THREE NOTES ON THE ORDER OF IDEALS DEFINING HYPERSURFACES

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MPI 87-15

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Let (R,M) be a regular local ring and I a proper ideal in R . Let e(R/I) be the Samuel multiplicity of R/I and ord(I) the order of I with respect to M . We describe three essential situations, where e(R/I) = ord(I) implies that I is principal. The results underscore a natural conjecture.

1. Introduction

Let (R,M) be a regular local ring and let I be a proper ideal in R . Let A = R/I and $\mathfrak{m}$ = M/I . For any ideal I in R we define the order of I to be ord(I) = $\max\{n \mid I \subset M^n\}$. It is well known that for $f \in M^n \setminus M^{n+1}$ the Samuel multiplicity is e(R/fR) = n . In these notes we give sufficient conditions on A and I respectively for the converse implication; more precise: we describe three situations where e(R/I) = ord(I) = n implies I = fR for some $f \in M^n \setminus M^{n+1}$. The first situation is based on the equimultiplicity of I , the second one (due to Ikeda) on the Buchsbaum-property of A where depth A > 0 . In the third situation we discuss the case of a homogeneous graded polynomial ring $R = k[X_1,\dots,X_r]$ over an algebraically closed field, where I is a homogeneous ideal. Here the methods of the proof were outlined by S. Ikeda and influenced in some parts by J. L. Vicente.

In the light of these results we ask if the following statement is true for R and I as above:

" Let $\dim(R/I) \geq 2$ and $\mathrm{ord}(I) \geq 2$. Assume that R/I satisfies Serre's condition S_2 . Then $I = f \cdot R$ if and only if $e(R/I) = \mathrm{ord}(I)$. "
If $\dim A = \mathrm{ord}(I) = 2$, then A is Cohen-Macaulay, and in that case the statement is correct, s. prop. 2.1. But the question seems to be open if A is any domain satisfying S_2 with $\dim(A) > 2$ and $e(A) = \mathrm{ord}(I) = 2$. Recall that if A is not a domain or if A contains a field then the conditions $e(A) = 2$ and S_2 imply that A is Cohen-Macaulay, in which case the statement is true by prop. 2.1.
The above statement is not true, if we omit the condition S_2 . This can be demonstrated by the following examples:

Example 1.1:
Let $A = k[[s^2, s^3, st, t]]$, where k is a field and s,t are indeterminates over k . Writing A is a quotient R/I of the power series ring $R = k[[X_1, X_2, X_3, X_4]]$, we see that $e(A) = 2 = \mathrm{ord}(I)$. Moreover A satisfies S_1 , but not S_2 . Clearly A is not Cohen-Macaulay.

Example 1.2:
Let $A = R/I$ be the following rational surface-germ $C \subset \mathbb{A}_k^4$:
$A = k[[X_1, X_2, X_3, X_4]]/(X_4X_3 - X_2X_1; X_2^2 - X_3^2 + X_1X_3^2; X_4^2 + X_1^3 - X_1^2; X_1X_3 - X_2X_4 - X_1^2X_3)$.
Let $V \subseteq \mathbb{A}_k^4$ be the complete intersection defined by the equations $X_4X_3 - X_2X_1 = 0$, $X_1X_3 - X_2X_4 - X_1^2X_3 = 0$ in $R = k[[X_1, X_2, X_3, X_4]]$. Finally let $L \subseteq \mathbb{A}_k^4$ be defined in $k[[X_1, X_2, X_3, X_4]]$ by the ideal $(X_1, X_4) \cap (X_2, X_3)$. Then $V = C \cup L$, i.e. C and L are linked by the complete intersection V . Since L is Buchsbaum but not Cohen-Macaulay, A is a non-Cohen-Macaulay Buchsbaum ring, satisfying S_1 . It is easy to check that again $e(A) = \mathrm{ord}(I) = 2$.

