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by

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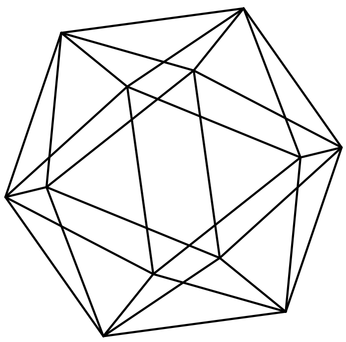

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ARITHMETIC DENSITY

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Abstract. We define arithmetic classes as a subset of $\mathbb{R}^n$ consisting of vectors which are not better approximated by the integer lattice than a given sequence. We give measure estimates for such sets.

1. Introduction

In the study of dynamical systems, one frequently consider deformations over completely discontinuous subsets. For instance, the KAM theorem asserts the existence of invariant tori parametrised by diophantine frequencies in hamiltonian systems. Such sets are usually themselves countable union of closed subsets [1, 9, 13]. Taken independently, some of these might have locally zero measure and even contain isolated points. The purpose of this note is to give density estimates in order to overcome this difficulty.

Our starting point is the observation that the Dani-Kleinbock-Margulis relating diophantine approximation to flows of discrete subgroups – via the Schmidt correspondance – can be adapted to this situation. In some respects, the case we consider is simpler and more explicit than that encountered originally by these authors [4, 5, 8, 12, 14]. For instance, it is sufficient to consider one parameter families for flows and the norm of the corresponding subgroups can be explicitly computed (Lemma 3 below).

2. Statement of the theorem

Let us now define the subsets of $\mathbb{R}^n$ that we wish to consider. Denote by $(\cdot,\cdot)$ the euclidean scalar product in $\mathbb{R}^n$. For any vector $\alpha \in \mathbb{R}^n$, we define the sequence $\sigma(\alpha)$ by :

$$\sigma(\alpha)_k := \min\{|(\alpha, i)| : i \in \mathbb{Z}^n \setminus \{0\}, \|i\| \leq 2^k\}.$$

Definition 1. *The arithmetical class in $\mathbb{R}^n$ associated to a real decreasing sequence $a = (a_k)$ is the set*

$$\mathcal{C}(a) := \{\alpha \in \mathbb{R}^n : \sigma(\alpha)_k \geq a_k\}.$$

Arithmetical classes are closed in $\mathbb{R}^n$. By Dirichlet's theorem, for any $C > 0,\ \tau \leq n$, the arithmetical class in $\mathbb{R}^n$ associated to the sequence

$(C2^{-\tau n})$ is empty [11] (see also [3]). On the other extreme, for $\tau > n$, the union of arithmetical classes associated to the sequence $(C2^{-\tau n})$ over different values of C defines a set of full measure. Similar dichotomy holds for submanifolds of $\mathbb{R}^n$ [7].

If instead of a countable union, we fix *one* arithmetical class then it is a closed subset. Therefore it cannot be of full measure unless it is equal to $\mathbb{R}^n$ itself. Our theorem states that it is nevertheless of positive measure near some points and that this property is preserved by mappings which are not flat at the given point.

For $\alpha \in \mathbb{R}^n$, we denote by $B(\alpha, r)$ the ball centred at α with radius r. Recall that the *density* of a measurable subset $K \subset \mathbb{R}^n$ at a point α is the limit (if it exists) :

$$\lim_{r \longrightarrow 0} \frac{\mathrm{Vol}(K \cap B(\alpha, r))}{\mathrm{Vol}(B(\alpha, r))}.$$

The density of a measurable subset is equal to 1 at almost all of its point [10]. For instance, sets of zero Lebesgue measure have density equal to one at *almost all points* and, in fact, equal to zero at *all points*.

If $u = (u_k)$ and $v = (v_k)$ are two real sequences, we denote by uv their product $(uv)_k = u_k v_k$.

