Dynamics of $\mathbb{CP}^1$ lumps on a cylinder

Nuno M. Romão *

Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstraße 22, D–04103 Leipzig, Germany

Max-Planck-Institut für Mathematik
Vivatsgasse 7, D–53111 Bonn, Germany

March 2004

Abstract

The slow dynamics of topological solitons in the $\mathbb{CP}^1$ σ-model, known as lumps, can be approximated by the geodesic flow of the L^2 metric on certain moduli spaces of holomorphic maps. In the present work, we consider the dynamics of lumps on an infinite flat cylinder, and we show that in this case the approximation can be formulated naturally in terms of regular Kähler metrics. We prove that these metrics are incomplete exactly in the multilump (interacting) case. The metric for two-lumps can be computed in closed form on certain totally geodesic submanifolds in terms of elliptic integrals; particular geodesics are determined and discussed in terms of the dynamics of interacting lumps.

1 Introduction

Many field theories possess topological solitons as classical solutions, and the study of their dynamics has long been an important research topic in mathematical physics. Exact results for this problem have only been obtained for rather special (integrable) models in $1+1$ dimensions; more generally, one has to resort to approximations based on truncations of the field theories to finite-dimensional configuration spaces of collective coordinates. One such scheme is the adiabatic approximation, first proposed by Manton in the context of BPS monopoles [1]. It has been applied to extract detailed information about the slow dynamics of solitons in a number of models in $1+2$, $1+3$ and more dimensions (notably gauged Ginzburg–Landau vortices [2, 3] and Yang–Mills–Higgs monopoles [4, 5]) and is believed to work well for a large class of field theories exhibiting self-duality.

*e-mail: `nromao@mpim-bonn.mpg.de`

One neat example of a self-dual field theory is the nonlinear σ-model with $\mathbb{CP}^1$ target on a Riemann surface Σ. This is the dynamical system for maps

$$W : \Sigma \to \mathbb{CP}^1$$

described by the wave equation associated to a specific riemannian metric on Σ and the usual round metric on the two-sphere $\mathbb{CP}^1$. Topological solitons in this model, usually referred to as lumps, will typically arise if Σ is compact or effectively compactified by suitable boundary conditions. Static solutions are harmonic maps, with energy given by the usual Dirichlet integral. In the adiabatic approximation, one constructs another dynamical system whose configuration space consists of the static solutions of minimal energy, the dynamics being defined by restricting the action functional of the original field theory. This space is stratified by homotopy classes, and the strata are usually referred to as the moduli spaces. For the $\mathbb{CP}^1$ σ-model, the Dirichlet energy is minimized exactly by the holomorphic or antiholomorphic maps within each homotopy class, labelled by the Brouwer degree $n \in \mathbb{Z}$ of W. The moduli spaces (if non-empty) then have the structure of finite-dimensional complex varieties [6], and the adiabatic dynamics is geodesic motion with respect to a metric on them. The Cauchy–Riemann equation, a first-order PDE, replaces the second-order static equations of motion as a description of the fields. This is the essential common feature to all self-dual theories. In the adiabatic programme, the moduli spaces are often smooth manifolds equipped with natural geometric structures (symplectic forms, metrics of special holonomy) that turn out to be interesting objects by themselves. In some instances, they have even been used to probe aspects of the quantum field theories underlying the original models [7, 8, 9].

The $\mathbb{CP}^1$ σ-model has applications to the physics of ferromagnets and as a high-energy effective model for vortices; however, its main interest has been as a toy-model displaying many of the features of more important field theories with gauge symmetry. The adiabatic approach to this model was first investigated by Ward for the case $\Sigma = \mathbb{R}^2$ in [10]; he found that the approximation is ill-defined, in the sense that the metric is infinite along certain directions that appear as frozen degrees of freedom. One way to regularise the metric is to place the vortices on a compact surface, and this was studied by Speight when Σ is a sphere [11] or a torus [12]. It has also been found that the metric for $\Sigma = \mathbb{R}^2$ regularises once a self-gravitating interaction is included in the lagrangian [13]. Determining these metrics in closed form is in general beyond reach, but some explicit formulae have been obtained in a number of nontrivial cases, namely for one-lumps on $\Sigma = S^2$ [11] and for certain totally geodesic submanifolds of two-lumps on $\mathbb{R}^2$ [10] and on the particular torus $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ [12]. Geodesic incompleteness of the moduli spaces was proved in [14]. There is also a general belief that the relevant metrics should be Kähler [15, 16]; this has been rigorised for $\Sigma = S^2$, and for $\Sigma = T^2$ and $n = 2$ [17]. The accuracy of the adiabatic approximation has been studied recently by Haskins and Speight [18] in the spirit of work by Stuart [3, 5] on the gauge theory models.

In this paper, the adiabatic dynamics of $\mathbb{CP}^1$ lumps is studied in some detail for the case where Σ is an infinite cylinder. We can say that this is an intermediate case between the situations $\Sigma = \mathbb{C}$ and Σ compact considered by previous authors. In the former, the metrics are ill-defined but explicit calculations of the metric are possible, whereas in the latter the metrics are regular but extremely hard to compute; the cylinder turns out to combine the advantages of both. So our study complements the existing literature in a setting that is unifying in some way, and our results will reflect this. Let us summarise how this paper is organised. We use the next section to fix the basic notation. In section 3, we obtain elementary properties of the moduli spaces; we formulate the adiabatic approximation in terms of regular riemannian metrics, which are shown to be Kähler. In section 4, we discuss the isometries of these metrics. The one-lump sector is studied in section 5. We then establish that all the multilump metrics are incomplete in section 6. In section 7, we address the two-lump dynamics and derive more explicit results about the metric, its geodesics and curvature properties. Finally, we discuss our results in section 8.

2 The $\mathbb{CP}^1$ σ-model on a cylinder

For the rest of the paper, we shall take Σ to be the infinite cylinder

$$\Sigma = \mathbb{C}/(2\pi i\mathbb{Z})$$

with local complex coordinates $z = x + iy$ and metric

$$ds^2_\Sigma = dx^2 + dy^2 \tag{1}$$

induced from the euclidean metric of its universal cover $\mathbb{C}$.

The action for the $\mathbb{CP}^1$ σ-model, whose objects are differentiable maps $W : \Sigma \to \mathbb{CP}^1$ dependent on time t, is given by

$$I[W] = \int_{\mathbb{R}} (T - V)\, dt \tag{2}$$

with kinetic and potential energies

$$T := 2\int_\Sigma \frac{|\dot{W}|^2}{(1+|W|^2)^2} d\mu_\Sigma, \tag{3}$$

$$V := 4\int_\Sigma \frac{|\partial_z W|^2 + |\partial_{\bar{z}} W|^2}{(1+|W|^2)^2} d\mu_\Sigma. \tag{4}$$

We shall only consider the dynamics of maps W for which the potential energy V above is finite. We represent W by means of an inhomogeneous coordinate taking values in $\mathbb{C}\cup\{\infty\}$ following usual practice; overdots denote time derivatives and $d\mu_\Sigma$ is the measure on Σ associated to (1). The variational principle yields the wave equation as equation of motion, and static solutions are harmonic maps from (Σ, ds^2_Σ) to $(\mathbb{CP}^1, ds^2_{S^2})$. Here, $ds^2_{S^2}$ is the riemannian metric on $\mathbb{CP}^1$ regarded as a two-sphere of unit radius; the Kähler $(1,1)$-form of this metric will be denoted by ω_{S^2}. Following an argument first presented by Belavin and Polyakov [19] (but already observed in a more general setting in the mathematical literature — cf. [20] p. 374), we write for a map $W : \Sigma \to \mathbb{CP}^1$

$$
\begin{aligned}
0 &\leq 2\int_\Sigma \frac{|(\partial_x \pm i\,\partial_y)W|^2}{(1+|W|^2)^2} d\mu_\Sigma \\
&= 4\int_\Sigma \frac{|\partial_z W|^2 + |\partial_{\bar{z}} W|^2}{(1+|W|^2)^2} d\mu_\Sigma \mp \int_\Sigma W^*(\omega_{S^2}) \\
&= V[W] \mp \deg(W)\,\mathrm{Vol}(S^2),
\end{aligned}
$$

where $\deg(W)$ is the Brouwer degree of the map. In the inequality above, it is useful to take the top signs if $n := \deg(W)$ is nonnegative, and the bottom signs otherwise. Then we learn that

$$V[W] \geq 4\pi|n|,$$

which implies that $n \in \mathbb{Z}$ provided V is finite. Moreover, we deduce that the potential (or Dirichlet) energy (4) is minimized to $4\pi|n|$ on each topological class by a solution of the Cauchy–Riemann equation

$$\partial_{\bar{z}} W = 0 \tag{5}$$

if $n \geq 0$, or $\partial_z W = 0$ if $n < 0$. To simplify our discussion, we will mostly be considering the case $n \geq 0$ only, but all the statements can be easily adapted to the $n < 0$ case.

In this paper, we shall be concerned exclusively with the adiabatic approximation to the dynamics (2). This takes place in the space of holomorphic maps from Σ to $\mathbb{CP}^1$, i.e. meromorphic functions on Σ. They are completely characterised by the following lemma.

Lemma 2.1. *Any meromorphic function $W : \Sigma \to \mathbb{CP}^1$ of degree $n \in \mathbb{Z}$ factorises uniquely as*

$$W = \tilde{W} \circ \exp,$$

where $\exp : \Sigma \to \mathbb{CP}^1$ is given by $\exp(z) = e^z$ and $\tilde{W} : \mathbb{CP}^1 \to \mathbb{CP}^1$ is a rational map of degree n.

Proof. Any meromorphic map W on Σ can be regarded as a meromorphic map on $\mathbb{C}$ of period $2\pi i$. We first claim that there is a unique meromorphic map $\tilde{W} : \mathbb{CP}^1 - \{0, \infty\} \to \mathbb{CP}^1$ such that $W(z) = \tilde{W}(e^z)$. This is true because $z \mapsto e^z$ is invertible in $\mathbb{C}$ modulo integer multiples of $2\pi i$, and this ambiguity does not change the value of $W(z)$; that $\tilde{W}$ is meromorphic (and thus a rational map) follows from $z \mapsto e^z$ being holomorphic and the inverse function theorem in one complex variable. Since exp has degree one and the degree is multiplicative with respect to composition, $\tilde{W}$ has degree n. Finally, $\tilde{W}$ can be extended to a meromorphic map $\tilde{W} : \mathbb{CP}^1 \to \mathbb{CP}^1$ in a unique way: it cannot have essential singularities at 0 or ∞, for then the (strong version of the) big Picard theorem (cf. [21] p. 210) would contradict $n \in \mathbb{Z}$. $\square$

A meromorphic (or antimeromophic) map on Σ of degree $n \in \mathbb{Z}$ will be called an *n-lump*. Lumps with $n < 0$ are sometimes called *antilumps*.

Corollary 2.2. For any lump $W : \Sigma \to \mathbb{CP}^1$, the limits

$$\ell_\pm(W) := \lim_{x \to \pm\infty} W(x + iy)$$

are well defined as points of the Riemann sphere.

Proof. The map $\tilde{W} : \mathbb{CP}^1 \to \mathbb{CP}^1$ determined from W by Lemma 2.1 is continuous, so this follows from the existence of $\ell_-(\exp) = 0$ and $\ell_+(\exp) = \infty$. $\square$

We shall call $\ell_-(W)$ and $\ell_+(W)$ the *endpoints* of W. It is easy to see that the existence of endpoints is a necessary condition for the Dirichlet energy (4) of any map $\Sigma \to \mathbb{CP}^1$ to be finite.

Remark 2.3. Lumps with $\ell_+(W) = \ell_-(W)$ can be interpreted as meromorphic functions on the pinched torus depicted in Figure 1 — an elliptic curve with a nodal singularity and flat metric (1). So the results that we shall obtain below for such maps can also be interpreted in the context of the $\mathbb{CP}^1$ σ-model defined on this singular space.

