SINGULARITIES OF ALGEBRAIC SURFACES AND CHARACTERISTIC NUMBERS

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ABSTRACT. The Chern numbers c_1^2 and c_2 of an algebraic surface of general type satisfy the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$. If the surface contains rational or elliptic curves, then $3c_2 - c_1^2$ is positive and one can give an estimate from below using the rational and elliptic curves. This has many geometric applications. I report on work of R. Kobayashi and Y. Miyaoka and use the Bonn dissertation of Th. Höfer and the Bonn Diplomarbeit of K. Ivinskis.

1. For every compact smooth algebraic surface X the Chern numbers c_2 and c_1^2 are defined. The Chern number c_2 equals the Euler number e of X, whereas c_1^2 is the selfintersection number of a canonical divisor of X. If X is a surface of general type, then $c_2 > 0$ and

$$c_1^2 \leq 3c_2 \quad .$$

The famous inequality $c_1^2 \leq 3c_2$ has a long history. It was proved by Y. Miyaoka [15] using ideas due to F.A. Bogomolov (compare A. Van de Ven [20]). It was proved independently by differential-geometric methods for the case that the canonical bundle of X is ample: According to T. Aubin and S.T. Yau there exists a unique Einstein-Kähler metric on X and by a result of H. Guggenheimer (1952) the difference $3c_2 - c_1^2$ is then given in terms of this metric by an integral over X with non-negative integrand which measures the deviation from constant holomorphic sectional curvature. Therefore $c_1^2 \leq 3c_2$ (Yau [22]). Furthermore (Yau [22]) the equation $c_1^2 = 3c_2$ implies that the universal cover of X is the ball

$B = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$. This important conclusion was proved by Yau and Miyaoka also in the case that the canonical bundle is not necessarily ample (see [16]).

We refer to the Bourbaki lecture of J.-P. Bourguignon [1] for further references to the literature.

2. As explained above, $c_1^2 \leqq 3c_2$ for a surface X of general type, and $c_1^2 = 3c_2$ if and only if the universal cover of X is the ball B . (The "if" part is the "proportionality" of [7].) If X contains rational or elliptic curves, then, clearly, the universal cover of X cannot be the ball and hence $c_1^2 < 3c_2$. Therefore, to a configuration E of rational or elliptic curves a positive number $m(E)$ should be assigned such that

$$3c_2 - c_1^2 \geqq m(E) \quad . \tag{1}$$

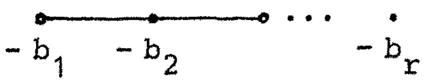
If E is a disjoint union of finitely many smooth elliptic curves C_j on X , then according to F. Sakai [19]

$$3c_2 - c_1^2 \geqq \sum(-C_j^2) \quad . \tag{2}$$

The selfintersection numbers C_j^2 are negative, since X is a surface of general type. Y. Miyaoka [17] has studied numbers $m(E)$ for configurations of rational curves. I discussed such questions with him during one of his visits in Bonn.

Let us consider an example. Suppose $S_1,\dots,S_r$ is a chain of smooth rational curves, where S_j intersects S_{j+1} transversally in exactly one point $(j=1,\dots,r-1)$ and $S_iS_j = 0$ for $|j-i| > 1$. We assume $-S_jS_j = b_j \geqq 2$. The dual graph of this chain is

(3)



For the continued fractions

$$b_1 - \frac{1|}{|b_2} - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_r} = \frac{n}{q}$$

$$b_r - \frac{1|}{|b_{r-1}} - \frac{1|}{|b_{r-2}} - \cdots - \frac{1|}{|b_1} = \frac{n}{q'}$$

we have

$$0 < q < n \quad , \quad 0 < q' < n \quad , \quad qq' \equiv 1 (\mathrm{mod}\, n) \quad .$$

In fact, this chain of rational curves is the resolution of the quotient singularity $\mathbb{C}^2/\mu_n$ where the group

$$\mu_n = \{\alpha \in \mathbb{C} : \alpha^n = 1\}$$

acts by $\alpha(z_1, z_2) = (\alpha z_1, \alpha^q z_2)$. Compare [6]. For the chain (3) of rational curves we can define the local Euler number e_{loc} which equals $r+1$. It is the Euler number of r twodimensional spheres $S_1, \ldots, S_r$ where a point on S_j is identified with a point of S_{j+1} for $j = 1, \ldots, r-1$. We also have a local canonical divisor $K_{loc} = \sum_{i=1}^{r} c_i S_i$ where the rational numbers c_i are defined by the linear equations

$$K_{loc} S_i + S_i S_i = -2$$

$$\sum_{i=1}^{r} c_i S_i S_j \qquad = b_j - 2$$

corresponding to the adjunction formula.

