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WITTEN'S D_4 INTEGRABLE HIERARCHIES CONJECTURE

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Abstract. We prove that the total descendant potential functions of the theory of Fan-Jarvis-Ruan-Witten for D_4 with symmetry group $\langle J\rangle$ and D_4^T with symmetry group G_{max}, respectively, are both tau-functions of the D_4 Kac-Wakimoto/Drinfeld-Sokolov hierarchy. This completes the proof, begun in [FJR], of the Witten Integrable Hierarchies Conjecture for all simple (ADE) singularities.

Contents

1. Introduction

Almost twenty years ago, Witten proposed a sweeping generalization of his famous conjecture connecting the KdV hierarchy to intersection numbers of the Deligne-Mumford stack of stable curves. The package includes (i) a first order nonlinear elliptic PDE (Witten equation) to replace the $\bar{\partial}$-equation in the Landau-Ginzburg setting; (ii) a conjecture asserting that the total potential function of an ADE-Landau-Ginzburg orbifold $(W, \langle J\rangle)$ is a tau-function of the corresponding integrable hierarchy. Here, $J = (\exp(2\pi i q_1), \dots, \exp(2\pi i q_N))$ is the so-called *exponential grading operator*, where the q_i are the weights of the quasi-homogeneous polynomial W. In a series of papers, Fan-Jarvis-Ruan have constructed the theory Witten expected and solved the conjecture for the D and E-cases except D_4, while the A_n-case was solved earlier by Faber-Shadrin-Zvonkin [FSZ].

But the story is still incomplete, and a resolution of the conjecture for the D_4 and D_4^T cases is needed. This is the main purpose of this article. When we say D_4, we mean the singularity $D_4 := x^3 + xy^2$. Recall that its exponential grading operator is $J = (\exp(2\pi i/3), \exp(2\pi i/3)$. Its dual or transpose singularity is $D_4^T = x^3y + y^2$, and the exponential grading operator of D_4^T is $J^T = (\exp(\pi/3), -1)$. Let $\mathscr{D}^{FJRW}_{D_4,\langle J\rangle}$ and $\mathscr{D}^{FJRW}_{D_4,G_{max}}$ (see notation in next section), denote the respective total descendant potential functions of

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the theory of Fan-Jarvis-Ruan-Witten for D_4 and D_4^T with symmetry groups $\langle J\rangle$, and G_{max}. The main theorem of this paper is the following.

Theorem 1.1.

(1) $\mathscr{D}^{FJRW}_{D_4,\langle J\rangle}$ *is a tau-function of the* D_4 *Kac-Wakimoto/Drinfeld-Sokolov hierarchy.*
(2) $\mathscr{D}^{FJRW}_{D_4,G_{max}}$ *is a tau-function of the* D_4 *Kac-Wakimoto/Drinfeld-Sokolov hierarchy.*

To explain the difficulty in the case of D_4 and D_4^T with symmetry group $\langle J\rangle$ and G_{max}, respectively, let's briefly review the proof in [FJR] of the conjecture for the other simple singularities. The idea of the proof is to identify the theory constructed by Fan, Jarvis, and Ruan for a given singularity W with the Saito-Givental theory of a related singularity W'. The later has been shown to give a tau-function of the corresponding hierarchy [FGM]. The proof consists of two steps: (1) prove two reconstruction theorems to completely determine Fan-Jarvis-Ruan-Witten theory and the Saito-Givental theory from the pairing, the three-point correlators, and certain special four-point correlators; (2) compute the three-point and special four-point correlators explicitly and match them. The reconstruction theorems apply to D_4 and its dual singularity D_4^T as well. But the computation of the special four-point correlators is much more difficult for the cases of D_4 with symmetry group $\langle J\rangle$ and D_4^T with symmetry group G_{max} than it is for all the other ADE singularities.[*]

In all cases except these two, the relevant insertions are what we call *narrow* (called *Neveu-Schwarz* in [FJR, JKV]). An early lemma of Witten asserts that the Witten equation with exclusively narrow insertions has only the zero solution. In such a case, the problem under study can be reduced to an algebro-geometric one and the computation can be carried out in a straightforward manner.

But these two cases of D_4 and D_4^T are entirely different, and instead of being narrow, some of the insertions are *broad* (called *Ramond* in [FJR]). Therefore, the algebro-geometric reduction does not apply. The problem under study is a PDE-problem and we do not yet have techniques to solve it explicitly in the case of broad insertions.

In the case of D_4^T we must compute two special correlators. One of the two correlators has only narrow insertions and its value was already computed in [FJR]. The final result follows (specifically, all the A-side correlators match those on the B-side) if and only if the other special correlator vanishes. Although the correlator has broad insertions, its vanishing is fairly simple to prove.

But in the case of D_4, those techniques do not work. Again we need to prove that one special four-point correlator vanishes and that the other does not vanish. The vanishing part is again simple, but the remaining special four-point correlator has broad insertions and cannot be computed easily. To prove that it does not vanish, we need a new idea, namely, to look for other correlators with only narrow insertions and use those to reconstruct the correlator with broad insertions. Indeed, there is a unique primary correlator with purely narrow insertions—the unique highest-point (seven) correlator for the D_4-theory.

We completely describe the genus-zero primary potential in terms of the one special four-point correlator. We further show that this correlator vanishes if and only if the the unique seven-point correlator vanishes. This part of the argument uses the full strength of the WDVV-equation. Then, with some work, we compute the seven-point correlator using algebro-geometric methods, and show that it is not zero, as required.

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[*] Note that for D_4^T the symmetry group $\langle J^T\rangle$ is equal to the maximal admissible symmetry group G_{max} for this theory, but for D_4 the group $\langle J\rangle$ has index 2 in the maximal admissible symmetry group of D_4.

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2. BACKGROUND AND NOTATION

In this section, we briefly review the theory of [FJR] for D_4 and D_4^T to set up the notation. Recall that for each quasi-homogeneous polynomial $W \in \mathbb{C}[x_1, \dots, x_N]$ of weights $q_1, \dots, q_N$, the *exponential grading operator* $J := (\exp(2\pi i q_1), \dots, \exp(2\pi i q_N))$ lies in the group $\mathrm{Aut}(W)$ of diagonal matrices γ such that $W(\gamma x) = W(x)$. Given any non-degenerate W and any subgroup $G \leq \mathrm{Aut}(W)$ such that $J \in G$, the results of [FJR] provide a cohomological field theory $(\mathscr{H}_W^G, \langle\,,\rangle^W, \{\Lambda_{g,k}^W\}, \mathbf{1})$ with flat identity.

The state space $\mathscr{H}_W^G$ is defined as follows. For each $\gamma \in G$, let $\mathbb{C}_\gamma^N$ be the fixed point set of γ, let N_γ denote the dimension of $\mathbb{C}_\gamma^N$, and let $W_\gamma = W|_{\mathbb{C}_\gamma^N}$. Let $\mathscr{H}_{\gamma,G}$ be the G-invariants of the middle-dimensional relative cohomology

$$\mathscr{H}_\gamma = H^{mid}(\mathbb{C}_\gamma^N, (\Re e W)^{-1}(M, \infty), \mathbb{C})^G$$

of $\mathbb{C}_\gamma^N$ for $M \gg 0$, as described in Section 3 of [FJR]. Wall's theorem [Wa1, Wa2] states that the cohomology group $H^{mid}(\mathbb{C}_\gamma^N, (\Re e W)^{-1}(M, \infty), \mathbb{C})$ is isomorphic, as a graded G-module, to the space $\mathscr{Q}_{W_\gamma}\omega_\gamma$, where $\mathscr{Q}_{W_\gamma}$ is the Milnor ring (local algebra) of W_γ and $\omega_\gamma = dx_{i_1} \wedge \cdots \wedge dx_{i_{N_\gamma}}$ is the canonical volume form on $\mathbb{C}^{N_\gamma}$. For computational purposes, it is generally much easier to use the Milnor ring than the middle cohomology, so we will assume from now on that an isomorphism

$$\mathscr{H}_\gamma = H^{mid}(\mathbb{C}_\gamma^N, (\Re e W)^{-1}(M, \infty), \mathbb{C})^G \cong \left(\mathscr{Q}_{W_\gamma}\omega_\gamma\right)^G \tag{1}$$

has been chosen for each γ, once and for all.

