu_\phi)=\lim_{\tau\to 0}\lim_{n\to\infty}\frac{1}{n}\int_\Lambda\log d_n(x,\mu_\phi,\tau)d\mu_\phi(x)$$

□

Corollary 1. *a) Let $f:\mathbb{T}^m\to\mathbb{T}^m, m\ge 2$ be a hyperbolic toral endomorphism, and ϕ be an arbitrary Hölder continuous potential on $\mathbb{T}^m$, with its associated equilibrium measure μ_ϕ. Then the entropy production of μ_ϕ is non-positive, i.e*

$$e_f(\mu_\phi)\le 0$$

In the same setting the entropy production of the Haar (Lebesgue) measure is equal to 0.

b) The same conclusions as above hold also for any Anosov endomorphism $f:\mathbb{T}^m\to\mathbb{T}^m$ with constant Jacobian with respect to the Riemannian metric, i.e for which $detDf$ is constant on $\mathbb{T}^m$.

Proof. a) In the case of a toral endomorphism f given by the integer-valued matrix A, the determinant of the derivative $\det Df$ is constant and equal to $\det A$. Thus

$$\int_{\mathbb{T}^m} \log |\det Df| d\mu_\phi = \log d,$$

where $d := |\det A|$. On the other hand, by looking at the area of $f(I \times \ldots \times I)$, it is easy to see that d is exactly the number of f-preimages that any point from $\mathbb{T}^m = I \times \ldots \times I$ (m times) has. Therefore, by taking $\Lambda = \mathbb{T}^m$ and by recalling Definition 3, one obtains that

$$d_n(x, \mu_\phi, \tau) \le d^n, \ \forall x \in \mathbb{T}^m, n > 0, \tau > 0$$

Hence from Theorem 2 it follows that

$$e_f(\mu_\phi) \le 0$$

For the last statement of a), we have that f invariates the Lebesgue measure m, that $|\det Df|$ is constant and equal to d and that $d_n(x, m, \tau)$ is constant in x and equal to d since the Lebesgue (Haar) measure is the unique measure of maximal entropy. Therefore the entropy production of the Lebesgue measure m with respect to f is equal to 0.

The last statement of a) can also be obtained from the fact that the entropy production of invariant absolutely continuous measures is non-negative (from [20]), combined with the first part of the proof.

b) The argument is the same as for a), namely if $\det Df$ is constant, then f invariates the Lebesgue measure m, and it is d-to-1, for $d = |\det Df|$. Then $d_n(x, \mu_\phi, \tau) \le d$ for any x, τ, n and $e_f(\mu_\phi) \le 0$.

□

However we will see later that Corollary 1 is no longer true for perturbations of a toral endomorphism f, and that there exist equilibrium measures of Hölder potentials which have in certain cases positive entropy production.

Theorem 2 also helps us calculate the folding entropy of the measure of maximal entropy for a general hyperbolic (hence non-expanding) endomorphism. Then by knowing this, one can calculate the **entropy production of the measure of maximal entropy**, from Definition 2.

Corollary 2. *In the setting of Theorem 2, denote by μ_0 the unique measure of maximal entropy for f on Λ. If $d_n(x)$ denotes the cardinality of $f^{-n}(f^n x) \cap \Lambda$ for $n \ge 1$, then:*

i) The folding entropy of μ_0 is given by

$$F_f(\mu_0) = \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log d_n(x) d\mu_0(x)$$

ii) The entropy production of μ_0 is given by

$$e_f(\mu_0) = \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log d_n(x) d\mu_0(x) - \int \log |det(Df)(x)| d\mu_0(x)$$

In particular if f is d-to-1 on the basic set Λ, then $e_f(\mu_0) = \log d - \int \log |det(Df)(x)| d\mu_0(x)$.

As a remark, while for toral endomorphisms the entropy production of the measure of maximal entropy (which in this case is the Haar measure) is zero, this fact is not necessarily clear (or true) for a hyperbolic endomorphism on a general fractal basic set Λ.

Let us now recall the notion of **inverse SRB measure**, introduced in [11]. These measures exist in the case of hyperbolic repellers (and in particular in the case of Anosov endomorphisms) and are physically relevant since they describe the *past trajectories* of Lebesgue almost all points in a neighbourhood of the repellor. We will show that there exist Anosov endomorphisms, whose respective inverse SRB measures have **negative entropy production**.