The intersection matrix $S_i S_j$ is negative-definite of determinant $(-1)^r n$.

Let E be the chain (3) . We define

$$m(E) = 3\, e_{loc} - K^2_{loc} - \frac{3}{n}$$

and expect an inequality (1). Why? Let X' be the surface obtained by collapsing E to a point. It has one singular point.

Suppose (as a "Gedankenexperiment") that there is a smooth surface Y with an action of μ_n which is free outside one point p such that $Y/\mu_n = X'$. Then

$$3c_2(Y) - c_1^2(Y) = n(3c_2(X) - c_1^2(X) - m(E))$$

and (1) follows for X because Y is smooth of general type and satisfies the Miyaoka-Yau inequality. This is no proof at all, but motivates the definition of $m(E)$ and indicates how to look for a proof.

In his recent Bonn Diplomarbeit K. Ivinskis [11] has given a nice formula for $m(E)$ for a chain (3). We have

$$m(E) = 1 + \sum_{i=1}^{r} (b_i+1) + \frac{q+q'-1}{n} \quad . \tag{4}$$

Similar formulas can be written down for all quotient singularities, i.e. for all configurations of rational curves arising from the resolution of quotient singularities (compare [2]). This becomes especially simple for the rational double points (see 8. below), i.e. for the configurations A_r, D_r, E_6, E_7, E_8. Here all curves have selfintersection number -2 which implies $K_{loc} = 0$ and $m(E) = 3e_{loc} - \frac{3}{|G|}$ where G is the finite group acting on $\mathbb{C}^2$ (and freely on $\mathbb{C}^2 - \{0\}$) such that $\mathbb{C}^2/G$ gives the quotient singularity. For example, A_r is the configuration (3) with $b_i = 2$ and

$$m(A_r) = 3(r+1) - \frac{3}{r+1} \tag{5}$$

which agrees with (4) since $n = r+1$, $q = q' = r$. For E_8 we have $e_{loc} = 9$ and

$$m(E_8) = 27 - \frac{3}{120} = 27 - \frac{1}{40} \quad .$$

3. THEOREM. Let X be a smooth surface of general type and $E_1, \ldots, E_k$ configurations (disjoint to each other) of rational curves (arising from quotient singularities) and $C_1, \ldots, C_p$ smooth elliptic curves (disjoint to each other and disjoint to the E_i). Let c_1^2, c_2 be the Chern numbers of X. Then

$$3c_2 - c_1^2 \geq \sum_{i=1}^{k} m(E_i) + \sum_{j=1}^{p} (-C_j^2) \quad . \tag{6}$$

This is a part of a theorem of Miyaoka [17]. It includes the older result of F. Sakai on elliptic curves. The inequality (6) is already true if the Kodaira dimension of X is non-negative.

4. R. Kobayashi ([13],[14]) has developped a theory of Einstein-Kähler metrics for V-manifolds (surfaces with quotient singularities) which are not necessarily complete (they are surfaces with elliptic curves removed). S.T. Yau told me that he also studied

these problems. R. Kobayashi was able to prove the inequality (6) and to prove a theorem for the case that equality holds in (6). He had to introduce some additional assumptions concerning (-1)-curves and (-2)-curves, i.e., smooth rational curves with selfintersection number -1 or -2 respectively. We simplify by making these assumptions unnecessarily strong.

THEOREM. *Let* X, E_i, C_j *be as in the above theorem. Suppose that* X *is minimal, i.e. does not contain* (-1)-*curves and that each* (-2)-*curve is contained in one configuration* E_i . *If equality holds in* (6), *then there exists a discrete subgroup* Γ *of the group of automorphisms of the ball, such that* $X' - \cup C_j = \Gamma \backslash B$. *Here* X' *is the singular surface obtained from* X *by blowing down the* E_i . *The group* Γ *has only isolated points in* B *with non-trivial isotropy group. They give the quotient singularities of* X' . *The group* Γ *has* p *"cusps". If one compactifies* $\Gamma \backslash B$ *to* $\overline{\Gamma \backslash B}$ *by adding* p *points at infinity for the* p *cusps, one gets* p *singular points which are resolved by the* p *elliptic curves* C_j . *Thus* X *is the smooth model* (*minimal desingularization*) *of* $\overline{\Gamma \backslash B}$.