Example 1.3:
Let
$$A = R/I = k[[X_1, \ldots, X_{2n}]]/(X_1, X_2) \cap (X_3, X_4) \cap \ldots \cap (X_{2n-1}, X_{2n}) = k[[x_1, \ldots, x_{2n}]] ,$$
where the X_i are indeterminates over a field k and

$n \geq 3$. Then:

$$e(A) = \mathrm{ord}(I) = n \quad \text{and} \quad \dim(A) = 2n-2 \geq 3 \quad .$$

Moreover depth $A \geq 2$, but A doesn't satisfy S_2 , since $\mathfrak{P} = (x_1,x_2,x_3,x_4) \subset A$ has $\mathrm{height}(\mathfrak{P}) = 2$, but A is of depth one . Therefore A is not even Buchsbaum. This example also shows that for any $n \geq 2$, there is always an unmixed homogeneous ideal I in a power series ring over k such that $n = \mathrm{ord}(I) = e(R/I)$, but R/I doesn't satisfy S_2 . (The same is true for a homogeneous graded polynomial ring $k[X_1,\dots,X_{2n}]$.)

Finally we remark that the corresponding statement for the dimension 1 case, which we will not consider in the sequel, is the following well known fact (which also follows from our prop. 2.1):

Proposition 1.4:

Let $A = R/I$ be a one-dimensional domain. Then A is a "plane curve" if and only if $e(A) = \mathrm{ord}(I)$.

This gives immediately the following corollary:

Corollary 1.5:

If $A = R/I$ is not a "plane curve" with $\mathrm{ord}(I) \geq 2$, then $e(A) \geq 3$.

2. Preliminaries

Throughout this paper (R,M) denotes a regular local ring with infinite residue field R/M .

Proposition 2.1:

Let I be an ideal of R with $e(R/I) = \mathrm{ord}(I) \geq 2$. If R/I is Cohen-Macaulay then $I = f\cdot R$ for some $f \in M^n \setminus M^{n+1}$.

Proof:

Let $A = R/I$, $\mathfrak{m} = M/I$, $d = \dim A$, $r = \dim R$ and $n = \mathrm{ord}(I)$. Since A is Cohen-Macaulay we find [0] a minimal reduction $a_1,\dots,a_d$ of $\mathfrak{m}$, such that $e(A) = e(A/a_1,\dots,a_d)$. Let $x_1,\dots,x_d$ inverse images of $a_1,\dots,a_d$ forming a part of a regular system of parameters in R . Let $\bar{R} = R/\underline{x}R$ and $\bar{I} = I+\underline{x}R/\underline{x}R$. Since $\bar{I} \subset \bar{M}^n$ we get:

$$e(A) = e(R/I+\underline{x}R) = (\bar{R}/\bar{I}) = 1 + \sum_{i\geq 1}^{n-1} \bar{M}^i/\bar{M}^{i+1} + l(\bar{M}^n/\bar{I}) \quad ,$$

hence (since $\bar{R}$ is regular)

$$(*) \quad e(A) = \binom{n-1+r-d}{r-d} + l(\bar{M}^n/\bar{I}) \quad ,$$

i.e. $e(A) \geq n-1+r-d$.

Therefore $e(A) = n$ if and only if $r-d = 1$ (and $\bar{I} = \bar{M}^n$) , i.e. I is principal in our case.

Question:

Do we get any additional information from the fact that $\bar{I} = \bar{M}^n$?

The following example shows that $e(A) = \mathrm{ord}(I)$ is essential.

Example 2.2:

Let $A = k[[x^2,xy,y^2,xz,yz,z]] \subset k[[x,y,z]]$, where x , y , z are indeterminates over a field k . A can be written as R/I with $R = k[[U,V,W,X,Y,Z]]$ and a suitable ideal I . A is a Cohen-Macaulay ring with $e(A) = 4 > \mathrm{ord}(I) = 2$. And A is not a hypersurface.

Proposition 2.3:

Let R/I be a non-Cohen-Macaulay Buchsbaum ring of $d = \dim R/I \geq 3$ satisfying S_2 . If $e(A) = d$ then $\mathrm{ord}(I) = 2$.

