QUADRATIC ENVELOPE OF THE CATEGORY OF CLASS TWO NILPOTENT GROUPS

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INTRODUCTION

In this paper we introduce a somewhat surprising extension of the category of class two nilpotent groups. It has the same objects but, unlike the latter, its morphisms are closed under pointwise addition of maps. At the same time the class of its morphisms is much smaller than the class of all maps between groups. In fact, the morphisms are quadratic maps of very special kind which we call q-maps. By definition a map $f : G \to H$ is called a q-map, if the expression $(x \mid y)_f = -(f(x) + f(y)) + f(x + y)$ lies in the commutator subgroup of H and is linear in x and y. Any homomorphism is a q-map, but as we said, the sum and composite of two q-maps is still a q-map and therefore one obtains the category **Niq**, with the objects all nilpotent groups of class two and morphisms all q-maps between them. The advantage of **Niq** is the fact that hom's in **Niq** are still nil_2-groups. Composition in **Niq** is left distributive, but not right distributive. Actually **Niq** is an example of a right quadratic category in the sense of [1]. Since the category **Niq** contains more morphisms than **Nil**, two nonisomorphic groups might be isomorphic as objects of **Niq**. Thus the classification problems (say of finite groups) in **Niq** are easier (but still highly nontrivial) than the corresponding problems in **Nil**.

We also indicate an approach to such classification questions using the notion of linear extension of categories from [3]. Namely, we construct several linear extensions connecting the category **Niq** to some simpler categories, among them some additive ones which might be susceptible to representation-theoretic classification methods. The point is that a linear extension induces bijection on isomorphism classes of objects.

Here is a short description of the contents of the paper. We begin by reproducing in a maximally elementary way some known facts about the category of class two nilpotent groups in section 1. In section 2 we present two variants of the notion of quadratic map for non-abelian groups and investigate some basic properties of such maps. Then in section 3 we introduce the new class of q-maps, lying strictly between homomorphisms and quadratic maps and obtain various key properties of this class. Then in the central section 4 using the q-maps we introduce the category **Niq** and give some of its features.

We then continue the study of **Niq** using linear extensions of categories. In the next section 5 we recall this notion and exhibit the category **Nil** of class two nilpotent groups and homomorphism as a linear extension of a simpler category $\mathbf{Nil}^{\sim}$ that up to equivalence can be described in terms of 2-cohomology classes of abelian groups. Then in section 6 we do a similar thing

Research supported by the RTN Network HPRN-CT-2002-00287. The second author supported by the Humboldt Foundation and the Deutsche Forschungsgemeinschaft.

with **Niq** in place of **Nil**; this time the simpler category $\mathbf{Niq}^{\sim}$ is even additive, unlike $\mathbf{Nil}^{\sim}$, and moreover is itself a linear extension of an even smaller additive category $\mathbf{Niq}^{\approx}$.

In the next section 7 we introduce a particular class of nil_2-groups which we call q-split. This class seems to be a simplest nontrivial one admitting classification modulo isomorphism in **Niq** in terms of abelian groups. At the same time, it is quite rich, and smallest examples of non-q-split groups are not quite trivial.

In section 8 we exhibit an analog of the notion of q-map and the category **Niq** for Lie algebras and prove that in the uniquely 2-divisible situation the classical Maltsev correspondence between nil_2 groups and Lie algebras extends to q-maps. This fact has some consequences for the classification questions in view of further linear extensions on the Lie algebra side. Finally in the last section 9 we, using methods of nonabelian cohomology, construct an obstruction to lifting homomorphisms to q-maps and in particular find an obstruction for a nil_2-group to be q-split.

1. The category **Nil**

The material in this section is well known and included for convenience of the reader and to compare with what follows next.

We fix some notation. Groups will be written additively. For a group G and elements $a, b \in G$ we let $[a, b] = -a - b + a + b$ be the commutator of a and b. If G_1 and G_2 are subgroups of G, then $[G_1, G_2]$ denotes the subgroup generated by elements $[a, b]$, where $a \in G_1$ and $b \in G_2$. An element $a \in G$ is called *central* if $[a, x] = 0$ for all $x \in G$. We denote by $\mathsf{Z}(G)$ the center of G, which is the subgroup consisting of all central elements of G.

For any group G we denote by G^{ab} the abelianization of G, that is, the quotient

$$G^{\mathrm{ab}} := G/[G, G].$$

For an element $x \in G$ we let $\hat{x}$ denote the class of x in G^{ab}. For any abelian group A one denotes by $\Lambda^2(A)$ the *second exterior power* of A, which is the quotient of $A \otimes A$ by the subgroup generated by elements of the form $a \otimes a$, $a \in A$.

A subgroup A of a group G is called *central* if $[G, A] = 0$, in other words $A \subset \mathsf{Z}(G)$. A short exact sequence of groups

$$\boldsymbol{E} = \left(0 \to A \xrightarrow{i} G \xrightarrow{p} Q \to 0\right)$$

is called a *central extension* of Q by A if $i(A)$ is a central subgroup of G. We refer to, e. g., [5] for details on the relationship between central extensions and the second cohomology.

A group G is of *nilpotence class two*, or is a *nil_2-group*, if all triple commutators of G vanish, $[[G, G], G] = 0$, i. e. one has $[G, G] \subseteq \mathsf{Z}(G)$.

The smallest nonabelian groups of nilpotence class two are the quaternion group Q_8 and the dihedral group $D_4 = \mathbf{Z}/4\mathbf{Z} \ltimes \mathbf{Z}/2\mathbf{Z}$, both of order 8. We denote by **Nil** the category of groups of nilpotence class two.

1.1. **Lemma.** *For any $G \in$ **Nil** one has:*

i) *There is a well-defined homomorphism $\Lambda^2(G^{\mathrm{ab}}) \to G$ given by $\hat{a} \wedge \hat{b} \mapsto [a, b]$.*
ii) *For any $a, b \in G$ one has $[a, b] = a + b - a - b$.*
iii) *One has the inclusion $[G, G] \subset \mathsf{Z}(G)$.*

iv) *For any* $a, b \in G$ *and any* $n \in \mathbf{Z}$ *one has*

$$na + nb = n(a+b) + \frac{n(n-1)}{2}[a,b].$$

□

The inclusion functor **Nil** $\subset$ **Groups** has a left adjoint, given by

$$G \mapsto G^{\mathrm{nil}} := G/[[G,G],G].$$

Since left adjoints preserve all existing colimits, one can obtain coproducts in **Nil** as $(-)^{\mathrm{nil}}$ of coproducts in **Groups**. But in fact coproducts in **Nil** are much easier to construct directly than those in **Groups**. Namely one has

1.2. **Proposition.** *For two nil*$_2$*-groups* G*,* H *let* $G \vee H$ *be the set* $G^{\mathrm{ab}} \otimes H^{\mathrm{ab}} \times G \times H$*. The equalities*

$$\begin{aligned}
&(\xi, g, h) + (\xi', g', h') = (\xi + \xi' - \hat{g'} \otimes \hat{h}, g + g', h + h'),\\
&-(\xi, g, h) = (-\xi - \hat{g} \otimes \hat{h}, -g, -h),\\
&0 = (0,0,0)
\end{aligned}$$

equip this set with a nil$_2$*-group structure such that there is a central extension*

$$0 \to G^{\mathrm{ab}} \otimes H^{\mathrm{ab}} \to G \vee H \to G \times H \to 0.$$

Moreover the maps $i_G : G \to G \vee H$*,* $i_H : H \to G \vee H$ *given by* $i_G(g) = (0, g, 0)$ *and* $i_H(h) = (0,0,h)$ *form a coproduct diagram in* **Nil**.

Proof. The map $(G \times H) \times (G \times H) \to G^{\mathrm{ab}} \otimes H^{\mathrm{ab}}$ given by $((g,h),(g',h')) \mapsto -\hat{g'} \otimes \hat{h}$ is easily seen to be a 2-cocycle, so it indeed defines a central extension as above, and the indicated maps are clearly homomorphisms. One calculates

$$[(\xi, g, h), (\xi', g', h')] = (\hat{g} \otimes \hat{h'} - \hat{g'} \otimes \hat{h}, [g, g'], [h, h']);$$

in particular, it follows that the elements $(0, [g, g'], 0)$ and $(0, 0, [h, h'])$, along with $(\xi, 0, 0)$, are central in $G \vee H$, so that the latter is a nil$_2$-group.