J.C. Hemperly [5] was the first to study the singularities at the cusps of surfaces $\overline{\Gamma \backslash B}$. An extensive study of the surfaces $\overline{\Gamma \backslash B}$ if Γ is a Picard modular group was carried out in many papers by R.-P. Holzapfel.

5. AN EXAMPLE. Consider the following surface Y in $P_4(\mathbb{C})$ with homogeneous coordinates $x_0, x_1, \ldots, x_4$.

$$\sum_{i=0}^{4} x_i^5 = 0 \quad , \quad \sum_{i=0}^{4} x_i^{15} = 0 \quad .$$

It covers the cubic surface S

$$\sum_{i=0}^{4} u_i = 0 \quad , \quad \sum_{i=0}^{4} u_i^3 = 0$$

which is the Clebsch diagonal surface (Clebsch 1871). It was studied in [8] in relation to Hilbert modular surfaces. The covering map $Y \longrightarrow S$ is given by $u_i = x_i^5$ and has degree 5^4 . The cubic surface S is smooth. The hyperplane section $u_i = 0$ is a cubic curve consisting of 3 lines intersecting each other in 3 points. This determines 15 distinguished points of S (the point 0:1:-1:1:-1 and all permutations) over which we have $15 \cdot 5^3 = 1875$ points of Y which are singular, in fact they are A_4-singularities. The three hyperplane sections $u_1 = 0$, $u_2 = 0$, $u_3 = 0$ intersect in exactly one point. We get 10 distinguished points ((0:0:0:1:-1) and permutations). They are the 10 Eckardt points (points of intersection of three lines on the surface). Over each Eckardt point we have 5 points of Y. They are singular. Each of these 50 singularities is a cone over a Fermat curve of degree 5 (genus 6). We now pass to the smooth model X of Y by resolving all singularities in the minimal way and want to calculate the Chern numbers of X. For a complete intersection X_0 of two hypersurfaces of degrees 5 and 15 in $P_4(\mathbb{C})$ in general position the total Chern class is given by

$$1 + c_1(X_0) + c_2(X_0) = (1+5g+10g^2)(1+5g)^{-1}(1+15g)^{-1}$$

where $g \in H^2(X_0, Z)$ is the cohomology class of a hyperplane section. This gives

$$1 + c_1(X_0) + c_2(X_0) = (1+10g^2)(1+15g)^{-1}$$

$$c_1(X_0) = -15g \ , \ c_2(X_0) = 235g^2$$

$$c_1^2(X_0) = 75 \cdot 225 \ , \ c_2(X_0) = 75 \cdot 235 \quad .$$

The A_4-singularities when resolved do not influence these values (cf. Brieskorn's theory on rational double points). Each of the remaining 50 singularities is resolved in a Fermat curve C of degree 5 with Euler number -10 and selfintersection number -5. The Milnor fibre of such a singularity has Euler number $1 + 4 \cdot 4 \cdot 4 = 65$. Therefore, each singularity reduces the Euler number by $65 + 10 = 75$. Each singularity reduces $c_1^2(X_0)$ by $-a^2C^2$ where a is determined by the adjunction formula

$$a\,C\,C + C^2 = -e(C) = 10 \quad .$$

Thus $a = -3$ and $-a^2C^2 = 45$. Hence,

$$c_1^2(X) = 16875 - 50 \cdot 45 = 14625$$

$$c_2(X) = 17625 - 50 \cdot 75 = 13875 \quad .$$

There are 1875 A_4-configurations of (-2)-curves on X. We have (5)

$$m(A_4) = 15 - \frac{3}{5}$$

and

$$3c_2(X) - c_1^2(X) = 27000 = 1875 \cdot 14\frac{2}{5} \quad .$$

Hence, the equality sign holds in (6). It follows from Kobayashi's theorem in 4. that Y with the 50 Fermat curve singularities resolved equals Γ/B where Γ acts on the ball with isolated fixed points with isotropy groups of order 5.

The surface Y was originally constructed in a different way - in relation to the icosahedral line arrangement - by Th. Höfer in his Bonn dissertation [10].