Proof:

Let $h^i = \dim(H^i_{\mathfrak{m}}(A))$ be the dimension of the local cohomology for $i = 0,\dots,d-1$. Let

$$J = \sum_i (y_1,\dots,\hat{y}_i,\dots,y_d) : y_i \quad ,$$

where $(y_1,\dots,y_d)$ is a minimal reduction of $\mathfrak{m}$.
Then [G1]

$$e(A) = 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot h^i + l(\mathfrak{m}/J) \quad .$$

By assumption $h^0 = h^1 = 0$ and $e(A) = d$. Therefore we get:

(1) $h^i = 0$ for $i \neq 2, d-1, d$,

(2) either $h^{d-1} = 1$ or $h^2 = 1$,

(3) $l(\mathfrak{m}/J) = 0$.

The last property implies that the reduction exponent of $\mathfrak{m}$ is two (i.e. $\mathfrak{m}^2 = \underline{y}\cdot\mathfrak{m}$ for suitable $y_1,\dots,y_d$) , hence $\operatorname{ord}(I) = 2$, since $\underline{y} \neq \mathfrak{m}$.

Remark:
In the case of proposition 2.3 the invariant $I(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot h^i$ of the Buchsbaum ring is 1 .
Since $\mathfrak{m}^2 = \underline{y}\mathfrak{m}$, we obtain [G1]:
$\operatorname{emb}(A) = e(A) + d - 1 + I = 2d$, i.e. $r - d = d$.
Moreover $e(A) = e(\underline{y}A) = l(R/I+\underline{\tilde{y}}R) - 1$, where $\underline{\tilde{y}} = (\tilde{y}_1,\dots,\tilde{y}_d)$ are inverse images of $y_1,\dots,y_d$ in R .
Therefore we have

$$d = e(A) = \binom{n-1+d}{d} + l(\bar{M}^n/\bar{I}) - 1 \quad ,$$

which also implies $n = \operatorname{ord}(I) = 2$, and $\bar{M}^2 = \bar{I}$.

Corollary 2.4:
Let R/I be a Buchsbaum ring satisfying S_2 . Assume that $e(R/I) = \dim(R/I) \geq 3$. Then we have:

a) If $\operatorname{ord}(I) \geq 3$, then R/I is Cohen-Macaulay.
b) If $\operatorname{ord}(I) = e(R/I)$, then R/I is a hypersurface.

Remark:

Statement b) of corollary 2.4 follows from a) and prop. 2.1. It is a special case of the following theorem 4.1; see also corollary 4.3. Note that in statement a) the multiplicity $e(R/I)$ can be strictly bigger than $ord(I)$. Therefore R/I need not be a hypersurface in this case.

3. Equimultiple ideals I

By $e(\underline{x},I,R)$ we denote the generalized multiplicity in the sense of [H-0-1], where $\underline{x}$ is a system of parameters mod I. $l(I)$ is the analytic spread of the ideal I.

Theorem 3.1:

Let I be an ideal of R with $ht(I) > 0$ and $ord(I) \geq 2$ such that:

(i) $ht(I) = l(I)$

(ii) $l(R_P/IR_P) = e(IR_P)$ for all $P \in Assh(R/I)$.

Then we have:

a) If $e(R/I) = ord(I) = n \Rightarrow ht(I) = 1$.

b) If I in a) is unmixed $\Rightarrow I = f \cdot R$ for some $f \in M^n \setminus M^{n+1}$.

Proof:

We put again $d = \dim R/I$ with $R/I = A$ and $r = \dim R$. We fix a system $y_1,\dots,y_d$ of parameters mod I which generates a minimal reduction of the maximal ideal $\mathfrak{m} \subset A$. So we have $e(A) = e(\underline{y},R/I)$, where $e(\underline{y},R/I) = e_R(\underline{y},R/I)$ in the sense of Northcott-Wright. For that given $\underline{y}$ we can construct by [H-0-1], lemma 1 a sequence $\underline{x} = (x_1,\dots,x_d)$ of superficial elements such that

$$(*) \quad \begin{aligned} &1)\ I + \underline{y}R = I + \underline{x}R \\ &2)\ e(\underline{y},I,R) = e(\underline{x},I,R) \\ &3)\ e(I+\underline{x}R) = e\left(\frac{I+\underline{x}R}{\underline{x}R}\right) \end{aligned}$$

