# $L^2$-cohomology and geometric group theory

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The purpose of this talk is to present recent developments about $L^2$-invariants and their applications to problems in other areas such as topology, group theory, $K$-theory, geometry and global analysis.

We begin with a list of certain results or open problems which do not a priori seem to be related to $L^2$-invariants but whose proof or at least proof in partially cases does require $L^2$-methods. For instance we mention the following results below due to Chang-Weinberger, Cheeger-Gromov, Donnelly-Xavier, Dodziuk, Gaboriau, Gromov and Lück.

- Let $G$ be a group which contains a normal infinite amenable subgroup. Suppose that there is a finite $CW$-model for its classifying space $BG$. Then its Euler characteristic $\chi(G) := \chi(BG)$ vanishes,

- Let $M$ be a closed manifold of even dimension $2m$. Suppose that $M$ is hyperbolic, or more generally, that its sectional curvature satisfies $-1 \leq \sec(M) < -(1-\frac{1}{m})^2$ .Then

$$(-1)^m \cdot \chi(M) > 0.$$

- Let $M$ be a closed Kähler manifold of (real) dimension $2m$. Suppose that $\pi_2(M)$ vanishes and $\pi_1(M)$ is word-hyperbolic. Then

$$(-1)^m \cdot \chi(M) > 0.$$

  and $M$ is a projective algebraic variety.

- Let $1 \to H \to G \to K \to 1$ be an extension of infinite groups such that $H$ is finitely generated and $G$ is finitely presented. Then

  1. The deficiency of $G$ satisfies $\mathrm{def}(G) \leq 1$.

2. If $M$ is a closed connected oriented 4-manifold with $\pi_1(M) \cong G$, then we get for its signature $\operatorname{sign}(M)$ and its Euler characteristic $\chi(M)$
$$|\operatorname{sign}(M)| \leq \chi(M).$$

- Let $G$ be a group and $\mathbb{C}G$ be its complex group ring. Let $G_0(\mathbb{C}G)$ be the Grothendieck group of finitely generated (not necessarily projective) $\mathbb{C}G$-modules. Then

  1. If $G$ is amenable, the class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ is an element of infinite order;
  2. If $G$ contains the free group $\mathbb{Z} * \mathbb{Z}$ of rank two, then $[\mathbb{C}G] = 0$ in $G_0(\mathbb{C}G)$.

- There are finitely generated groups which are quasi-isometric but not measurably equivalent.

- Let $M^{4k+3}$ be a closed oriented smooth manifold for $k \geq 1$ whose fundamental group has torsion. Then there are infinitely many pairwise non-homeomorphic smooth manifolds which are homotopy equivalent to M (and even simply and tangentially homotopy equivalent to $M$) but not homeomorphic to M.

- Open Problem: Are the group von Neumann algebras of two finitely generated free groups isomorphic if and only if the groups are isomorphic?

The $L^2$-Betti numbers of the universal covering $\widetilde{M}$ of a closed Riemannian manifold $M$ were originally defined by Atiyah in terms of the heat kernel on the universal covering and a fundamental domain $F$ of the $\pi_1(M)$-action by

$$b_p^{(2)}(\widetilde{M}) \;:=\; \lim_{t\to\infty} \int_F \operatorname{tr}\left(e^{-t\widetilde{\Delta}_p}(\widetilde{x},\widetilde{x})\right) \operatorname{dvol}_{\widetilde{M}}.$$

One can extend this definition to arbitrary $G$-spaces $X$ by

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; \dim_{\mathcal{N}(G)}\left(H_p(C_*^{\mathrm{sing}}(X) \otimes_{\mathbb{Z}} \mathcal{N}(G)\right),$$

where $\mathcal{N}(G)$ is the group von Neumann algebra and $\dim_{\mathcal{N}(G)}$ is the generalized dimension function of Lück. In particular one can define for every group $G$ its $p$-th $L^2$-Betti number $b_p^{(2)}(G)$ by $b_p^{(2)}(EG;\mathcal{N}(G))$.