Moreover in $G \vee H$ one obviously has the identities

$$(\xi, g, h) = (\xi, 0, 0) + (0, g, 0) + (0, 0, h)$$

and

$$(\hat{g} \otimes \hat{h}, 0, 0) = [(0, g, 0), (0, 0, h)].$$

Hence if we want to extend some homomorphisms $u : G \to X$, $v : H \to X$ with $X \in$ **Nil** to a homomorphism $(u, v) : G \vee H \to X$ along i_G and i_H, by the above identities we have a unique choice, namely to put

$$(\xi, g, h) \mapsto [u, v](\xi) + u(g) + v(h),$$

where $[u, v] : G^{\mathrm{ab}} \otimes H^{\mathrm{ab}} \to X$ is determined by $[u, v](\hat{g} \otimes \hat{h}) = [u(g), v(h)]$. Since X is in **Nil**, the expression $[u(g), v(h)]$ factors through $G^{\mathrm{ab}} \times H^{\mathrm{ab}}$ and is bilinear, so we indeed have a correctly defined map $G \vee H \to X$. Then using the fact that the elements $[u(g), v(h)]$ are also central in X, it is easy to see that this map is in fact a homomorphism. □

The forgetful functor $\mathsf{Nil} \to \mathsf{Sets}$ has a left adjoint, whose value on a set S is known as *the free nilpotent group of class two generated by* S and is denoted by $\mathbf{Z}_{\mathsf{Nil}}[S]$. If F_S is the free group spanned by S, then $\mathbf{Z}_{\mathsf{Nil}}[S] = (F_S)^{\mathrm{nil}}$. Moreover since left adjoints preserve coproducts, and S is the coproduct of S copies of a singleton in Sets, one has

$$\mathbf{Z}_{\mathsf{Nil}}[S] = \bigvee_S \mathbf{Z}$$

in Nil. Using Proposition 1.2, we obtain the following particular case of the famous result of Witt, which asserts that the graded Lie ring obtained by the lower central series of a free group is a free Lie ring.

1.3. **Corollary.** *For a free nil$_2$-group G one has the following central extension*

$$0 \to \Lambda^2(G^{\mathrm{ab}}) \to G \to G^{\mathrm{ab}} \to 0$$

Proof. It suffices to prove the lemma for $G = \mathbf{Z}_{\mathsf{Nil}}[S]$ with S finite. Indeed every S is a directed colimit of its finite subsets, all functors under consideration preserve colimits, and a directed colimit of short exact sequences is short exact.

For finite S we use induction on the number of elements, the case of one element, i. e. $G = \mathbf{Z}$, being trivially true. In other words, we have to show that if the above sequence is short exact for G and H, then it also is for $G \vee H$. Now this is clear from the following diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G^{\mathrm{ab}} \oplus H^{\mathrm{ab}} \arrow[r, "="] \arrow[d] & G^{\mathrm{ab}} \oplus H^{\mathrm{ab}} \arrow[d] & & \\
 & \Lambda^2((G \vee H)^{\mathrm{ab}}) \arrow[r] \arrow[d] & G \vee H \arrow[r] \arrow[d] & (G \vee H)^{\mathrm{ab}} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \Lambda^2(G^{\mathrm{ab}}) \times \Lambda^2(H^{\mathrm{ab}}) \arrow[r] \arrow[d] & G \times H \arrow[r] \arrow[d] & G^{\mathrm{ab}} \times H^{\mathrm{ab}} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

taking into account that for any G, H the canonical homomorphism $G^{\mathrm{ab}} \oplus H^{\mathrm{ab}} \to (G \vee H)^{\mathrm{ab}}$ is an isomorphism since $(-)^{\mathrm{ab}}$ is a left adjoint and thus preserves all colimits. □

2. Quadratic maps between nonabelian groups

Let G and H be arbitrary groups. Call a map $f : G \to H$ *weakly quadratic* if for any $a, b \in G$ the *cross-effect*

$$(a \mid b)_f := -(f(a) + f(b)) + f(a + b)$$

commutes with $f(c)$ for all $c \in G$ and is linear in a and b. Thus we have

$$f(a+b) = f(a) + f(b) + (a \mid b)_f,$$

and the equalities

$$\begin{aligned}(a_1 + a_2 \mid b)_f &= (a_1 \mid b)_f + (a_2 \mid b)_f,\\ (a \mid b_1 + b_2)_f &= (a \mid b_1)_f + (a \mid b_2)_f,\\ (a \mid b)_f + f(c) &= f(c) + (a \mid b)_f\end{aligned}$$

hold for any $a, a_1, a_2, b_1, b_2, b, c \in G$.

A weakly quadratic map $f : G \to H$ is *quadratic* [1] if in fact $(a \mid b)_f \in \mathsf{Z}(H)$ for all $a, b \in G$.

Obviously every weakly quadratic map to an abelian group is quadratic. We denote the set of all weakly quadratic maps from G to H by $\mathsf{wQuad}(G, H)$ and that of quadratic maps by $\mathsf{Quad}(G, H)$. It is clear that a map $f : G \to H$ is a homomorphism iff $(- \mid -)_f = 0$. Thus

$$\mathsf{Hom}(G, H) \subseteq \mathsf{Quad}(G, H) \subseteq \mathsf{wQuad}(G, H).$$

2.1. **Lemma.** *For $f \in \mathsf{wQuad}(G, H)$ the following assertions are true:*

i) *The cross-effect yields a well-defined homomorphism $(- \mid -)_f : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to H$.*

ii) $f(0) = 0$.

iii) $f(-a) = -f(a) + (a \mid a)_f$.

iv) *If $c \in [G, G]$, then for any $a \in G$ one has $f(a + c) = f(a) + f(c)$. In particular the restriction of f to the commutator subgroup is a homomorphism.*

v) *For any $a, b \in G$ one has*

$$f([a, b]) = -f(b + a) + f(a + b) = [f(a), f(b)] + (a \mid b)_f - (b \mid a)_f.$$

vi) *For any $a, b, c \in G$ one has $f([a, [b, c]]) = [f(a), [f(b), f(c)]]$.*

Proof. i) Since the elements $(a \mid b)_f$ commute with everything in the image of f, they centralize the subgroup generated by this image. But they belong to this subgroup themselves, so commute with each other. Thus for each $a \in G$ the map $(a \mid -)_f : G \to H$ is a homomorphism with abelian image, hence it factors trough G^{ab}. Similarly for $(- \mid b)_f$.

ii) $0 = (0 \mid 0)_f = -f(0) - f(0) + f(0) = f(0)$.

iii) By ii),

$$0 = f(0) = f(a - a) = f(a) + f(-a) + (a \mid -a)_f$$

and the statement follows.

iv) By i), $(a \mid c)_f = 0$.

v) We have

$$\begin{aligned}f([a, b]) &= f(-(b + a) + a + b)\\ &= f(-(b + a)) + f(a + b) + (-(b + a) \mid a + b)_f\\ &= -f(b + a) + (b + a \mid b + a)_f + f(a + b) + (-(b + a) \mid a + b)_f && \text{(by iii))}\\ &= -f(b + a) + f(a + b) && \text{(by i))}\\ &= -(f(b) + f(a) + (b \mid a)_f) + f(a) + f(b) + (a \mid b)_f\\ &= [f(a), f(b)] + (a \mid b)_f - (b \mid a)_f.\end{aligned}$$

vi) We have

$$
\begin{aligned}
f([a,[b,c]]) &= [f(a), f([b,c])] + (a \mid [b,c])_f - ([b,c] \mid a)_f && \text{(by v))}\\
&= [f(a), f([b,c])] && \text{(by i))}\\
&= [f(a), [f(b), f(c)] + (b \mid c)_f - (c \mid b)_f] && \text{(by v))}\\
&= [f(a), [f(b), f(c)]] && \text{(by i)).}
\end{aligned}
$$

□

2.2. **Corollary.** *Let $f : G \to H$ be a weakly quadratic map. If H is a nilpotent group of class two, then f factors through $G^{\mathrm{nil}} = G/[G,[G,G]]$. Thus*

$$
\begin{aligned}
\mathsf{wQuad}(G,H) &\cong \mathsf{wQuad}(G^{\mathrm{nil}},H)\\
\mathsf{Quad}(G,H) &\cong \mathsf{Quad}(G^{\mathrm{nil}},H).
\end{aligned}
$$

Proof. Indeed, if $c \in [G,[G,G]]$ then $f(c) = 0$ thanks to vi) of Lemma 2.1. Thus $f(a+c) = f(a)$ by iv) of Lemma 2.1. □

The set of quadratic maps for nilpotent groups of class two has some remarkable properties. First of all unlike $\mathsf{Hom}(G,H)$ or $\mathsf{wQuad}(G,H)$ the set $\mathsf{Quad}(G,H)$ is a group with respect to the pointwise addition of maps. This is the subject of the following Lemma.