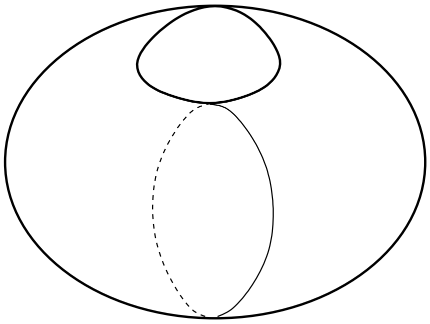


Figure 1: The pinched torus

3 Moduli spaces of lumps

The moduli space of n-lumps will be denoted by $\mathcal{M}_n$. By Lemma 2.1, any $W \in \mathcal{M}_n$ can be written as

$$W(z) = \frac{B(e^z)}{A(e^z)} \tag{6}$$

where

$$A(w) = \sum_{k=0}^{n} c_k w^{n-k}, \qquad B(w) = \sum_{k=0}^{n} c_{n+k+1} w^{n-k} \tag{7}$$

are complex polynomials with no common roots, and such that c_n and c_{2n+1} are not both equal to zero. These conditions are expressed algebraically by the nonvanishing of the resultant of A and B,

$$\delta(c_0,\dots,c_{2n+1}) := \mathrm{Res}(A,B) = \begin{vmatrix} c_0 & c_1 & \cdots & c_n & & & \\ & c_0 & \cdots & c_{n-1} & c_n & & \\ & & \ddots & & & \ddots & \\ & & & c_0 & c_1 & \cdots & c_n \\ c_{n+1} & c_{n+2} & \cdots & c_{2n+1} & & & \\ & c_{n+1} & \cdots & c_{2n} & c_{2n+1} & & \\ & & \ddots & & & \ddots & \\ & & & c_{n+1} & c_{n+2} & \cdots & c_{2n+1} \end{vmatrix}. \tag{8}$$

Notice that the polynomials A and B are not uniquely determined from W, but subject to the ambiguity of simultaneous multiplication by an element of $\mathbb{C}^\times$. Thus we can regard $\mathcal{M}_n$ as a subset of $\mathbb{CP}^{2n+1}$ through the injection that maps an n-lump W given by (6)–(7) to the point

$$[c_0 : c_1 : \cdots : c_{2n+1}] \in \mathbb{CP}^{2n+1}.$$

The image of this map is the complement of the hypersurface of degree $n+1$ associated to the homogeneous polynomial δ in (8),

$$\mathcal{V}(\delta) = \{[c_0 : \cdots : c_{n+1}] \in \mathbb{CP}^{n+1} : \delta(c_0,\dots,c_{n+1}) = 0\},$$

and is therefore an open subset in the Zariski topology of $\mathbb{CP}^{2n+1}$. So we have shown:

Proposition 3.1. $\mathcal{M}_n$ is a smooth complex quasiprojective variety of dimension

$$\dim_{\mathbb{C}} \mathcal{M}_n = 2n+1.$$

Let $\mathbb{CP}^1_\Delta$ denote the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$. For our purposes, it will be useful to give the following description of the moduli spaces:

Proposition 3.2. There exists a morphism for $n > 0$

$$\ell : \mathcal{M}_n \to \mathbb{CP}^1 \times \mathbb{CP}^1 \tag{9}$$

whereby $\ell = (\ell_-, \ell_+)$ associates to each lump its endpoints; for $n > 1$, this is a fibration by smooth irreducible closed subvarieties of $\mathcal{M}_n$ with complex dimension $2n-1$. Moreover, $\ell : \mathcal{M}_1 \to \mathbb{CP}^1 \times \mathbb{CP}^1 - \mathbb{CP}^1_\Delta$ is an algebraic principal fibre bundle with structure group $\mathbb{C}^\times$.

Proof. We define ℓ as the restriction of the rational map $\mathbb{CP}^{2n+1} \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ given by

$$[c_0 : \cdots : c_{2n+1}] \mapsto ([c_n : c_{2n+1}], [c_0 : c_{n+1}]). \tag{10}$$

On the complement of the hypersurface $\mathcal{V}(\delta)$, (c_{n+1}, c_0) and (c_{2n+1}, c_n) are never equal to $(0,0)$; thus (10) is regular on $\mathcal{M}_n = \mathbb{CP}^n - \mathcal{V}(\delta)$, and ℓ is a morphism of algebraic varieties.

The fibre of ℓ over $(p,q) = ([p_0 : p_1], [q_0 : q_1])$ is obtained by intersecting $\mathcal{M}_n$ with the algebraic subset of $\mathbb{CP}^{2n+1}$ defined by the homogeneous polynomials

$$p_0 c_{2n+1} - p_1 c_n \qquad \text{and} \qquad q_0 c_{n+1} - q_1 c_0.$$

Since these polynomials are linear in different variables and nonzero, it is clear that

$$\mathbb{C}[c_0, \ldots, c_{2n+1}] / \left(p_0 c_{2n+1} - p_1 c_n, q_0 c_{n+1} - q_1 c_0\right)$$

is isomorphic to the ring of complex polynomials in $2n-1$ variables, whose homogeneous maximal spectrum is $\mathbb{CP}^{2n-1}$. If $n > 1$, it straightforward to verify that this $\mathbb{CP}^{2n-1}$ intersects $\mathcal{V}(\delta)$ transversely independently of (p,q), and this shows that all the fibres are irreducible, smooth and of dimension $2n+1$.

For $n = 1$, it is easily checked that δ belongs to the ideal

$$\left(p_0 c_{2n+1} - p_1 c_n, q_0 c_{n+1} - q_1 c_0\right)$$

exactly when $p = q$, so that the range of ℓ in this case is the complement of the diagonal $\mathbb{CP}^1_\Delta$. To show that $\ell : \mathcal{M}_1 \to \mathbb{CP}^1 \times \mathbb{CP}^1 - \mathbb{CP}^1_\Delta$ is a principal fibre bundle, we start by pointing out that the identification of one-lumps with rational maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ of degree one given by Lemma 2.1 endows $\mathcal{M}_1$ with an algebraic group structure, namely

$$\mathcal{M}_1 \cong \mathrm{PGL}_2\mathbb{C}.$$

More precisely, we identify one-lumps with Möbius transformations of $w = e^z$. The subgroup generated by rotations and dilations is isomorphic to $\mathbb{C}^\times$, and it is an easy task to verify that the quotient map

$$\mathrm{PGL}_2\mathbb{C} \to \mathrm{PGL}_2\mathbb{C}/\mathbb{C}^\times$$

can be identified with ℓ. $\square$

The pre-image of $(p,q) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ under the map ℓ considered in Proposition 3.2 will be denoted by $\mathcal{M}_n^{(p,q)}$; we shall also write $\mathcal{M}_n^p := \mathcal{M}_n^{(p,0)}$. The adiabatic approximation consists of endowing each of these spaces with a riemannian metric γ and studying its geodesic flow, which is a dynamical system on $\mathcal{M}_n$ by automorphisms of the fibration ℓ. The geodesics on the fibres can be interpreted physically as a slow motion of lumps of degree n preserving the endpoints labelling the fibre. Physically, it makes sense to constrain the motion of the endpoints because we know that in the model it costs an infinite amount of energy to move them, which is not available to a lump that starts moving with a finite velocity.

The metric γ on each $\mathcal{M}_n^{(p,q)}$ is obtained from the kinetic energy (3) of the σ-model. This means that, if ζ_k $(k = 1, \ldots, 2n-1)$ are local complex coordinates for $\mathcal{M}_n^{(p,q)}$, $\gamma = \gamma_{i\bar{j}} d\zeta_i d\bar{\zeta}_j$ is defined such that

$$T = \tfrac{1}{2} \gamma_{i\bar{j}}|_{(\zeta_1, \ldots, \zeta_{2n-1})} \dot{\zeta}_i \dot{\bar{\zeta}}_j. \tag{11}$$

Here we allow the ζ_k, regarded as parameters specifying W, to depend on time and apply the chain rule. Geometrically, γ can be interpreted as the restriction of the L^2 metric on the infinite-dimensional manifold of smooth maps $(\Sigma, ds^2_\Sigma) \to (\mathbb{CP}^1, ds^2_{S^2})$ to a finite-dimensional submanifold of holomorphic maps with suitable boundary conditions. Thus given $W \in \mathcal{M}_n^{(p,q)}$ and two vectors X, Y of the tangent space

$$T_W \mathcal{M}_n^{(p,q)} = \{X \in H^0(\Sigma, W^*(T^{(1,0)}\mathbb{CP}^1)) : \lim_{x\to-\infty} X|_{x+iy} = 0 = \lim_{x\to+\infty} X|_{x+iy}\},$$

the metric at W is evaluated as

$$\gamma|_W(X,Y) = \int_\Sigma (W^* ds^2_{S^2})(X,Y)\, d\mu_\Sigma \tag{12}$$

whenever this integral exists. The main result of this section is the following

Theorem 3.3. The riemannian metric γ on $\mathcal{M}_n^{(p,q)}$ relevant for the adiabatic approximation is regular for $n \geq 1$. Moreover, it is a Kähler metric with respect to the complex structure induced by $\mathcal{M}_n^{(p,q)} \hookrightarrow \mathbb{CP}^{2n+1}$.

Proof. Since we still have the freedom of choosing the inhomogeneous coordinate on the $\mathbb{CP}^1$ target, we may assume without loss of generality that $q = 0$. This means that we can restrict our attention to maps W for which $c_0 \neq 0$. This condition defines an affine piece of $\mathbb{CP}^{2n+1}$ where

$$\zeta_k := \frac{c_k}{c_0}, \qquad k = 1, \ldots, 2n+1 \tag{13}$$

are good complex coordinates. Now $q = 0$ implies $\zeta_{n+1} = 0$ on $\mathcal{M}_n^p$. There is one more redundant coordinate on $\mathcal{M}_n^p$ among (13), and it can be eliminated through the equation

$$p_0 \zeta_{2n+1} - p_1 \zeta_n = 0, \tag{14}$$

where $p =: [p_0 : p_1]$. Suppose first that $p_0 \neq 0$ holds, so that ζ_{2n+1} can be eliminated. Then a map $W \in \mathcal{M}_n^p$ can be expressed as

$$W(z) = \frac{\displaystyle\sum_{k=1}^{n-1} \zeta_{k+n+1} e^{(n-k)z} + p\zeta_n}{\displaystyle e^{nz} + \sum_{k=1}^{n} \zeta_k e^{(n-k)z}}, \tag{15}$$

where we write $p = p_1/p_0$. (If $n = 1$, the sum in the numerator should be ignored and $p \neq 0$.) According to (11), the components of the metric in these coordinates can be read off as

$$\begin{aligned} \gamma_{i\bar{j}} &= \int_{-\pi}^{\pi}\int_{-\infty}^{+\infty} \frac{4}{(1+|W|^2)^2} \frac{\partial W}{\partial \zeta_i} \frac{\partial \bar{W}}{\partial \bar{\zeta}_j}\, dx\, dy \\ &= 2i \int_{\mathbb{C}} \frac{1}{(1+|\tilde{W}|^2)^2} \frac{\partial \tilde{W}}{\partial \zeta_i} \frac{\partial \bar{\tilde{W}}}{\partial \bar{\zeta}_j} \frac{dw \wedge d\bar{w}}{|w|^2} \end{aligned} \tag{16}$$

where the indices run from 1 to $2n$ and we have used the change of variables $w = e^z$. After differentiating (15), it is not hard to check that the integrand in (16) (with respect to the euclidean measure $\frac{i}{2} dw \wedge d\bar{w}$) is a rational function of w and $\bar{w}$, with the only singularity occurring at $w = 0$ and being of the form $\mathcal{O}(|w|^{-1})$ as $|w| \to 0$, and with the asymptotic behaviour $\mathcal{O}(|w|^{-3})$ as $|w| \to \infty$. So we conclude that the integral in (16) is finite for all i and j, which means that the metric γ is regular.

To show that the metric is Kähler, we start by observing that it is hermitian with respect to the complex structure associated to the coordinates ζ_k,

$$\gamma_{i\bar{j}} = \overline{\gamma_{j\bar{i}}}.$$

The closure of the corresponding $(1,1)$-form can then be seen to be equivalent to the conditions

$$\frac{\partial \gamma_{i\bar{j}}}{\partial \zeta_k} = \frac{\partial \gamma_{k\bar{j}}}{\partial \zeta_i}, \qquad i, j = 1, 2, \ldots, n, n+2, \ldots, 2n.$$

For these to hold, it is sufficient that integration and differentiation with respect to ζ_k may be interchanged in $\partial \gamma_{ij}/\partial \zeta_k$ But this follows from a standard result on Lebesgue integration of differentiable maps (cf. e.g. [22], p. 226) once we observe that the integral

$$2i \int_{\mathbb{C}} \frac{\partial}{\partial \zeta_k} \left(\frac{1}{(1+|\tilde{W}|^2)^2} \frac{\partial \tilde{W}}{\partial \zeta_i} \frac{\partial \bar{\tilde{W}}}{\partial \bar{\zeta}_j} \right) \frac{dw \wedge d\bar{w}}{|w|^2}$$

exists and is finite by an argument analogous to the one used to establish regularity of γ.

It remains to address the case $p_0 = 0$, i.e. $p = \infty$. Then we necessarily have $p_1 \neq 0$ and (14) yields $\zeta_n = 0$. A map $W \in \mathcal{M}_n^\infty$ is now expressed as

$$W(z) = \frac{\displaystyle\sum_{k=1}^{n} \zeta_{k+n+1} e^{(n-k)z}}{e^{nz} + \displaystyle\sum_{k=1}^{n-1} \zeta_k e^{(n-k)z}} \tag{17}$$

(where the sum in the denominator should be ignored if $n = 1$). The rest of the argument follows essentially unchanged from the case $p_0 \neq 0$ above. □

4 Isometries of $\mathcal{M}_n$

Our major goal is to compute explicitly the metrics γ describing the slow motion of lumps in some special situations and interpret their geodesics. Not surprisingly, a central part of this study is concerned with the exploration of isometries, to which we shall now turn. In this section, n will not necessarily be taken as nonnegative.