These invariants share basic properties we are used to for the classical Betti numbers $b_p(X)$ for a finite $CW$-complex $X$ plus some extra features.

If $X$ is a finite $CW$-complex, let $b_p^{(2)}(\widetilde{X})$ be the $L^2$-Betti number of the $\pi_1(X)$-spaces given by the universal covering. Then

- Homotopy invariance
  If $X \simeq Y$, then $b_p^{(2)}(\widetilde{X}) = b_p^{(2)}(\widetilde{Y})$.

- Euler-Poincaré formula
  $\chi(X) \;=\; \sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(\widetilde{X})$.

- Poincaré duality
  Let $M$ be a closed orientable $n$-dimensional manifold. Then
  $$b_p^{(2)}(\widetilde{M}) \;=\; b_{n-p}^{(2)}(\widetilde{M}).$$
- Künneth formula.
- Morse inequalities.
- $L^2$-Hodge-deRham Theorem.
- Zero-th $L^2$-Betti number
  $$b_0^{(2)}(X) \;=\; \frac{1}{|\pi_1(X)|}.$$
- Multiplicativity
  If $X \to Y$ is a $d$-sheeted covering, then
  $$b_p^{(2)}(\widetilde{X}) \;=\; d \cdot b_p^{(2)}(\widetilde{Y}).$$

For instance Multiplicativity implies $b_p^{(2)}(\widetilde{S^1}) = 0$ for all $p \geq 0$.

The Euler-Poincaré formula is the only general relation between the Betti numbers $b_p(X)$ and the $L^2$-Betti numbers $b_p^{(2)}(\widetilde{X})$ for a finite $CW$-complex. The $L^2$-Betti numbers can be viewed as asymptotic Betti numbers by the following result due to Lück which was conjectured by Gromov.

**Theorem (Approximation Theorem).**
*Let $X$ be a finite $CW$-complex. Suppose that $G$ is residually finite, i.e. there is a nested sequence*
$$\pi_1(X) = G_0 \supset G_1 \supset G_2 \supset \dots$$
*of normal subgroups of finite index with $\cap_{n\geq 1} G_n = \{1\}$. Then for any such sequence $(G_n)_{n\geq 1}$*
$$b_p^{(2)}(\widetilde{X}) \;=\; \lim_{n\to\infty} \frac{b_p(X_n)}{[\pi_1(X) : G_n]},$$
*where $X_n \to X$ is the $[\pi_1(X) : G_n]$-sheeted covering associated to $G_n \subset \pi_1(X)$.*

We discuss the following further results of Lück.

**Theorem ($L^2$-Betti numbers and aspherical $S^1$-manifolds).**
*Let $M$ be an aspherical closed manifold with non-trivial $S^1$-action. Then all $L^2$-Betti numbers $b_p^{(2)}(\widetilde{M})$ are trivial and $\chi(M) = 0$.*

**Theorem (Vanishing of $L^2$-Betti numbers of mapping tori).**
*Let $f\colon X \to X$ be a cellular selfhomotopy equivalence of a finite $CW$-complex $X$. Then we get for all $p \geq 0$*
$$b_p^{(2)}(\widetilde{T_f}) \;=\; 0.$$

**Theorem (First $L^2$-Betti number and group extensions).**
*Let $1 \to H \to G \to K \to 1$ be an extension of infinite groups such that $H$ is finitely generated and $G$ is finitely presented. Then*

$$b_1^{(2)}(G) = 0.$$

We explain how some of the results mentioned in the beginning follow from the theorems above.