Let Λ be a connected hyperbolic repeller for a smooth endomorphism $f : M \to M$ defined on a Riemannian manifold M, and assume f has no critical points in Λ. Let V be a neighbourhood of Λ in M and for any $z \in V$ define the measures

$$\mu_n^z := \frac{1}{n} \sum_{y \in f^{-n}z \cap V} \frac{1}{d(f(y)) \dots d(f^n(y))} \sum_{i=1}^{n} \delta_{f^i y}, \tag{24}$$

where $d(y)$ is the number of f-preimages belonging to V of a point $y \in V$ ($d(\cdot)$ is called also the degree function).

Then we proved in [11] that there exists an f-invariant measure μ^- on Λ, a neighbourhood V of Λ and a borelian set $A \subset V$ with $m(V \setminus A) = 0$ (where m is the Lebesgue measure on M) and a subsequence $n_k \to \infty$ such that for any $z \in A$,

$$\mu_{n_k}^z \underset{k\to\infty}{\to} \mu^- \tag{25}$$

The measure μ^- is called the **inverse SRB measure** of the hyperbolic repeller. We showed in [11] that μ^- is the equilibrium measure of the stable potential $\Phi^s(x) := \log |\det Df_s(x)|, x \in \Lambda$, with respect to f (where we recall the notation from (3)). The difficulty is that the map f is non-invertible, hence μ^- is **not** simply the SRB measure for the inverse f^{-1}. Moreover from the hyperbolicity condition in the case of endomorphisms, the unstable manifolds may intersect each other both in Λ and outside Λ and through any point of Λ there may pass infinitely many (even uncountably many, as shown in [10]) unstable manifolds.

We also proved that this inverse SRB measure μ^- is the unique f-invariant measure μ satisfying an *inverse Pesin entropy formula;* in the case when f is d-to-1 on Λ we have:

$$h_{\mu^-}(f) = \log d - \int_\Lambda \sum_{i, \lambda_i(\mu^-, x) < 0} \lambda_i(\mu^-, x) m_i(\mu^-, x) d\mu^-(x), \tag{26}$$

where $\lambda_i(\mu^-, x)$ are the Lyapunov exponents of the measure μ^- at x and $m_i(\mu^-, x)$ are the respective multiplicities of these Lyapunov exponents. In addition if f is d-to-1 on the connected hyperbolic repeller Λ, then the inverse SRB measure μ^- has *absolutely continuous conditional measures on local stable manifolds* (see [11]).

Also for an Anosov endomorphism f on M, we know from [16], [15] that there exists a unique SRB measure μ^+ which satisfies a Pesin entropy formula and which is the projection π_* of the equilibrium measure of the unstable potential $\Phi^u(\hat{x}) := -\log|\det Df_u(\hat{x})|, \hat{x} \in \hat{M}$ (with the notation for the unstable derivative from (3)).

We prove now that the entropy production of the respective inverse SRB measure μ_g^-, of a perturbation g of a hyperbolic toral endomorphism, is less than or equal to 0. Cohomological conditions guaranteeing that the entropy production $e_g(\mu_g^-)$ is zero are also given. In particular we identify the cases when $e_g(\mu_g^-) = 0$ as exactly those cases when μ_g^- is **absolutely continuous** on $\mathbb{T}^m$. For all other perturbation cases, one has **negative entropy production**, i.e $e_g(\mu_g^-) < 0$.

Theorem 3. *Let f be a hyperbolic toral endomorphism on $\mathbb{T}^m, m \geq 2$ given by an integer-valued matrix A without zero eigenvalues, and let g be a $\mathcal{C}^1$ perturbation of f. Consider μ_g^- the inverse SRB measure of g and μ_g^+ the (usual forward) SRB measure. Then:*

a) $e_g(\mu_g^-) \leq 0$ and $F_g(\mu_g^-) = \log d$. Moreover $e_g(\mu_g^+) \geq 0$.

b) $e_g(\mu_g^-) = 0$ if and only if $|detDg|$ is cohomologous to a constant on $\mathbb{T}^m$.

Also, $e_g(\mu_g^+) = 0$ if and only if $|detDg|$ is cohomologous to a constant on $\mathbb{T}^m$. In either case we obtain $\mu_g^- = \mu_g^+$, and the common value is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^m$.