The *state space* of our theory is the sum

$$\mathscr{H}_{W,G} = \bigoplus_{\gamma \in G} \mathscr{H}_\gamma.$$

The state space $\mathscr{H}_{W,G}$ admits a grading and a non-degenerate pairing $\langle\,\rangle^W$. The pairing is essentially the residue pairing on the underlying Milnor rings $\mathscr{Q}_{W_\gamma}$, but with the elements of $\mathscr{H}_\gamma$ only pairing with elements of $\mathscr{H}_{\gamma^{-1}} \cong \mathscr{H}_\gamma$. The grading is more subtle, as we now describe.

Definition 2.1. The *central charge* of the singularity W_γ is denoted $\hat{c}_\gamma$:

$$\hat{c}_\gamma := \sum_{i:\Theta_i^\gamma = 0} (1 - 2q_i). \tag{2}$$

Definition 2.2. Suppose that $\gamma = (e^{2\pi i \Theta_1^\gamma}, \dots, e^{2\pi i \Theta_N^\gamma})$ for rational numbers $0 \leq \Theta_i^\gamma < 1$.

We define the *degree shifting number*

$$\iota_\gamma = \sum_i (\Theta_i^\gamma - q_i) \tag{3}$$

$$= \frac{\hat{c}_W - N_\gamma}{2} + \sum_{i:\Theta_i^\gamma \neq 0} (\Theta_i^\gamma - 1/2) \tag{4}$$

$$= \frac{\hat{c}_\gamma - N_\gamma}{2} + \sum_{i:\Theta_i^\gamma \neq 0} (\Theta_i^\gamma - q_i). \tag{5}$$

For a class $\alpha \in \mathscr{H}_\gamma$, we define

$$\deg_W(\alpha) = \deg(\alpha) + 2\iota_\gamma = N_\gamma + 2\iota_\gamma. \tag{6}$$

The classes $\Lambda^W_{g,k} \in \operatorname{Hom}(\mathscr{H}_W^{\otimes k}, H^*(\overline{\mathscr{M}}_{g,k}))$ satisfy the usual axioms of a cohomological field theory with flat identity, including symmetry, and the composition axioms (see [FJR, §4.2] for details). Here $\overline{\mathscr{M}}_{g,k}$ is the stack of stable, k-pointed, genus-g curves.

When an element $\gamma \in G$ fixes only $0 \in \mathbb{C}^N$, we say that the sector $\mathscr{H}_\gamma$ is *narrow* (called Neveu-Schwarz in [FJR, JKV]), and when the element γ fixes a non-trivial subspace $\mathbb{C}^{N_\gamma} \subset \mathbb{C}^N$ with $N_\gamma > 0$, we say that the sector $\mathscr{H}_\gamma$ is *broad* (called *Ramond* in [FJR]). For each narrow sector $\mathscr{H}_\gamma$, the isomorphism (1) defines an element $\mathbf{e}_\gamma$ as the image of $1 \in \mathbb{C} \cong \mathscr{H}_\gamma$ in $\mathscr{H}_\gamma$. The exponential grading operator J is narrow, and the isomorphism (1) can be chosen so that the element $\mathbf{e}_J$ is the flat identity $\mathbf{1}$. The elements $\mathbf{e}_\gamma$ also play an important role in one additional axiom that allows us to compute the classes $\Lambda^W_{g,k}$ in some special cases, namely the *Concavity Axiom*, which we now briefly review.

As mentioned above, in the case that all the insertions are narrow, the construction of the classes $\Lambda^W_{g,k}$ reduces to an algebro-geometric problem on the moduli of W-curves $\mathscr{W}_{W,g,k}$. The universal W-structure on the universal W-curve $\mathscr{C} \xrightarrow{\pi} \mathscr{W}_{W,g,k}$ corresponds to a choice of orbifold line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_N$ on $\mathscr{C}$ with certain properties described in [FJR]. In the special case that these bundles are *concave*, i.e., when the pushforward $\pi_*\left(\bigoplus_{i=1}^N \mathscr{L}_i\right) = 0$, then the class $\Lambda^W_{g,k}(\mathbf{e}_{\gamma_1}, \ldots, \mathbf{e}_{\gamma_k})$ is given by

$$\Lambda^W_{g,k}(\mathbf{e}_{\gamma_1}, \ldots, \mathbf{e}_{\gamma_k}) = \frac{|G|^g}{\deg(st)} st_* \left(c_{top} \left(R^1\pi_* \bigoplus_{i=1}^N \mathscr{L}_i \right)^* \right),$$

where $st : \mathscr{W}_{W,g,k} \longrightarrow \overline{\mathscr{M}}_{g,k}$ is the canonical morphism from the stack of W-curves to the stack of stable curves, where c_{top} is the top Chern class, or Euler class, and where $\left(R^1\pi_* \bigoplus_{i=1}^N \mathscr{L}_i\right)^*$ is the dual of the bundle $R^1\pi_* \bigoplus_{i=1}^N \mathscr{L}_i$.

Another essential property of the classes $\Lambda^W_{g,k}$ is that the cohomology classes have complex dimension equal to $D + \sum \frac{1}{2} N_\gamma$, where $-D$ is the sum of the indices of the W-structure bundles:

$$D := -\sum_{i=1}^N \operatorname{index}(\mathscr{L}_i) = \hat{c}_W(g-1) + \sum_{j=1}^k \iota_{\gamma_j}, \tag{7}$$

and N_γ is the complex dimension of the fixed locus $(\mathbb{C}^N)^\gamma$ of γ.

Definition 2.3. For the cohomological field theory $(\mathscr{H}_W^G, \langle\,,\rangle^W, \{\Lambda_{g,k}^W\}, \mathbf{1})$, we define correlators in the standard manner, as

$$\langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_k}(\alpha_k)\rangle_g^W := \int_{[\overline{\mathscr{M}}_{g,k}]} \Lambda_{g,k}^W(\alpha_1, \ldots, \alpha_k) \prod_{i=1}^k \psi_i^{l_i}.$$

The genus-zero, three-point correlators of any cohomological field theory define an associative multiplication $\star$ on the state space, making the state space into a Frobenius algebra. In our case the new (degree-shifted) grading on $\mathscr{H}_{W,G}$ is also compatible with the multiplication, making $\mathscr{H}_{W,G}, \star$ into a graded Frobenius algebra.

The main selection rules in this theory are

- $\langle \alpha_1, \ldots, \alpha_k\rangle_g^W = 0$ unless
$$\hat{c}(g-1) + \sum_{i=1}^k \frac{1}{2}\deg_W(\alpha_i) = \dim(\overline{\mathscr{M}}_{g,k}) = 3g-3+k, \tag{8}$$
- $\langle \alpha_1, \ldots, \alpha_k\rangle_g^W = 0$ unless for each of the coarse line bundles $|\mathscr{L}_j|$ underlying the W-structure, the degree
$$\deg(|\mathscr{L}_j|) = q_j(2g-2+k) - \sum_{\ell=1}^k \Theta_j^{\gamma_\ell} \tag{9}$$
is integral.

Furthermore, the selection rule (9) must hold on each irreducible component of a stable curve.

Remark 2.4. When the group $G \leq \mathrm{Aut}(W)$ is not equal to the maximal group $\mathrm{Aut}(W)$ of diagonal symmetries, then $\mathrm{Aut}(W)$ acts on the A-model state space, and the correlators are all invariant under this action. In particular, this property forces many correlators to vanish.