Recently $L^2$-Betti numbers have successfully been applied to problems about von Neumann algebras. Besides Popa's results on fundamental groups of factors we mention the following notion due to Connes and Shlyakhtenko. They define for a finite von Neumann algebra $\mathcal{A}$ its $L^2$-Betti numbers by

$$b_p^{(2)}(\mathcal{A}) \;=\; \dim_{\mathcal{A}\overline{\otimes}\mathcal{A}^{op}}\left(HH_p(\mathcal{A};\mathcal{A}\overline{\otimes}\mathcal{A}^{op})\right).$$

The main still open problem is to decide whether for a group $G$ the equality

$$b_p^{(2)}(G) \;=\; b_p^{(2)}(\mathcal{N}(G)).$$

holds. A positive answer would imply a positive answer to the problem whether the group von Neumann algebras of two finitely generated free groups are isomorphic if and only if the groups are isomorphic.

This definition is motivated by a certain dictionary due to Connes, the following equalities

$$\begin{aligned}
\mathbb{C}G \otimes (\mathbb{C}G)^{op} &= \mathbb{C}[G \times G];\\
\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op} &= \mathcal{N}(G \times G).
\end{aligned}$$

and the calculation

$$\begin{aligned}
b_p^{(2)}(G) &= \dim_{\mathcal{N}(G)}\left(H_p\left(C_*^{\mathrm{sing}}(EG) \otimes_{\mathbb{C}} \mathcal{N}(G)\right)\right)\\
&= \dim_{\mathcal{N}(G)}\left(\mathrm{Tor}_p^{\mathbb{C}G}(\mathbb{C};\mathcal{N}(G)\right)\\
&= \dim_{\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op}}\left(\mathrm{Tor}_p^{\mathbb{C}G}(\mathbb{C};\mathcal{N}(G) \otimes_{\mathcal{N}(G)} \mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op}\right)\\
&= \dim_{\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op}}\left(\mathrm{Tor}_p^{\mathbb{C}G}(\mathbb{C};\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op})\right)\\
&= \dim_{\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op}}\left(\mathrm{Tor}_p^{\mathbb{C}G\otimes(\mathbb{C}G)^{op}}(\mathbb{C}G;\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op})\right)\\
&= \dim_{\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op}}\left(HH_p(\mathbb{C}G;\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{op})\right).
\end{aligned}$$

Notice that in the last line one has to replace $\mathbb{C}G$ by $\mathcal{N}(G)$ to obtain the definition of Connes and Shlyakhtenko.

In connection with group theory the following result of Gaboriau is interesting. If two groups $G_0$ and $G_1$ are measurably equivalent, then there exists a constant $C$ such that

$$b_p^{(2)}(G_0) \;=\; C \cdot b_p^{(2)}(G_1)$$

holds for all $p \geq 0$. This result does play an important result in Popa's work on the fundamental group of factors. These developments seem also to indicate that one should use measure theoretic methods to make further progress about $L^2$-invariants. This concerns also the following open conjecture due to Gromov.

**Conjecture**. If $M$ is an aspherical oriented manifold whose simplicial volume is zero, then $b_p^{(2)}(\widetilde{M}) = 0$ for all $p \geq 0$.

There are the following open outstanding problems concerning $L^2$-invariants:

**Atiyah Conjecture**. Let $M$ be a closed Riemannian manifold. Then all $L^2$-Betti numbers $b_p^{(2)}(\widetilde{M})$ are rational numbers. If the fundamental group is torsionfree, then all $L^2$-Betti numbers $b_p^{(2)}(\widetilde{M})$ are integers.

**Singer Conjecture**. If $M$ is an aspherical closed manifold, then

$$b_p^{(2)}(\widetilde{M}) \;=\; 0 \qquad \text{if } 2p \neq \dim(M).$$

If $M$ is a closed Riemannian manifold with negative sectional curvature, then

$$b_p^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2p \neq \dim(M); \\ > 0 & \text{if } 2p = \dim(M). \end{cases}$$

Finally mention that there are further more sophisticated $L^2$-invariants such as Novikov-Shubin invariants and $L^2$-torsion.