2.3. **Lemma.** *Let G be a group and let H be a nilpotent group of class two. If the maps $f, g : G \to H$ are quadratic, then $f+g$ and $-f$ are also quadratic and*

$$
\begin{aligned}
(a \mid b)_{f+g} &= (a \mid b)_f + (a \mid b)_g + [f(b), g(a)],\\
(a \mid b)_{-f} &= [f(b), f(a)] - (a \mid b)_f.
\end{aligned}
$$

Proof. The above formulæ for $(- \mid -)_{f+g}$ and $(- \mid -)_{-f}$ can be easily checked. Since the commutators are central, it remains to show that $[f(b), g(a)]$ is linear in a and b for any quadratic f and g. But this is clear, because $[-,-]$ is central, bilinear and vanishes on central elements. □

2.4. **Example** ([2])**.** For any group G there exists a *universal weakly quadratic map* $p_2 : G \to \mathrm{P}_2G$ such that for any other weakly quadratic map $q : G \to H$ there is a unique homomorphism $f_q : \mathrm{P}_2G \to H$ with $q = f_q p_2$. Thus

$$
\mathsf{wQuad}(G,H) \cong \mathsf{Hom}(\mathrm{P}_2G, H).
$$

One defines P_2G by the pullback square

$$
\begin{tikzcd}
\mathrm{P}_2G \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, tail, "\text{diagonal}"] \\
G \vee G \arrow[r, two heads] & G \times G.
\end{tikzcd}
$$

Thus by Proposition 1.2 there is a central extension

$$
0 \to G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \xrightarrow{\iota} \mathrm{P}_2G \xrightarrow{\pi} G \to 0. \tag{1}
$$

and P_2G is isomorphic to the set $G^{\mathrm{ab}}\otimes G^{\mathrm{ab}}\times G$ with the group structure given by

$$(\xi,g)+(\xi',g')=(\xi+\xi'-\hat{g}\otimes\hat{g}',g+g').$$

The universal weakly quadratic map $p_2:G\to\mathrm{P}_2G$ is given by $p_2(g)=(0,g)$. Indeed in P_2G one then has

$$(\hat{x}\otimes\hat{y},0)=-((0,x)+(0,y))+(0,x+y)=(x\mid y)_{p_2}$$

and

$$(\xi,g)=(\xi,0)+p_2(g),$$

so to factor a weakly quadratic map $q:G\to H$ through p_2 via a homomorphism $f_q:\mathrm{P}_2G\to H$ one is forced to put

$$f_q\,(\xi,g)=(-\mid-)_q(\xi)+q(g).$$

One then checks easily that this indeed gives the required factorization.

Note that the universal weakly quadratic map $p_2:G\to\mathrm{P}_2G$ is not only weakly quadratic but actually also quadratic.

2.5. **Lemma.** *For any $G\in\mathsf{Niq}$ and $A\in\mathsf{Ab}$ one has an exact sequence:*

$$0\to\mathsf{Hom}(G,A)\to\mathsf{Quad}(G,A)\to\mathsf{Hom}(G^{\mathrm{ab}}\otimes G^{\mathrm{ab}},A)\to H^2(G,A)$$

where the last homomorphism is given by $f\mapsto f_*([G])$. Here $f:G^{\mathrm{ab}}\otimes G^{\mathrm{ab}}\to A$ is a homomorphism and $[G]\in H^2(G,G^{\mathrm{ab}}\otimes G^{\mathrm{ab}})$ is the class represented by the central extension (1).

Proof. The result is an immediate consequence of the 5-term exact sequence in group cohomology (see for example [5], page 15) applied to the central extension (1) and from the fact that for abelian A one has $\mathsf{Hom}(\mathrm{P}_2G,A)\cong\mathsf{Quad}(G,A)$. □

2.6. **Lemma.** *For any groups $(G_i)_{i\in I}$ and H one has natural bijections*

$$\mathsf{Quad}(H,\prod_i G_i)\approx\prod_i\mathsf{Quad}(H,G_i).$$

If moreover $H\in\mathsf{Nil}$ then there is a central extension

$$0\to\mathsf{Hom}(G_1^{\mathrm{ab}}\otimes G_2^{\mathrm{ab}},\mathsf{Z}(H))\xrightarrow{\alpha}\mathsf{Quad}(G_1\times G_2,H)\to\mathsf{Quad}(G_1,H)\times\mathsf{Quad}(G_2,H)\to 0$$

where $(\alpha(\xi))(g_1,g_2)=\xi(\hat{g}_1,\hat{g}_2)$ for $\xi\in\mathsf{Hom}(G_1^{\mathrm{ab}}\otimes G_2^{\mathrm{ab}},\mathsf{Z}(H))$ and $g_k\in G_k$, $k=1,2$.

Proof. The first assertion is clear. For the second, take any elements $f_k\in\mathsf{Quad}(G_k,H)$, $k=1,2$. Then the composite maps f_kp_k are again quadratic, where $p_k:G_1\times G_2\to G_k$ are the projections. Thus $f=f_1p_1+f_2p_2:G_1\times G_2\to H$ is a quadratic map. It is clear that $fi_k=f_k$, where $i_k:G_k\to G_1\times G_2$ are the standard inclusions. This shows that the map $\mathsf{Quad}(G_1\times G_2,H)\to\mathsf{Quad}(G_1,H)\times\mathsf{Quad}(G_2,H)$ is surjective. Let us compute the kernel of the latter homomorphism. Take an f from the kernel. Then $f:G_1\times G_2\to H$ is a quadratic map such that $f(g_1,0)=0=f(0,g_2)$ for all $g_k\in G_k$. Define $\xi:G_1^{\mathrm{ab}}\times G_2^{\mathrm{ab}}\to\mathsf{Z}(H)$ by $\xi(\hat{g_1},\hat{g_2}):=((g_1,0)\mid(0,g_2))_f$. Then one has

$$f(g_1,g_2)=f((g_1,0)+(0,g_2))=((g_1,0)\mid(0,g_2))_f=\xi(\hat{g}_1,\hat{g}_2)$$

and the lemma follows. □

In the rest of the paper we will assume that all groups under consideration are nilpotent of class two.

2.7. **Lemma.** *Let $f : G \to H$ be a weakly quadratic map. For any homomorphism $h : G_1 \to G$ the composite $fh : G_1 \to H$ is also weakly quadratic and*

$$(a \mid b)_{fh} = (h(a) \mid h(b))_f, \ \ a, b \in G_1;$$

moreover if f is quadratic then so is fh.

For any homomorphism $g : H \to H_1$ the composite $gf : G \to H_1$ is also weakly quadratic and

$$(a \mid b)_{gf} = g((a \mid b)_f).$$

If moreover f is quadratic then gf will be quadratic provided g carries central elements to central elements.

□

Thus for any $N \in \mathbf{Nil}$, one obtains functors

$$\begin{aligned} \mathsf{wQuad}(-, N) &: \mathbf{Nil}^{\mathrm{op}} \to \mathbf{Sets}, \\ \mathsf{Quad}(-, N) &: \mathbf{Nil}^{\mathrm{op}} \to \mathbf{Nil}, \\ \mathsf{wQuad}(N, -) &: \mathbf{Nil} \to \mathbf{Sets}. \end{aligned}$$

In fact by Example 2.4 the last functor is representable, i. e. one has

$$\mathsf{wQuad}(N, -) \approx \mathsf{Hom}_{\mathbf{Nil}}(\mathrm{P}_2 N, -).$$

However the mapping $\mathsf{Quad}(N, -)$ is NOT functorial.

2.8. **Examples.** i) For a fixed $n \in \mathbf{Z}$, consider the map $n : G \to G$ given by $a \mapsto na$. Then

$$(a \mid b)_n = -\frac{n(n-1)}{2}[a, b].$$

Thus $n \in \mathsf{Quad}(G, G)$.

ii) Let $+ : G \times G \to G$ be the map given by $(a, b) \mapsto a + b$. Then

$$((a, b) \mid (c, d))_+ = [c, b].$$

In particular $+ \in \mathsf{Quad}(G \times G, G)$.

iii) For any elements $a \in G$ and $b \in \mathsf{Z}(G)$ we put

$$f_{a,b}(n) = na + \frac{n(n-1)}{2} b.$$

The map $f_{a,b} : \mathbf{Z} \to G$ is a quadratic map with $(n \mid m)_{f_{a,b}} = nmb$ for any $n, m \in \mathbf{Z}$. We claim that any quadratic map $f : \mathbf{Z} \to G$ is of this form. Indeed, one puts $a = f(1)$, $b = (1 \mid 1)_f$ and considers $g = f - f_{a,b}$. Then one has $g(1) = 0 = (1 \mid 1)_g$. Since $(- \mid -)_g$ is bilinear it follows that $(n \mid m)_g = nm(1 \mid 1)_g = 0$. Hence g is a homomorphism and the condition $g(1) = 0$ shows that $g = 0$ and the claim is proved. One easily computes that

$$f_{a,b} + f_{a',b'} = f_{a+a', b+b'+[a,a']}.$$

Thus pointwise sum of quadratic maps $\mathbf{Z} \to G$ is quadratic, so that $\mathsf{Quad}(\mathbf{Z}, G)$ has a group structure and one has the following central extension:

$$0 \to \mathsf{Z}(G) \to \mathsf{Quad}(\mathbf{Z}, G) \xrightarrow{\mathsf{ev}(1)} G \to 0$$

where $\mathsf{ev}(1)(f) = f(1)$. A 2-cocycle $G \times G \to \mathsf{Z}(G)$ corresponding to this central extension is given by the commutator map.