Proof. a) If f is given by an integer valued matrix A, then f is d-to-1 on $\mathbb{T}^m$, where $d = |\det A|$ (for example [23]). If g is a $\mathcal{C}^1$ perturbation of the hyperbolic toral endomorphism f, then it is clear that g is also hyperbolic on $\mathbb{T}^m$. Thus from [16] we can construct the SRB measure of g, denoted by μ_g^+, which is the projection by π_* of the equilibrium measure of $\Phi_g^u(\hat{x}) = -\log|\det Dg_u(\hat{x})|, \hat{x} \in \hat{\mathbb{T}^m}$. In particular μ_g^+ is ergodic, hence its Lyapunov exponents are constant μ_g^+-a.e.

From the discussion above, since f has no critical points, we can construct the inverse SRB measure μ_g^- which is the equilibrium measure of the stable potential $\Phi_g^s(x) = \log|\det Dg_s(x)|, x \in \mathbb{T}^m$; thus μ_g^- is ergodic too, and its Lyapunov exponents are constant μ_g^--a.e on $\mathbb{T}^m$.

Now since g is a perturbation of f, it follows that every point in $\mathbb{T}^m$ has exactly d g-preimages, where $d = |\det A|$. Thus from [11], it follows that μ_g^- is the weak limit of a sequence of measures of type (24), where the degree function $d(\cdot)$ is constant and equal to d everywhere on $\mathbb{T}^m$. This implies then that the Jacobian of μ_g^- is constant and equal to d, since for any small borelian set B, we have that a point $x \in g(B)$ if and only if there is exactly one g-preimage x_{-1} of x in B, and we use this fact in the above convergence (25) of measures towards μ^-. Hence

$$F_g(\mu_g^-) = \int \log J_g(\mu_g^-)(x) d\mu_g^-(x) = \log d$$

And from (26) we have that

$$h_{\mu_g^-}(g) = \log d - \sum_{\lambda_i(\mu_g^-)<0} \lambda_i(\mu_g^-)$$

Thus if $e_g(\mu_g^-) > 0$, it would follow from the g-invariance of μ_g^-, that

$$F_g(\mu_g^-) > \int \log|\det Dg| d\mu_g^- = \frac{1}{n}\int \log|\det Dg^n| d\mu_g^-, n \geq 1$$

Hence from the last two displayed formulas and Birkhoff Ergodic Theorem, we obtain $h_{\mu_g^-}(g) > \sum\limits_{\lambda_i(\mu_g^-)>0} \lambda_i(\mu_g^-)$, which gives a contradiction with Ruelle's inequality. Therefore for any perturbation g we have that

$$e_g(\mu_g^-) \le 0$$

Now for the SRB measure μ_g^+: if the entropy production $e_g(\mu_g^+)$ were strictly negative, then $F_g(\mu_g^+) < \int \log|\det Dg| d\mu_g^+$. Since from [9], $h_{\mu_g^+}(g) \le F_g(\mu_g^+) - \sum\limits_{\lambda_i(\mu_g^+)<0} \lambda_i(\mu_g^+)$, it would follow that $h_{\mu_g^+}(g) < \sum\limits_{\lambda_i(\mu_g^+)>0} \lambda_i(\mu_g^+)$, which is a contradiction to the fact that the SRB measure satisfies Pesin entropy formula. Consequently,

$$e_g(\mu_g^+) \ge 0$$

b) If $e_g(\mu_g^-) = 0$, then $F_g(\mu_g^-) = \int \log|\det Dg| d\mu_g^-$; hence from the Birkhoff Ergodic Theorem and [9] we obtain:

$$h_{\mu_g^-}(g) = \int \log|\det Dg| d\mu_g^- - \sum_{\lambda_i(\mu_g^-)<0} \lambda_i(\mu_g^-) = \sum_{\lambda_i(\mu_g^- >0} \lambda_i(\mu_g^-)$$

Therefore from the uniqueness of the g-invariant measure satisfying Pesin entropy formula, we obtain that $\mu_g^- = \mu_g^+$.

Recalling from above that μ_g^- is the equilibrium measure of the stable potential Φ^s and μ_g^+ is the equilibrium measure of the unstable potential Φ^u, we see from Livshitz Theorem (see [6]), that $\mu_g^- = \mu_g^+$ if and only if $\det Dg$ is cohomologous to a constant.

Assume now that $\mu_g^+ = \mu_g^-$; then since μ_g^+ has absolutely continuous conditional measures associated to a partition subordinated to local unstable manifolds ([16], [15]) and μ_g^- has absolutely continuous conditional measures associated to a partition subordinated to local stable manifolds (from [11]) and g has local product structure on $\mathbb{T}^m$, we obtain that μ_g^+ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^m$.