Recall that γ is determined from both the metric ds^2_Σ on space Σ and the metric $ds^2_{S^2}$ on the target, cf. (12). These have isometry groups

$$\mathrm{Iso}(\Sigma) = V_4 \ltimes \mathbb{C}^\times \tag{18}$$

and

$$\mathrm{Iso}(S^2) = \mathrm{O}(3) \cong \mathbb{Z}_2 \times \mathrm{SO}(3),$$

where V_4 denotes the Vierergruppe $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. They act on these spaces as follows. The factor $\mathbb{C}^\times$ of $\mathrm{Iso}(\Sigma)$ refers to the translation group of the cylinder,

$$T_\lambda : z \mapsto z - \log \lambda, \qquad \lambda \in \mathbb{C}^\times,$$

whereas the Vierergruppe is generated by any two of the three transformations

$$\sigma_1 : \quad z \mapsto -\bar{z}, \tag{19}$$

$$\sigma_2 : \quad z \mapsto \bar{z}, \tag{20}$$

$$\sigma_3 : \quad z \mapsto -z, \tag{21}$$

which also define the semidirect product in (18). Notice that both σ_1 and σ_2 reverse the orientation of Σ, whereas σ_3 preserves the orientation. On the target, the proper rotations in $\mathrm{SO}(3)$ can be represented in terms of Möbius transformations of the coordinate W by

$$R : W \mapsto \frac{\alpha W - \bar{\beta}}{\beta W + \bar{\alpha}}, \qquad |\alpha|^2 + |\beta|^2 \neq 0, \tag{22}$$

and we can take a reflection across any great circle as the generator of the $\mathbb{Z}_2$ factor, say

$$\sigma : W \mapsto \bar{W}.$$

It is natural to expect isometries of $(\mathcal{M}_n^{(p,q)}, \gamma)$ to be produced from the induced action of $\mathrm{Iso}(\Sigma) \times \mathrm{Iso}(S^2)$ on $C^\infty(\Sigma, \mathbb{CP}^1)$:

$$(g,h) : W(z) \mapsto h(W(g^{-1}(z))), \qquad (g,h) \in \mathrm{Iso}(\Sigma) \times \mathrm{Iso}(S^2).$$

In general, these transformations do not preserve the spaces $\mathcal{M}_n^{(p,q)}$, but it is straightforward to show from the representation (12) that they still act isometrically as follows:

$$\begin{aligned}
T_\lambda \equiv (T_\lambda, \mathrm{id}) : &\quad \mathcal{M}_n^{(p,q)} \to \mathcal{M}_n^{(p,q)}, \quad \lambda \in \mathbb{C}^\times \\
\sigma_1 \equiv (\sigma_1, \mathrm{id}) : &\quad \mathcal{M}_n^{(p,q)} \to \mathcal{M}_{-n}^{(q,p)} \\
\sigma_2 \equiv (\sigma_2, \mathrm{id}) : &\quad \mathcal{M}_n^{(p,q)} \to \mathcal{M}_{-n}^{(p,q)} \\
\sigma_3 \equiv (\sigma_3, \mathrm{id}) : &\quad \mathcal{M}_n^{(p,q)} \to \mathcal{M}_n^{(q,p)} \\
R \equiv (\mathrm{id}, R) : &\quad \mathcal{M}_n^{(p,q)} \to \mathcal{M}_n^{(R(p),R(q))}, \quad R \in \mathrm{SO}(3) \\
\sigma \equiv (\mathrm{id}, \sigma) : &\quad \mathcal{M}_n^{(p,q)} \to \mathcal{M}_{-n}^{(\sigma(q),\sigma(p))}.
\end{aligned}$$

The next proposition shows how these isometries can be used to simplify the study of the metrics γ.

Proposition 4.1. Each fibre of $\ell : \mathcal{M}_n \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is isometric to a fibre of the form $\mathcal{M}_{|n|}^p$ with $p \in [0, \infty] \subset \mathbb{R}$. Moreover, the isometry groups $\mathrm{Iso}(\mathcal{M}_n^p)$ of these spaces always contain a subgroup isomorphic to

$$V_4 \ltimes \mathbb{C}^\times;$$

if $n > 1$, $\mathrm{Iso}(\mathcal{M}_n^p)$ contains a subgroup isomorphic to $(V_4 \ltimes \mathbb{C}^\times) \times \mathrm{SO}(2)$ if $p = 0$ or $p = \infty$.

Proof. Consider the fibre $\mathcal{M}_n^{(p',q')}$ of ℓ over arbitrary (p',q'). If $n<0$, we can use σ_2 to map it isometrically to $\mathcal{M}_{|n|}^{(p',q')}$. If $q'\neq 0$, we then use the transformation

$$Q: W \mapsto \frac{W-q'}{1+\bar{q}'W}$$

(to be read as $W\mapsto -W^{-1}$ if $q'=\infty$) to map it to a fibre of the form $\mathcal{M}_{|n|}^{Q(p')}$. Finally, if $Q(p')\notin[0,\infty]$, we use a rotation $W\mapsto e^{i\vartheta}W$ by $\vartheta=-\arg(Q(p'))$. The composition of these isometries then takes $\mathcal{M}_n^{(p',q')}$ to $\mathcal{M}_{|n|}^{p}$ for some $p\in[0,\infty]$.

To prove the second part, we start by recalling from above that the translations of Σ preserve each $\mathcal{M}_n^p$, so that $\mathbb{C}^\times\subset\mathrm{Iso}(\mathcal{M}_n^p)$. This is not the case for the transformations induced by the generators of $V_4\subset\mathrm{Iso}(\Sigma)$, but they can be combined with target transformations to produce $\tilde{\sigma}_j\in\mathrm{Iso}(\mathcal{M}_n^p)$ from the $\sigma_j\in\mathrm{Iso}(\Sigma)$ in (19)–(21). Specifically, we take

$$\begin{aligned}
\tilde{\sigma}_1 &:= R\circ\sigma\circ\sigma_1, \\
\tilde{\sigma}_2 &:= \sigma\circ\sigma_2, && (23)\\
\tilde{\sigma}_3 &:= R\circ\sigma_3, && (24)
\end{aligned}$$

where $R\in\mathrm{SO}(3)$ is defined by

$$R: W\mapsto \begin{cases} W\;, & \text{if} \quad p=0,\\ \dfrac{W-p}{1+pW}, & \text{if} \quad 0<p<\infty,\\ -W^{-1}\;, & \text{if} \quad p=\infty. \end{cases} \tag{25}$$

It is clear that the proper rotations of the target giving rise to isometries of $\mathcal{M}_n^p$ must fix the set $\{0,p\}$, so they are either R in (25) (leading to $\tilde{\sigma}_3$ above) or an element of $\mathrm{Stab}_0\cap\mathrm{Stab}_p$, and this group is trivial for $0<p<\infty$ and $\mathrm{SO}(2)$ for $p=0$ and $p=\infty$. However, the case $n=1$ is exceptional: we necessarily have $p\neq 0$, and in the case of $p=\infty$ target rotations about the endpoints act as translations (by an imaginary quantity) on the whole fibre, so they do not lead to new isometries. Finally, the target reflections have to be combined with σ_1 or σ_2 to produce a degree-preserving transformation, so no more isometries arise from them. □

The proof above also shows that no further isometries of $(\mathcal{M}_n^p,\gamma)$ can be constructed by combining space and target isometries.

We now state a fundamental lemma relating isometries of a riemannian manifold and its totally geodesic submanifolds; these are the submanifolds whose geodesics (in the induced metric) are also geodesics of the ambient metric (cf. [23], p. 132).

Lemma 4.2. Let $S\subset\mathrm{Iso}(M)$ be any set of isometries of a riemannian manifold (M,g), and $F\subset M$ the set of points that are fixed by all the elements of S. If F is a manifold, it is a totally geodesic submanifold of (M,g).

This elementary result has been used rather crucially in studies of soliton dynamics, in the case where F is taken to be a finite set (or the subgroup generated by it), but is also true more generally; we include a proof in Appendix A. The main interest of totally geodesic submanifolds in the context of soliton dynamics is of course that if their dimension is small enough it may be possible to compute the restriction of the relevant metric to them. The geodesics of such manifolds can sometimes be determined (in particular, they already are geodesics if their dimension is one), and they typically describe soliton scattering processes for which the energy density has some degree of symmetry. This approach has been exceptionally fruitful in the study of BPS monopoles in $\mathbb{R}^3$ (see [24] for an overview), although in this context it is often more convenient to impose the relevant symmetries on certain geometric objects parametrised by the same moduli spaces as the solutions of the Bogomol'nyĭ equations, rather than on the metrics directly.

Using the isometries in Proposition 4.1 and Lemma 4.2, it is not hard to find nontrivial totally geodesic submanifolds for the spaces $(\mathcal{M}_n^p, \gamma)$. For instance, if we take $S = \{\tilde{\sigma}_2\}$ (cf. (23)) we find that F is a subvariety of real dimension $2n-1$ if $n=2$ and $p \neq 0$, or $n > 2$, whereas it has real dimension 4 for $n=2$ and $p=0$. Moreover, images of totally geodesic submanifolds under isometries are again totally geodesic submanifolds. A much harder problem is to find totally geodesic submanifolds on which the metric and its geodesics can be computed explicitly. We shall give examples of such in Section 7.1.

5 Degree-one lumps

For $n=0$, the moduli space is trivially a copy of $\mathbb{CP}^1$; this follows from Lemma 2.1 and the fact that the only rational maps of degree zero are the constants. The map $\ell : \mathcal{M}_0 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ analogous to (9) is of course just the embedding of the diagonal $\mathbb{CP}^1_\Delta$. The adiabatic dynamics as we have defined it in Section 3 is trivial in this degenerate case, because the level sets of ℓ are either empty or just one point. We regard $\mathcal{M}_0 = \mathbb{CP}^1$ as a moduli space of classical vacua.

The moduli space of one-lumps is potentially more interesting. Recall that we established in Proposition 3.2 that $\mathcal{M}_1$ has the structure of a principal fibre bundle:

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, hook] & \mathcal{M}_1 \arrow[d, "\ell"] \\ & \mathbb{CP}^1 \times \mathbb{CP}^1 - \mathbb{CP}^1_\Delta \end{tikzcd} \qquad (26)$$

It is easy to understand that this is just the complexification of the familiar description of a two-sphere as a homogeneous space,

$$S^2 = \mathrm{SO}(3)/\,\mathrm{SO}(2).$$

The fact that the diagonal $\mathbb{CP}^1_\Delta$ is absent from the range of ℓ means of course that there is no one-lump W with $\ell_-(W) = \ell_+(W)$; in particular, one-lumps do not exist on a pinched torus, cf. Remark 2.3. This is also a feature of the $\mathbb{CP}^1$ σ-model on a smooth torus [12].

On each fibre $\mathcal{M}_1^{(p,q)} \cong \mathbb{C}^\times$ of (26), the structure group acts (transitively and freely) by spatial translations, which we know to be isometries of the metric γ. It follows from the local isotropy of (1) that the metric on $\mathcal{M}_1^{(p,q)}$ is completely described by a constant $m(p,q)$. By fixing a one-lump $W_0 \in \mathcal{M}_1^{(p,q)}$, we can introduce a global coordinate $\zeta \in \mathbb{C}^\times \cong \Sigma$ and parametrise any other $W_0 \in \mathcal{M}_1^{(p,q)}$ as $W(z) = W_0(z-\zeta)$; if we define c locally by $c := \log\zeta$, we can write down the metric conveniently as

$$\gamma = m(p,q)\, dc\, d\bar{c}.$$

This constant $m(p,q)$ can be interpreted as the mass of a lump with endpoints p and q, and adiabatic motion in $\mathcal{M}_1^{(p,q)}$ is just rigid motion on Σ with inertia given by this constant.

It is straightforward to verify that two one-lumps with endpoints p_1, q_1 and p_2, q_2 at the same distance $d(p_1,q_1) = d(p_2,q_2)$ have the same shape, in the sense that their energy densities $\mathcal{E} \in L^2(\Sigma, d\mu_\Sigma)$, given by

$$\mathcal{E}(z) = \frac{4|\partial_z W(z)|^2}{(1+|W(z)|^2)^2} \tag{27}$$

are related by a spatial translation. The possible shapes of one-lumps are classified through d by the points of the interval $]0,\pi]$. It follows from these observations that on each $\mathcal{M}_1^{p,q}$ a one-lump moves without altering its shape, and that $m(p,q)$ can be expressed as a function of $d(p,q)$. In Figure 2, we plot the energy density profiles of one-lumps with different shapes. For $d = \pi$, the lump profile has circular symmetry, which is hardly surprising – it is readily checked that (27) is invariant under any global target rotation (22), and we know from the proof of Proposition 4.1 that for $n = 1$ and antipodal endpoints a translation of z by an imaginary quantity is equivalent to a target rotation about the endpoints. When we decrease d, the profile acquires a peak, which becomes more and more pronounced as the endpoints approach each other.