Theorem 1. *Let $a = (a_k), \rho = (\rho_k), \rho_k < 1$ be two real positive sequences and*

$$f = (f_1, \dots, f_n) : \mathbb{R}^d \longrightarrow \mathbb{R}^n, \ f(0) \neq 0$$

a C^l-mapping. If the l-th order Taylor expansion of f is not constant and if

$$\sum_{k \geq 0} (2^{(k+1)n+1} \sqrt{\rho_k}) < +\infty$$

then the density of the set $f^{-1}(\mathcal{C}(\rho a))$ is equal to 1 at each point of $f^{-1}(\mathcal{C}(a))$.

3. Functions of class (C, τ).

For a subset $K \subset \mathbb{R}^d$ and a function

$$f : K \longrightarrow \mathbb{R}$$

we define

$$\|f\|_K := \sup_{x \in K} |f(x)|$$

(which might be infinite) and use the convention $1/0 = +\infty$. In the sequel, we denote by $U \subset \mathbb{R}^d$ an open neighbourhood of the origin.

Definition 2 ([8]). *A map $f : U \longrightarrow \mathbb{R}$ is of (C,τ)-class if for any open ball $B \subset U$ and any $\varepsilon > 0$, the following estimate holds :*

$$\mathrm{Vol}(\{x \in B : |f(x)| \leq \varepsilon\}) \leq C \left(\frac{\varepsilon}{\|f\|_B} \right)^{\tau} \mathrm{Vol}(B).$$

Functions of class (C,τ) define a cone : if f is of (C,τ)-class then so is λf for any $\lambda \in \mathbb{R}$.

Lets us denote by $x_1, x_2, \ldots, x_d$ the coordinates in $\mathbb{R}^d$. We shall use multi-index notations

$$\partial^{\beta} := \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \ldots \partial_{x_d}^{\beta_d}$$

and put $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_d$.

A compact $K \subset \mathbb{R}^d$ will be called a *hypercube* if it is of the type

$$K := [a_1, a_1 + \delta] \times [a_n, a_n + \delta] \times \cdots \times [a_d, a_d + \delta].$$

for some real numbers $a_1, a_2, \ldots, a_d, \delta$ with δ positive. The volume of such a subset is δ^d.

Lemma 1 ([8]). *Let $f : U \longrightarrow \mathbb{R}$ be a C^l function. Assume that there exists $M, m > 0$ such that for any multi-index β with $|\beta| \leq l$, we have :*

i) $\inf_{x \in U} \|\partial_{x_i}^{l} f(x)\| > m$ *for* $i = 1, \ldots, d$ *;*
ii) $\sup_{x \in U} \|\partial^{\beta} f(x)\| < M$.

For any hypercube K contained in U, we have

$$\mathrm{Vol}(\{x \in K : |f(x)| \leq \varepsilon\}) \leq C \left(\frac{\varepsilon}{\|f\|_K} \right)^{1/dl} \mathrm{Vol}(K)$$

with

$$C := dl(l+1) \left(\frac{M}{m} (l+1)(2l^l + 1) \right)^{1/l}.$$

Corollary 1. *Let $f : U \longrightarrow \mathbb{R}$ be a C^l function. Assume that the l-th order Taylor expansion of f at the origin is not constant. Then there exist a neighbourhood of the origin and constants C, τ such that the restriction of f to this neighbourhood is of (C,τ)-class.*

Indeed, up to a rotation, we may assume that the Taylor expansion of f at the origin is of the type

$$f(x) = \sum_{|i|=k} a_i x^i + o(|x|^k),\ k \leq l$$

with $\partial_i^k f(0) \neq 0$ for all $i = 1, \ldots, d$.

Choose r sufficiently small so that there exists m, M with

$$\|\partial_i^k f(x)\| \geq m,\ \forall x \in B(0,r), \forall i = 1, \ldots, d.$$

Now, any ball with radius ρ lying inside $B(0, r/\sqrt{d})$ is contained in a circunscribed hypercube whose sides have length 2ρ, itself contained inside the ball $B(0,r)$. Thus, the estimate of the previous lemma gives constants C, τ for which the restriction of f to $B(0, r/\sqrt{d})$ is of (C,τ) class.