6. LINE ARRANGEMENTS. We consider the complex projective plane $P_2(\mathbb{C})$ with homogeneous coordinates z_0, z_1, z_2. An arrangement of k lines is a set of k distinct lines in $P_2(\mathbb{C})$. They can be given by linear forms $\ell_1, \ldots, \ell_k$ in z_0, z_1, z_2. Let $t_r (r \geq 2)$ be the number of r-fold points, i.e., the number of points lying on exactly r lines of the arrangement. Then we have

$$\frac{k(k-1)}{2} = \sum_{r \geq 2} t_r \frac{r(r-1)}{2} \quad .$$

For an arrangement $\ell_1 = 0, \ldots, \ell_k = 0 (k \geq 3,\ t_k = 0)$ of lines we consider the function field

$$\mathbb{C}(z_1/z_0, z_2/z_0)((\ell_2/\ell_1)^{1/2}, \ldots, (\ell_k/\ell_1)^{1/2})$$

which is an abelian extension (Kummer extension) of the function field $\mathbb{C}(z_1/z_0, z_2/z_0)$ of $P_2(\mathbb{C})$ of degree 2^{k-1} and Galois group $(\mathbb{Z}/2\mathbb{Z})^{k-1}$. It determines an algebraic surface X with normal singularities which ramifies over the plane with the arrangement as locus of ramification. If the point $p \in P_2(\mathbb{C})$ lies on r lines of the arrangement $(r \geq 0)$, then there are 2^{k-1-r} points of X over p which are an orbit of the Galois group. For $r \geq 3$ these points are singular. The minimal resolution of such a point replaces the point by a smooth curve C of Euler number $2^{r-2}(4-r)$ and selfintersection number -2^{r-2}. We obtain a smooth surface Y associated to the arrangement.

Line arrangements and related algebraic surfaces were studied in [9]. Many more investigations were carried out in Th. Höfer's thesis [10] where he emphasized the relation to the work of P. Deligne and G.D. Mostow [3]. In this lecture we consider only some examples whose discussion we continue now.

As shown in [9] p. 132 we have

$$3c_2(Y) - c_1^2(Y) = 2^{k-3}(t_2+3t_3+t_4-k-\sum_{r\geq 5}(2r-9)t_r) \quad . \tag{7}$$

The surface Y contains $t_3 \cdot 2^{k-4}$ rational curves E_i of self-intersection number -2. They lie over the 3-fold points of the arrangement. The surface Y contains $t_4 \cdot 2^{k-5}$ elliptic curves C_j. They lie over the 4-fold points of the arrangement. We have $m(E_i) = \frac{9}{2}$ (see (5)) and $C_j^2 = -4$. Therefore,

$$\begin{aligned} &3c_2(Y) - c_1^2(Y) - \sum m(E_i) - \sum(-C_j^2) \\ &= 2^{k-3}(t_2 + \tfrac{3}{4}t_3 - k - \sum_{r\geq 5}(2r-9)t_r) \quad . \end{aligned} \tag{8}$$

It was shown in [9] that Y is of general type if $k \geqq 7$ and $t_k = t_{k-1} = t_{k-2} = t_{k-3} = 0$, it is of non-negative Kodaira dimension if $k \geqq 6$ and $t_k = t_{k-1} = t_{k-2} = 0$. In this talk, for simplicity, we always assumed general type. However, Miyaoka's inequality (6) is true if the surface has non-negative Kodaira dimension. From (6) and (8) we get

THEOREM. <u>For an arrangement of</u> k <u>lines in the complex projective plane we have</u>

$$t_2 + \frac{3}{4}t_3 \geqq k + \sum_{r\geq 5}(2r-9)t_r \tag{9}$$

<u>provided</u> $t_k = t_{k-1} = t_{k-2} = 0$.

The inequality (9) is an improvement of an inequality mentioned in [9] p. 140. It does not seem to be known to experts in the theory of arrangements (see the literature quoted in [9]). There is an arrangement of 8 lines with $t_2 = 10$, $t_3 = 1$, $t_6 = 1$ for which (9 is wrong.

We mention some arrangements for which (9) is sharp and for

which the surface Y is related to the ball by the theorem in 4.
a) the nine inflection points of a smooth cubic surface determine the 12 lines of the Hesse arrangement with $k = 12$, $t_2 = 12$, $t_4 = 9$, $t_r = 0$ otherwise

b) the nine inflection points define 9 lines in the dual projective plane. This gives an arrangement with $k = 9$, $t_3 = 12$, $t_r = 0$ otherwise.

c) the simple group of order 168 operates on the complex projective plane (F. Klein [12]p. 101). It has 21 involutions. Each involution leaves a line pointwise fixed. We get an arrangement with $k = 21$, $t_3 = 24$, $t_4 = 21$ and $t_r = 0$ otherwise.

d) $(z_0^m - z_1^m)(z_1^m - z_2^m)(z_2^m - z_0^m) = 0$, $m \geq 4$, defines $3m$ lines with $t_2 = 0$, $t_3 = m^2$, $t_m = 3$ and $t_r = 0$ otherwise. We have the equality sign in (9) if and only if $m = 4$ or $m = 6$.