Definition 2.5. Let $\{\alpha_0, \ldots, \alpha_s\}$ be a basis of the state space $\mathscr{H}_W$ such that $\alpha_0 = \mathbf{1} := \mathbf{e}_J$, and let $\mathbf{t} = (\mathbf{t}_0, \mathbf{t}_1, \ldots)$ with $\mathbf{t}_l = (t_l^{\alpha_0}, t_l^{\alpha_1}, \ldots, t_l^{\alpha_s})$ be formal variables. Denote by $\Phi^W(\mathbf{t}) \in \lambda^{-2}\mathbb{C}[[\mathbf{t}, \lambda]]$ the (large phase space) potential of the theory:

$$\Phi^W(\mathbf{t}) := \sum_{g\geq 0} \Phi_g^W(\mathbf{t}) := \sum_{g\geq 0} \lambda^{2g-2} \sum_k \frac{1}{k!} \sum_{l_1,\ldots,l_k} \sum_{\alpha_{i_1},\ldots,\alpha_{i_k}} \langle \tau_{l_1}(\alpha_{i_1}) \cdots \tau_{l_k}(\alpha_{i_k})\rangle_g^W t_{l_1}^{\alpha_{i_1}} \cdots t_{l_k}^{\alpha_{i_k}}.$$

Theorem 2.6 (See [FJR, Thm 4.2.8])**.** *The potential $\Phi^W(\mathbf{t})$ satisfies analogues of the string and dilaton equations and the topological recursion relations.*

3. Frobenius Algebras

3.1. **Frobenius Algebra for the B-model of D_4.** The Frobenius Algebra for D_4 is the local algebra (Milnor ring)

$$\mathscr{Q}_{D_4} = \mathbb{C}[X,Y]/(3X^2+Y^2, 2XY)$$

with the residue pairing. That is, if

$$fg = \alpha\frac{\operatorname{Hess} D_4}{\mu} + \text{lower order terms} = \alpha\frac{24X^2}{4} + \text{l.o.t} = \alpha\frac{-8Y^2}{4} + \text{l.o.t},$$

then

$$\langle f, g\rangle = \alpha.$$

We have $\hat{c} = 2/3$ and $\deg(X) = \deg(Y) = 1/3$.

3.2. **Frobenius Algebra for the A-model of** (D_4^T, G_{max})**.** The singularity D_4^T is defined by the polynomial $W = x^3y + y^2$, and the exponential grading operator J^T is

$$J^T = (\xi, \xi^3), \qquad \text{where } \xi = \exp(2\pi i/6).$$

The element J^T has order 6 and generates the maximal group G_{max} of diagonal symmetries of D_4^T. It is straightforward to check that the set $\{\mathbf{e}_1, x^2\mathbf{e}_0, \mathbf{e}_3, \mathbf{e}_5\}$ is a basis of $\mathscr{H}^{G_{max}}_{D_4^T}$, where

$$\begin{aligned} \mathbf{e}_0 &:= dx\wedge dy \in H^{mid}(\mathbb{C}^N_{J^n}, W^\infty_{J^n}, \mathbb{Q}) = H^{mid}(\mathbb{C}^N_{J^0}, W^\infty_{J^0}, \mathbb{Q}), \\ \mathbf{e}_1 &= 1 \in H^{mid}(\mathbb{C}^N_{J^1}, W^\infty_{J^1}, \mathbb{Q}) \cong \mathbb{C} \\ \mathbf{e}_3 &= 1 \in H^{mid}(\mathbb{C}^N_{J^3}, W^\infty_{J^3}, \mathbb{Q}) \cong \mathbb{C} \\ \mathbf{e}_5 &= 1 \in H^{mid}(\mathbb{C}^N_{J^5}, W^\infty_{J^5}, \mathbb{Q}) \cong \mathbb{C}. \end{aligned}$$

The Frobenius algebra structure on $\mathscr{H}_{D_4^T, G_{max}}$ was determined in [FJR, §5.2.4], where it was shown that the structure is given by an isomorphism to $\mathscr{Q}_{D_4}$ as follows: $\mathbb{C}[X, Y]/(3X^2 + Y^2, 2XY) \longrightarrow \mathscr{H}_{D_4^T, G_{max}}$, with

$$\begin{aligned} 1 &\mapsto \mathbf{e}_1 & Y &\mapsto 3\alpha x^2\mathbf{e}_0 \\ X &\mapsto \alpha\mathbf{e}_3 & X^2 &\mapsto \alpha^2\mathbf{e}_5, \end{aligned}$$

where $\alpha^2 = 1/6$.

As in the previous case, we have $\hat{c} = 2/3$ and $\deg(X) = \deg(Y) = 1/3$.

3.3. **Frobenius Algebra for the A-model of** $(D_4, \langle J\rangle)$**.** The singularity D_4 is defined by the polynomial $W = x^3 + xy^2$, and the exponential grading operator J is

$$J = (\xi^2, \xi^2), \qquad \text{where } \xi = \exp(2\pi i/6).$$

The element J has order 3 in the group $\mathrm{Aut}(D_4) = \langle\lambda\rangle \cong \mathbb{Z}/6\mathbb{Z}$, where $\lambda := (\xi^{-2}, \xi)$. And it is straightforward to check that the set $\{\mathbf{1}, x\mathbf{e}_0, y\mathbf{e}_0, \mathbf{e}_2\}$ is a basis of $\mathscr{H}^{\langle J\rangle}_{D_4}$, where

$$\begin{aligned} \mathbf{e}_0 &:= dx\wedge dy \in H^{mid}(\mathbb{C}^N_{J^n}, W^\infty_{J^n}, \mathbb{Q}) = H^{mid}(\mathbb{C}^N_{J^0}, W^\infty_{J^0}, \mathbb{Q}), \\ \mathbf{1} = \mathbf{e}_1 &= 1 \in H^{mid}(\mathbb{C}^N_{J^1}, W^\infty_{J^1}, \mathbb{Q}) \cong \mathbb{C} \\ \mathbf{e}_2 &= 1 \in H^{mid}(\mathbb{C}^N_{J^2}, W^\infty_{J^2}, \mathbb{Q}) \cong \mathbb{C}. \end{aligned}$$

The Frobenius algebra structure on $\mathscr{H}_{D_4, \langle J\rangle}$ was determined in [FJR, §5.2.4], where it was shown that the primitive classes are $x\mathbf{e}_0, y\mathbf{e}_0$. The D_4 case is unusual because the primitive classes are all broad.

It is shown in [FJR, §5.2.4] (and is straightforward to check directly) that the pairing is given by

$$\langle x\mathbf{e}_0, x\mathbf{e}_0\rangle = \frac{1}{6}, \langle x\mathbf{e}_0, y\mathbf{e}_0\rangle = 0, \langle y\mathbf{e}_0, y\mathbf{e}_0\rangle = \frac{-1}{2}. \tag{10}$$

We note that the "obvious" isomorphism of Frobenius algebras $\mathscr{H}_{D_4, \langle J\rangle} \longrightarrow \mathscr{Q}_{D_4} := \mathbb{C}[X, Y]/(3X^2 + Y^2, 2XY)$ given in [FJR, §5.2.4] does not turn out to be the right one for our purposes in this paper. Instead, we will use the following isomorphism of Frobenius algebras:

$$\begin{aligned} \mathbf{e}_1 &\mapsto \mathbf{1} & x\mathbf{e}_0 &\mapsto \frac{Y}{\sqrt{-3}} \\ \mathbf{e}_2 &\mapsto 6X^2 & y\mathbf{e}_0 &\mapsto \sqrt{-3}X. \end{aligned}$$

Via this isomorphism we also have,

$$\langle X,X\rangle = \frac{1}{6}, \quad \langle X,Y\rangle = 0, \quad \langle Y,Y\rangle = \frac{-1}{2}. \tag{11}$$

As in the previous two cases, we have $\hat{c} = 2/3$ and $\deg(X) = \deg(Y) = 1/3$.

Definition 3.1. Hereafter, we will take $\{\mathbf{1}, X, Y, X^2\}$ as the standard basis for all three cases: the A-models $\mathscr{H}_{D_4^T,G_{max}}$, and $\mathscr{H}_{D_4,\langle J\rangle}$, and the B-model $\mathscr{Q}_{D_4}$. The dual basis is $\{6X^2, 6X, -2Y, 6\mathbf{1}\}$.

4. Shared Properties of the Correlators and Potentials

In this section we will discuss some properties that all three of our theories share, including various relations among the primary correlators and the potentials.

From the Frobenius algebra structure, we can see immediately that in all three theories, all the three-point correlators vanish except $\langle X,X,\mathbf{1}\rangle = \langle X^2,\mathbf{1},\mathbf{1}\rangle = \frac{1}{6}$, and $\langle Y,Y,\mathbf{1}\rangle = -\frac{1}{2}$.