Then using assumption (ii) and the associativity law for multiplicities we have:

$$e(\underline{y},I,R) = \sum_{P \in \mathrm{Assh}(R/I)} e(\underline{y},R/P) \cdot e(IR_P)$$

$$= \sum_{P} e(\underline{y},R/P) \cdot l(R_P/IR_P)$$

(**) $$= e(\underline{y},R/I)$$

$$= e(A) \quad ,$$

and in the same way :

$e(\underline{x},I,R) = e(\underline{x},R/I)$.

Assumption (i) implies $e(\underline{y},I,R) = e(I+\underline{y}R)$ by [H-0-2], theorem 0 . Using the properties 1) and 3) in (*) we get:

$$e(A) = e(I+\underline{y}R) = e(I+\underline{x}R) = e\left(\frac{I+\underline{x}R}{\underline{x}R}\right) \quad .$$

Now we choose a system $z_1,\ldots,z_t$ of parameters in I mod $\underline{x}R$, where $t = ht(I)$, such that:

$$e\left(\frac{I+\underline{x}R}{\underline{x}R}\right) = e\left(\frac{\underline{z}R+\underline{x}R}{\underline{x}R}\right) \quad .$$

[Note that $z_1,\ldots,z_t$ generate a minimal reduction of I , s. [H-0-1], Korollar, p. 655 .]

We put $\tilde{R} = R/\underline{x}R$, $\tilde{M} = M/\underline{x}R$ and $\tilde{I} = I+\underline{x}R/\underline{x}R$.

Since $e(A) = e(\underline{x},R/I)$ by (**) and property 2), we know that $\underline{x}$ mod I is a minimal reduction of $\mathfrak{m}$, hence $\tilde{R}$ is regular.

We denote by $\tilde{z}_1,\ldots,\tilde{z}_t$ the images of $z_1,\ldots,z_t$ in $\tilde{R}$.

Then $e(A) = e(\underline{\tilde{z}}\tilde{R},\tilde{R}) = l(\tilde{R}/\underline{\tilde{z}}\tilde{R})$.

Since $\underline{\tilde{z}}\tilde{R} \subset \tilde{M}^n$ and $\tilde{R}/\underline{\tilde{z}}\tilde{R}$ is Cohen-Macaulay, we obtain by (*) in section 2:

$$e(A) \geq \binom{(n-1)+r-d}{r-d} \geq n-1+r-d \quad .$$

Hence $e(A) = n$ if and only if $r-d = ht(I) = 1$, which proves a) and b) of theorem 3.1.

Corollary 3.2:

Let $P \neq M$ be an equimultiple prime ideal in R of height $P > 0$. If $e(R/P) = \mathrm{ord}(P)$, then $P = f \cdot R$ for some $f \in M^n \setminus M^{n+1}$.

Remark:

Since R is regular, one can show by [H-I], prop. 1.2, that this equimultiple prime ideal P is already generated by a regular sequence. Using this fact, proposition 2.1 yields the claim of the corollary too.

4. Buchsbaum ideals

A key strategy here will be to study the possible orders of I, if R/I is a non-Cohen-Macaulay Buchsbaum ring of depth > 0 and $\dim R/I \geq 2$. This will tell us what we need to know about I in case that $e(R/I) = \mathrm{ord}(I) \geq 3$.

Theorem 4.1:

Let I be an ideal of (R,M) with $e(R/I) = \mathrm{ord}(I) = n \geq 3$. Assume that $\mathrm{depth}(R/I) > 0$. If R/I is Buchsbaum, then $I = fR$ for some $f \in M^n \setminus M^{n+1}$.

Remark:

The assumption $n \geq 3$ is necessary in the above statement:

Example 4.2:

Let $R = k[[X_1,X_2,X_3,X_4]]$, where X_1,X_2,X_3,X_4 are indeterminates over a field k. Let $I = (X_1,X_2) \cap (X_3,X_4)$, hence $\mathrm{ord}(I) = 2$. It is well known that R/I is a Buchsbaum ring with $e(R/I) = 2$.