7. DOUBLE POINTS ON HYPERSURFACES. Let F_d be a smooth hypersurface of degree d in $P_3(\mathbb{C})$. It is easy to calculate

$$(10) \qquad 3c_2(F_d) - c_1^2(F_d) = 2d(d-1)^2 \quad .$$

Now we admit that F_d has ordinary nodes (double points) and is otherwise smooth. These nodes are points in whose neighborhood F_d can be given by $u^2 + v^2 + w^2 = 0$ with respect to a local analytic coordinate system of $P_3(\mathbb{C})$. If we pass to the minimal resolution $\tilde{F}_d$ of F_d , then each node is replaced by a smooth rational curve of selfintersection number -2 . By Brieskorn's theory $\tilde{F}_d$ belongs to the same "family" as F_d and (10) holds for $\tilde{F}_d$. The surface $\tilde{F}_d$ is of general type for $d \geq 5$ (it is a K3-surface, Kodaira dimension 0 , for $d = 4$). Let $\mu(d)$ be the maximum number of nodes on a hypersurface of degree d . For $d \geq 4$ we get by (5), (6) and (10)

$$(11) \qquad \mu(d) \leq \frac{4}{9} d(d-1)^2 \quad .$$

Miyqoka [17] discusses this inequality in relation to many classical and more recent results. (For example $\mu(5) = 31$ by A. Beauville.)

V.I. Arnold drew my attention to other results which are not mentioned by Miyaoka. See A. Varchenko [21]. S.V. Čmutov (cf.[21]) used the Čebyšev polynomials to define surfaces F_d with many nodes: The Čebyšev polynomial $T_d(x)$ is defined by

$$T_d(\cos\alpha) = \cos(d\alpha)$$

We have

$$T_d(x) = \sum (-1)^j \binom{d}{2j} x^{d-2j}(1-x^2)^j \quad .$$

The derivative $T_d'(x)$ has only simple zeros, they give the $d-1$ maxima and minima of $T_d(x)$. The maxima all have the value 1, the minima the value -1. We use affine coordinates x_1, x_2, x_3 for $P_3(\mathbb{C})$ and consider the Čmutov surface

$$T_d(x_1) + T_d(x_2) + T_d(x_3) = -1$$

which has no singularities on the infinite plane of $P_3(\mathbb{C})$. A node of the Čmutov surface has coordinates x_1, x_2, x_3 with $T_d'(x_i) = 0$, where among x_1, x_2, x_3 we must have two numbers for which T_d has a minimum. If $c(d)$ is the number of nodes, then

$$c(d) = 3 \cdot \left(\frac{d-1}{2}\right)^3 \quad \text{for } d \text{ odd}$$

(12)

$$c(d) = 3 \cdot \left(\frac{d}{2}\right)^2 \left(\frac{d}{2} - 1\right) \quad \text{for } d \text{ even} \quad .$$

By (11) and (12)

(13) $$\frac{3}{8} \leq \overline{\lim_{d\to\infty}}\, \mu(d)/d^3 \leq \frac{4}{9} \quad .$$

This seems to be all what is know about $\overline{\lim_{d\to\infty}}\, \mu(d)/d^3$.

8. SINGULARITIES ON PLANAR CURVES. Let C be a smooth curve of even degree $2k$ in $P_2(\mathbb{C})$. Let X_C be the 2-fold cover of the plane branched along C. Then

(14) $$3c_2(X_C) - c_1^2(X_C) = k(10k-6) \quad .$$

Now we admit that C has simple singularities. A singular point is simple if, with respect to a local analytic coordinate system, the curve can be given as follows

$A_k (k \geq 1)$	$x^2 + y^{k+1} = 0$
$D_k (k \geq 4)$	$y(x^2 + y^{k-2}) = 0$
E_6	$x^3 + y^4 = 0$
E_7	$x(x^2 + y^3) = 0$
E_8	$x^3 + y^5 = 0$.