4. The Kleinbock-Margulis theorem

Denote by $e_1, e_2, \dots, e_{n+1}$ the standard basis of the vector space $\mathbb{R}^{n+1}$. For $i = (i_1, i_2, \dots, i_k)$, $i_j < i_{j+1}$, we put

$$e_i := e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$$

and endow the exterior algebra $\Lambda^\bullet \mathbb{R}^n$ of a scalar product as follows. First define the *Hodge operator*

$$* : \Lambda^p \mathbb{R}^n \longrightarrow \Lambda^{n-p} \mathbb{R}^n$$

by the condition

$$*u \wedge u = e_1 \wedge e_2 \wedge \dots \wedge e_n$$

and the scalar product in $\Lambda^\bullet \mathbb{R}^n$ by

$$*u \wedge v = (u, v) e_1 \wedge e_2 \wedge \dots \wedge e_n.$$

This endows the exterior algebra of an euclidean structure for which the e_i's define an orthonormal basis.

The map

$$\Lambda^\bullet \mathbb{R}^n \longrightarrow \Lambda^\bullet \mathbb{R}^n, \ v \mapsto -v$$

defines an action of the group $\mathbb{Z}/2\mathbb{Z}$ on the exterior algebra $\Lambda^\bullet \mathbb{R}^n$. There is a well-defined injective map

$$\Gamma \mapsto \overline{u_1 \wedge u_2 \wedge \cdots \wedge u_r}$$

which sends a discrete subgroup Γ generated by $u_1, u_2, \dots, u_r$ to the class $u_1 \wedge u_2 \wedge \cdots \wedge u_r$ in the quotient space $\Lambda^\bullet \mathbb{R}^n / (\mathbb{Z}/2\mathbb{Z})$.

We say that the vector $u_1 \wedge u_2 \wedge \cdots \wedge u_r$ *represents* the discrete subgroup $\Gamma \subset \mathbb{R}^n$ and we define

$$\|\Gamma\| := \|u_1 \wedge u_2 \wedge \cdots \wedge u_r\|$$

A discrete subgroup is called *primitive* if it is not a proper subgroup of a discrete subgroup with the same rank. We denote by $\mathcal{L}^r$ the primitive subgroups of $\mathbb{Z}^r$ and by $L_0(\mathbb{R}^r, \mathbb{R}^{n+1})$ the vector space of rank r linear mappings from $\mathbb{R}^r$ to $\mathbb{R}^n$.

Theorem 2. *Let $h : \mathbb{R}^d \supset B(0, 3R) \longrightarrow L_0(\mathbb{R}^r, \mathbb{R}^{n+1})$ be such that for any $\Gamma \in \mathcal{L}^r$ the mapping*

$$\psi_\Gamma : B(0, 3^r R) \longrightarrow \mathbb{R}, \ x \mapsto \|h(x)\Gamma\|$$

is of (C,τ) class. Choose $\rho \leq 1$ such that the inequality

$$\|\psi_\Gamma\|_{B(0,R)} \geq \rho$$

holds for any $\Gamma \in \mathcal{L}^r$. There exists a constant C' which depends only on C, d, r such that

$$\mathrm{Vol}(\{x \in B(0,R) : \delta(h(x)\mathbb{Z}^r) \leq \varepsilon\}) \leq C' \left(\frac{\varepsilon}{\rho}\right)^{\tau} R^d$$

for any $\varepsilon \leq \rho$.

The value of C' is given in the paper of Kleinbock and Margulis who proved the theorem for the case $r = n+1$ [8, Theorem 5.2]. Theorem 2 is given by Kleinbock and the proof is essentially the same as for $r = n+1$ [6, Theorem 2.6]. There also exists a more general statement due to Bernick, Kleinbock and Margulis [2, Theorem 6.2].