Proof of theorem 4.1:

Let $d = \dim(R/I) = \dim(A)$. Fix a minimal reduction $a_1,\dots,a_d$ of $\mathfrak{m} \subset A$, then $e(A/(a_1,\dots,a_{d-1})A) = e(A)$, since A is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Assh}(A/(a_1,\dots,a_{d-1})A)$ in our case. It is known that the 1-dimensional ring $A/(a_1,\dots,a_{d-1}) : a_d$ is Cohen-Macaulay. [For the proof of this fact the main point being the exact sequence

$$0 \to \frac{(a_1,\dots,a_{d-1}):a_d}{(a_1,\dots,a_{d-1})} \to \frac{A}{(a_1,\dots,a_{d-1})} \to A/(a_1,\dots,a_{d-1}):a_d \to 0$$

and as a consequence

$$0 \to \frac{(a_1,\dots,a_{d-1}):a_d}{(a_1,\dots,a_{d-1})} \to H^0_{\mathfrak{m}}\left(\frac{A}{(a_1,\dots,a_{d-1})}\right)$$

$$\to H^0_{\mathfrak{m}}(A/((a_1,\dots,a_{d-1}):a_d)) \to 0 \quad .$$

Since

$$l_A(H^0_{\mathfrak{m}}(A/(a_1,\dots,a_{d-1})A) = l_A\left(\frac{(a_1,\dots,a_{d-1}):a_d}{(a_1,\dots,a_{d-1})}\right) \quad ,$$

which comes again from the Buchsbaum-property of A [G2], we compute $l(H^0_{\mathfrak{m}}(A/(a_1,\dots,a_{d-1}):a_d)) = 0$, which gives the claim.] Hence

$$\begin{aligned} e(A) &= e(A/(a_1,\dots,a_{d-1})A) \\ &= e(a_d;A/(a_1,\dots,a_{d-1})A) \\ &= e(a_d;A/(a_1,\dots,a_{d-1}):a_d) \\ &= l(A/(a_1,\dots,a_{d-1}):a_d+(a_d)A) \quad ; \end{aligned}$$

i.e. in our case

$$n = l(A/((a_1,\dots a_{d-1}):a_d+(a_d)A)) \quad .$$

Assuming that A is not Cohen-Macaulay, we have $\dim(R) = \operatorname{emb}(A) \geq d+2$. Then we can extend $a_1,\dots,a_d$ to a system of generators $a_1,\dots,a_d,a_{d+1},a_{d+2},\dots,a_r$ of $\mathfrak{m}$, such that the inverse images $x_1,\dots,x_r$ form a minimal system of generators of M . We put:

$y = x_{d+1}$ and $z = x_{d+2}$. Let $J = (I,x_1,\dots,x_{d-1}):x_d+x_dR$. Then $gr_{M/J}(R/J) = gr_M(R)/gr_M(J,R)$, where $gr_M(J,M) = \Sigma\,(J\cap M^n) + M^{n+1}/M^{n+1}$. From this we can quickly compute $l(gr_{M/J}(R/J)) = l(A/(a_1,\dots,a_{d-1}):a_d+a_dA)$. To find some lower bound of $l(A/(a_1,\dots,a_{d-1}):a_d+a_dA)$, we take the initialforms of y,z in $gr_M(R)$ and consider their images $\bar{y},\bar{z}$ in $gr_{M/J}(R/J)$. We claim that under the hypothesis made before, the elements

$$\{\bar{y}^i\cdot\bar{z}^j \mid i+j \leq n-2\}$$

are linearly independent over $R/M = k$: Suppose that for some l with $0 \leq l \leq n-2$ there is a non-trivial relation in $gr_{M/J}(R/J)$, say

$$\sum_{i+j=l\leq n-2} \alpha_{ij}y^iz^j = 0 \quad , \quad \alpha_{ij} \in k \quad .$$