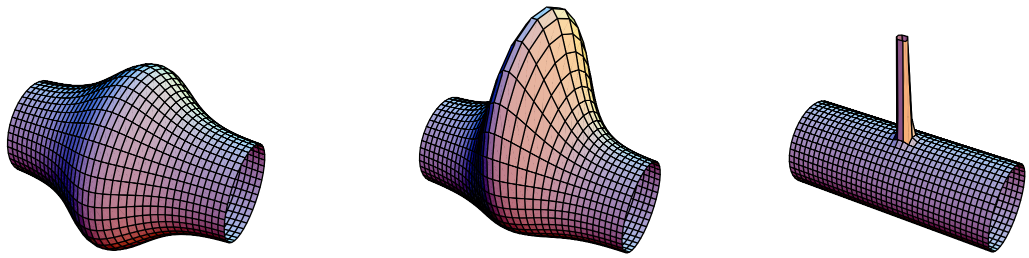
In fact, we can show that even one-lumps of different shapes have the same mass in the adiabatic approximation. This is just a special case of the following fact:

Proposition 5.1. The mass of any n-lump is $4\pi n$. In particular, the L^2 metric on $\mathcal{M}_1^{(p,q)}$ is $\gamma = 4\pi\, dc\, d\bar{c}$.

Proof. Let W be any n-lump and introduce a coordinate $c = \log\zeta$, $\zeta \in \mathbb{C}^*$ as above. According to (16), the mass of W is then given by the integral

$$\begin{aligned} m &= \int_\Sigma \frac{4}{(1+|W(z)|^2)^2} \left| \frac{\partial}{\partial c}\bigg|_{c=0} W(z-c) \right|^2 d\mu_\Sigma \\ &= 2i \int_\Sigma \frac{4}{(1+|W(z)|^2)^2} \left(\left|\frac{\partial W}{\partial z}(z)\right|^2 + \left|\frac{\partial W}{\partial \bar{z}}(z)\right|^2 \right) d\mu_\Sigma \\ &= V[W] = 4\pi n, \end{aligned}$$

where we made use of the Bogomol'nyĭ equation (5) to complete the potential energy (4). □



(a) $d = \pi$ (b) $d = 2\pi/3$ (c) $d = \pi/250$

Figure 2: *One-lumps with different shapes:* (a) $W(z) = e^{-z}$*;* (b) $W(z) = \sqrt{3}/(2ie^z + 1)$*;* (c) $W(z) = \tan(\frac{\pi}{500})/(ie^z + 1)$*. The energy density profiles are plotted radially on top of a cylinder of unit radius for* $|\,\mathrm{Re}\, z| \leq 5/2$*.*

6 Incompleteness of multilump metrics

As we have seen, the adiabatic dynamics of one-lumps can be described as constant motion on the translation group of Σ; the shape of the one-lump is fixed by the initial endpoints and the dynamical moduli can be interpreted as a centre of mass. For $n \geq 2$ however, typical dynamical processes will include relative motion determined by the interactions among the individual solitons entwined within a given field configuration, and the metrics are correspondingly more complicated. In this section, we shall establish an important property of the multilump metrics, which accounts for the possiblity of lump collapse in finite time:

Theorem 6.1. For $n \geq 2$, the L^2 metric (12) on $\mathcal{M}_n^{(p,q)}$ is incomplete for any $(p,q) \in \mathbb{CP}^1 \times \mathbb{CP}^1$.

Proof. By Proposition 4.1, we do not loose generality by taking $p \in [0, \infty]$ and $q = 0$. Our strategy will be to exhibit (for each $n \geq 2$) particular paths $\gamma_p : [a, b[\subset \mathbb{R} \to \mathcal{M}_n^p$ such that:

- $\lim\limits_{t \to b} \gamma_p(t) \notin \mathcal{M}_n^p$;
- γ_p has finite length in the metric (12).

According to (16), the length of γ_p is given by

$$L(\gamma_p) = \int_a^b \left(\int_{\mathbb{C}} \Phi(w,t)\, d^2 w \right)^{1/2} dt, \qquad (28)$$

where

$$\Phi(w,t) = \frac{4}{(1+|\tilde{W}_t|^2)^2}\left|\frac{\partial \tilde{W}_t}{\partial t}\right|^2 \frac{1}{|w|^2} \tag{29}$$

and we denote by $w \mapsto \tilde{W}_t(w)$ the rational map $\mathbb{CP}^1 \to \mathbb{CP}^1$ corresponding to $\gamma_p(t)$. It is convenient to consider three separate cases: (i) $p=0$, (ii) $0<p<\infty$ and (iii) $p=\infty$. In each case, we choose γ_p having future convenience in mind.

(i) $p = 0$:

We define γ_0 on $[1,+\infty[$ by

$$\gamma_0(t) : w \mapsto \frac{2w}{t(w^n+1)} = \tilde{W}_t(w). \tag{30}$$

Notice that, for $t \in [1,+\infty[$, $\tilde{W}_t$ is a rational map of degree n with the required boundary conditions $\tilde{W}_t(0) = 0 = \tilde{W}_t(\infty)$, thus γ_0 is well defined. Moreover, $\tilde{W}_\infty := \lim_{t\to+\infty} \tilde{W}_t$ is not a rational map:

$$\tilde{W}_\infty(w) = \begin{cases} 0\,, & \text{if} \quad w^n \neq -1 \\ \infty, & \text{if} \quad w^n = -1; \end{cases}$$

so indeed $\gamma_0(t)$ leaves $\mathcal{M}_n^0$ as $t \to +\infty$.

We now set to prove that γ_0 has finite length. This is given by (28) with

$$\Phi(w,t) = \frac{4}{(t^2+|\tilde{W}_1|^2)^2}\frac{|\tilde{W}_1|^2}{|w|^2}.$$

We denote by $\{w_j\}_{j=1}^n$ the set of nth roots of -1, which are all the (simple) poles of $\tilde{W}_1$, and fix $\varepsilon \in]0, \frac{1}{2}\sin\frac{\pi}{n}[$. It is now convenient to write

$$\int_{\mathbb{C}} \Phi(\cdot,t) = \left(\int_{B_{1/\varepsilon}(0)-\cup_{j=1}^\infty B_\varepsilon(w_j)} + \int_{\mathbb{C}-B_{1/\varepsilon}(0)} + \sum_{j=1}^n \int_{B_\varepsilon(w_j)}\right)\Phi(\cdot,t)$$

and estimate each of the integrals separately. In the following, C, C_0, etc. denote positive constants (dependent on ε only).

Since $w \mapsto \tilde{W}_1(w)/w$ has modulus bounded on $B_{1/\varepsilon}(0) - \cup_{j=1}^\infty B_\varepsilon(w_j)$ (say by a constant C_0) independently of t, we may write

$$\int_{B_{1/\varepsilon}(0)-\cup_{j=1}^\infty B_\varepsilon(w_j)} \Phi(\cdot,t) < \int_{B_{1/\varepsilon}(0)-\cup_{j=1}^\infty B_\varepsilon(w_j)} \frac{4|\tilde{W}_1|^2 d^2w}{t^4|w|^2} < \int_{B_{1/\varepsilon}(0)} \frac{4C_0^2 d^2 w}{t^4} = \frac{4\pi C_0^2}{\varepsilon^2 t^4}.$$

Similarly, $w \mapsto w\,\tilde{W}_1(w)$ is also bounded in modulus for $|w| > 1/\varepsilon$ (by C_∞, say), hence

$$\int_{\mathbb{C}-B_{1/\varepsilon}(0)} \Phi(\cdot,t) < \int_{\mathbb{C}-B_{1/\varepsilon}(0)} \frac{4|\tilde{W}_1|^2 d^2w}{t^4|w|^2} < \int_{\mathbb{C}-B_{1/\varepsilon}(0)} \frac{4C_\infty^2 d^2w}{t^4|w|^4} = \frac{4\pi C_\infty^2}{\varepsilon^2 t^4}.$$

On $B_\varepsilon(w_j)$, the function $w \mapsto (w-w_j)\tilde{W}(w)$ has no poles or zeroes, so there is a constant $C_j > 1$ satisfying

$$C_j^{-1} < |(w-w_j)\tilde{W}(w)| < C_j.$$

Therefore,

$$\int_{B_\varepsilon(w_j)} \Phi(\cdot,t) < \frac{4}{(1-\varepsilon)^2}\int_{B_\varepsilon(w_j)} \frac{\frac{C_j^2}{|w-w_j|^2}}{\left(t^2+\frac{C_j^{-2}}{|w-w_j|^2}\right)^2}\, d^2w.$$

We now make the change of variable $w \mapsto v := tC_j(w-w_j)$ and estimate the right-hand-side of the inequality above as follows:

$$\begin{aligned}
\frac{4C_j^2}{(1-\varepsilon)^2t^4}\int_{B_{t\varepsilon C_j}(0)} \frac{|v|^2\, d^2v}{(|v|^2+1)^2} &= \frac{8\pi C_j^2}{(1-\varepsilon)^2t^4}\int_0^{t\varepsilon C_j} \frac{|v|^3\, d|v|}{(|v|^2+1)^2}\\
&= \frac{8\pi C_j^2}{(1-\varepsilon)^2t^4}\int_0^{1+(t\varepsilon C_j)^2} \frac{u-1}{u^2}\, du\\
&< \frac{8\pi C_j^2}{(1-\varepsilon)^2t^4}\log(1+(t\varepsilon C_j)^2).
\end{aligned}$$

There are constants C_j' and C_j'' such that the last expression above is bounded by

$$C_j' t^{-4} + C_j'' t^{-4}\log t.$$

Putting together all the estimates above, we conclude that there is an overall constant C such that the length of γ_0 satisfies the inequality

$$L(\gamma_0) < C\int_1^\infty \frac{1}{t^2}\sqrt{1+\log t}\, dt,$$

in which the right-hand-side is finite.

Before we proceed, we would like to remark that we defined γ_0 as a target scaling of $W_1 \in \mathcal{M}_n^0$ in (30), and that any other choice of W_1 would lead (through scaling) to a path in $\mathcal{M}_n^0$ that would suit our purposes (cf. [14]). However, the scaling of a given map within a fibre $\mathcal{M}_n^p$ does not yield paths of finite length if $p \neq 0$.

(ii) $0 < p < \infty$:

We now take γ_p with domain $[\frac{1}{2}, 1[$ and defined by

$$\gamma_p(t) : w \mapsto \frac{tp(tw+1)}{(1-w)^{n-1}(w+t)} = \tilde{W}_t(w).$$

It is easy to check that $W_t \in \mathcal{M}_n^p$ for each $t \in [\frac{1}{2}, 1[$. Now $W_1 \notin \mathcal{M}_n^p$ because $\tilde{W}_1$ is a map of degree $n-1$.