5. Discrete subgroups and arithmetic classes

To the vector $\alpha \in \mathbb{R}^n$, we associate the discrete subgroup $[\alpha]$ in $\mathbb{R}^{n+1}$ of rank n defined by

$$[\alpha] := \{(i, (\alpha, i)) \in \mathbb{R}^{n+1} : i \in \mathbb{Z}^n\}$$

where $(\cdot,\cdot)$ denotes the euclidean scalar product.

Consider the linear map

$$g_t : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$$

whose matrix in the standard basis is diagonal with coefficients :

$$(e^{-t}, e^{-t}, \ldots, e^{-t}, e^t).$$

Given a discrete subgroup $\Gamma \subset \mathbb{R}^{n+1}$, we use the notation

$$\delta(\Gamma) := \inf_{\gamma \in \Gamma} \|\gamma\|$$

where $\|\cdot\|$ denotes the euclidean norm.

Lemma 2. *Let $i \in \mathbb{Z}^n$ be such that $|(\alpha, i)| \leq a$ then*

$$\delta(g_t[\alpha]) \leq \varepsilon$$

where ε, t are defined by

$$\begin{cases} \varepsilon &= \sqrt{a\|i\|}\,; \\ t &= \frac{1}{2}\log\frac{\|i\|}{a} \end{cases}$$

Proof. For any $x \in \mathbb{R}^n$ and any $y \in \mathbb{R}$, we have :

$$\|(x,y)\| \leq \sqrt{2}\max\left(\|x\|, |y|\right).$$

Consequently, the estimates $|(\alpha, i)| \leq a$ gives :

$$|g_t(i, (\alpha, i)| \leq \sqrt{2}\max\left(e^{-t}\|i\|, e^t a\right) = \varepsilon$$

□

Consider the map

$$h : \mathbb{R}^d \longrightarrow L_0(\mathbb{R}^n, \mathbb{R}^{n+1}),\ x \mapsto [u \mapsto e^{-t}u + e^t(u, f(x))e_{n+1}].$$

In particular, if Γ is the lattice represented by $e_1 \wedge e_2 \wedge \ldots \wedge e_n$, then $h(x)\Gamma$ is the discrete group associated to $f(x)$:

$$h(x)\Gamma = [f(x)].$$

Lemma 3. *We have the equality*

$$\|h(x)\Gamma\| = \sqrt{e^{-2rt} + \|f(x)\|^2}\, \|\Gamma\|.$$

Proof. Let $u_1, \ldots, u_r$ be a basis of a primitive subgroup $\Gamma \subset \mathbb{Z}^n$. The subgroup $h(x)\Gamma$ is represented by the vector

$$e^{-rt}u_1 \wedge u_2 \wedge \ldots \wedge u_r + \sum_{i=1}^{r} (-1)^i (u_i, f(x)) u_1 \wedge \ldots \wedge u_{i-1} \wedge \hat{u}_i \wedge u_{i+1} \ldots \wedge u_r \wedge e_{n+1}.$$

Choose orthonormal vectors $b_1, \ldots, b_r$ of $\mathbb{R}^n$ which span the r-dimensional vector space containing Γ and which define the same orientation as $u_1, \ldots, u_r$, that is :

$$u_1 \wedge u_2 \wedge \ldots \wedge u_r = \|\Gamma\|\, b_1 \wedge b_2 \wedge \ldots \wedge b_r$$

I assert that

$$v(x) := \sum_{i=1}^{r} (-1)^i (u_i, f(x)) u_1 \wedge \ldots \wedge u_{i-1} \wedge \hat{u}_i \wedge u_{i+1} \ldots \wedge u_r$$

and

$$v'(x) := \|\Gamma\| \sum_{i=1}^{r} (-1)^i f_i(x) b_1 \wedge \ldots \wedge b_{i-1} \wedge \hat{b}_i \wedge b_{i+1} \ldots \wedge b_r$$

are equal. Indeed for any $i = 1, \ldots, r$, we have

$$u_i \wedge v = (-1)^i (u_i, f) \|\Gamma\|\, b_1 \wedge \ldots \wedge b_r = u_i \wedge v'.$$