There is also a selection rule for both A- and B-model correlators, namely:

$$\langle \varkappa_1,\ldots,\varkappa_k\rangle = 0 \qquad \text{unless } \hat{c}(g-1) + \sum \deg(\varkappa_i) = 3(g-1)+k,$$

which, in all three of our cases, gives

$$\langle \varkappa_1,\ldots,\varkappa_k\rangle = 0 \qquad \text{unless } \sum \deg(\varkappa_i) = \frac{3k-7}{3}. \tag{12}$$

Moreover, we have

$$\langle \varkappa_1,\ldots,\varkappa_{k-1},\mathbf{1}\rangle = 0 \qquad \text{unless } k = 3. \tag{13}$$

Proposition 4.1. *For all three of our cases, the only genus-zero primary correlators that are not* a priori *forced to vanish by the selection rules (12) and (13) are*

$$\begin{gathered}\langle \mathbf{1},\mathbf{1},X^2\rangle, \langle \mathbf{1},Y,Y\rangle, \langle \mathbf{1},X,X\rangle,\\ \langle Y,Y,Y,X^2\rangle, \langle Y,Y,X,X^2\rangle, \langle Y,X,X,X^2\rangle, \langle X,X,X,X^2\rangle,\\ \langle Y,Y,X^2,X^2,X^2\rangle, \langle Y,X,X^2,X^2,X^2\rangle, \langle X,X,X^2,X^2,X^2\rangle,\\ \langle Y,X^2,X^2,X^2,X^2,X^2\rangle, \langle X,X^2,X^2,X^2,X^2,X^2\rangle,\\ \langle X^2,X^2,X^2,X^2,X^2,X^2,X^2\rangle.\end{gathered}$$

Proof. This is a straightforward computation. □

A key tool that we will need in this paper is the following generalization of the *Reconstruction Lemma* of [FJR].

Lemma 4.2. *For both the A- and B-models, given any genus-zero, k-point correlator of the form $\langle \gamma_1,\ldots,\gamma_{k-3},\alpha,\beta,\varepsilon\star\phi\rangle$, choose a basis $\{\delta_i\}$ of $\mathscr{H}_{W,G}$ such that $\delta_0 = \varepsilon\star\phi$, and let δ'_i be the dual basis with respect to the pairing (i.e., $\langle\delta_i,\delta'_j\rangle = \delta_{ij}$).*

The correlator $\langle \gamma_1,\ldots,\gamma_{k-3},\alpha,\beta,\varepsilon\star\phi\rangle$ can be rewritten as

$$\begin{aligned}\langle \gamma_1,\ldots,\gamma_{k-3},\alpha,\beta,\varepsilon\star\phi\rangle = &\sum_{k-3=I\sqcup J}\sum_{\ell}\langle \gamma_{i\in I},\alpha,\varepsilon,\delta_\ell\rangle\langle\delta'_\ell,\phi,\beta,\gamma_{j\in J}\rangle \\ &- \sum_{\substack{k-3=I\sqcup J\\ J\neq\emptyset}}\sum_{\ell}\langle \gamma_{i\in I},\alpha,\beta,\delta_\ell\rangle\langle\delta'_\ell,\phi,\varepsilon,\gamma_{j\in J}\rangle.\end{aligned} \tag{14}$$

Proof. This is a straightforward computation using WDVV and the definition of $\star$. □

Using the reconstruction lemma and a genus reduction argument, one can show the following:

Theorem 4.3. [FJR, Thm 6.2.10] *In both the A- and B-model, the total descendant potential function $\mathscr{D}$ for our three cases (Saito for D_4, FJRW for D_4^T, G_{max}, and FJRW for $D_4, \langle J\rangle$) are completely determined by the pairing, the three-point correlators (i.e., the Frobenius algebra structure) and the four-point correlators.*

Theorem 4.4. *In all three cases (Saito for D_4, FJRW for (D_4^T, G_{max}), and FJRW for $(D_4, \langle J\rangle)$) the four-point correlators $\langle Y, Y, Y, X^2\rangle$ and $\langle Y, X, X, X^2\rangle$ vanish.*

Proof. For the Saito Frobenius manifold this was computed in [FJR, Prop 6.4.4].

For the FJRW theory of (D_4^T, G_{max}), the selection rule of Equation (9) can be applied; namely, we will show that for the two correlators in question, the degree of the line bundle $|\mathscr{L}_x|$ is not integral, and hence the correlator must vanish. To do this, we need only the fact that $\Theta_x^{(J^T)^a} = a/6$, which is straightforward to check. We have for $\langle Y, X, X, X^2\rangle$

$$\begin{aligned}\deg(|\mathscr{L}_x|) &= q_x(2g-2+k) - \sum_{\ell=1}^{k} \Theta_x^{\gamma_\ell} \\ &= \frac{1}{6}(-2+4) - (3/6+3/6+0/6+5/6) = -\frac{3}{2} \notin \mathbb{Z}.\end{aligned}$$

This shows that the correlator $\langle Y, X, X, X^2\rangle$ must vanish. A similar computation show that the correlator $\langle Y, Y, Y, X^2\rangle$ must also vanish.

Finally, for the FJRW theory of $(D_4, \langle J\rangle)$ we note that the maximal group of symmetries contains an element $\lambda = (\xi^{-2}, \xi)$ which is not contained in $\langle J\rangle$. A simple computation gives the action of λ on the basis:

$$\begin{gathered}\lambda\mathbf{1} = \mathbf{1}, \quad \lambda X = Y, \quad \lambda X^2 = X^2, \\ \text{but} \quad \lambda Y = -Y.\end{gathered} \tag{15}$$

As observed in Remark 2.4, all correlators must be λ-invariant. This forces all correlators with an odd number of Y-insertions to vanish, as desired. □

Theorem 4.5. *Denote the four-point correlator*

$$a := \langle X, X, X, X^2\rangle,$$

For all three cases (Saito for D_4, FJRW for D_4^T, G_{max}, and FJRW for $D_4, \langle J\rangle$), the primary genus-zero potential has the following form

$$\Phi(\mathbf{t}) = \frac{1}{12}t_X^2 t_{\mathbf{1}} - \frac{1}{4}t_Y^2 t_{\mathbf{1}} + \frac{1}{12}t_{X^2} t_{\mathbf{1}}^2 + \frac{a}{6}t_X^3 t_{X^2} + \frac{3a}{2}t_X t_Y^2 t_{X^2} + \frac{a^2}{2}t_X^2 t_{X^2}^3 - \frac{3a^2}{2}t_Y^2 t_{X^2}^3 + \frac{3a^4}{70}t_{X^2}^7.$$

Proof. This follows by running the reconstruction lemma "in reverse." First we apply the Reconstruction Lemma (4.2) to the four-point correlator

$$\begin{gathered}\langle X, Y, Y, X^2\rangle = \sum_\delta \langle X, Y, \delta\rangle\langle \delta', X, Y, X\rangle + \\ \sum_\delta \langle X, Y, X, \delta\rangle\langle \delta', Y, X\rangle - \sum_\delta \langle X, X, X, \delta\rangle\langle \delta', Y, Y\rangle,\end{gathered}$$

where the sums run over a basis δ of the state space, and δ' is the element dual to δ. This yields

$$\langle X, Y, Y, X^2\rangle = -\langle X, X, X, X^2\rangle\langle 6, Y, Y\rangle = 3a.$$

Now we apply the Reconstruction Lemma (4.2) to the 5-point correlators:

$$\begin{aligned}\langle X, X, X^2, X^2, X^2\rangle &= \sum_l \langle X, X^2, X, \delta_l\rangle\langle\delta_l', X, X^2, X\rangle\\ &= \langle X^2, X, X, X\rangle\langle 6X, X, X, X^2\rangle + \langle X^2, X, X, Y\rangle\langle -2Y, X, X, X^2\rangle\\ &= 6a^2.\end{aligned}$$

A similar computation for $\langle Y, Y, X^2, X^2, X^2\rangle$ gives us

$$\langle X^2, X^2, X^2, Y, Y\rangle = -2\langle Y, Y, X, X^2\rangle^2 = -18a^2$$

and

$$\langle X^2, X^2, X^2, X, Y\rangle = 0.$$

Applying the Reconstruction Lemma to the six-point correlator $\langle X^2, X^2, X^2, X^2, X, X^2\rangle = -\frac{1}{3}\langle X^2, X^2, X^2, X^2, X, Y^2\rangle$ yields

$$\begin{aligned}-3\langle X^2, X^2, X^2, X^2, X, X^2\rangle &= \sum_l \langle X^2, X^2, X^2, Y, \delta_l\rangle\langle\delta_l', X, Y, X^2\rangle + \langle X^2, Y, Y, \delta_l\rangle\langle\delta_l', X, X^2, X^2, X^2\rangle\\ &\qquad - 2\langle X^2, X^2, X^2, X, \delta_l\rangle\langle\delta_l', Y, Y, X^2\rangle\\ &= 0,\end{aligned}$$

and a similar calculation shows that $\langle X^2, X^2, X^2, X^2, Y, X^2\rangle = 0$. Thus both the 6-point correlators are zero.