This gives a non-trivial relation

$$\sum_{i+j=l} a_{ij}y^iz^j \in J\cap M^l + M^{l+1}$$

where at least one of the a_{ij} is a unit in R and moreover $n = \mathrm{ord}(I) \geq l+2$. Since $J = (I, x_1, \ldots, x_{d-1}) : x_d + x_d R$ we have

$$x_d(\Sigma\, a_{ij} y^i z^j) = t + \sum_{i=1}^{d-1} b_i x_i + x_d^2 \cdot c$$

for some $t \in I$ and $c, b_i \in R$. Then

$$w = x_d(\Sigma\, a_{ij} y^i z^j - c x_d) - \sum_{i=1}^{d-1} b_i x_i \in I + M^{l+2} \subset M^{l+2} \quad .$$

The elements $x_1, \ldots, x_d$ form a part of a regular system of parameters in R , hence

$$w \in (x_1, \ldots, x_d) \cap M^{l+2} = (x_1, \ldots, x_d) M^{l+1} \quad .$$

This implies

$$\Sigma\, a_{ij} y^i z^j - c x_d - f \in (x_1, \ldots, x_{d-1}) R$$

for some $f \in M^{l+1}$, which is impossible since one of the a_{ij} is a unit. This proves the claim.

Note that there are $t+1$ elements of degree t among the $\bar{y}^i \bar{z}^j$, where $0 \leq t \leq n-2$, i.e. we have $1+2+\ldots+(n-1)$ independent elements $\bar{y}^i \bar{z}^j$ with $i+j \leq n-2$, hence

$$n = l(A/(a_1, \ldots, a_{d-1}) : a_d + a_d A) \geq \frac{n(n-1)}{2}$$

i.e. under the assumptions of the theorem and the hypothesis, that A is not Cohen-Macaulay, we get $n = 3$. Now let $K = \Sigma_{i=1}^{d} (a_1, \ldots, \check{a}_i, \ldots, a_d) : a_i$, then [G-1]

$$(*) \qquad 3 = e(A) = 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot h^i + l(\mathfrak{m}/K) \quad .$$

Since $\mathrm{ord}(I) \geq 3$ by assumption, we have $(a_1, \ldots, \check{a}_i, \ldots, a_d) : a_d \setminus (a_1, \ldots, \check{a}_i, \ldots, a_d) \in \mathfrak{m}^2$, i.e. $K \subset (a_1, \ldots, a_d) + \mathfrak{m}^2$, hence:

$l(\mathfrak{m}/K) \geq l(\mathfrak{m}/(a_1, \ldots, a_d + \mathfrak{m}^2) = \mathrm{emb}(A/(a_1, \ldots, a_d)) \geq 2$.
Therefore we conclude from (*):

$$3 \geq 1 + 2 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot h^i \quad ,$$

i.e. $h^1 = h^2 = \ldots = h^{d-1} = 0$. Also $h^0 = 0$, since depth $A > 0$. So A would be Cohen-Macaulay, which contradicts the hypothesis that A is not.
This means that under the assumption of theorem 4.1, A must be Cohen-Macaulay. Then proposition 2.1 implies the claim of the theorem.

The following corollary to theorem 4.1 is closely related to corollary 2.4.

Corollary 4.3:
Let I be an ideal of the regular ring (R,M) . Assume that $\operatorname{ord}(R/I) \geq 2$. Then the following conditions are equivalent:

(1) R/I is a hypersurface and $e(R/I) \leq d$
(2) $\mathcal{R}(\mathfrak{m}) := \bigoplus_{n \geq 0} \mathfrak{m}^n$ is Cohen-Macaulay and $e(R/I) = \operatorname{ord}(I)$.

Proof:
(1) $\Rightarrow$ (2): Assumption (1) implies that $\mathcal{R}(\mathfrak{m})$ is Cohen-Macaulay by [GHO], Cor. 5.5.
(2) $\Rightarrow$ (1): Since $\mathcal{R}(\mathfrak{m})$ is Cohen-Macaulay, A is Buchsbaum satisfying S_2 by [I-1]. Now we consider two cases:

Case 1: $\operatorname{ord}(I) \geq 3$. Then R/I is a hypersurface by theorem 4.1, and $e(R/I) \leq d$ by [GHO], Cor. 5.5.
Case 2: $\operatorname{ord}(I) = 2$. Then by the multiplicity formula for the Buchsbaum ring $A = R/I$ and the fact that A satisfies S_2 , one knows that A must be Cohen-Macaulay.