Let us next investigate quadratic maps of the form $f : G_1 \vee G_2 \to H$. For such a map, denote

$$f_i = f|_{G_i} : G_i \to H,$$

$i = 1, 2$ and

$$f_\otimes = f|_{G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}} : G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \to H,$$

where the inclusion $G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \subset G_1 \vee G_2$ is as in Proposition 1.2. Since $G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}$ is contained in the commutator subgroup of $G_1 \otimes G_2$, the map $f_\otimes$ is a homomorphism, and its image lies in the center of H (by v) of Lemma 2.1). As for f_i, they are quadratic maps. Since every element of $G \vee H$ has the form $(\xi, g_1, g_2) = (\xi, 0, 0) + (0, g_1, 0) + (0, 0, g_2)$ with $\xi \in G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}$ and

$$f(\xi, g_1, g_2) = f_\otimes(\xi) + f_1(g_1) + f_2(g_2) + (\hat{g}_1 \mid \hat{g}_2)_f,$$

it follows that f is uniquely reconstructed from the maps $f_\otimes$, f_1, f_2 and the homomorphism

$$(- \mid -)_f|_{G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}} : G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \to \mathsf{Z}(H),$$

which we will denote by $\hat{f}$.

Conversely, for any given maps

$$f_i \in \mathsf{Quad}(G_i, H), \quad i = 1, 2, \quad f_\otimes, \hat{f} \in \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H))$$

define the map $f : G_1 \vee G_2 \to H$ by

$$f(\xi, g_1, g_2) = f_\otimes(\xi) + f_1(g_1) + f_2(g_2) + \hat{f}(\hat{g}_1 \otimes \hat{g}_2).$$

Then

$$\begin{aligned}
&f((\xi, x_1, x_2) + (\eta, y_1, y_2)) = f(\xi + \eta - \hat{y}_1 \otimes \hat{x}_2, x_1 + y_1, x_2 + y_2)\\
&= f_\otimes(\xi + \eta - \hat{y}_1 \otimes \hat{x}_2) + f_1(x_1 + y_1) + f_2(x_2 + y_2) + \hat{f}((\hat{x}_1 + \hat{y}_1) \otimes (\hat{x}_2 + \hat{y}_2))\\
&= f_\otimes(\xi) + f_\otimes(\eta) - f_\otimes(\hat{y}_1 \otimes \hat{x}_2)\\
&\qquad + f_1(x_1) + f_1(y_1) + (x_1 \mid y_1)_{f_1} + f_2(x_2) + f_2(y_2) + (x_2 \mid y_2)_{f_2}\\
&\qquad + \hat{f}(\hat{x}_1 \otimes \hat{x}_2) + \hat{f}(\hat{x}_1 \otimes \hat{y}_2) + \hat{f}(\hat{y}_1 \otimes \hat{x}_2) + \hat{f}(\hat{y}_1 \otimes \hat{y}_2)\\
&= f(\xi, x_1, x_2) + f(\eta, y_1, y_2)\\
&\qquad - f_\otimes(\hat{y}_1 \otimes \hat{x}_2) + [f_1(y_1), f_2(x_2)] + (x_1 \mid y_1)_{f_1} + (x_2 \mid y_2)_{f_2} + \hat{f}(\widehat{x_1 \otimes y_2}) + \hat{f}(\hat{y}_1 \otimes \hat{x}_2).
\end{aligned}$$

It follows that f is a quadratic map, so that indeed any choice of $f_\otimes$, f_1, f_2 and $\hat{f}$ as above is valid.

Now suppose given two quadratic maps $f, f' : G_1 \vee G_2 \to H$. Then for their sum clearly one has $(f + f')_i = f_i + f'_i$, $i = 1, 2$, and $(f + f')_\otimes = f_\otimes + f'_\otimes$. Moreover one calculates

$$\begin{aligned}\widehat{f+f'}(\hat{g}_1 \otimes \hat{g}_2) &= ((0, g_1, 0) \mid (0, 0, g_2))_{f+f'}\\ &= ((0, g_1, 0) \mid (0, 0, g_2))_f + ((0, g_1, 0) \mid (0, 0, g_2))_{f'} + [f(0, 0, g_2), f'(0, g_1, 0)]\\ &= \hat{f}(\hat{g}_1 \otimes \hat{g}_2) + \hat{f}'(\hat{g}_1 \otimes \hat{g}_2) + [f_2(g_2), f'_1(g_1)].\end{aligned}$$

Thus identifying f with the quadruple $(f_1, f_2, f_\otimes, \hat{f})$ as above one has

$$(f_1, f_2, f_\otimes, \hat{f}) + (f'_1, f'_2, f'_\otimes, \hat{f}') = (f_1 + f'_1, f_2 + f'_2, f_\otimes + f'_\otimes, \hat{f} + \hat{f}' - [f'_1, f_2]),$$

where

$$[f'_1, f_2](\hat{g}_1 \otimes \hat{g}_2) = [f'_1(g_1), f_2(g_2)].$$

We thus have proved

2.9. **Lemma.** *For any nil$_2$-groups G_1, G_2, H there is a central extension*

$$\begin{aligned}0 \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H)) \to \mathsf{Quad}(G_1 \vee G_2, H) \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H))&\\ \times \mathsf{Quad}(G_1, H) \times \mathsf{Quad}(G_2, H) \to 0.&\end{aligned}$$

A cocycle defining this extension is given by

$$\begin{aligned}((f_\otimes, f_1, f_2), (f'_\otimes, f'_1, f'_2)) \mapsto \alpha((f_\otimes, f_1, f_2), (f'_\otimes, f'_1, f'_2)) : G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \to \mathsf{Z}(H),&\\ \hat{g}_1 \otimes \hat{g}_2 \mapsto [f_2(g_2), f'_1(g_1)].&\end{aligned}$$

□

2.10. **Corollary.** *Let G be a free nil$_2$-group on $x_1, ..., x_n$. Then for any nil$_2$-group H and any elements $a_1, ..., a_n \in H$, $a_{ij}, b_{ij} \in \mathsf{Z}(H)$, $i < j$ there exists a unique quadratic map $f : G \to H$ such that*

$$\begin{aligned} f(x_i) &= a_i, \quad 1 \leqslant i \leqslant n,\\ f([x_i, x_j]) &= a_{ij}, \quad i < j,\\ (x_i \mid x_j)_f &= b_{ij}, \quad i < j.\end{aligned}$$

□

3. q-MAPS

The last identity of Lemma 1.1 suggests the following definition:

3.1. **Definition.** A weakly quadratic map $f : G \to H$ between nil$_2$-groups is a *q-map* if one has $(a \mid b)_f \in [H, H]$ for all $a, b \in G$.

We denote by $\mathsf{Q}(G, H)$ the collection of all q-maps from G to H, so that

$$\mathsf{Hom}(G, H) \subseteq \mathsf{Q}(G, H) \subseteq \mathsf{Quad}(G, H).$$

3.2. **Lemma.** *The set $\mathsf{Q}(G, H)$ is a normal subgroup of $\mathsf{Quad}(G, H)$. In particular any linear combination of homomorphisms is a q-map.*

Proof. The first identity of Lemma 2.3 shows that $\mathsf{Q}(G, H)$ is a subgroup of $\mathsf{Quad}(G, H)$. By the same Lemma for any $f \in \mathsf{Quad}(G, H)$ and $g \in \mathsf{Q}(G, H)$ we have

$$(a \mid b)_{f+g-f} = (a \mid b)_g + [fb, ga] - [gb, fa]$$

and the result follows. □

3.3. **Lemma.** *A weakly quadratic map is in $\mathsf{Q}(G, H)$ iff its composite with $H \twoheadrightarrow H^{\mathrm{ab}}$ is a homomorphism. In particular any q-map $f : G \to H$ yields a well-defined homomorphism $f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ such that the diagram*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & H \arrow[d] \\ G^{\mathrm{ab}} \arrow[r, "f^{\mathrm{ab}}"] & H^{\mathrm{ab}} \end{tikzcd}$$

commutes.

Proof. Indeed $(a \mid b)_f \in [H, H]$ iff the image of $(a \mid b)_f$ vanishes in H^{ab}. □

Obviously one has an embedding

$$\mathsf{Quad}(G, [H, H]) \subset \mathsf{Q}(G, H)$$

as a central subgroup.

3.4. **Lemma.** *For an abelian group H one has*

$$\mathsf{Q}(G, H) = \mathsf{Hom}(G, H)$$

*for any $G \in$ **Nil**.*

Proof. Since $[H, H] = 0$, a map $f : G \to H$ is a q-map iff $(- \mid -)_f = 0$. □

3.5. **Examples.** The first two quadratic maps considered in Examples 2.8 are actually q-maps. Also the map

$$\delta = i_1 + i_2 : G \to G \vee G$$

is a q-map, with $(x \mid y)_\delta = [i_1(y), i_2(x)]$.

On the other hand, the quadratic map $f_{a,b} : \mathbf{Z} \to G$ associated to elements $a \in G$ and $b \in \mathsf{Z}(G)$ as in iii) of Examples 2.8 is a q-map iff $b \in [G, G]$. Thus for any $G \in$ **Nil** one has the following central extension:

$$0 \to [G, G] \to \mathsf{Q}(\mathbf{Z}, G) \xrightarrow{\mathsf{ev}(1)} G \to 0.$$

A 2-cocycle $G \times G \to [G, G]$ corresponding to this central extension is given by the commutator map.

