TWO POINT HOMOGENEOUS SPACES ARE SYMMETRIC

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Short topological proof

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Two point homogeneous spaces were studied by H. Busemann [2], G. Birkhoff [1], H.C. Wang [9] and J. Tits [8]. The last two authors have given a classification for these spaces and it turned out (just from this list) that these spaces are symmetric. They started with a connected locally compact metric space which could be also our starting point because the transformation groups acting transitively on such spaces are Lie groups furthermore the isotropy group describes a sphere as an indicatrix at any tangent space, i.e. the space is a Riemannian space.

For the direct proof (not using classification) of symmetricity see the works: J. Wolf [10], S. Helgason [4], T. Nagano [6] and H. Matsumoto [7]. All of these authors use group theoretic methods.

Our proof is simple topological. The main tool is the following

Lemma Let $X(\underline{m})$ be a continuous tangent vector field on the euclidean unit sphere $S_0^n \subset \mathbb{R}^{n+1}$. Then an antipodal point–paar $\underline{m}_0$, $-\underline{m}_0$ on S_0^n exists such that $X(\underline{m}_0) = -X(-\underline{m}_0)$.

We have to prove that the symmetric part

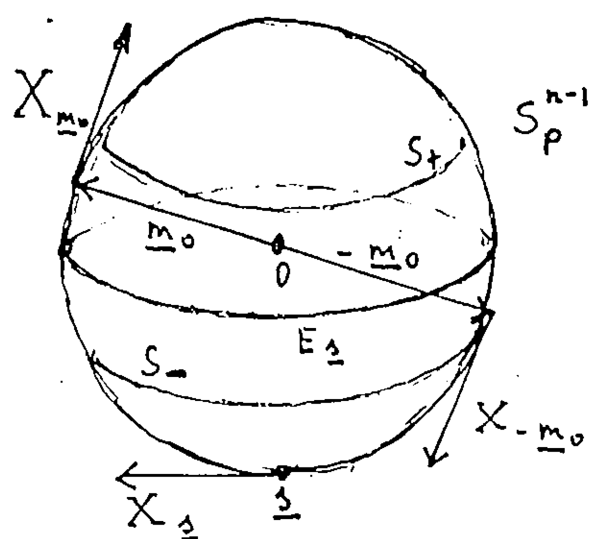
$$X_{sym}(\underline{m}) = \tfrac{1}{2}(X(\underline{m}) + X(-\underline{m}))$$

of X vanishes at least at one antipodal point–paar $\underline{m}_0$, $-\underline{m}_0$. Assume the contrary, the normalized vector field $f = X_{sym}/\|X_{sym}\|$ defines a fixpoint free map $f : S^n \longrightarrow S^n$. Therefore $\deg f = (-1)^{n+1}$ follows ([5] p. 219). On the other hand $f(\underline{m}) = f(-\underline{m})$ for any $\underline{m} \in S^n$, so $\deg f$ is an even number ([3] p. 127). This contradiction proves the Lemma completely.

We turn to the proof of the statement indicated in the title.

We consider at a fixed point p of a two point homogeneous space M^n the field of the self–adjoint operators $A_{\underline{m}}(.) := \nabla_m R(.,\underline{m})\underline{m}$ acting in the tangent spaces of the unit sphere $S_p^{n-1} \subset T_p(M^n)$ at the points $\underline{m} \in S_p^{n-1}$. Here $R(.,\underline{m})\underline{m}$ means the curvature tensor and $\nabla_m R(.,\underline{m})\underline{m}$ is it's covariant derivative w.r.t. $\underline{m}$.

The eigenvalues of $A_{\underline{m}}$ are constant along S_p^{n-1} because $A_{\underline{m}}$ is invariant under the action of the isotropy group (which group is transitive on S_p^{n-1} in this case considered). The $A_{\underline{m}}$ is skew in the sense: $A_{-\underline{m}} = -A_{\underline{m}}$. So if λ is an eigenvalue then also $-\lambda$ is an eigenvalue with the same multiplicity. Let us assume that we have a non–zero eigenvalue λ. We show that this leads to a contradiction and therefore $A_{\underline{m}} = 0$, $\nabla R = 0$ follows by a well known argument, which proves the statement completely.



Pick up a "south" point $\underline{s}$ on S_p^{n-1} and let $E_{\underline{s}}$ be the equator–sphere w.r.t. $\underline{s}$. We take also a smaller sphere S_+ above $E_{\underline{s}}$ which is parallel to $E_{\underline{s}}$ furthermore S_- is the reflected sphere: $S_- = -S_+$ under $E_{\underline{s}}$.

Notice, that the eigensubspaces of $A_{\underline{m}}$ decompose the tangent space $T(S_p^{n-1})$ into continuous distributions. Furthermore the part of S_p^{n-1} under S_+ is contractible, therefore any eigenvector $X_{\underline{s}}$ at $\underline{s}$ with the eigenvalue λ can be extended into a continuous vector field X of S_p^{n-1} such that:

(1) $X_{\underline{m}}$ is an eigenvector with the eigenvalue λ at any point,

(2) It does not vanish under S_+ and it vanish above S_+ .

Now we apply the Lemma to this vectorfield X . The points $\underline{m}_0$, $-\underline{m}_0$ could lay only between S_+ and S_- because X is non–vanishing under S_- and it is vanishing above S_+ everywhere. At the point $\underline{m}_0$ the non–zero vectors $X_{\underline{m}_0}$ and $X_{-\underline{m}_0} = -X_{\underline{m}_0}$ are eigenvectors of $A_{\underline{m}_0}$ with the eigenvalue λ , therefore $X_{-\underline{m}_0}$ is an eigenvector of $A_{-\underline{m}_0} = -A_{\underline{m}_0}$ with the eigenvalue $-\lambda$. This is a contradiction, and so any eigenvalue of $A_{\underline{m}}$ is zero.

Remark The odd dimensional case can be proved easier. In this case the eigenvalues of $R(.,\underline{m})\underline{m}$ are constant on S_p^{n-1} . These eigenvalues must be equal because in the opposite case the eigensubspaces split the tangent space $T(S_p^{n-1})$ into non–trivial continuous distributions: $T(S_p^{n-1}) = \xi_1 \oplus \xi_2 \oplus \ldots \oplus \xi_k$. This is impossible because the Euler classes $\chi(\xi_i)$ are zeros, furthermore the Euler class $\chi(T(S^{n-1})) = \chi(\xi_1) \vee \chi(\xi_2) \vee \ldots \vee \chi(\xi_k)$ of an even dimensional sphere is non–zero. Therefore the space is of constant sectional curvature.

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