Then X_C has singularities (rational double points) which blow up to configurations of (-2)-curves, namely to the configurations A_k, D_k, E_6, E_7, E_8 , on the smooth model $\tilde{X}_C$. For each simple singularity we define $m(p) = m(E)$ if p is blown up to the configuration E . We can apply (6) using that $\tilde{X}_C$ belongs to the same family as X_C by Brieskorn's theory and also satisfies (14).

THEOREM. <u>Let</u> C <u>be a curve of degree</u> $2k$ <u>having only simple singularities</u> $(k \geq 3)$. <u>Then</u>

(15) $$\sum_{p \in \operatorname{sing}(C)} m(p) \leq k(10k-6) .$$

If d is the number of ordinary double points (A_1) and s the number of ordinary cusps (A_2) then by (15) and (5)

(16) $$\frac{9}{2} d + 8 s \leq k(10k-6) .$$

The paper of Ivinski's [11] contains many examples and references to the literature, in particular one finds a result of Varchenko [21]p. 164, related to (16). If $s(n)$ is the maximal number of cusps on a curve of degree n (with simple singularities only), then by (16)

$$\overline{\lim_{n\to\infty}} \frac{s(n)}{n^2} \leq \frac{5}{16} .$$

Varchenko (loc. cit.) obtains

$$\overline{\lim_{n\to\infty}} \frac{s(n)}{n^2} \leq \frac{23}{72} .$$

Examples show that this limes superior is not less than $\frac{1}{4}$.

The estimate

$$\frac{1}{4} \leq \overline{\lim_{n\to\infty}} \frac{s(n)}{n^2} \leq \frac{5}{16}$$

seems to be all what is known about it. Since a cusp reduces the class $n(n-1)$ of a smooth curve of degree n by 3, it is known classically that

$$\overline{\lim_{n\to\infty}} \frac{s(n)}{n^2} \leq \frac{1}{3} \quad .$$

9. ARRANGEMENTS OF CONICS. Let $C_1,\ldots,C_k$ be distinct smooth conics in the complex projective plane $(k \geq 4)$. We assume that the curve $C = C_1 \cup \ldots \cup C_k$ of degree $2k$ has only singularities of type A_1 or A_3, which means that a point $p \in C$ lies on one or two conics, in the latter case p is a transversal intersection of the two conics or the two conics touch each other in p with intersection multiplicity 2 (tacnode). Let d be the number of transversal intersections and t the number of tacnodes. Then obviously,

$$d + 2t = 2k(k-1) \quad ,$$

whereas by (15) and (5)

$$\frac{9}{2}d + \frac{45}{4}t \leq k(10k-6) \quad .$$

Eliminating d gives

$$t \leq \frac{4}{9}(k^2 + 3k) \quad .$$

For $k = 4,5,6$ we get $t \leq 12,17,24$. As U. Persson remarked there are 4 conics with $t = 12$ (and $d = 0$), there is a (projectively unique) arrangement of 5 conics with $t = 17$ given by I. Naruki [18]p. 1144, but the maximal t for $k \geq 6$ is not known. There is a beautiful arrangement of 12 conics (Gerbaldi 1882; see Fricke-Klein [4]p. 648-649). The alternating group A_6 operates on the complex projective plane (Valentiner and Wiman). There are six canonical subgroups of A_6 isomorphic to A_5. They are icosahedral groups. Each leaves a conic fixed. Under an outer automorphism of

A_6 we get another system of six groups isomorphic to A_5 and again six conics. Each conic of the first system of six conics touches each conic of the second system in two points. We have $k = 12$ and $t = 72$, the above estimate gives $t \leq 80$.

A similar (classical) example with $k = 14$ and $t = 98$ can be obtained from the action of the simple group G_{168} on $P_2(\mathbb{C})$, see F. Klein [12] p. 106. There are two systems of seven subgroups of G_{168} isomorphic to the octahedral group S_4 . Each subgroup leaves a conic invariant. Each conic of one system touches each concic of the other system in 2 points. Of the 98 tacnodes, 56 lie on the invariant curve f of degree 4, they are the 56 touching points of the 28 double tangents of f , and 42 lie on the invariant curve of degree 6 .

For $k = 14$, the above estimate gives $t \leq 105$. I do not know whether $\overline{\lim\limits_{k\to\infty}} \frac{t(k)}{k^2}$ is positive ($t(k)$ = maximal number of tacnodes for an arrangement of k conics).

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