□

Remark. There exist also examples of non-invertible hyperbolic repellors (f, Λ) which are **not** Anosov endomorphisms, but which still retain the property that they are d-to-1, for some integer d; some of them were given in [11] as perturbations of certain product basic sets. For those examples as well, we can apply the fact that the Jacobian of the inverse SRB measure μ^- is constant, and the inverse Pesin entropy formula (26), in order to obtain the same result as in Theorem 3.

□

Corollary 3. *In the setting of Theorem 3, let g be a perturbation of the hyperbolic toral endomorphism f s.t $|detDg|$ is not cohomologous to a constant. Then its unique inverse SRB measure μ_g^- is not a weak limit of a sequence of type (2).*

Proof. As was proved in [20], the entropy production of any limit of measures of type (2) is non-negative. On the other hand, if $|\mathrm{det}Dg|$ is not cohomologous to a constant, then $e_g(\mu_g^-) < 0$. Thus in our case μ_g^- is not a weak limit of measures of type (2).

□

We show now that the set of maps with negative entropy production for their respective inverse SRB measures, is **open and dense** in a neighbourhood of a hyperbolic toral endomorphism f.

Corollary 4. *a) Let f be a hyperbolic toral endomorphism on $\mathbb{T}^m, m \geq 2$. Then there exists a neighbourhood V of f in $\mathcal{C}^1(\mathbb{T}^m, \mathbb{T}^m)$ and a set $W \subset V$ such that W is open and dense in the $\mathcal{C}^1$ topology in V and s.t for any $g \in W$ we have $e_g(\mu_g^-) < 0$.*

b) Consider the hyperbolic toral endomorphism on $\mathbb{T}^2$ given by $f(x, y) = (2x+2y, 2x+3y)$ $(mod\, 1)$ and its smooth perturbation

$$g(x, y) = (2x + 2y + \varepsilon sin2\pi y,\ 2x + 3y + 2\varepsilon sin2\pi y)\ (mod\ 1)$$

Then the inverse SRB measure of g has negative entropy production, while the SRB measure of g has positive entropy production, i.e

$$e_g(\mu_g^-) < 0 \text{ and } e_g(\mu_g^+) > 0$$

Proof. a) If f is a hyperbolic toral endomorphism on $\mathbb{T}^m$ then there exists a neighbourhood V of f in $\mathcal{C}^1$ topology, so that any $g \in V$ is hyperbolic and d-to-1, where $d = |\mathrm{det}Df|$.

We showed in Theorem 3 that $e_g(\mu_g^-) < 0$ unless $|\mathrm{det}Dg|$ is cohomologous to a constant. But from the Livshitz Theorem (see for instance [6]) it follows that this is equivalent to the existence of a constant c such that for any $n \geq 1$,

$$S_n(|\mathrm{det}Dg|)(x) = nc,\ \forall x \in \mathrm{Fix}(g^n)$$

As the set of g's *not satisfying* the above equalities is open and dense in V, we obtain the conclusion of part a).

b) First of all we notice that f is given by an integer valued matrix A which has one eigenvalue larger than 1 and another eigenvalue in $(0, 1)$, so f is hyperbolic. Thus for $\varepsilon > 0$ small enough, we have that g (which is well defined as an endomorphism on $\mathbb{T}^m$) is hyperbolic as well.

We calculate now the determinant of the derivative of g as

$$\mathrm{det}Dg(x, y) = 2 + 4\pi\varepsilon cos2\pi y$$

Now, from Theorem 3 we see that $e_g(\mu_g^-) < 0$ if and only if the function $|\mathrm{det}Dg|$ is cohomologous to a constant. But this is equivalent from the Livshitz conditions ([6]) to the fact that there exists a constant c such that

$$S_n(|\mathrm{det}Dg|)(x) = nc,\ x \in \mathrm{Fix}(g^n), n \geq 1$$

In our case, notice that both $(0,0)$ and $(0,\frac{1}{2})$ are fixed points for g. But $|\det Dg(0,0)| = 2+4\pi\varepsilon$, whereas $|\det Dg(0,\frac{1}{2})| = 2-4\pi\varepsilon$. So the Livshitz condition above is not satisfied, and $|\det Dg|$ is not cohomologous to a constant. Hence according to Theorem 3 we obtain

$$e_g(\mu_g^-) < 0 \text{ and } e_g(\mu_g^+) > 0$$

□

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