This proves the assertion. As the vectors $b_1, \ldots, b_r$ are orthonormal, we have

$$\|v'(x)\| = \|\Gamma\|\, \|f(x)\|.$$

This concludes the proof of the lemma. □

Assume that the Taylor series of f at the origin is not constant. In such a case, Corollary 1 implies that there exist C, τ such that the restriction of the function

$$x \mapsto \frac{1}{\|\Gamma\|} \|h(x)\Gamma\| = \sqrt{e^{-2rt} + \|f(x)\|^2}$$

to an appropriate neighbourhood of the origin is of class (C, τ). As the (C, τ)-class functions define a cone, the maps

$$x \mapsto \|h(x)\Gamma\|$$

are also of class (C, τ) for any discrete subgroup Γ. As $f(0) \neq 0$ and $\|\Gamma\| \geq 1$, in a sufficiently small neighbourhood of the origin, these functions are bounded from below independently on the value of t. Thus, the lemma shows that the assumptions of the Kleinbock-Margulis are satisfied, consequently :

Proposition 1. *Let*

$$f = (f_1, \ldots, f_n) : B(0, R) \longrightarrow \mathbb{R}^n, f(0) \neq 0$$

be a C^l-map having a non-constant l-th order Taylor expansion at the origin. There exist constants $C', \rho > 0$ and such that

$$\mathrm{Vol}(\{x \in B(0, r) : \delta([g_t f(x)]) \leq \varepsilon\}) \leq C' \left(\frac{\varepsilon}{\rho}\right)^\tau r^d$$

for $r \leq R$ and any $\varepsilon \leq \rho$.

6. Proof of Theorem 1

Denote by $[\cdot]$ the integer value and consider the map

$$\varphi : \mathbb{Z}^n \longrightarrow \mathbb{N}, \ i \mapsto [\log_2 \|i\|] + 1.$$

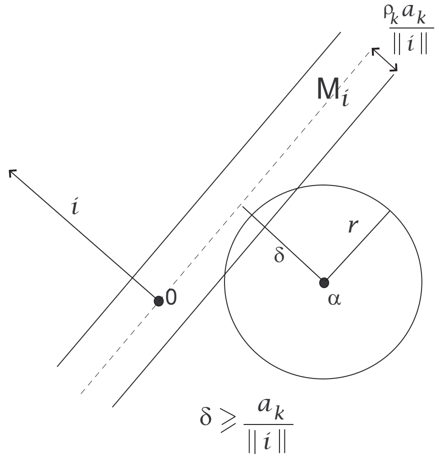
For $i \in \mathbb{Z}^n$, $\varphi(i)$ is the smallest natural number such that i is contained in the ball of radius $2^{\varphi(i)}$ centred at origin.

Fix $i \in \mathbb{Z}^n$ and put $k := \varphi(i)$. The set

$$M_i := \{\beta \in \mathbb{R}^n : |(\beta, i)| < \rho_k a_k\}$$

is a band of width $2\rho_k a_k / \|i\|$ and the union over the i's of the subsets M_i is the complement of the arithmetic class $\mathcal{C}(\rho a)$:

$$\mathbb{R}^n \setminus \mathcal{C}(\rho a) = \bigcup_{i \in \mathbb{Z}^n} M_i.$$