Exactly as for Lemma 2.6 one has

3.6. **Lemma.** *For any groups $(G_i)_{i \in I}$, H one has natural bijections*

$$\mathsf{Q}(H, \prod_i G_i) \cong \prod_i \mathsf{Q}(H, G_i).$$

If moreover $H \in \mathsf{Nil}$ then there is a central extension

$$0 \to \mathsf{Q}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, [H,H]) \to \mathsf{Q}(G_1 \times G_2, H) \to \mathsf{Q}(G_1, H) \times \mathsf{Q}(G_2, H) \to 0.$$

□

Moreover one has exactly as in Lemma 2.9

3.7. **Lemma.** *For any nil$_2$-groups G_1, G_2, H there is a central extension*

$$\begin{aligned} 0 \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, [H,H]) \to \mathsf{Q}(G_1 \vee G_2, H) \to \mathsf{Hom}(&G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, [H,H]) \\ &\times \mathsf{Q}(G_1, H) \times \mathsf{Q}(G_2, H) \to 0. \end{aligned}$$

A cocycle defining this extension is given by

$$\begin{aligned} ((f_\otimes, f_1, f_2), (f'_\otimes, f'_1, f'_2)) \mapsto \alpha((f_\otimes, f_1, f_2), (f'_\otimes, f'_1, f'_2)) : G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} &\to [H,H], \\ \hat{g}_1 \otimes \hat{g}_2 &\mapsto [f_2(g_2), f'_1(g_1)]. \end{aligned}$$

In particular, if G is a free nil$_2$-group on $x_1, \cdots, x_n$ then for any nil$_2$-group H and any elements $a_1, \cdots, a_n \in H$, $a_{ij}, b_{ij} \in [H,H], i < j$ there exists the unique q-map $f : G \to H$ such that

$$\begin{aligned} f(x_i) &= a_i, \quad 1 \leqslant i \leqslant n, \\ f([x_i, x_j]) &= a_{ij}, \quad i < j, \\ (x_i \mid x_j)_f &= b_{ij}, \quad i < j. \end{aligned}$$

□

By Lemma 3.3 any q-map $f : G \to H$ yields a homomorphism $f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. We now associate two more homomorphisms to any q-map.

3.8. **Proposition.** *Let $f : G \to H$ be a q-map. Then $f([G,G]) \subset [H,H]$ and the restriction of f to $[G,G]$ yields a homomorphism $[f,f] : [G,G] \to [H,H]$, which fits in the following commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & {[G,G]} \arrow[r] \arrow[d, "{[f,f]}"] & G \arrow[r] \arrow[d, "f"] & G^{\mathrm{ab}} \arrow[r] \arrow[d, "f^{\mathrm{ab}}"] & 0 \\ 0 \arrow[r] & {[H,H]} \arrow[r] & H \arrow[r] & H^{\mathrm{ab}} \arrow[r] & 0 \end{tikzcd}$$

Moreover, there exists a unique homomorphism

$$\beta(f) : \mathsf{Ker}(f^{\mathrm{ab}}) \to \mathsf{Coker}([f,f])$$

such that

$$\beta(f)(\hat{a}) = f(a) \mod f([G,G])$$

for any $a \in f^{-1}([H,H])$. Furthermore if f is injective then $[f,f]$ and $\beta(f)$ are monomorphisms and if f is surjective then $\beta(f)$ and f^{ab} are epimorpisms.

Proof. If f is a q-map then it follows from v) of Lemma 2.1 that $f[G,G] \subseteq [H,H]$. Hence by iv) of Lemma 2.1, $[f,f] : [G,G] \to [H,H]$ is a homomorphism and obviosly the diagram commutes. We claim that $\beta(f)$ is well-defined. One observes that if $\hat{a}_1 = \hat{a}$ then $a_1 = a + b$ with $b \in [G,G]$. It follows by iv) of Lemma 2.1 that $f(a_1) = f(a) \mod f([G,G])$ and the claim is proved. The rest is just diagram chase. □

4. THE CATEGORY **Niq**

In this section, our main character enters. This is the category **Niq**. The definition is based on the following result.

4.1. **Proposition.** *Any composite of q-maps is a q-map. More precisely, for q-maps $f : G \to H$ and $g : G_1 \to G$ the cross-effect of their composite is given by*

$$(a \mid b)_{fg} = f((a \mid b)_g) + (g(a) \mid g(b))_f, \ \ a, b \in G_1.$$

Proof. One has

$$\begin{aligned} fg(a+b) &= f(g(a) + g(b) + (a \mid b)_g) \\ &= f(g(a) + g(b)) + f((a \mid b)_g) && \text{(by iv) of Lemma 2.1)} \\ &= f(g(a)) + f(g(b)) + (g(a) \mid g(b))_f + f((a \mid b)_g), \end{aligned}$$

which proves the equality above. □

Hence there is a well-defined category **Niq** whose objects are nil$_2$-groups and morphisms are all q-maps between them. The hom-sets

$$\mathsf{Hom}_{\mathbf{Niq}}(G, H) = \mathsf{Q}(G, H)$$

are equipped with structures of nilpotent groups of class two. **Nil** is a subcategory of **Niq**, with the same objects. The hom-functor of **Niq** (with values in sets) gives rise to a well-defined bifunctor

$$\mathsf{Q}(-,-) : \mathbf{Nil}^{\mathrm{op}} \times \mathbf{Nil} \to \mathbf{Nil}.$$

Moreover there are well-defined functors **Niq** $\to$ **Ab** given respectively by $G \mapsto G^{\mathrm{ab}}$ and $G \mapsto [G, G]$.

Composition in **Niq** is left distributive,

$$(f + f')g = fg + f'g,$$

but not right distributive; rather it is right quadratic, in the following sense. First of all one has

$$f(g + g') = fg + fg' + (g \mid g')_f, \ \ f \in \mathsf{Q}(G, H), \ \ g, g' \in \mathsf{Q}(G_1, G), \tag{2}$$

where $(g \mid g')_f : G_1 \to H$ is given by

$$(g \mid g')_f(x) = (g(x) \mid g'(x))_f. \tag{3}$$

Secondly $(g \mid g')_f$ lies in the center of $\mathsf{Q}(G_1, H)$ and it is bilinear in g, g' and quadratic in f — more precisely, one has

$$(g \mid g')_{f+f'}(x) = (g \mid g')_f(x) + (g \mid g')_{f'}(x) + [fg'(x), f'g(x)].$$

The category **Niq** possesses all products and both the inclusion **Nil** $\hookrightarrow$ **Niq** and the forgetful functor **Niq** $\to$ **Sets** respect products.

Every object in **Niq** has a canonical internal group structure. However a morphism in **Niq** is compatible with the corresponding internal group structures iff it lies in **Nil**, i. e. is a homomorphism.

If $p_k : G_1 \times G_2 \to G_k$ are the standard projections and $i_k : G_k \to G_1 \times G_2$ are the standard inclusions then one has $p_k i_k = \mathsf{Id}_{G_k}$, $i_1 p_1 + i_2 p_2 = \mathsf{Id}_{G_1 \times G_2}$ and $p_2 i_1 = 0$, $p_1 i_2 = 0$. Therefore **Niq** is a right quadratic category in the terminology of [1]. Trivial groups are zero objects in **Niq**.

Note also that it follows from Lemma 3.4 that

4.2. **Proposition.** *Any group isomorphic in* **Niq** *to an abelian group is itself abelian.*

□

4.3. **Example.** Let $f : \mathbf{Z}^3 \to \mathbf{Z} \vee \mathbf{Z}$ be the map given by

$$f(l, m, n) = l[x, y] + mx + ny,$$

where x and y are the generators of $\mathbf{Z} \vee \mathbf{Z}$. One then has

$$((l, m, n) \mid (l', m', n'))_f = m'n[x, y],$$

so that f is a q-map. It is obviously a bijection. However it cannot be an isomorphism in **Niq** because of Proposition 4.2.

In fact,

$$(l[x, y] + mx + ny \mid l'[x, y] + m'x + n'y)_{f^{-1}} = (-m'n, 0, 0),$$

so that f^{-1} is quadratic, but not a q-map.

Let us point out that there exist nil$_2$-groups isomorphic in **Niq** but not in **Nil**. We will see such examples below (see Example 7.4).

Let us recall that a *weak coproduct* of objects X_1 and X_2 of some category is an object W together with morphisms $i_k : X_k \to W$ such that for any morphisms $f_k : X_k \to Z$ there exists a morphism (not necessarily unique) $f : W \to Z$ with $f_k = f i_k$, $k = 1, 2$.