Example 4.4:
$R = k[[X_1,X_2,X_3,Y_1,Y_2,Y_3]]$, k a field,
$I = (X_1Y_1+X_2Y_2+X_3Y_3, (Y_1,Y_2,Y_3)^2)$.
Then $e(R/I) = d = 3$, $\operatorname{ord}(I) = 2$ and $\mathcal{R}(\mathfrak{m})$ is Cohen-Macaulay, but R/I is not a hypersurface.

Remark 4.5:
The condition " $\mathcal{R}(\mathfrak{m})$ is Cohen-Macaulay" in Cor. 4.3 cannot be replaced by "Proj($\mathcal{R}(\mathfrak{m})$) is Cohen-Macaulay" as

we can see from the previous example 1.1, where $e(R/I) = d = ord(I) = 2$ and $Proj(\mathcal{R}(m))$ is Cohen-Macaulay.

5. Graded rings with (S_2) .

Let $R = k[X_1,\ldots,X_r]$ be a homogeneous graded polynomial ring over an algebraically closed field k . Let M be the maximal homogeneous ideal of R . For an homogeneous ideal $I = \bigoplus_{n\geq 0} I_n$ of R we define

$e(R/I) = e(R_M/IR_M)$ and $ord(I) = \min\{n \mid I_n \neq 0\}$.
Since $ord(I) = ord(IR_M)$, the condition $e(R/I) = ord(I)$ is equivalent to the condition $e(R_M/IR_M) = ord(IR_M)$ for the local ring R_M .

Theorem 5.1:

Let I be an homogeneous ideal in R with $e(R/I) = ord(I) = n \geq 1$. Assume that R/I satisfies S_2 . Then R/I is a hypersurface.

Proof:

Since R/I is catenarian and satisfies S_2 , I is unmixed by [Gr], 5.10.9. We may assume that $X_1,\ldots,X_d$ is a homogeneous system of parameters mod I . We put $S = k[X_1,\ldots,X_d]$. Then $A = R/I$ is a finite S-module and $e(R/I) = rank_S(A)$, s. [Hu]. We want to show that $r = \dim R = d+1$.

Case 1: R/I is not a domain. Assume that $r \geq d+2$. Put $Y = X_{d+1}$ and $Z = X_{d+2}$. We consider the following rings

$$B = k[X_1,\ldots,X_d,Y]/I \cap k[X_1,\ldots,X_d,Y]$$
$$C = k[X_1,\ldots,X_d,Z]/I \cap k[X_1,\ldots,X_d,Z] \quad .$$

It is well known that any unmixed ideal J in a polynomial ring over a field k with $ord(J) = n > 0$ has a quotient ring R/J with $e(R/J) \geq n$. Therefore we have

$$n \leq e(B) \leq e(A) = n$$
$$n \leq e(C) \leq e(A) = n \quad ,$$

i.e. $e(B) = e(C) = n$. Hence, regarding the multiplicities $e(B)$, $e(C)$ and $e(A)$ as the S-ranks of $k[X_1,\ldots,X_d,Y]$, $k[X_1,\ldots,X_d,Z]$ and R respectively, we see that

$$(*) \quad A \otimes_S Q(S) \simeq B \otimes_S Q(S) \simeq C \otimes_S Q(S) \quad ,$$

where $Q(S)$ is the quotient field of $S = k[X_1,\ldots,X_d]$. Let P be a minimal prime of $I \subset R$. Since $\mathfrak{a}_1 = k[X_1,\ldots,X_d,Y] \cap I$ and $\mathfrak{a}_2 = k[X_1,\ldots,X_d,Z] \cap I$ are unmixed homogeneous ideals of height 1 , we get