Now applying reconstruction to the seven-point correlator $\langle X^2, X^2, X^2, X^2, X^2, X^2, X^2\rangle$ gives us

$$\langle X^2, X^2, X^2, X^2, X^2, X^2, X^2\rangle = 6\langle X^2, X^2, X^2, X, X\rangle^2 = 6^3a^4.$$

Inserting these correlators into the potential

$$\Phi(\mathbf{t}) = \sum_{k\geq 3}\sum_{\varkappa_1,\ldots,\varkappa_k}\langle\varkappa_1,\ldots,\varkappa_k\rangle\frac{\prod_{i=1}^k t_{\varkappa_i}}{k!}$$

gives the desired result. □

Corollary 4.6. *The potential $\Phi^S_{D_4}$ for the Saito Frobenius manifold of $D_4 = x^3 + xy^2$ with primitive form $dx \wedge dy$ is*

$$\Phi^S_{D_4}(\mathbf{t}) = \frac{1}{12}t_X^2 t_{\mathbf{1}} - \frac{1}{4}t_Y^2 t_{\mathbf{1}} + \frac{1}{12}t_{X^2}t_{\mathbf{1}}^2 - \frac{1}{216}t_X^3 t_{X^2} - \frac{1}{24}t_X t_Y^2 t_{X^2} + \frac{1}{2592}t_X^2 t_{X^2}^3 - \frac{1}{864}t_Y^2 t_{X^2}^3 + \frac{1}{3919140}t_{X^2}^7.$$

Proof. In [FJR, Prop 6.4.4] we computed that for the primitive form $6dx \wedge dy$ we have $\langle\mathbf{1}, Y, Y\rangle = -3$, $\langle\mathbf{1}, X, X\rangle = \langle\mathbf{1}, \mathbf{1}, X^2\rangle = 1$, $\langle X, X, X, X^2\rangle = -\frac{1}{6}$, and $\langle Y, Y, X, X^2\rangle = -\frac{1}{2}$. Rescaling the primitive form by a non-zero β rescales the entire potential (and hence also the metric) by β. The choice of primitive form $dx \wedge dy$, corresponding to $\beta = \frac{1}{6}$, gives us the same metric and three-point correlators we computed earlier in this paper. Substituting the value of $a = -\frac{\beta}{6}$ gives the result. □

We also have the fundamental result of Frenkel-Givental-Milanov.

Theorem 4.7 ([GM, FGM]). *For any ADE-singularity W, the Saito-Givental (B-side) total descendant potential $\mathscr{D}^{SG}_W$ is a τ-function of the corresponding Drinfeld-Sokolov/Kac-Wakimoto ADE-hierarchy.*

Theorems 4.3 and 4.7 will allow us to prove the following lemma.

Lemma 4.8. *For both the FJRW theory of D_4^T with symmetry group G_{max} and the FJRW theory for D_4 with symmetry group $\langle J\rangle$, if the correlator $\langle X, X, X, X^2\rangle$ is non-zero, then the corresponding total descendant potential function $\mathscr{D}^{FJRW}_{D_4,J}$ or $\mathscr{D}^{FJRW}_{D_4^T,J^T}$ is a tau-function of the D_4 Kac-Wakimoto/Drinfeld-Sokolov hierarchy.*

Proof. For each $\varkappa$ in the standard basis, the change of variables $t_\varkappa \mapsto \sigma^{deg(\varkappa)} t_\varkappa$ rescales the degree-three part of the Saito genus-zero potential (that is, the metric and all three-point correlators) by $\sigma^{\hat{c}_W}$, and it rescales the degree-four part of the potential (the four point correlators) by $\sigma^{\hat{c}_W+1}$, but it leaves the flat identity element **1** unchanged.

Re-scaling the primitive form by a non-zero scalar β also leaves the identity element **1** unchanged, but it rescales the entire potential by the same element β. If we choose $\beta = \sigma^{-\hat{c}_W}$, then the composition of the change of variables followed by rescaling the primitive form leaves the degree-three part of the potential (i.e., the entire Frobenius algebra structure) completely unchanged, and it rescales the degree-four part of the potential by σ.

If the correlator $\langle X, X, X, X^2\rangle$ is non-zero, we can change variables and rescale the primitive form to make the Saito (B-model) correlator $\langle X, X, X, X^2\rangle$ precisely match the FJRW (A-model) correlator. Since all the the four- and higher-point correlators are completely determined by $\langle X, X, X, X^2\rangle$, this means that the genus-zero potentials Φ^{FJRW} and Φ^{Saito} will match exactly.

By Theorem 4.3, this shows that the total descendant potential function $\mathscr{D}^{FJRW}_{D_4,\langle J\rangle}$ or $\mathscr{D}^{FJRW}_{D_4^T,G_{max}}$ will precisely match the Givental-Saito total descendant potential function $\mathscr{D}^{GS}_{D_4}$. The desired result now follows from Theorem 4.7. □

5. Proof of the Main Theorem

In this section we will finish the proof of Theorem 1.1. The hardest part of this proof boils down to proving that the seven-point correlator $\langle X^2, X^2, X^2, X^2, X^2, X^2, X^2\rangle$ is non-zero in the FJRW theory for D_4 with symmetry group $\langle J\rangle$. We do this by a long computation in Lemma 5.2.

Theorem 1.1.

(1) $\mathscr{D}^{FJRW}_{D_4,\langle J\rangle}$ *is a tau-function of the D_4 Kac-Wakimoto/Drinfeld-Sokolov hierarchy.*

(2) $\mathscr{D}^{FJRW}_{D_4,G_{max}}$ *is a tau-function of the D_4 Kac-Wakimoto/Drinfeld-Sokolov hierarchy.*

Proof. By Lemma 4.8 all that is required is to show that the correlator $\langle X, X, X, X^2\rangle$ does not vanish for the FJRW D_4^T and D_4 theories.

In [FJR, §6.3.7] the correlator $\langle X, X, X, X^2\rangle$ for the FJRW theory of D_4^T with symmetry group G_{max} is computed to be $\alpha/6^3$, where $\alpha^2 = 1/6$. This proves the theorem for the case of D_4^T.

As noted in the introduction, computing the correlator $a = \langle X, X, X, X^2\rangle$ in the D_4 case directly is very difficult because three of the insertions are broad. Computing the correlator in such a case is a difficult PDE problem and we do not yet have techniques to solve it explicitly. However, by Theorem 4.5 we know that the seven point correlator $\langle X^2, X^2, X^2, X^2, X^2, X^2, X^2\rangle$ satisfies

$$\langle X^2, X^2, X^2, X^2, X^2, X^2, X^2\rangle = 6^3 a^4,$$

so showing that a is non-zero is equivalent to showing that $\langle X^2, X^2, X^2, X^2, X^2, X^2, X^2\rangle$ is non-zero. This is proved in Lemma 5.2, below. □

Lemma 5.2. *In the FJRW (A-model) theory of D_4 with symmetry group $\langle J\rangle$, the genus-zero seven-point correlator $\langle X^2, X^2, X^2, X^2, X^2, X^2, X^2\rangle$ is non-zero.*

Proof. This correlator has only narrow insertions and has $\deg(|\mathscr{L}_x|) = \deg(|\mathscr{L}_y|) = -3$; therefore, it is concave and can be computed directly using the concavity axiom [FJR, Axiom 5.a]. Because $\mathscr{L}_x$ and $\mathscr{L}_y$ are concave, the pushforwards satisfy $R^\bullet\pi_*(\mathscr{L}_x) = R^1\pi_*(\mathscr{L}_x)$ and $R^\bullet\pi_*(\mathscr{L}_y) = R^1\pi_*(\mathscr{L}_y)$ and are vector bundles. It is straighforward to check that the (complex) rank of each of these push-forward bundles is 2. By the concavity axiom we have

$$\begin{aligned}\Lambda^{D_4}_{0,7}(X^2,X^2,X^2,X^2,X^2,X^2,X^2) &= \frac{1}{\deg(st)} c_4\left(R^1\pi_*(\mathscr{L}_x)\oplus R^1\pi_*(\mathscr{L}_y)\right)\\ &= \frac{1}{\deg(st)} c_2\left(R^1\pi_*(\mathscr{L}_x)\right) c_2\left(R^1\pi_*(\mathscr{L}_y)\right). \end{aligned} \tag{16}$$