The length of γ_p is given by (28), with

$$\Phi(w,t) = \frac{4p^2|2tw+t^2+1|^2|w-1|^{2n-2}}{(|w-1|^{2n-2}|w+t|^2+|tp|^2|tw+1|^2)^2}.$$

We fix now $\varepsilon \in]0, \frac{1}{2}[$ and write

$$\int_{\mathbb{C}} \Phi(\cdot,t) = \left(\int_{B_\varepsilon(0)} + \int_{\mathbb{C}-(B_\varepsilon(0)\cup B_\varepsilon(-1))} + \int_{B_\varepsilon(-1)} \right) \Phi(\cdot,t).$$

The first integrals do not cause problems, as we can write for suitable constants C_0 and C_∞ and all $t \in [\frac{1}{2}, 1[$

$$\int_{B_\varepsilon(0)} \Phi(\cdot,t) < \int_{B_\varepsilon(0)} C_0 \, d^2w = \pi\varepsilon^2 C_0$$

and

$$\int_{\mathbb{C}-(B_\varepsilon(0)\cup B_\varepsilon(-1))} \Phi(\cdot,t) < \int_{\mathbb{C}-B_\varepsilon(0)} \frac{C_\infty \, d^2w}{|w|^{2n}} = \frac{\pi C_\infty}{(n-1)\varepsilon^{2n-2}}.$$

As $t \to 1$ however, the integral over $B_\varepsilon(-1)$ becomes unbounded, but we shall show that the lenght $L(\gamma_p)$ remains finite. We change variables as $w \mapsto v := (1-t)^{-1}(w+1)$ and estimate

$$\begin{aligned}
\int_{B_\varepsilon(-1)} \Phi(\cdot,t) &< \int_{B_{\frac{\varepsilon}{(1-t)}}(0)} \frac{C_1|(1-t)+2tv|^2}{(|v+1|^2+C_2|tv+1|^2)^2} \, d^2v \\
&< C_1 \int_{B_{\frac{\varepsilon}{(1-t)}}(0)} \frac{(1-t)^2+4(1-t)t|v|+4t^2|v|^2}{(|v+1|^2+C_2|tv+1|^2)^2} \, d^2v \\
&< C_3 \int_{B_{\frac{\varepsilon}{(1-t)}}(0)} \frac{(1-t)^2+4(1-t)t|v|+4t^2|v|^2}{(|v|^2+1)^2} \, d^2v \\
&= 2\pi C_3 \int_0^{\frac{\varepsilon}{(1-t)}} \frac{(1-t)^2|v|+4(1-t)t|v|^2+4t^2|v|^3}{(|v|^2+1)^2} d|v| \\
&= 2\pi C_3 \left[\frac{(1-t)^2}{2} \frac{\varepsilon^2}{\varepsilon^2+(1-t)^2} \right. \\
&\quad + 2(1-t)t \left(\arctan \frac{\varepsilon}{1-t} - \frac{\varepsilon(1-t)}{\varepsilon^2+(1-t)^2} \right) \\
&\quad \left. + 2t^2 \left(\log \frac{\varepsilon^2+(1-t)^2}{(1-t)^2} - \frac{\varepsilon^2}{\varepsilon^2+(1-t)^2} \right) \right] \\
&< 2\pi C_3 \log\left(1+\frac{\varepsilon^2}{(1-t)^2}\right) \\
&< 4\pi C_3 \log \frac{1}{1-t},
\end{aligned}$$

where again C_1, C_2 and C_3 are suitable constants dependent on ε only. Hence the length of γ_p is bounded above by a finite quantity:

$$L(\gamma_p) < C \int_{1/2}^{1} \left(1 + \log \frac{1}{1-t}\right)^{1/2} dt. \tag{31}$$

(iii) $p = \infty$:

Finally, we define the path γ_∞ with domain $[\frac{1}{2}, 1[$ by

$$\gamma_\infty(t) : w \mapsto \frac{tw + 1}{w^{n-1}(w+t)} = \tilde{W}_t(w).$$

Again, it is easy to check that this defines a path on the fibre $\mathcal{M}_n^\infty$ with $\tilde{W}_1$ having degree $n-1$. In the formula (28) for the length of γ_∞ we now have

$$\Phi(w,t) = \frac{4|w - w^{-1}|^2}{(|w|^{n-1}|w+t|^2 + |tw+1|^2|w|^{-n+1})^2}.$$

The rest of the argument is completely analogous (if somewhat easier) to case (ii) above, and we are again led to a finite bound for $L(\gamma_\infty)$ identical to (31). ☐

7 Dynamics of degree-two lumps

To have some insight on the nontrivial scattering of multilumps, and in particular how the lump collapse established in Theorem 6.1 may be realised, we may hope to compute particular geodesics of the multilump metrics. This is a very difficult task, but in this section we show that some results can be obtained in the simplest case of $n = 2$ lumps.

7.1 Symmetric two-lumps

Even for $n = 2$, $\dim_{\mathbb{R}} \mathcal{M}_2^p = 6$ is too large to render the computation of the total metric feasible. We start by restricting our attention to a totally geodesic submanifold.

Lemma 7.1. The following are totally geodesic submanifolds for the metrics γ:

(i) $\tilde{\Xi}_0 := \{z \mapsto \frac{\alpha}{\cosh z + \beta} : \alpha \in \mathbb{C}^\times, \beta \in \mathbb{C}\} \subset \mathcal{M}_2^0$

(ii) $\Xi_\infty := \{z \mapsto \frac{e^{-z}+\alpha}{e^z+\alpha} : \alpha \in \mathbb{C} - \{-1, 1\}\} \subset \mathcal{M}_2^\infty$.

Proof. Part (i) follows from the direct application of Lemma 4.2 to the set S consisting of the isometry $\tilde{\sigma}_3$ defined in (24), and using the parametrisation (15) for $W \in \mathcal{M}_2^p$. Part (ii) follows from the same argument (now using (17) to express W), combined with the application of isometries of the form $W(z) \mapsto -W(z)$ and $W(z) \mapsto W(-z)$ discussed in Proposition 4.1. ☐

We shall now focus on the two cases $p = 0$ and $p = \infty$ separately.

7.1.1 $p = 0$

The submanifold $\tilde{\Xi}_0$ in Lemma 7.1 has real dimension four. Computing the restriction of the metric to it is still too complicated, but we can achieve this in certain submanifolds of codimension two. If they are totally geodesic in $\tilde{\Xi}_0$, they will also be in $(\mathcal{M}_2^0, \gamma)$.

We start by applying again Lemma 4.2 to $\tilde{\Xi}_0$ with S consisting of the isometry

$$W(z) \mapsto -W(z - i\pi).$$

The fixed point set in $\tilde{\Xi}_0$ is

$$\Xi_0 := \{z \mapsto \alpha \operatorname{sech} z : \alpha \in \mathbb{C}^\times\} \subset \mathcal{M}_2^0,$$

which is a two-dimensional totally geodesic submanifold. The SO(2) isometry subgroup of target rotations acts on Ξ_0, and this implies that the restriction of the metric to this submanifold is independent of $\vartheta := \arg \alpha$. It can be written as

$$\gamma|_{\Xi_0} = I(a)(da^2 + a^2 d\vartheta^2) \tag{32}$$

where $a := |\alpha|$ and $I(a)$ is given from (16) as

$$I(a) = 4 \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \frac{|\operatorname{sech} z|^2}{(1 + a^2 |\operatorname{sech} z|^2)^2}\, dx\, dy. \tag{33}$$

We shall now explain how to compute the integral (33) in closed form. We introduce the quantity

$$J_\Lambda(a) := 2i \int_\Sigma \left(\frac{1}{1 + a^2 |\operatorname{sech} z|^2} - \Lambda(|\operatorname{sech} z|^2) \right) dz \wedge d\bar{z}$$

dependent on a *regulator* Λ, which we define to be an integrable function $\Lambda : [0, +\infty[\to \mathbb{R}$ with $\operatorname{supp}(\Lambda) \subset [0, 1]$ such that $J_\Lambda(a)$ above exists as a real number. Our aim will be to compute $J_\Lambda(a)$ for suitable Λ, and then determine $I(a)$ as

$$I(a) = -\frac{1}{2a} \frac{d}{da} J_\Lambda(a). \tag{34}$$

It should be noted that the value of $I(a)$ is then independent of the regulator; more precisely, it will become clear (cf. (36) below) that $I(a)$ as given by (34) is invariant under any of the transformations

$$\Lambda(r) \mapsto \Lambda(r) + r\lambda(r)$$

where λ is an element of $L^2([0, +\infty[)$ supported on a subset of $[0, 1]$.

To calculate $J_\Lambda(a)$, we start by changing variables using $z \mapsto u = \operatorname{sech}^2 z$; this is a map $\Sigma \to \mathbb{CP}^1$ of degree four, so we obtain

$$J_\Lambda(a) = 2i \int_{\mathbb{C}} \left(\frac{1}{1 + a^2 |u|} - \Lambda(|u|) \right) \frac{du \wedge d\bar{u}}{|u|^2 |u - 1|}.$$

In terms of polar coordinates r and θ for the u-plane,

$$\begin{aligned} J_\Lambda(a) &= 4\int_0^\infty \left(\int_0^\pi \frac{d\theta}{\sqrt{r^2+1-2r\cos\theta}} + \int_0^\pi \frac{d\theta}{\sqrt{r^2+1+2r\cos\theta}}\right)\left(\frac{1}{1+a^2r} - \Lambda(r)\right)\frac{dr}{r} \\ &= 16\int_0^\infty \left(\frac{1}{1+a^2r} - \Lambda(r)\right)\frac{K(k(r))dr}{r(r+1)}. \end{aligned}$$

Here, K is Legendre's complete elliptic integral of the first kind, and we have made use of the standard formulas (289.00) and (291.00) in [25], with

$$k(r) = \frac{2\sqrt{r}}{r+1}. \tag{35}$$

To proceed, we change the variable of integration from r to

$$c := \frac{1-k'}{1+k'},$$

whereby $k' := \sqrt{1-k^2}$ as usual. This requires some care, since (35) is not injective, but can be inverted as

$$\begin{aligned} r &= \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} = c \quad \text{for} \quad r \in [0,1), \\ r &= \frac{1+\sqrt{1-k^2}}{1-\sqrt{1-k^2}} = \frac{1}{c} \quad \text{for} \quad r \in [1,\infty). \end{aligned}$$

By making use of Landen's transformation (cf. e.g. [26], p. 238)

$$K(k) = \frac{2}{1+k'}K\left(\frac{1-k'}{1+k'}\right),$$

we then arrive at

$$J_\Lambda(a) = 16\int_0^1 \left(\frac{c}{c+a^2} - \frac{a^2}{1+a^2c} + \frac{1-\Lambda(c)}{c}\right)K(c)dc. \tag{36}$$

At this stage, we choose the particular regulator

$$\Lambda(c) = \begin{cases} c+1, & \text{if} \quad 0 \le c \le 1, \\ 0\ , & \text{if} \quad c > 1, \end{cases}$$

and drop the Λ subscript in J_Λ to obtain

$$J(a) = -16a^2\int_0^1 \left(\frac{1}{c+a^2} + \frac{1}{1+a^2c}\right)K(c)dc. \tag{37}$$

The integral above can be evaluated in closed form by making use of the following result, which we prove in Appendix B:

Lemma 7.2. The integral

$$f(t) := \int_0^1 \left(\frac{1}{k+t} + \frac{1}{1+tk} \right) K(k) dk \tag{38}$$

defines an analytic function on $]0, +\infty[$ which satisfies

$$f(t) = \begin{cases} \dfrac{\pi}{2} K\left(\sqrt{1-t^2}\right), & \text{if} \quad 0 < t \leq 1, \\ \dfrac{\pi}{2t} K\left(\sqrt{1-t^{-2}}\right), & \text{if} \quad t > 1. \end{cases} \tag{39}$$

Thus we can write (37) as

$$J(a) = -16a^2 f(a^2)$$

and determine from (34) the conformal factor in the metric (32) as

$$\begin{aligned} I(a) &= 16f(a^2) + 16a^2 f'(a^2) \\ &= \begin{cases} \dfrac{8\pi}{a^4-1} \left(E\left(\sqrt{1-a^4}\right) - K\left(\sqrt{1-a^4}\right) \right) & , \quad \text{if} \quad 0 < a \leq 1, \\ \dfrac{8\pi}{a^4-1} \left(a^2 E\left(\sqrt{1-a^{-4}}\right) - a^{-2} K\left(\sqrt{1-a^{-4}}\right) \right), & \text{if} \quad a > 1, \end{cases} \end{aligned} \tag{40}$$

where E is Legendre's complete elliptic integral of the second kind; here, we made use of

$$\frac{dK}{dk} = \frac{E(k) - (1-k^2)K(k)}{k(1-k^2)}.$$

Notice that the function I is smooth on $]0, \infty]$; we plot a section of its graph in Figure 3.

We now show that the conformal factor $I(a)$ given by (40) is such that Ξ_0 can be embedded in euclidean $\mathbb{R}^3$:

Lemma 7.3. The riemannian manifold Ξ_0 with metric given by equations (32) and (40) can be isometrically embedded in $\mathbb{R}^3$ as a surface of revolution.