4.4. **Lemma.** *The category* **Niq** *possesses weak coproducts.*

Proof. We claim that $W = X_1 \times X_2$ does the job. Indeed, for any $f_k : X_k \to Z$ put $f = f_1 p_1 + f_2 p_2$. Then one has

$$f i_k = (f_1 p_1 + f_2 p_2) i_k = f_1 p_1 i_k + f_2 p_2 i_k = f_k.$$

□

5. The category **Nil** as a linear extension

We start with recalling the definition of a linear extension of a small category by a bifunctor [3]

5.1. **Definition.** A *linear extension* of a small category $\boldsymbol{C}$ by a bifunctor $D : \boldsymbol{C}^{\mathrm{op}} \times \boldsymbol{C} \to \mathbf{Ab}$

$$0 \to D \to \boldsymbol{E} \xrightarrow{P} \boldsymbol{C} \to 0$$

is a functor P with the following properties: $\boldsymbol{C}$ and $\boldsymbol{E}$ have the same objects and P is a full functor which is the identity on objects. For each pair of objects i and j the abelian group $D(i, j)$ acts transitively and effectively on the set $\mathsf{Hom}_{\boldsymbol{E}}(i, j)$. We write $\alpha + a$ for the action of $a \in D(i, j)$ on $\alpha \in \mathsf{Hom}_{\boldsymbol{E}}(i, j)$. The action satisfies the linear distributivity law :

$$(\alpha + a)(\beta + b) = \alpha\beta + P(\alpha)_* b + P(\beta)^* a.$$

It is known and easy to prove that in any linear extension the functor q reflects isomorphisms and yields a bijection on isomorphism classes of objects.

Our aim is to obtain the category **Nil** as a linear extension. To do so we first recall some classical results on group (co)homology.

5.2. **Proposition.**

i) *For a central extension*

$$\boldsymbol{E} = \left(0 \to A \xrightarrow{i} G \xrightarrow{p} Q \to 0\right) \tag{4}$$

there is a well-defined class $\langle \boldsymbol{E} \rangle \in H^2(Q; A)$ *and in this way one obtains a one-to-one correspondence between the equivalence classes of central extensions of* Q *by* A *and elements of the group* $H^2(Q; A)$. *If* $\boldsymbol{E}'$ *is also a central extension of a group* Q' *by* A' *and* $f : Q \to Q'$, $g : A \to A'$ *are group homomorphisms then* $g_* \langle \boldsymbol{E} \rangle$ *and* $f^* \langle \boldsymbol{E}' \rangle$ *are the same elements in* $H^2(Q; A')$ *iff there is a group homomorphism* $h : G \to G'$ *such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "g"'] & G \arrow[r] \arrow[d, "h"'] & Q \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A' \arrow[r] & G' \arrow[r] & Q' \arrow[r] & 0
\end{tikzcd}$$

commutes.

ii) *Let* Q *be a group and* A *be an abelian group, considered as a* Q*-module via the trivial action of* Q *on* A. *Then one has the universal coefficient exact sequence*

$$0 \to \mathsf{Ext}(Q^{\mathrm{ab}}, A) \to H^2(Q; A) \xrightarrow{\mu} \mathsf{Hom}(H_2 Q, A) \to 0.$$

iii) *For the central extension* (4) *one has the following Ganea exact sequence*

$$G^{\mathrm{ab}} \otimes A \to H_2 G \to H_2 Q \xrightarrow{\nu} A \to G^{\mathrm{ab}} \to Q^{\mathrm{ab}} \to 0,$$

where $\nu = \mu \langle \boldsymbol{E} \rangle$, *with* $\langle \boldsymbol{E} \rangle$ *as in i) above.*

iv) *If* B *is an abelian group then* $H_2(B) \cong \Lambda^2 B$ *and the homomorphism* $\mu \langle \boldsymbol{E} \rangle : \Lambda^2 B \to A$ *corresponding to a central extension*

$$\boldsymbol{E} = \left(0 \to A \xrightarrow{i} G \xrightarrow{p} B \to 0\right)$$

is determined by

$$i\left(\mu \langle \boldsymbol{E} \rangle (p(x) \wedge p(y))\right) = [x, y], \quad x, y \in G.$$

Proof. These results are well known, see for example [5]. □

The class of the central extension

$$0 \to [G, G] \to G \to G^{\mathrm{ab}} \to 0 \tag{5}$$

in $H^2(G^{\mathrm{ab}}; [G, G])$ is denoted by $\mathsf{e}(G)$.

5.3. **Lemma.** *The homomorphism* $\mu(\mathsf{e}(G)) : \Lambda^2(G^{\mathrm{ab}}) \to [G,G]$ *is surjective.*

Proof. This follows from iv) of Proposition 5.2 applied to the central extension (5). $\square$

The exact sequence (5) is functorial on G, meaning that if $f : G \to H$ is a homomorphism, then one has the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {[G,G]} \arrow[r, "i"] \arrow[d, "{[f,f]}"'] & G \arrow[r, "p"] \arrow[d, "f"'] & G^{\mathrm{ab}} \arrow[r] \arrow[d, "f^{\mathrm{ab}}"] & 0 \\
0 \arrow[r] & {[H,H]} \arrow[r, "j"] & H \arrow[r, "q"] & H^{\mathrm{ab}} \arrow[r] & 0
\end{tikzcd}$$

If $g : G \to H$ is another homomorphisms, then we write $f \sim g$ provided $f^{\mathrm{ab}} = g^{\mathrm{ab}}$ and $[f,f] = [g,g]$. It is clear that $f \sim g$ iff there exists a homomorphism $k : G^{\mathrm{ab}} \to [H,H]$ such that $f - g = jkp$. We can consider the corresponding quotient category $\mathbf{Nil}^{\sim}$. Objects are the same as of $\mathbf{Nil}$. Two homomorphisms $f, g : G \to H$ defines the same morphism in $\mathbf{Nil}^{\sim}$ provided $f \sim g$. Comparing with the notion of linear extension of categories (see Definition 5.1) we obtain the following result.

5.4. **Theorem.** *One has the following linear extension of categories*

$$0 \to D \to \mathbf{Nil} \to \mathbf{Nil}^{\sim} \to 0$$

where the bifunctor

$$D : (\mathbf{Nil}^{\sim})^{\mathrm{op}} \times \mathbf{Nil}^{\sim} \to \mathbf{Ab}$$

is given by

$$D(G,H) = \mathsf{Hom}(G^{\mathrm{ab}}, [H,H]).$$

$\square$

Our next aim is to describe the quotient category $\mathbf{Nil}^{\sim}$ in cohomological terms. Define the category $\mathbf{Nil}^{\mathrm{ab}}$ as follows. The objects of $\mathbf{Nil}^{\mathrm{ab}}$ are triples (A,B,e), where A and B are abelian groups and $e \in H^2(A,B)$ is such an elements that $\mu(e) : \Lambda^2(A) \to B$ is an epimorphism. A morphism from (A,B,e) to (A',B',e') is a pair (f,g), where $f : A \to A'$ and $g : B \to B'$ are homomorphisms such that the equation

$$f^*(e') = g_*(e)$$

holds in $H^2(A,B')$. Thus for any $G \in \mathbf{Nil}$ the triple

$$\mathsf{ch}(G) = (G^{\mathrm{ab}}, [G,G], \mathsf{e}(G))$$

is an object of $\mathbf{Nil}^{\mathrm{ab}}$. Moreover, if $f : G \to H$ is homomorphism of groups, then $(f^{\mathrm{ab}}, [f,f]) : \mathsf{ch}(G) \to \mathsf{ch}(H)$ is a morphism. In this way one obtains the functor

$$\mathsf{ch} : \mathbf{Nil}^{\sim} \to \mathbf{Nil}^{\mathrm{ab}}.$$

5.5. **Theorem.** *The functor* $\mathsf{ch} : \mathbf{Nil}^{\sim} \to \mathbf{Nil}^{\mathrm{ab}}$ *is an equivalence of categories.*

Proof. We claim that for any object $(A, B, e) \in \textsf{\textbf{Nil}}^{\mathrm{ab}}$ there exist an object $G \in \textsf{\textbf{Nil}}$ and an isomorphism $\mathsf{ch}(G) \to (A, B, e)$ in $\textsf{\textbf{Nil}}^{\mathrm{ab}}$. Indeed, consider a central extension

$$0 \to B \to G \to A \to 0$$

corresponding to the element e. The exact sequence iii) of Proposition 5.2 in our case has the following form

$$H_2G \to H_2A \to B \to G^{\mathrm{ab}} \to A \to 0$$

Since $H_2A \to B$ is an epimorphism, it follows that $G^{\mathrm{ab}} \cong A$. Therefore $[G, G] \cong B$ and the claim is proved.