Let α be a vector in $\mathcal{C}(a)$. Denote by δ_i the distance from α to the hyperplane orthogonal to the vector $i \in \mathbb{Z}^n$. The intersection of the set M_i with the ball $B(\alpha, r)$ can possibly be non-empty only if

$$r > \delta_i - \frac{\rho_k a_k}{\|i\|}.$$

As $\alpha \in \mathcal{C}(a)$, we have

$$|(\alpha, i)| \geq a_k$$

thus

$$\delta_i \geq \frac{a_k}{\|i\|} \geq \frac{a_k}{2^k}$$

and therefore

$$\frac{(1-\rho_k)a_k}{2^k} < r.$$

As the sequence ρ is summable, there exists an integer N such that

$$\rho_k < \frac{1}{2},\ \forall k \geq N.$$

Choose

$$r < \inf\left\{\frac{(1-\rho_k)a_k}{2^k} : k \leq N\right\}$$

then

$$\varphi(i) < N \implies M_i \cap B(\alpha, r) = \emptyset.$$

This shows that if M_i intersects the ball $B(\alpha, r)$ then the vector $i \in \mathbb{Z}^n$ belongs to the set

$$I_r := \{i \in \mathbb{Z}^n : \frac{a_k}{2^{k+1}} < r,\ k = \varphi(i)\}.$$

Put

$$\begin{cases} \varepsilon_k & := & \sqrt{2^{k+1} a_k \rho_k}\ ; \\ t_k & := & \frac{1}{2}\log\frac{\|i\|}{a_k}. \end{cases}$$

Lemma 2 implies that

$$f(x) \in M_i \implies \delta(g_{t_k}[f(x)]) \leq \varepsilon_k.$$

Consequently, according to Proposition 1, there exists some constants $C, \gamma > 0$ such that

$$\mathrm{Vol}(B(0,r) \cap f^{-1}(M_i)) \leq C\varepsilon_k^\gamma Vol(B(0,r)).$$

As the map

$$f : \mathbb{R}^d \longrightarrow \mathbb{R}^n$$

is differentiable, by the mean-value theorem, there exists a constant κ such that for any sufficiently small r

$$f(B(0,r)) \subset B(\alpha, \kappa r),\ f(0) = \alpha.$$

In particular

$$f^{-1}(M_i) \cap B(0,r) \neq \emptyset \implies i \in I_{\kappa r},$$

for any sufficiently small r.

This shows that the measure of the complement to $f^{-1}(\mathcal{C}(\rho a))$ in $B(0,r)$ is bounded from above by

$$C\,\mathrm{Vol}(B(0,r))\left(\sum_{i\in I_{\kappa r}} \sqrt{2^{\varphi(i)+1} a_{\varphi(i)}\rho_{\varphi(i)}}\right)^{\gamma}.$$

By definition of $I_{\kappa r}$, we have :

$$\sum_{i\in I_{\kappa r}} \sqrt{2^{\varphi(i)+1} a_{\varphi(i)}\rho_{\varphi(i)}} < 2\sqrt{r}\sum_{i\in I_{\kappa r}} 2^{\varphi(i)}\sqrt{\rho_{\varphi(i)}}$$

and

$$\sum_{i\in\mathbb{Z}^n} 2^{\varphi(i)}\sqrt{\rho_{\varphi(i)}} = \sum_{k\geq 0}\sum_{\varphi(i)=k} 2^{\varphi(i)}\sqrt{\rho_{\varphi(i)}}.$$

We have

$$\#\{\varphi(i)=k\} = \#\{\varphi(i)\leq k\} - \#\{\varphi(i)\leq k-1\} \leq 2^{(k+1)n}$$

where the symbol $\#-$ stands for the cardinal. This shows that

$$\sum_{i\in\mathbb{Z}^n} 2^{\varphi(i)}\sqrt{\rho_{\varphi(i)}} \leq \sum_{k\geq 0} 2^{(k+1)n+k}\sqrt{\rho_k}.$$

By assumption, the series is in the right hand side is convergent therefore the sums

$$\sum_{i\in I_{\kappa r}} \sqrt{2^{\varphi(i)+1} a_{\varphi(i)}\rho_{\varphi(i)}}$$

goes to 0 as r becomes smaller. This concludes the proof of the theorem.

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