Proof. A general surface of revolution in $\mathbb{R}^3$ is described by an embedding (in cartesian coordinates)

$$(a, \vartheta) \mapsto (a\, u(a) \cos\vartheta, a\, u(a) \sin\vartheta, v(a)),$$

where ϑ is a standard local coordinate on the circle and we take $a > 0$. Under this map, the euclidean metric of $\mathbb{R}^3$ pulls back as

$$\left(\big(u(a) + au'(a)\big)^2 + v'(a)^2 \right) da^2 + a^2 u(a)^2 d\vartheta^2.$$

In order for this to be the polar isothermal form (as in equation (32)) of a metric in two dimensions, one must set the coefficient of da^2 equal to $u(a)^2$, yielding

$$v(a) = \int^a \sqrt{-s^2 u'(s)^2 - s(u(s)^2)'} ds + \text{const}. \tag{41}$$

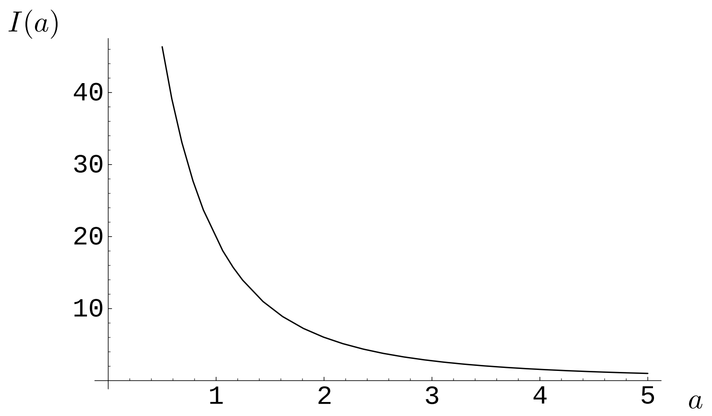


Figure 3: *The conformal factor $I(a)$ for the metric on Ξ_0.*

This determines a real function if and only if the condition

$$a(u'(a))^2 \le (u(a)^2)' \tag{42}$$

is satisfied for all a. In our case of interest, we should take

$$u(a) = \sqrt{I(a)}.$$

It is easy to check that $u'(a) < 0$ for all $a > 0$ and (42) can be expressed as

$$\frac{d}{da} \log I(a) \ge -\frac{4}{a},$$

which can be verified to hold for $a \in]0, +\infty[$. $\square$

The embedded surface has cylindrical topology and provides a good picture of the geometry of Ξ_0; we plot a section of it in Figure 4, using the construction in Lemma 7.3. It follows from

$$\lim_{a\to+\infty} \left(a\sqrt{I(a)} \right) = \sqrt{8\pi} \tag{43}$$

that this surface is asymptotic to a cylinder of radius $\sqrt{8\pi}$ for large a. Moreover,

$$\lim_{a\to 0^+} \left(a\sqrt{I(a)} \right) = 0$$

implies that it can be completed to a simply-connected surface by adding the single point at $a = 0$. However, this completion fails to be smooth. One way to see this is to consider the scalar curvature of the surface; it depends only on a, and can be easily calculated in terms of elliptic integrals from the formula (cf. [27])

$$R(a) = -\frac{1}{2aI(a)} \frac{d}{da} \left(a \frac{d}{da} \log I(a) \right).$$

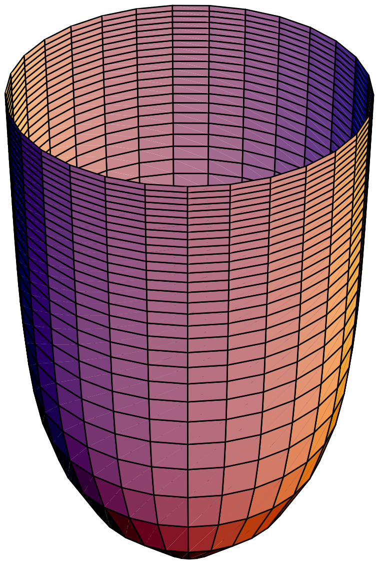


Figure 4: *The surface Ξ_0 embedded as a convex surface of revolution in $\mathbb{R}^3$.*

We find that this is a positive function on $]0, +\infty[$, monotonically decreasing, and with limits

$$\lim_{a\to 0^+} R(a) = +\infty, \qquad \lim_{a\to +\infty} R(a) = 0;$$

a plot of $R(a)$ is shown in Figure 5. Thus Ξ_0 is asymptotically flat for large a, which fits with the asymptotics already mentioned. The unboundedness of the curvature as $a \to 0$ implies that the one-point completion is not smooth at the tip $a = 0$. A rather surprising fact is that this occurs even though the profile curves of the surface $\Xi_0 \hookrightarrow \mathbb{R}^3$ approach the symmetry axis at right angles:

$$\begin{aligned} \theta_0 &= \arctan \lim_{a\to 0^+} \frac{d(au(a))}{dv(a)} \\ &= \arctan \lim_{a\to 0^+} \frac{2\sqrt{I(a)}(2I(a) - aI'(a))}{aI'(a)(4I(a) - aI'(a))} = \frac{\pi}{2} \end{aligned} \tag{44}$$

(here u and v are as defined in the proof of Lemma 7.3). Now the limit (43) implies that, as $a \to \infty$, tangents to the profile curves make an angle of $\theta_\infty = 0$ with the direction of the symmetry axis. An elementary result on differential geometry of surfaces of revolution in $\mathbb{R}^3$ then allows us to compute the total curvature of Ξ_0 from (44) as

$$\int_{\Xi_0} R = 2\pi(\sin\theta_0 - \sin\theta_\infty) = 2\pi.$$

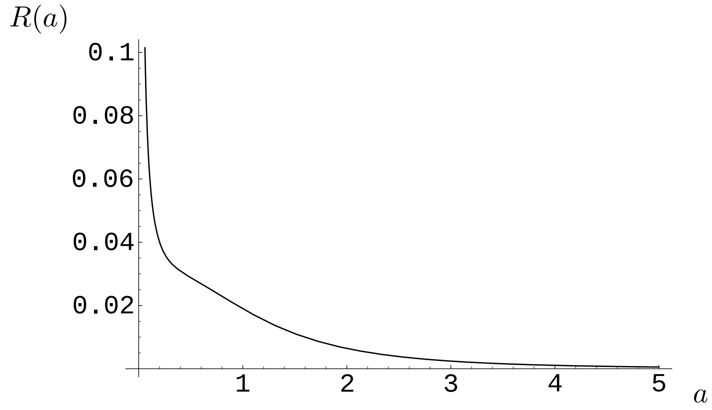


Figure 5: *The scalar curvature $R(a)$ on Ξ_0.*

This result agrees with what one would obtain for any embedded surface of revolution in $\mathbb{R}^3$ asymptotic to a cylinder at one end and smooth at the other end, by the theorem of Gauß–Bonnet.

Any meridian (given by equating ϑ to a constant) is a geodesic of the surface Ξ_0. It follows from our proof of Theorem 6.1 that the meridians are incomplete geodesics – any point on them is at finite distance from the tip $a = 0$. This can also be checked from the explicit formulas (32) and (40). It is also easy to show that the integrand in (41) never vanishes, and this implies that none of the parallels (circles of constant a) is a geodesic (cf. [28], p. 182). More general geodesics on Ξ_0 are straightforward to find as an application of Clairaut's theorem (cf. [28], pp. 183–185).

7.1.2 $p = \infty$

The metric on the totally geodesic submanifold Ξ_∞ introduced in Lemma 7.1 can be written as

$$\gamma|_{\Xi_\infty} = \gamma_{\alpha\bar{\alpha}}(\alpha)|d\alpha|^2 \tag{45}$$

where

$$\gamma_{\alpha\bar{\alpha}}(\alpha) = 2i\int_{\mathbb{C}} \frac{|w - w^{-1}|^2}{(|w+\alpha|^2 + |w^{-1}+\alpha|^2)^2}\frac{dw \wedge d\bar{w}}{|w|^2}.$$

Notice that the prefactor $\gamma_{\alpha\bar{\alpha}}$ diverges at the points $\alpha = 1$ and $\alpha = -1$, where the degree of the maps $\frac{e^{-z}+\alpha}{e^{z}+\alpha}$ jump to one. Notice also that $\gamma_{\alpha\bar{\alpha}}$ depends on both the modulus and the argument of α, which makes the computation of the integral in closed form a more difficult task. However, we can calculate some geodesics of $\gamma_{\alpha\bar{\alpha}}$ even without performing the integral.

Lemma 7.4. The intervals $]-\infty,-1[$, $]-1,1[$, $]1,+\infty[$ and $i\mathbb{R}$ in the complex plane parametrised by α are all geodesics of the metric (45).

Proof. This follows again from Lemma 4.2. Invariance of the maps

$$W(z) = \frac{e^{-z}+\alpha}{e^{z}+\alpha} \quad \in \Xi_\infty$$

with respect to the isometry $\tilde{\sigma}_2$ defined in (23) imposes the relation

$$\alpha = \bar{\alpha};$$

on Ξ_∞, this is the equation for the union of the three intervals $]-\infty,-1[$, $]-1,1[$ and $]1,+\infty[$, which are therefore geodesics of $\gamma|_{\Xi_\infty}$. Similarly, considering $R : w \mapsto w^{-1}$, invariance under the isometry

$$T_{i\pi} \circ R \circ \tilde{\sigma}_2 : \frac{e^{-z}+\alpha}{e^{z}+\alpha} \mapsto \frac{e^{z}+\bar{\alpha}}{e^{-z}+\bar{\alpha}}$$

leads to the constraint

$$\alpha = -\bar{\alpha},$$

and this shows that $i\mathbb{R}$ is also a geodesic. □

The proof of Theorem 6.1 implies that the geodesic segment in Ξ_∞ corresponding to $\alpha \in [\frac{1}{2}, 1[$, has finite length with respect to the metric (45), and the same is true for any other piece of the intervals in Lemma 7.4 that accumulates at $\alpha = 1$ or -1.

Analogously to the case of Ξ_0 above, we can prove that the scalar curvature of Ξ_∞ becomes unbounded in the neighbourhood of the points where the metric becomes singular:

Lemma 7.5. The scalar curvature R of Ξ_∞ satisfies

$$\lim_{\alpha \to \pm 1} R(\alpha) = +\infty.$$

Proof. We focus on the limit $\alpha \to +1$ without loss of generality. Consider the paths $\gamma_{\infty,u} : [\frac{1}{2}, 1[\to \Xi_\infty$, with u in the unit circle of $\mathbb{C}$, given by

$$\gamma_{\infty,u}(t) : w \mapsto \frac{(ut-u+1)w+1}{w(w+ut-u+1)} = \tilde{W}_{u,t}(w).$$

These paths parametrise radial segments tending to $\alpha = 1$. (Notice that $\gamma_{\infty,1}$ coincides with γ_∞ defined in the proof of Theorem 6.1 (iii)). An analogous argument to the one in Theorem 6.1, and which we shall not reproduce here, leads to the following estimate for the curvature of each $\gamma_{\infty,u}$:

$$
\begin{aligned}
k_u(t) &= \frac{\dfrac{\partial}{\partial t}\displaystyle\int_{\mathbb{C}} \Phi_u(w,t)\, d^2w}{2\left(\displaystyle\int_{\mathbb{C}} \Phi_u(w,t)\, d^2w\right)^{1/2}} \\
&> C_u \frac{\dfrac{1}{1-t} - \dfrac{(1-t)^3}{(\varepsilon^2+(1-t)^2)^2}}{\left(\log \dfrac{\varepsilon^2+(1-t)^2}{(1-t)^2} - \dfrac{\varepsilon^2}{\varepsilon^2+(1-t)^2}\right)^{1/2}}.
\end{aligned}
$$

Here, $\Phi_u(w,t)$ is again determined from $\tilde{W}_{u,t}(w)$ by (29) and C_u denote positive constants dependent on u and on a fixed $\varepsilon \in]0, \frac{1}{2}[$. Since the right-hand-side is strictly positive when $t \to 1^-$, the scalar curvature must be positive in some neighbourhood of $\alpha = 1$ by a continuity argument. The inequality above also implies that the minimal curvature becomes unbounded as $t \to 1^-$, and the result follows. $\square$

7.2 Two-lump scattering

Now that we have found some geodesics on $\mathcal{M}_2$, we can interpret them in terms of soliton scattering by plotting the energy density (27) along them.

7.2.1 $p = 0$

We have seen that Ξ_0 is a surface of revolution. The meridians of this surface define a one-parameter family of geodesics; one of them is

$$\Gamma_0 := \{z \mapsto \alpha \operatorname{sech} z : \alpha \in]0, +\infty[\}.$$

All the other meridians are related to Γ_0 through a fixed target rotation, under which the energy density (27) does not change, and therefore describe the same type of process. This process can be interpreted as a frontal collision of two lumps as we let α decrease from a large value to zero; a plot of the energy densities is shown in Figure 6. For large α, the configuration can be roughly described as a superposition of two single lumps with SO(2) symmetry (thus having antipodal endpoints) which are far apart. As α decreases, these lumps approach each other (meaning that the regions of large $\mathcal{E}$ come closer together on the cylinder), and at close distance the approximate SO(2) symmetry of the energy density breaks down. At this stage, energy density peaks form over antipodal points of a circle transverse to the axis of the cylinder; these peaks become more and more pronounced, with a singularity forming in the limit $\alpha \to 0^+$. As we have seen, this is achieved in finite time, which is a symptom of the incompleteness of the metric. This type of phenomenon is not surprising for the $\mathbb{CP}^1$ model; it has been reproduced in numerical studies of scattering lumps on the plane [29].