In order to compute these Chern classes, we will use Chiodo's formula [Ch, Thm 1.1.1] for the Chern character of the pushforward of an rth root of an sth power of ω_{log}. To do this, we note first that the stack $\mathscr{W}_{g,k,\langle J\rangle}(D_4)$ can be identified with the open and closed substack of $\mathscr{W}_{g,k,G_{max}}(D_4)$ corresponding to $(D_4 + x^2y)$-curves (see [FJR, §2.3]). That means that $\mathscr{L}_x^{\otimes 3} \cong \omega_{log}$, and $\mathscr{L}_x \cong \omega_{\log} \otimes \mathscr{L}_y^{\otimes -2}$ and $\mathscr{L}_y \cong \omega_{log} \otimes \mathscr{L}_x^{\otimes -2}$. This implies that

$$\mathscr{L}_x \cong \mathscr{L}_y \quad \text{and} \quad \mathscr{L}_x^3 = \omega_{log}.$$

This shows that the stack $\mathscr{W}_{g,k,\langle J\rangle}(D_4)$ is canonically isomorphic to the stack $\overline{\mathscr{M}}^{1/3}_{g,k}$ of third roots of ω_{log}.

Chiodo's formula states that for the universal rth root $\mathscr{L}$ of ω^s_{log} on the universal family of pointed orbicurves $\pi : \mathscr{C} \longrightarrow \overline{\mathscr{M}}^{s/r}_{g,k}(\gamma_1,\dots,\gamma_k)$ and with local group $\langle\gamma_i\rangle$ of order m_i at the ith marked point, we have

$$\mathrm{Ch}(R^\bullet\pi_*(\mathscr{L}) = \sum_{d\geq 0}\left[\frac{B_{d+1}(s/r)}{(d+1)!}\kappa_d - \sum_{i=1}^{k}\frac{B_{d+1}(\Theta^{\gamma_i})}{(d+1)!}\psi_i^d + \frac{1}{2}\sum_{\Gamma_{\mathrm{cut}}}\frac{rB_{d+1}(\Theta^{\gamma_+})}{(d+1)!}\tilde{\rho}_{\Gamma_{\mathrm{cut}}*}\Big(\sum_{\substack{i+j=d-1\\ i,j\geq 0}}(-\psi_+)^i\psi_-^j\Big)\right],$$

where the second sum is taken over all decorated stable graphs Γ_{cut} with one pair of tails labelled + and −, respectively, so that once the + and − edges have been glued, we get a single-edged, n-pointed, connected, decorated graph of genus g and with additional decoration (γ_+ and γ_-) on the internal edge. Each such graph Γ_{cut} has the two cut edges, decorated with group elements γ_+ and γ_-, respectively, and the map $\tilde{\rho}_{\Gamma_{\mathrm{cut}}}$ is the corresponding gluing map $\overline{\mathscr{M}}^{r/s}_{\Gamma_{\mathrm{cut}}} \longrightarrow \overline{\mathscr{M}}^{r/s}_{g,k}(\gamma_1,\dots,\gamma_k)$.

In our case, we have $\mathscr{L}_x \cong \mathscr{L}_y$, and we can use Chiodo's formula for both $\mathscr{L}_x$ and $\mathscr{L}_y$. We have, therefore, $r = 3$, $s = 1$, and $\Theta^{\gamma_i} = 2/3$ for every $i \in \{1,\dots,7\}$. Expressing c_2 in terms of the Chern character, we have

$$\begin{aligned}
c_2\left(R^1\pi_*(\mathscr{L})\right) &= \frac{1}{2}\operatorname{Ch}_1^2(\mathscr{L}) - \operatorname{Ch}_2(\mathscr{L})\\
&= \frac{1}{4}\left(B_2(1/3)\kappa_1 - \sum_{i=1}^{7} B_2(2/3)\psi_i + \frac{1}{2}\sum_{\Gamma_{\mathrm{cut}}} rB_2(\Theta^{\gamma_+})\tilde{\rho}_{\Gamma_{\mathrm{cut}}*}(1)\right)^2\\
&\qquad - \frac{1}{6}\left(B_3(1/3)\kappa_2 - \sum_{i=1}^{7} B_3(2/3)\psi_i^2 + \frac{1}{2}\sum_{\Gamma_{\mathrm{cut}}} rB_3(\Theta^{\gamma_+})\tilde{\rho}_{\Gamma_{\mathrm{cut}}*}(\psi_- - \psi_+)\right)\\
&= \frac{1}{4}\left(-\frac{1}{18}\kappa_1 + \sum_{i=1}^{7}\frac{1}{18}\psi_i + \frac{1}{2}\sum_{\Gamma_{\mathrm{cut}}} 3B_2(\Theta^{\gamma_+})\widetilde{\Delta}_\Gamma\right)^2\\
&\qquad - \frac{1}{6}\left(\frac{1}{27}\kappa_2 + \sum_{i=1}^{7}\frac{1}{27}\psi_i^2 + \frac{1}{2}\sum_{\Gamma_{\mathrm{cut}}} 3B_3(\Theta^{\gamma_+})\tilde{\rho}_{\Gamma_{\mathrm{cut}}*}(\psi_- - \psi_+)\right), \qquad (17)
\end{aligned}$$

where $\widetilde{\Delta}_\Gamma$ is the boundary divisor in $\overline{\mathscr{M}}_{0,7}^{1/3}$ corresponding to $\tilde{\rho}_{\Gamma_{\mathrm{cut}}}(1)$.

To push forward from $\overline{\mathscr{M}}_{0,7}^{1/3}$ to the stack of stable curves $\overline{\mathscr{M}}_{0,7}$, note that the κ_d and the ψ^d are pullbacks of their counterparts on $\overline{\mathscr{M}}_{0,7}$, and since the morphism $st : \overline{\mathscr{M}}_{0,7}^{1/3} \longrightarrow \overline{\mathscr{M}}_{0,7}$ has three-fold ramification along the locus corresponding to Γ, we have $3\tilde{\rho}_{\Gamma_{\mathrm{cut}}*}(\psi_\pm^j) = st^*(\rho_{|\Gamma_{\mathrm{cut}}|*}(\psi_\pm^j))$ and $3\widetilde{\Delta}_\Gamma := 3\tilde{\rho}_{\Gamma_{\mathrm{cut}}*}(1) = \rho_{|\Gamma_{\mathrm{cut}}|*}(1) =: \Delta_\Gamma$, where $|\Gamma_{\mathrm{cut}}|$ is the undecorated graph underlying Γ_{cut}, and $\rho_{|\Gamma_{\mathrm{cut}}|} : \overline{\mathscr{M}}_{|\Gamma_{\mathrm{cut}}|} \longrightarrow \overline{\mathscr{M}}_{0,7}$ is the associated gluing map.

Applying Equation (16) we have

$$\begin{aligned}
&\Lambda_{0,7}^{D_4}(X^2, X^2, X^2, X^2, X^2, X^2, X^2) \qquad (18)\\
&\quad = \left[\frac{1}{4}\left(-\frac{1}{18}\kappa_1 + \sum_{i=1}^{7}\frac{1}{18}\psi_i + \frac{1}{2}\sum_{\Gamma_{\mathrm{cut}}} B_2(\Theta^{\gamma_+})\Delta_\Gamma\right)^2\right.\\
&\qquad\qquad \left. - \frac{1}{6}\left(\frac{1}{27}\kappa_2 + \sum_{i=1}^{7}\frac{1}{27}\psi_i^2 + \frac{1}{2}\sum_{\Gamma_{\mathrm{cut}}} B_3(\Theta^{\gamma_+})\rho_{\Gamma_{\mathrm{cut}}*}(\psi_- - \psi_+)\right)\right]^2.
\end{aligned}$$