$$P \cap k[X_1,\ldots,X_d,Y] = f(Y)\cdot k[X_1,\ldots,X_d,Y]$$

and

$$P \cap k[X_1,\ldots,X_d,Z] = g(Z)\cdot k[X_1,\ldots,X_d,Z] \quad ,$$

where

$$f(Y) = Y^v + a_1Y^{v-1} + \ldots + a_v$$

$$g(Z) = Z^m + b_1Z^{m-1} + \ldots + b_m$$

with $m,v < n = \mathrm{ord}(I)$. [This strict inequality comes from the fact that $I \neq P$ in case 1; in particular: $\mathrm{ord}(\mathfrak{a}_1) = \sum_i \mathrm{ord}(f_i(Y))$, where the $f_i(Y)$ generate the associated primes of $\mathfrak{a}_1$, i.e. $n \leq \mathrm{ord}(\mathfrak{a}_1) < \mathrm{ord}(f_i(Y))$ for each i.] Since $g(Z) \bmod I$ is in C, we conclude from (*) that there is a non-zero element $s \in S$ such that

$$(**) \quad s\cdot g(Z) \equiv c_lY^l + c_{l-1}Y^{l-1} + \ldots + c_o \bmod I$$

for some $c_i \in S$ and $i = 0,\ldots,l$. Taking the isomorphism

$$(k[X_1,\ldots,X_d,Z] \,/\, I \cap k[X_1,\ldots,X_d,Z]) \otimes_S Q(S) \simeq R/I \otimes_S Q(S)$$

modulo P, we get via (**) (up to isomorphism):

$$\begin{aligned} R/P \otimes_S Q(S) &= (k[X_1,\ldots,X_d,Z] \,/\, P \cap k[X_1,\ldots,X_d,Z]) \otimes_S Q(S) \\ &= Q(S)[Z] \,/\, s\cdot g(Z)\cdot Q(S)[Z] \\ &= Q(S)[Y] \,/\, (c_lY^l+\ldots+c_o)Q(S)[Y] \quad , \end{aligned}$$

i.e. $l = m = v$.

Then we have

$$c_mY^m + \ldots + c_m \in P \cap S[Y] = f(Y)\cdot S[Y] \quad .$$

Hence for some homogeneous element $t \in S$, $t \neq 0$, we get

$$s\cdot g(Z) \equiv t\cdot f(Y) \bmod I \quad ,$$

where $\deg s = \deg t$. Note that every non-zero homogeneous element of S is a non-zero divisor on R/I . Therefore we may assume that s and t have no common divisor. If $\deg s = \deg t = 0$, the element $s \cdot g(Z) - t \cdot f(Y)$ is of degree $m < n$ in I , which contradicts to $\mathrm{ord}(I) = n$. If $\deg s = \deg t > 0$, then the images of s and t mod I form a regular sequence in R/I (since R/I satisfies S_2), hence $f(Y) \equiv s \cdot h \bmod I$ for some $h \in R$. But this gives again an element of degree $m < n$ in I . Therefore we have $r = \dim R = d+1$ in case 1.

<u>Case 2</u>: $R/P = k[x_1,\ldots,x_r]$ is a domain. Since k is algebraically closed, we may assume by [A] that $x_1,\ldots,x_{r-1}$ generate a prime ideal $\mathfrak{g} \in k[x_1,\ldots,x_r]$ with $\mathrm{ht}(\mathfrak{g}) = r-1$. For $D = k[x_1,\ldots,x_{r-1}] \subset R/I$ we get by [A], 12.3.4:

(1) $e(D) = e(R/P)$ if $\dim D \lneqq d$,

(2) $e(D) < e(R/P)$ if $\dim D = d$.

Since $D = k[X_1,\ldots,X_{r-1}]/q$ with $q \subset P$ prime and $\mathrm{ord}(q) \geq \mathrm{ord}(P) = n$, we know that $e(D) \geq n = e(R/P)$. Therefore (2) cannot occur. So we have $\dim D < d$, i.e. x_r is algebraically independent over D . Moreover we have $\mathrm{ord}(q) = \mathrm{ord}(P) = n$. Therefore we can use induction on r since the assertion is clear for $r = 3$.

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