Now, we show that for any morphism $(f, g) : \mathsf{ch}(G) \to \mathsf{ch}(G_1)$ in $\textsf{\textbf{Nil}}^{\mathrm{ab}}$, there is a unique homomorphism $h : G \cong G_1$ in $\textsf{\textbf{Nil}}^{\sim}$, such that $\mathsf{ch}(h) = (f, g)$. Indeed, by definition of morphisms in $\textsf{\textbf{Nil}}^{\mathrm{ab}}$ and by part i) of Proposition 5.2 there exist $h : G \to G_1$ in $\textsf{\textbf{Nil}}$, such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & {[G,G]} \arrow[r, "i"] \arrow[d, "g"'] & G \arrow[r, "p"] \arrow[d, "h"'] & G^{\mathrm{ab}} \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & {[G_1,G_1]} \arrow[r, "i_1"] & G_1 \arrow[r, "p_1"] & G_1^{\mathrm{ab}} \arrow[r] & 0
\end{tikzcd}$$

If $h' : G \to G'$ is another such homomorphism, then clearly $h^{\mathrm{ab}} = (h')^{\mathrm{ab}}$ as well as $[h, h] = [h', h']$ and result is proved. $\square$

6. The category $\textsf{\textbf{Niq}}$ as a linear extension

For $\alpha \in \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H, H])$ and $f \in \mathsf{Q}(G, H)$ define $f + \alpha \in \mathsf{Q}(G, H)$ to be the map given by

$$(f + \alpha)(g) = f(g) + \alpha(\hat{g}, \hat{g}).$$

It is clear that for any $\alpha, \beta \in \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H, H])$ and $f \in \mathsf{Q}(G, H)$ one has

$$f + (\alpha + \beta) = (f + \alpha) + \beta$$

and therefore the group $\mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H, H])$ acts on the set $\mathsf{Q}(G, H)$. In particular, this gives the following equivalence relation: for q-maps $f, g \in \mathsf{Q}(G, H)$ we put $f \sim g$ provided $g = f + \alpha$, for some homomorphism $\alpha : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H, H]$.

6.1. **Lemma.**

i) *Let $f_1, f_2, g_1, g_2 : G \to H$ be q-maps. If $f_1 \sim g_1$ and $f_2 \sim g_2$, then*

$$f_1 + f_2 \sim g_1 + g_2.$$

ii) *Let $f, g : G \to H$ be q-maps. Then*

$$f + g \sim g + f.$$

iii) *Let $f : G \to H$ and $g_1, g_2 : G_1 \to G$ be q-maps. Then*

$$f(g_1 + g_2) \sim fg_1 + fg_2.$$

iv) *Let $f : G \to H$ and $g : G_1 \to G$ be q-maps. Then for any homomorphisms $\alpha : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H,H]$ and $\beta : G_1^{\mathrm{ab}} \otimes G_1^{\mathrm{ab}} \to [G,G]$ one has*

$$(f+\alpha)(g+\beta) = fg + f_*\beta + g^*\alpha$$

where $f_*\beta$ and $g^*\alpha$ are homomorphisms $G_1^{\mathrm{ab}} \otimes G_1^{\mathrm{ab}} \to [H,H]$ given by $f_*\beta(x,y) = f(\beta(x,y))$ and $g^*\alpha(\hat{x},\hat{y}) = \alpha(\widehat{g(x)},\widehat{g(y)})$.

v) *Let $f_1, f_2 : G \to H$ and $g_1, g_2 : G_1 \to G$ be q-maps. If $f_1 \sim f_2$ and $g_1 \sim g_2$, then*

$$f_1 g_1 \sim f_2 g_2.$$

vi) *Let $f, g : G \to H$ be q-maps. If $f \sim g$ then they induce the same homomorphisms $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ and $[G,G] \to [H,H]$.*

Proof. i) We have $g_i = f_i + \alpha_i$, where $\alpha_i : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H,H]$ is a homomorphism $i = 1,2$. Since the values of α_i are in the center, we obtain $g_1 + g_2 = f_1 + f_2 + (\alpha_1 + \alpha_2)$.

ii) It suffices to observe that $f + g = g + f + \alpha$, where $\alpha(\hat{x},\hat{y}) = [f(x), g(y)]$.

iii) Thanks to equation (3) one has $f(g_1 + g_2) = fg_1 + fg_2 + \alpha$, where $\alpha(\hat{x}_1,\hat{x}_2) = (g_1(x_1) \mid g_2(x_2))_f$.

iv) We have $(f+\alpha)(g+\beta)(x) = f(g(x) + \beta(\hat{x},\hat{x})) + \alpha(\widehat{g(x)+\beta(\hat{x},\hat{x})}, \widehat{g(x)+\beta(\hat{x},\hat{x})})$. Since the values of β lie in the commutator subgroup of H and α is defined on the abelization, we get $\alpha(\widehat{g(x)+\beta(\hat{x},\hat{x})}, \widehat{g(x)+\beta(\hat{x},\hat{x})}) = \alpha(\widehat{g(x)})$. Thus the result follows from iv) of Lemma 2.1.

v) This property is an immediate consequence of iv).

vi) By assumption $g = f + \alpha$, for some homomorphism $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H,H]$. If $c \in [G,G]$, then $\hat{c} = 0$ in G^{ab}, thus $\alpha(\hat{c},\hat{c}) = 0$ and hence $g(c) = f(c)$. On the other hand, for any $x \in G$ the class of $\alpha(\hat{x},\hat{x})$ in H^{ab} vanishes, hence $f^{\mathrm{ab}} = g^{\mathrm{ab}}$. □

6.2. **Corollary.** *There is a well-defined category $\mathsf{Niq}^\sim$, with objects nil$_2$-groups, and morphisms $\sim$-equivalence classes of q-maps. The category $\mathsf{Niq}^\sim$ is an additive category.*

Proof. By v) $\mathsf{Niq}^\sim$ is a well-defined category. By i) and ii) hom's in $\mathsf{Niq}^\sim$ are abelian groups. Since Niq was left distributive, it follows from iii) that the composition in $\mathsf{Niq}^\sim$ is distributive. One easily sees that the product in Niq remains also a product in $\mathsf{Niq}^\sim$ and therefore Niq is an additive category with products. □

For q-maps $f, g \in \mathsf{Q}(G,H)$ we put $f \approx g$ provided both of them yield the same homomorphisms $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ and $[G,G] \to [H,H]$. The corresponding quotient category is denoted by $\mathsf{Niq}^\approx$. By iv) in Lemma 6.1 the quotient functor $\mathsf{Niq} \to \mathsf{Niq}^\approx$ factors trough $\mathsf{Niq}^\sim$.

For nil$_2$-groups G and H we let $D^\sim(G,H)$ be the quotient of $\mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H,H])$ by the subgroup spanned by such $\alpha \in \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H,H])$ that $\alpha(\hat{x},\hat{x}) = 0$ for all $x \in G$. In this way one obtains a bifunctor $D^\sim : (\mathsf{Niq}^\approx)^{\mathrm{op}} \times \mathsf{Niq}^\approx \to \mathsf{Ab}$. We also need another bifunctor $D^\approx : (\mathsf{Niq}^\approx)^{\mathrm{op}} \times \mathsf{Niq}^\approx \to \mathsf{Ab}$ given by $D^\approx(G,H) = \mathsf{Quad}(G^{\mathrm{ab}}, [H,H])$. There is a natural transformation $\rho : D^\sim \to D^\approx$, which takes $\alpha : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H,H]$ to the quadratic map $\rho(\alpha) : G^{\mathrm{ab}} \to [H,H]$ given by $\rho(\alpha)(\hat{x}) = \alpha(\hat{x},\hat{x})$. It follows from the definition of $D^\sim$, that ρ is a monomorphism. We define $\tilde{D} := \mathsf{Coker}(\rho)$. Using the quotient functors $\mathsf{Niq} \twoheadrightarrow \mathsf{Niq}^\sim \twoheadrightarrow \mathsf{Niq}^\approx$ one considers $D^\sim$, $D^\approx$ also as bifunctors on $\mathsf{Niq}^\sim$, or Niq.

6.3. **Proposition.** *One has the following commutative diagram of linear extensions:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^{\sim} \arrow[r, "\rho"] \arrow[d, equal] & D^{\approx} \arrow[r] \arrow[d] & \tilde{D} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D^{\sim} \arrow[r] & \mathsf{Niq} \arrow[r] \arrow[d] & \mathsf{Niq}^{\sim} \arrow[r] \arrow[d] & 0 \\
 & & \mathsf{Niq}^{\approx} \arrow[r, equal] \arrow[d] & \mathsf{Niq}^{\approx} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

In particular $\mathsf{Niq}^{\approx}$ *is also an additive category and the quotient functors* $\mathsf{Niq} \to \mathsf{Niq}^{\sim} \to \mathsf{Niq}^{\approx}$ *reflect isomorphisms and yield bijections on isomorphism classes of objects.*

Proof. The operation $\mathsf{Q}(G,H) \times \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H,H]) \to \mathsf{Q}(G,H)$ given by $(f,\alpha) \mapsto f + \alpha$ yields the action of $D^{\sim}$ on the category Niq and by the property iv) one obtains a linear extension of categories

$$0 \to D^{\sim} \to \mathsf{Niq} \to \mathsf{Niq}^{\sim} \to 0.$$

By Proposition 3.8 for q-maps $f, g : G \to H$ one has $f \approx g$ iff there is a quadratic map $h : G^{\mathrm{ab}} \to [H,H]$ such that $f - g$ factors trough h. This shows that

$$0 \to D^{\approx} \to \mathsf{Niq} \to \mathsf{Niq}^{\approx} \to 0$$

is a linear extension of categories. The rest follows from the properties of linear extensions. □