We use Carel Faber's Maple code [Fa] to complete the calculation. For simplicity, and to match Faber's usage, we index the graphs Γ_{cut} by the subsets $I \subset [7] := \{1, \ldots, 7\}$ such that $1 \in I$ and $2 \leq |I| \leq 5$. That is, Δ_I is the same as $\Delta_{\Gamma_{\mathrm{cut}}}$ where the graph Γ_{cut} has its tails on one vertex labeled by the elements of I and the symbol "+," whereas the tails on the second vertex are labeled by the elements of $I^c = [7] \setminus I$ and the symbol "−." This indexing includes exactly half of all the graphs—the other half come from swapping the positions of the + and − labels. If we denote by Γ'_{cut} the graph obtained from Γ_{cut} by swapping the + and − labels, then it is easy to see that the following hold:

$$\Delta_{\Gamma_{\mathrm{cut}}} = \Delta_{\Gamma'_{\mathrm{cut}}} \qquad (19)$$

$$\rho_{\Gamma_{\mathrm{cut}}*}(\psi_+) = \rho_{\Gamma'_{\mathrm{cut}}*}\psi_- \qquad (20)$$

$$\rho_{\Gamma_{\mathrm{cut}}*}(\psi_-) = \rho_{\Gamma'_{\mathrm{cut}}*}\psi_+. \qquad (21)$$

Moreover, since for any decorated graph we have $\gamma_+^{-1} = \gamma_-$, we have either $\Theta^{\gamma_-} = 1 - \Theta^{\gamma_+}$ if $\Theta_{\gamma_+} \neq 0$, and we have $\Theta^{\gamma_-} = \Theta^{\gamma_+} = 0$ otherwise. And the Bernoulli numbers $B_n(t)$ satisfy the well-known relation

$$B_n(1-t) = (-1)^n B_n(t).$$

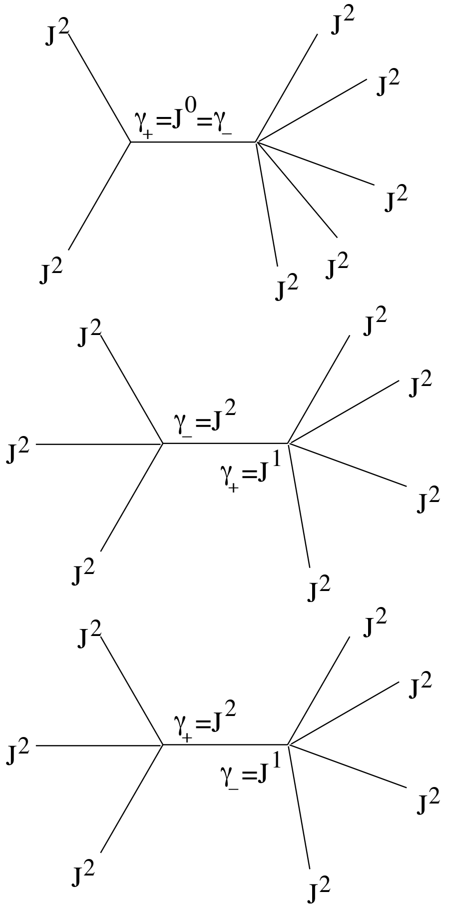

Figure 1. The three forms that a one-edge degeneration can take for a seven-point graph $\Gamma \in \mathbf{\Gamma}_{0,7,D_4}(J^2, J^2, J^2, J^2, J^2, J^2, J^2)$. The first corresponds to both $|I| = 2$ and $|I| = 5$, the second corresponds to $|I| = 4$ and the third to $|I| = 3$.

Therefore, we have

$$\begin{aligned}\Lambda_{0,7}^{D_4}&(X^2, X^2, X^2, X^2, X^2, X^2, X^2) \qquad (22)\\ &= \left[\frac{1}{4}\left(-\frac{1}{18}\kappa_1 + \sum_{i=1}^{7}\frac{1}{18}\psi_i + \sum_{1\in I\subset[7]} B_2(\Theta^{\gamma_+})\Delta_I\right)^2\right.\\ &\qquad\left. - \frac{1}{6}\left(\frac{1}{27}\kappa_2 + \sum_{i=1}^{7}\frac{1}{27}\psi_i^2 + \sum_{1\in I\subset[7]} B_3(\Theta^{\gamma_+})\rho_{I*}(\psi_- - \psi_+)\right)\right]^2.\end{aligned}$$

We now need to compute the values of γ_+ that occur for the various choices of I. The graph Γ_{cut} must be one of the three forms depicted in Figure 1. There are $\binom{7}{2} = 21$ choices of ways to label the external edges of the first graph and $\binom{7}{3} = 35$ choices for the second and third. In the first case, we have $B_2(0) = 1/6$ and $B_3(0) = 0$. For the second type, we have

$B_2(1/3) = -1/18$ and $B_3(1/3) = 1/27$, while in the last case we have $B_2(2/3) = -1/18$ and $B_3(2/3) = -1/27$.

Faber's code computes intersections of divisors on the stack of stable curves, so we must express every one of the classes in Equation (22) in terms of divisor classes. The only classes that are not already products of divisors are κ_2 and $\rho_{I*}\psi_{\pm}$. To rewrite the first, we use the fact that on $\overline{\mathscr{M}}_{0,5}$ we have

$$\kappa_{(0,5),2} = \kappa_{(0,5),1}\Delta_{1,2,3}.$$

This follows, for example, from [KK, Cor 2.2]. Moreover, κ_2, ψ_i and Δ_I pull back (see [AC98, Lm 1.2]) along the forgetting tails map $\tau : \overline{\mathscr{M}}_{0,n+1} \longrightarrow \overline{\mathscr{M}}_{0,n}$ as

$$\begin{aligned}
\tau^*(\kappa_{(0,n),a}) &= \kappa_{(0,n+1),a} - \psi_{n+1}^a\\
\tau^*(\psi_i) &= \psi_i - \Delta_{i,n+1}\\
\tau^*(\Delta_I) &= \Delta_I + \Delta_{I\cup\{n+1\}}.
\end{aligned}$$

Using the pullback twice, we get

$$\begin{aligned}
\kappa_{(0,7),2} &= \tau^*(\kappa_{(0,6),2}) + \psi_7^2 \qquad (23)\\
&= \tau^*(\tau^*(\kappa_{(0,5),2}) + \psi_6^2) + \psi_7^2\\
&= \tau^*(\tau^*(\kappa_{(0,5),1}\Delta_{1,2,3}) + \psi_6^2) + \psi_7^2\\
&= \tau^*\left((\kappa_{(0,6),1} - \psi_6)(\Delta_{1,2,3} + \Delta_{1,2,3,6}) + \psi_6^2\right) + \psi_7^2\\
&= (\kappa_{(0,7),1} - \psi_7 - (\psi_6 - \Delta_{6,7}))(\Delta_{1,2,3} + \Delta_{1,2,3,7} + \Delta_{1,2,3,6} + \Delta_{1,2,3,6,7}) + (\psi_6 - \Delta_{6,7})^2 + \psi_7^2.
\end{aligned}$$

To match Faber's notation, we further rewrite the boundary divisors in this equation for κ_2 such that they all have 1 in their index set. This can easily be done by means of the obvious equality

$$\Delta_I = \Delta_{I^c}.$$

It is more messy to rewrite the classes $\rho_{I*}(\psi_{\pm})$ in terms of divisors. To do this, we use the following well-known relation: for any distinct $a, b, i \in \{1, \ldots, k\}$, we have

$$\psi_i = \sum_{\substack{i\in I\\ a,b\notin I}} \Delta_I. \qquad (24)$$

And a straightforward computation now yields the following lemma.