It is a consequence of Clairaut's theorem that the geodesics of Ξ_0 other than the meridians are complete and do not involve singular peaking. A simple way to understand

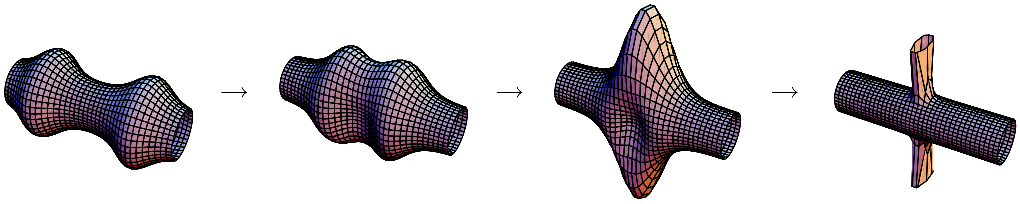


Figure 6: *Frontal colision of a symmetric configuration of 2-lumps on $\mathcal{M}_2^0$, corresponding to the geodesic Γ_0. Energy densities of lumps of the form $W(z) = \alpha \operatorname{sech} z$ are plotted on top of a cylinder of unit radius for $|\mathrm{Re}\, z| \leq 4$:* (a) $\alpha = 5$; (b) $\alpha = 2$; (c) $\alpha = 1$; (d) $\alpha = 0.05$.

them (cf. [11, 12]) is to interpret the geodesic flow on Ξ_0 as the dynamics of a particle in $]0, +\infty[$ with position-dependent mass $I(a)$ and lagrangian

$$L = \frac{1}{2} I(a)\dot{a}^2 + p_\vartheta^2\, U_{\text{eff}}(a),$$

where

$$U_{\text{eff}}(a) = \frac{1}{2a^2 I(a)}. \tag{46}$$

Here, $p_\vartheta = a^2 I(a)\dot{\vartheta}$ is the (conserved) momentum conjugate to the cyclic coordinate ϑ, which can be interpreted as a coupling to the effective potential (46). We plot $U_{\text{eff}}(a)$ in Figure 7; it is a monotonically decreasing function and has a horizontal asymptote at $\frac{1}{16\pi}$ as $a \to +\infty$, corresponding to the limit (43). From the plot, it is immediately clear that the complete geodesics of Ξ_0, corresponding to taking $p_\vartheta \neq 0$, describe reflection collisions. In these processes, the peaking of the energy density as the lumps approach each other is reversed at a certain instant, after which the lump separation grows to infinity. For instance, the sequence (a)→(b)→(c)→(b)→(a) of configurations in Figure 6 represents snapshots of one such reflection. Processes of this type are accompanied by a rotation of the overall phase of the field configuration and are generic among the motions on the submanifold Ξ_0.

7.2.2 $p = \infty$

The geodesics we have found on $\mathcal{M}_2^\infty$ give three qualitatively distinct two-lump motions:

$$\Gamma_j := \left\{ z \mapsto \frac{e^{-z} + \alpha}{e^z + \alpha} : \alpha \in I_j \right\}, \qquad j = 1, 2, 3$$

with

$$I_1 :=]1, +\infty[,$$
$$I_2 := i\mathbb{R},$$
$$I_3 :=]-1, 1[.$$

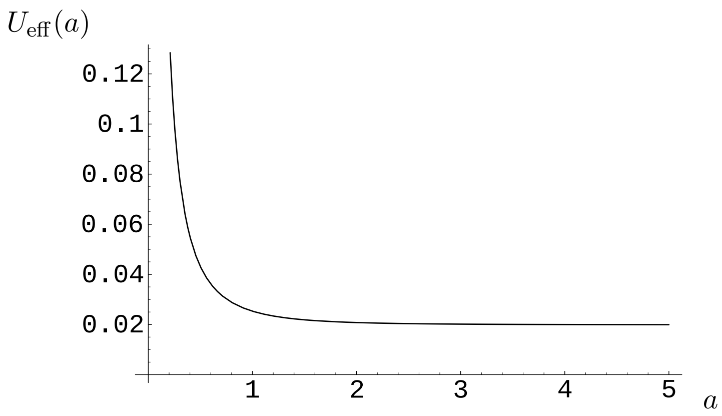


Figure 7: *The effective potential* $U_{\text{eff}}(a)$*.*

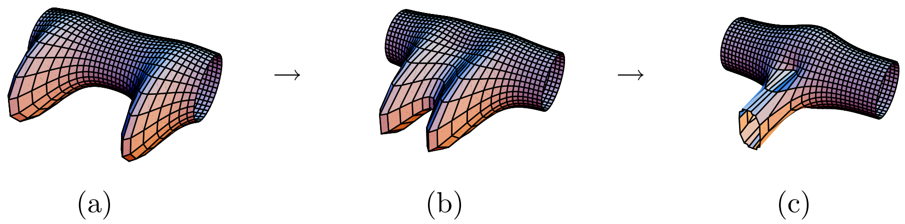
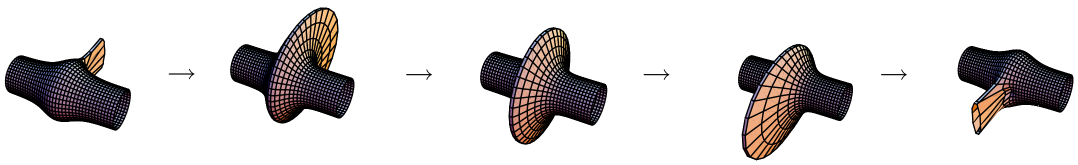


Figure 8: *Frontal collision of lumps, corresponding to the geodesic* Γ_1 *of* $\mathcal{M}_2^\infty$*. Energy densities of lumps of the form* $W(z) = (e^{-z}+\alpha)/(e^{z}+\alpha)$ *are plotted on a cylinder of unit radius for* $|\mathrm{Re}\, z| \leq 3$*:* (a) $\alpha = 5$*;* (b) $\alpha = 2$*;* (c) $\alpha = 1.1$.

Energy densities of the process described by Γ_1 are plotted in Figure 8. We can say it consists of a frontal collision of two peaked single lumps of the same shape along a longitudinal (straight) line. At collision, the two peaks coalesce and develop a singularity over the midpoint of their initial positions (local maxima of $\mathcal{E}$) in the limit $\alpha \to 1^+$.

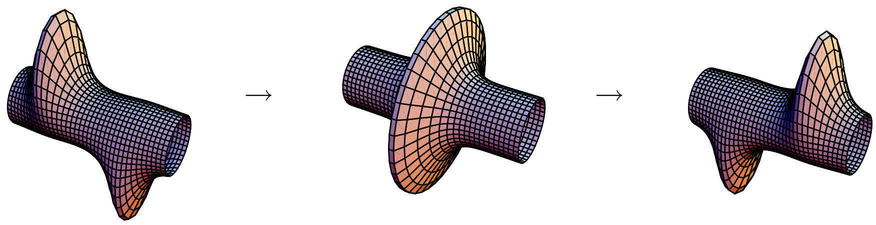
The geodesic Γ_2 describes processes related to the decay of the two-lump $W(z) = e^{-2z}$ (whose energy density exhibits SO(2) symmetry) to peaked configurations that become singular in the limit $\alpha \to \mp 1^{\pm}$. Strictly speaking, this process alone is not of scattering nature because it does not connect configurations of asymptotically well-separated maxima of energy density. Alternatively, one could interpret the geodesic as a tunneling of single-peaked two-lumps through the cylinder, passing through the SO(2)-symmetric configuration. This is illustrated in Figure 9.

Finally, the geodesic Γ_3 may be interpreted as a scattering process of two single lumps with the same shape along longitudinal lines positioned antipodally on the cylinder;



(a) (b) (c) (d) (e)

Figure 9: *Tunneling of 2-lumps through the cylinder, corresponding to the geodesic* Γ_2 *of* $\mathcal{M}_2^\infty$*. Energy densities of lumps of the form* $W(z) = (e^{-z} + \alpha)/(e^z + \alpha)$ *are plotted on a cylinder of unit radius for* $|\mathrm{Re}\, z| \leq 3$*:* (a) $\alpha = -0.95$*;* (b) $\alpha = -1/3$*;* (c) $\alpha = 0$*;* (d) $\alpha = 1/3$*;* (e) $\alpha = 0.95$*.*



(a) (b) (c)

Figure 10: *Scattering of antipodal lumps, corresponding to the geodesic* Γ_3 *of* $\mathcal{M}_2^\infty$*. Energy densities of lumps of the form* $W(z) = (e^{-z} + \alpha)/(e^z + \alpha)$ *are plotted on a cylinder of unit radius for* $|\mathrm{Re}\, z| \leq 3$*:* (a) $\alpha = 3i$*;* (b) $\alpha = 0$*;* (c) $\alpha = -3i$*.*

energy densities are plotted in Figure 10. If the lumps travel past each other, there is again an instant for which the energy density of the configuration has SO(2) symmetry, and after that each individual lump continues its motion along the longitudinal line with no significant distortion of shape.

8 Discussion

In this paper, we have considered the adiabatic approximation to the dynamics of solitons in the $\mathbb{CP}^1$ σ-model on an infinite cylinder Σ. As in previous studies of this model on other surfaces, for each degree $n \in \mathbb{Z}$ there is a smooth, finite-dimensional moduli space $\mathcal{M}_n$ parametrising static solutions (n-lumps); in our case, this space is modelled on the space of rational maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ and is therefore a complex manifold. We have found that the approximation defines a dynamical system by automorphisms of a natural map $\ell : \mathcal{M}_n \to \mathbb{CP}^1 \times \mathbb{CP}^1$ specifying the boundary values of the fields. On each fibre, these automorphisms are defined by the geodesic flow of the L^2 metric, which is regular and Kähler. By means of this fibration, we avoid making reference to degenerate metrics as in [10]. Although our language could be adapted to deal with lumps on the plane, in that case the boundary values of the fields alone are still not enough to specify a sufficiently fine fibration of the moduli spaces to render the metrics regular.

Lumps of degree one are characterised by a shape function d taking values in $]0, \pi]$ (the distance of their endpoints), together with a location (a point on Σ if $d \neq 0$ or a transversal circle if $d = 0$) and a physically irrelevant global phase. Their adiabatic dynamics is trivial: it reduces to uniform motion of their location on the cylinder with shape-independent inertial mass. This is similar to the $\mathbb{CP}^1$ σ-model on the plane, where the only possible adiabatic motion of one-lumps is also uniform motion along geodesics, i.e. straight lines [10].

The dynamics of multilumps is more interesting to study. We established that all the metrics for multilumps on the cylinder are incomplete. Again, this parallels an analogous result for lumps on the plane, as put forward by Sadun and Speight in [14]. Incompleteness of the metric translates into the possibility of lump collapse in finite time in the adiabatic approximation. Using standard symmetry considerations, we have found totally geodesic submanifolds for the metrics on two types of fibres $\mathcal{M}_2^{(p,q)} = \ell^{-1}(\{p,q\})$, namely for $p = q$ and p, q antipodal, and some geodesics on them. We have also found explicit formulae for the metric on one two-dimensional totally geodesic submanifold of $\mathcal{M}_2^0 \cong \mathcal{M}_2^{(p,p)}$, which involves elliptic integrals. This metric is incomplete, and the corresponding lump collapse can be plotted with no difficulty (Figure 6). Similarly, some of the geodesics we found for p and q antipodal exhibit lump collapse (Figures 8 and 9). It is still an unsettled question how to interpret finite-time collapse (which is understood as a feature of the adiabatic approximation) at the full field theory level. As the metric becomes singular, one may expect the approximation to break down; on the other hand, numerical simulations of the field theory seem to support the claim that collapse in finite time should also occur in the full dynamics. Another question is whether the dynamics is well defined beyond collapse. A rather striking feature of the geodesics describing collapse that we found is that they all have natural prolongations on the relevant moduli spaces; in particular, for the scattering processes we described on Ξ_∞, the whole real α-axis can be interpreted as a process of double scattering at 90^{o} of two one-lumps approaching first along a generatrix of the cylinder, then travelling along a transversal circle, and finally separating along the opposite generatrix, which is very natural to expect from our intuition on (second-order) soliton dynamics in two dimensions.

The scattering processes corresponding to the geodesics that we found explicitly turn out to be rather unique when compared to previous results on other surfaces, a fact that is due to the different topology of the cylinder. It should be expected that more generic geodesics will give rise to more familiar processes, in particular the frontal scattering at 90^{o}. In fact, we have found curves on the submanifold $\tilde{\Xi}_0$ which are close to geodesics (in the sense that the Christoffel symbols related to transverse motion are small in some region) and that describe processes of this type.

There is some belief that lump configurations at collapse are supressed at the quantum level. Following Gibbons and Manton [7], the quantum-mechanical version of the adiabatic dynamics should be based on a Schrödinger equation on each $\mathcal{M}_n^{(p,q)}$ using the covariant laplacian of the L^2 metric, but as a correction one expects an effective potential term given by the scalar curvature of the moduli space [30]. Accordingly, wavefunctions should be given zero boundary values whenever the scalar curvature diverges. In our examples, we found that the scalar curvatures of Ξ_0 and Ξ_∞ blow up as the boundary of the moduli space is approached upon collapse. In [17], Speight also found a divergence of the scalar curvature of the moduli space of one-lumps on S^2 preventing collapse, but our results directly refer to the interacting case and therefore give more substantial support to the hope that the degree of lumps should be conserved in the quantum field theory.