Sub-Lemma 5.3. *For any subset K with $1 \in K$ and for any distinct $r, s \in K \setminus \{1\}$ and any distinct $t, u \notin K$ we have*

$$\rho_{K*}(\psi_+) = 0 \text{ if } |K| \leq 2 \qquad (25)$$

$$\rho_{K*}(\psi_+) = \sum_{\{1,r,s\}\subseteq I\subset K} \Delta_K\Delta_I + \sum_{1\in I\subseteq K\setminus\{r,s\}} \Delta_K\Delta_{I\cup K^c} \qquad (26)$$

$$\rho_{K*}(\psi_-) = 0 \text{ if } |K| \geq 5 \qquad (27)$$

$$\rho_{K*}(\psi_-) = \sum_{\substack{\emptyset\neq I\subset K^c\\ t,u\notin I}} \Delta_K\Delta_{I\cup K}. \qquad (28)$$

To complete the computation, we make all these substitutions into Equation (22) and then use Faber's code [Fa] to integrate the resulting (enormous) sum of degree-four monomials of divisors. The code which we used to make this computation is included in the appendix of this paper. We find from this that

$$\langle X^2, X^2, X^2, X^2, X^2, X^2, X^2 \rangle_0^{D_4} = \frac{221}{6561},$$

which is not zero. □

Appendix A. Code Used to Compute the D_4 Seven-Point Correlator

```
restart:
with(combinat):
read("MgnF"):

##Set constants
k := 7: r := 3: s := 1: q := s/r: Theta := [‘$‘(2/3, 7)]:
AmbientSet:={‘$‘(1..7)}:

##Choose divisors in Mbar_{0,n} which are stable and have the mark 1
StableChoose:=proc(numpts::integer)
option remember;
local initialList,finalList,A;
initialList:=choose(numpts): finalList:=[]:
for A in initialList do
    if (nops(A)<numpts-1 and nops(A)>1 and ‘in‘(1,A))
    then
       finalList:=[op(finalList),convert(A,set)]:
    end if
end do:
return(finalList):
end proc:

##Assign values of Theta(gamma+/-)
ThetaGamma := proc (A::set)
local tpm;
    tpm := [[0, 0], [2/3, 1/3], [1/3, 2/3], [0, 0]]:
    if (1 < nops(A) and nops(A) < 6)
    then return(tpm[nops(A)-1]):
    else error "invalid divisor":
    end if:
end proc:

##Compute a single boundary term when given a list of tails containing 1
BoundaryTerm := proc (dim::integer, A::set)
global alpha, beta, Delta;
    if (dim = 1)
    then return(bernoulli(dim+1, ThetaGamma(A)[1])*Delta[A]/factorial(dim+1)):
    else if (dim = 2)
         then
           return bernoulli(dim+1, ThetaGamma(A)[1])*(psiminus[A]-psiplus[A])/factorial(dim+1)
         else error "dim out of range":
         end if:
    end if:
end proc:

##Collect all the boundary terms together
```

```
BoundaryContrib := proc (dim::integer)
local Bdry;
global k;
    Bdry := map(x-> BoundaryTerm(dim, x), StableChoose(k)):
    return foldl(`+`, 0, op(Bdry)):
end proc:

##Return the components of the chern char of -R^1\pi_*L, per Chiodo's formula
ChernChar := proc (dim::integer)
global  Theta, i, k, kappa, psi, q;
    return (simplify(bernoulli(dim+1, q)*kappa[dim]/factorial(dim+1)
           -(add(bernoulli(dim+1, Theta[i])*psi[i]^dim/factorial(dim+1),
             i = 1 .. k))+BoundaryContrib(dim)))
end proc:

## Now compute the second chern class of -R^1\pi_* L
cTwo := simplify(expand((1/2)*ChernChar(1)^2-ChernChar(2))):

##The class Lambda is the square of cTwo
Lambda := expand(cTwo^2):

##Now  rewrite Kappa2 as a quadratic in divisors
kappa[2]:=(kappa[1]-psi[7]-(psi[6]-Delta[{1,7}]))*(Delta[{1,2,3,6,7}]
+Delta[{1,2,3,6}] + Delta[{1,2,3,7}] + Delta[{1,2,3}])
+(psi[6] -Delta[{1,2,3,4,5}])^2 + \psi[7]^2:

##Rewrite psi+/psi-  as intersections of boundary divisors
rewritepsi := proc (A::set)  ##A=index set of the divisors
local AlistOne, AlistTwo, Blist, Apair, Bpair, B,  SmallA, SmallB,i:
global alpha, beta, Delta, psi:
    B := `minus`({1, 2, 3, 4, 5, 6, 7}, A): #B is complement of A
    if (`in`(1, A) and `subset`(A, {1, 2, 3, 4, 5, 6, 7}) and 1 < nops(A) and nops(A) < 6)
    then
    ##First rewrite psi+
        if (nops(A) < 3 or 5 < nops(A)) ##psi+ on a 3-point component
                                         ##vanishes, and 2-point
                                         ##components aren't stable.
         then psiplus[A] := 0
         else if nops(A) = 3
              then psiplus[A] := psi[1]*Delta[A] ## all psis are equivalent
                                            ##on M04bar so replace psi+ by psi[1]
              else  ##now consider partitions with 4 or 5 points
                  Apair := choose(`minus`(A, {1}), 2)[1]: ##Choose one pair of
                                                     ##tails in A--doesn't matter which
                  SmallA:=`minus`(A, `union`({1}, Apair)): ##Remove one pair and {1} from A
                  AlistOne := choose(SmallA): ##All collections of tails from SmallA
                           ##Next same as above but remove the biggest subset from the list.
                  AlistTwo := `minus`(AlistOne, choose(SmallA, nops(SmallA))):
  ##Actually we want 1 and B in each of the AlistOne sets
                  AlistOne := map(x-> ((x union B) union {1}), AlistOne):
                   ##Actually we want 1 and Apair in each of the AlistTwo sets
                  AlistTwo := map(x-> ((x union {1}) union Apair), AlistTwo):
                  psiplus[A] := add(Delta[A]*Delta[AlistTwo[i]],
```

```
                              i = 1 .. nops(AlistTwo)) +
                              add(Delta[A]*Delta[AlistOne[i]],
                              i = 1 .. nops(AlistOne)):
              end if
         end if:
##Now rewrite psi-
        if 4 < nops(A) or nops(A) < 2
        then psiminus[A] := 0
        else if nops(A) = 4
             then
                 ##again replace psi- by psi[j] for any j in B
                psiminus[A] := psi[B[1]]*Delta[A]:
               else Bpair := choose(B, 2)[1]:
                 SmallB:= B minus Bpair:
                           #Remove emptyset from the set of subsets
                 Blist := `minus`(choose(SmallB), choose(SmallB, 0)):
                 Blist := map(x-> (x union A) , Blist):
                 for i to nops(Blist) do
                     if Blist[i] = A
                     then error "Bad self intersection (B)"
                     end if
                 end do:unassign('i'):
                 psiminus[A] := add(Delta[A]*Delta[Blist[i]], i = 1 .. nops(Blist))
             end if
        end if
    else error "Input set must be subset of {1..7}
            containing 1 and having between 2 and 5 elements"
    end if
end proc:
##Now actually execute the rewritepsi proc.
LL := StableChoose(7):
for AA in LL do rewritepsi(AA)
end do:

##Now separate out the terms from coeffs, so we can integrate the terms.

LambdaIndets := convert(indets(Lambda), list): #indeterminates
Lambda:=collect(Lambda,LambdaIndets,'distributed'):
LambdaCoefs := [coeffs(Lambda, LambdaIndets, 'NLatt')]: #coefs
LambdaTerms := [NLatt]: #terms

##To integrate, first expand each term into a list of 4 divisors and
##  then convert each of those to a Faber number to feed
##  to Faber's mgn procedure.

ExpPowers := proc (Z) ##expands a  monomial with powers into
                      ##a list of individual variables
local exper,ZZ;
    exper := proc (z)
        if op(0, z) = `^` ##if a power,return the appropriate
                          ##number of copies.
        then
            return(`$`(op(1, z), op(2, z)));
```

```
        else
            return(z)
        end if:
    end proc:
    if type(Z,‘ˆ‘)  ##if the whole thing is a 4th power, return 4 copies.
    then
        return([‘$‘(op(1, Z), op(2, Z))]);
    else
        ZZ:=convert(Z,list):
        return(map(exper, ZZ)):
    end if:
end proc:

##Convert divisors into Faber numbers
Faberify:=proc(z)
 if op(0,z)=’Delta’ then return(num(0,7,[0,op(1,z)])):
 elif op(0,z)=’psi’ then return(op(1,z)): Faber number of psi[i]=i
 elif op(0,z)=’kappa’ and op(1,z) = 1 then return(8) ##8 = Faber(ka_1)
 else
  error "Cannot Faberify Divisor %1",z:
 end if:
end proc:

##Use Faber’s mgn to compute the intersection
## of a(degree-4) monomial of divisors on M07bar
Intersector:=proc(X)
local x;
    x:=ExpPowers(X):
    return(mgn(0,map(Faberify,x))):
end proc:

##Apply the Intersector to the list of terms to get a list of integrated monomials.
Intersections:=map(Intersector, LambdaTerms):

##Finally, multiply by the coefficients and add them up:
FinalAnswer:=add(LambdaCoefs[i]*Intersections[i],i=1..nops(Intersections));
```

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