Acknowledgements

I would like to thank Klaus Kirsten, Nick Manton, Avijit Mukherjee, Oliver Schnürer and especially Martin Speight for discussions. This work was supported by the Max-Planck-Gesellschaft, Germany.

Appendix A: Proof of Lemma 4.2

Totally geodesic submanifolds $N \subset M$ can be characterised by the property that (the continuation of) any geodesic of the ambient metric starting tangent to N will never leave N. Suppose, for a contradiction, that there is a geodesic $r :]-\varepsilon, \varepsilon[\to M$ of M such that

$$r(t) \in F \quad \forall t \in]-\varepsilon, 0] \tag{47}$$

and

$$r(t) \notin F \quad \forall t \in]0, \varepsilon[. \tag{48}$$

It follows from (47) that $r'(0) \in T_{r(0)}F \subset T_{r(0)}M$, and as such it can be regarded as an equivalence class of paths through $r(0)$ containing paths that lie entirely on F. These paths are fixed by S, so $f_{*r(0)}r'(0) = r'(0)$ for all $f \in S$. Now (48) implies that for any $\tilde{t} \in]0, \varepsilon[$ there is at least one element $\tilde{f} \in S$ such that $\tilde{f}(r(\tilde{t})) \neq r(\tilde{t})$. Since $\tilde{f} \in \mathrm{Iso}(M)$, $\tilde{f} \circ r$ is also a solution to the equation of the geodesics for (M, g); it satisfies the Cauchy data

$$\begin{cases} (\tilde{f} \circ r)(0) = r(0) \\ (\tilde{f} \circ r)'(0) = r'(0) \end{cases}$$

and is distinct from r. This contradicts the Picard–Lindelöf theorem ensuring uniqueness of solutions of ODEs. Hence F must be a totally geodesic submanifold. □

Appendix B: Proof of Lemma 7.2

The fact that f in (38) is analytic for positive real values of t follows from the (complex) analyticity of the integrand as a function of t, the integrability of the integrand as a function of k and the Leibniz rule. To compute $f(t)$ in closed form, we use an argument based on the idea that f can be extended by analytic continuation to a neighbourhood of the set

$$\{t \in \mathbb{C}^* : \operatorname{Re} t \geq 0, \operatorname{Im} t \geq 0\}.$$

We start by rewriting

$$\begin{aligned} f(t) &= \int_0^1 \left(\frac{k-t}{k^2-t^2} + \frac{1-tk}{1-t^2k^2} \right) K(k)dk \\ &= \int_0^1 \left(\frac{k}{k^2-t^2} + \frac{1}{1-t^2k^2} \right) K(k)dk + \frac{1}{t} \int_0^1 \left(\frac{k}{k^2-t^{-2}} + \frac{1}{1-t^{-2}k^2} \right) K(k)dk; \end{aligned}$$

here, the last two integrals must be interpreted as Cauchy principal values, which are easily seen to exist. To evaluate their sum, we first Wick-rotate t to it, which leads to

$$\int_0^1 \left(\frac{k}{k^2+t^2} + \frac{1}{1+t^2k^2} \right) K(k)dk - \frac{i}{t} \int_0^1 \left(\frac{k}{k^2+t^{-2}} + \frac{1}{1+t^{-2}k^2} \right) K(k)dk.$$

Each of the two terms above can be evaluated in closed form using the result (cf. formula I.(5) in [31])

$$\int_0^1 \left(\frac{k}{k^2+z^2} + \frac{1}{1+z^2k^2} \right) K(k)dk = \frac{1}{\sqrt{1+z^2}} K\left(\frac{1}{\sqrt{1+z^2}} \right).$$

This yields

$$\frac{1}{\sqrt{1+t^2}} K\left(\frac{1}{\sqrt{1+t^2}} \right) - \frac{i}{t} \frac{1}{\sqrt{1+t^2}} K\left(\frac{1}{\sqrt{1+t^2}} \right). \tag{49}$$

We have to now undo the Wick rotation in the expression above to obtain the values of f we are interested in. This must be done carefully, since the analytic continuation of K branches at the singular point $k = 1$ and the square root branches at the origin. Recall that $K(k)$ can be represented as a hypergeometric series for $0 < k < 1$ (cf. (900.00) in [25]):

$$K(k) = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2 \right) = \frac{\pi}{2} \sum_{j=0}^{\infty} \left(\frac{(2j-1)!!}{2^j j!} \right)^2 k^2. \tag{50}$$

(Here $(2n-1)!! := (2n-1)(2n-3)\cdots 1$ and $(-1)!! := 1$.) So the properties of the analytic continuation of K can be deduced from those of Gauß's ${}_2F_1$. Following common practice, we introduce a branch cut on the real axis from 1 to $+\infty$. On $\mathbb{C} - [1, +\infty]$, K is single-valued, and it commutes with complex conjugation,

$$K(\overline{k}) = \overline{K(k)}, \tag{51}$$

because the coefficients of the series in (50) are real. Across the branch cut, there is a nontrivial monodromy that accounts for a discontinuity

$$K(k) = \frac{1}{k} \left(K\left(\frac{1}{k} \right) \pm iK\left(\frac{\sqrt{k^2-1}}{k} \right) \right), \qquad k \in]1, +\infty[, \tag{52}$$

where the top/bottom signs correspond to the limits obtained when k approaches the cut from above/below. This result can be obtained by relating Kummer's solutions of hypergeometric differential equations (cf. [32], Chap. 2). By a similar argument (and (51)), one can show that

$$K(ik) = \frac{1}{\sqrt{1+k^2}} K\left(\frac{1}{\sqrt{1+k^2}}\right), \qquad k \in \mathbb{R}. \tag{53}$$

We now want to Wick-rotate t back to $-it$ in (49); one way to keep track of the branching of the functions involved is to substitute the final t by $e^{\frac{i\epsilon}{2}}t$, with ϵ small, real and positive, evaluate in terms of continuous quantities and let $\epsilon \to 0^+$ at the end. It is convenient to consider the cases $0 < t < 1$ and $t > 1$ separately. In the first case, we find that the argument of the first K in (49) after substution, $(1 - e^{i\epsilon}t^2)^{-1/2}$, has positive imaginary part, and according to (52) above we should then evaluate

$$K\left(\frac{1}{\sqrt{1-e^{i\epsilon}t^2}}\right) \to \sqrt{1-t^2}\left(K\left(\sqrt{1-t^2}\right) + iK(t)\right), \qquad 0 < t < 1.$$

On the other hand, the second term in (49) is free from branching, and we can evaluate using (51) and (53)

$$K\left(-\frac{i}{\sqrt{t^{-2}-1}}\right) = t\sqrt{t^{-2}-1}K(t), \qquad 0 < t < 1.$$

In the case $t > 1$, the argument of the first K in (49) does not lie on the branch cut after rotation, but the square root itself branches on the negative real half-axis, meaning that we should take

$$\frac{1}{\sqrt{1-e^{i\epsilon}t^2}} \to +\frac{i}{t\sqrt{1-t^{-2}}}, \qquad t > 1$$

(where of course $\sqrt{\cdot}$ always denotes the principal branch of the square root); we then find using (53)

$$K\left(\frac{i}{\sqrt{t^2-1}}\right) = \sqrt{1-t^{-2}}K\left(\frac{1}{t}\right), \qquad t > 1.$$

However, this time the second K does branch; since $(1 - e^{-i\epsilon}t^{-2})^{-1/2}$ has negative imaginary part, we should take the lower sign in (52) and find

$$K\left(\frac{1}{\sqrt{1-e^{-i\epsilon}t^{-2}}}\right) \to \sqrt{1-t^{-2}}\left(K\left(\sqrt{1-t^{-2}}\right) - iK\left(\frac{1}{t}\right)\right).$$

Adding the two terms in each of the two cases $0 < t < 1$ and $t > 1$, we finally obtain the result (39). □

References

[1] N.S. Manton: A remark on the scattering of BPS monopoles. *Phys. Lett. B* **110** (1982) 54–56.

[2] T.M. Samols: Vortex scattering. *Commun. Math. Phys.* **145** (1992) 149–180.

[3] D. Stuart: Dynamics of Abelian Higgs vortices in the near Bogomolny regime. *Commun. Math. Phys.* **159** (1994) 51–91.

[4] M.F. Atiyah and N.J. Hitchin: The Geometry and Dynamics of Magnetic Monopoles. Princeton University Press, 1988.

[5] D.M.A. Stuart: The geodesic approximation for the Yang–Mills–Higgs equations. *Commun. Math. Phys.* **166** (1994) 149–190.

[6] M. Namba: Families of Meromorphic Functions on Compact Riemann Surfaces (Lecture Notes in Mathematics 767). Springer-Verlag, 1979.

[7] G.W. Gibbons and N.S. Manton: Classical and quantum dynamics of BPS monopoles. *Nucl. Phys. B* **274** (1986) 183–224.

[8] B.J. Schroers: Quantum scattering of BPS monopoles at low energy. *Nucl. Phys. B* **367** (1991) 177–214.

[9] N.M. Romão: Quantum Chern–Simons vortices on a sphere. *J. Math. Phys.* **42** (2001) 3445–3469, `hep-th/0010277`.

[10] R.S. Ward: Slowly-moving lumps in the $\mathbb{CP}^1$ model in (2+1) dimensions. *Phys. Lett. B* **158** (1985) 424–428.

[11] J.M. Speight: Low-energy dynamics of a $\mathbb{CP}^1$ lump on the sphere. *J. Math. Phys.* **36** (1995) 796–813, `hep-th/9712089`.

[12] J.M. Speight: Lump dynamics in the $\mathbb{CP}^1$ model on the torus. *Commun. Math. Phys.* **194** (1998) 513–539, `hep-th/9707101`.

[13] J.M. Speight and I.A.B. Strachan: Gravity thaws the frozen moduli of the $\mathbb{CP}^1$ lump. *Phys. Lett. B* **457** (1999) 17–22, `hep-th/9903264`.

[14] L.A. Sadun and J.M. Speight: Geodesic incompleteness in the $\mathbb{CP}^1$ model on a compact Riemann surface. *Lett. Math. Phys.* **43** (1998) 329–334, `hep-th/9707169`.

[15] A.M. Din and W.J. Zakrzewski: Skyrmion dynamics in 2+1 dimensions. *Nucl. Phys. B* **259** (1985) 667–676.

[16] P.J. Ruback: Sigma model solitons and their moduli space metrics. *Commun. Math. Phys.* **116** (1988) 645–658.

[17] J.M. Speight: The L^2 geometry of spaces of harmonic maps $S^2 \to S^2$ and $\mathbb{RP}^2 \to \mathbb{RP}^2$,. *J. Geom. Phys.* **47** (2003) 343–368, `math.DG/0102038`.

[18] M. Haskins and J.M. Speight: The geodesic approximation for lump dynamics and coercivity of the Hessian for harmonic maps. *J. Math. Phys.* **44** (2003) 3470–3494, `hep-th/0301148`.

[19] A.A. Belavin and A.M. Polyakov: Metastable states of two-dimensional isotropic ferromagnets. *JETP Lett.* **22** (1975) 245–247.

[20] A. Lichnerowicz: Applications harmoniques et variétés kähleriennes. *Symp. Math. Bologna* **3** (1970) 341–402.

[21] R. Remmert: Funktionentheorie 2. Springer-Verlag, 2nd edition, 1995.

[22] S. Lang: Real and Functional Analysis. Springer-Verlag, 3rd edition, 1993.

[23] M.P. do Carmo: Riemannian Geometry. Birkhäuser, 1992.

[24] P.M. Sutcliffe: BPS monopoles. *Int. J. Mod. Phys. A* **12** (1997) 4663–4705, `hep-th/9707009`.

[25] P.F. Byrd and M.D. Friedman: Handbook of Elliptic Integrals for Engineers and Physicists. Springer-Verlag, 1954.

[26] F. Tricomi: Elliptische Funktionen. Akademische Verlagsgesellschaft Geest & Portig, Leipzig, 1948.

[27] J. Jost: Differentialgeometrie und Minimalflächen. Springer-Verlag, 1994.

[28] A. Pressley: Elementary differential geometry. Springer-Verlag, 2001.

[29] R.A. Leese: Low-energy scattering of solitions in the $\mathbb{CP}^1$ model. *Nucl. Phys. B* **344** (1990) 33–72.

[30] I.G. Moss and N. Shiiki: Quantum mechanics on moduli spaces. *Nucl. Phys. B* **565** (2000) 345–362, `hep-th/9904023`.

[31] M.L. Glasser: Definite integrals of the complete elliptic integral K. *J. Res. Nat. Bur. Standards Sect. B* **80B** (1976) 313–323.

[32] K. Iwasaki, H. Kimura, S. Shimomura and Y. Masaaki: From Gauss to Painlevé. Vieweg, 1991.