6.4. **Remark.** For an abelian group A the group $A \otimes A$ has a canonical involution $(a \otimes b)^{\sigma} = b \otimes a$. We put $\tilde{\Gamma}^2(A) := \{x \in A \otimes A \mid x^{\sigma} = x\}$. Then one has an exact sequence

$$0 \to \tilde{\Gamma}^2(A) \to A \otimes A \to \Lambda^2(A) \to 0$$

One easily sees that the class of abelian groups for which this sequence splits is closed under direct sums and contains all cyclic groups (and all uniquely 2-divisible groups). In particular the sequence splits, provided A is a direct sum of cyclic groups. The exact sequence yields the following exact sequence

$$0 \to \mathsf{Hom}(\Lambda^2(A), B) \to \mathsf{Hom}(A \otimes A, B) \xrightarrow{\xi_{A,B}} \mathsf{Hom}(\tilde{\Gamma}^2(A), B)$$

for all abelian group B. It follows from the definition that $D^{\sim}(G,H) \cong \mathsf{Im}(\xi_{G^{\mathrm{ab}},[H,H]})$. In particular, if G^{ab} is a direct sum of cyclic groups, then $D^{\sim}(G,H) = \mathsf{Hom}(\tilde{\Gamma}^2(G^{\mathrm{ab}}), [H,H])$.

6.5. **Definition.** For abelian groups A and B we denote by $H_b^2(A,B)$ be the subgroup of $H^2(A,B)$ generated by bilinear 2-cocycles. Thus by definition one has the following exact sequence

$$\mathsf{Quad}(A,B) \xrightarrow{(-|-)_?} \mathsf{Hom}(A \otimes A, B) \to H_b^2(A,B) \to 0$$

where the first map assigns to a quadratic map f its cross-effect $(-\mid -)_f$.

We now define the category $\mathsf{Niq}^{\mathrm{ab}}$, which has the same objects as the category $\mathsf{Nil}^{\mathrm{ab}}$. Thus objects are triples (A,B,e) where A and B are abelian groups, and $e\in H^2(A,B)$ is such an element that $\mu(e):\Lambda^2(A)\to B$ is an epimorphism. A morphism from (A,B,e) to (A',B',e') in $\mathsf{Niq}^{\mathrm{ab}}$ is a pair (f,g), where $f:A\to A'$ and $g:B\to B'$ are homomorphisms such that

$$f^*(e')-g_*(e)\in H^2_b(A,B').$$

6.6. **Theorem.** *The functor* $\mathsf{ch}:\mathsf{Nil}^{\sim}\to\mathsf{Nil}^{\mathrm{ab}}$ *has a canonical extension*

$$\mathsf{ch}:\mathsf{Niq}^{\approx}\to\mathsf{Niq}^{\mathrm{ab}}$$

which is an equivalence of categories.

Proof. On objects one puts

$$\mathsf{ch}(G)=(G^{\mathrm{ab}},[G,G],\mathsf{e}(G)).$$

If $f:G\to H$ is a q-map, then one puts

$$\mathsf{ch}(G\xrightarrow{f}H)=(f^{\mathrm{ab}},[f,f]).$$

We claim that one has

$$\left(f^{\mathrm{ab}}\right)^*(\mathsf{e}(H))-[f,f]_*(\mathsf{e}(G))\in H^2_b(G^{\mathrm{ab}},[H,H]).$$

Let α (resp. β) be a 2-cocycle representing the class $\mathsf{e}(G)$ (resp. $\mathsf{e}(H)$). Thus $G=G^{\mathrm{ab}}\times[G,G]$ (resp. $H=H^{\mathrm{ab}}\times[H,H]$) as a set, with the following group structure $(a,u)+(b,v)=(a+b,\alpha(a,b)+u+v)$, where $a,b\in G^{\mathrm{ab}}$ and $u,v\in[G,G]$ (resp. $(c,x)+(d,y)=(c+d,\beta(c,d)+x+y)$, $c,d\in H^{\mathrm{ab}}$, $x,y\in[H,H]$). Any q-map $f:G\to H$ has the form $f(a,u)=(f^{\mathrm{ab}}(a),[f,f](u)+\gamma(a))$, where $f^{\mathrm{ab}}:G^{\mathrm{ab}}\to H^{\mathrm{ab}}$ and $[f,f]:[G,G]\to[H,H]$ are homomorphisms, while $\gamma:G^{\mathrm{ab}}\to[H,H]$ is a map. One has

$$\begin{aligned}f((a,u)+(b,v))=&f(a+b,\alpha(a,b)+u+v)\\ =&(f^{\mathrm{ab}}(a)+f^{\mathrm{ab}}(b),\gamma(a+b)+[f,f](\alpha(a,b))+[f,f](u)+[f,f](v)).\end{aligned}$$

On the other hand we have $f((a,u)+(b,v))=f((a,u))+f((b,v))+((a,u),(b,v))_f$. Since the cross-effect of f factors through $\delta:G^{\mathrm{ab}}\otimes G^{\mathrm{ab}}\to[H,H]$ we obtain

$$\begin{aligned}&f((a,u)+(b,v))\\ &=(f^{\mathrm{ab}}(a),[f,f](u)+\gamma(a))+(f^{\mathrm{ab}}(b),[f,f](v)+\gamma(b))+(0,\delta(a,b))\\ &=(f^{\mathrm{ab}}(a)+f^{\mathrm{ab}}(b),\beta(f^{\mathrm{ab}}(a),f^{\mathrm{ab}}(b))+\gamma(a)+\gamma(b)+\delta(a,b)+[f,f](u)+[f,f](v)).\end{aligned}$$

Comparing these expressions we obtain

$$\gamma(a+b)+[f,f](\alpha(a,b))=\beta(f^{\mathrm{ab}}(a),f^{\mathrm{ab}}(b))+\gamma(a)+\gamma(b)+\delta(a,b).$$

Thus the class $\left(f^{\mathrm{ab}}\right)^*(\mathsf{e}(H))-[f,f]_*(\mathsf{e}(G))$ in the group $H^2_b(G^{\mathrm{ab}},[H,H])$ coincides with the class of $-\delta$ and the claim is proved. It follows that ch is a well-defined functor $\mathsf{Niq}\to\mathsf{Niq}^{\mathrm{ab}}$, which obviously factors trough the category $\mathsf{Niq}^{\approx}$. By our construction and by definition of $\mathsf{Niq}^{\approx}$ the induced map

$$\mathrm{Hom}_{\mathsf{Niq}^{\approx}}(G,H)\to\mathrm{Hom}_{\mathsf{Niq}^{\mathrm{ab}}}(\mathsf{ch}(G),\mathsf{ch}(H))$$

is an injection. Let us show that this map is surjective as well. Take any morphism $(g, h) : \mathsf{ch}(G) \to \mathsf{ch}(H)$ in $\mathbf{Niq}^{\mathrm{ab}}$. Then $g : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ and $h : [H, H] \to [G, G]$ are homomorphisms such that

$$h(\alpha(a, b)) - \beta(g(a), g(b)) = -\gamma(a + b) + \gamma(a) + \gamma(b) + \delta(a, b)$$

where $\delta : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to [H, H]$ is a homomorphism, $\gamma : G^{\mathrm{ab}} \to [H, H]$ is a map, while α and β are as above. Define the map $f : G \to H$ via $f(a, u) = (g(a), \gamma(a) + h(u))$. Then one has

$$((a, u), (b, v))_f = \delta(a, b).$$

Thus f is a q-map with $f^{\mathrm{ab}} = g$ and $[f, f] = h$. Therefore ch is full and faithful. By Theorem 5.5 the functor ch is surjective on isomorphism classes of objects and the result follows. □

7. q-SPLIT GROUPS

We start with the following definitions.

7.1. **Definition.** Call nil$_2$-groups *similar* if they have isomorphic abelianizations and isomorphic commutator subgroups.

7.2. **Definition.** Call a nil$_2$-group G *q-split* if the quotient map $G \twoheadrightarrow G^{\mathrm{ab}}$ has a quadratic section. It is easy to see that this section is then a q-map.

7.3. **Lemma.** *The class of q-split groups contains all abelian groups and is closed under products and coproducts.*

Proof. For products and abelian groups this is obvious. For coproducts, note that the central extension

$$0 \to G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \to G_1 \vee G_2 \to G_1 \times G_2 \to 0$$

has a quadratic section s given by $s(g_1, g_2) = (0, g_1, g_2)$. One easily checks that

$$((x_1, x_2) \mid (y_1, y_2))_s = (y_1 \otimes x_2, 0, 0) = [(0, y_1, 0), (0, 0, x_2)].$$

Thus s is a q-map. Since $(G_1 \vee G_2)^{\mathrm{ab}} = (G_1 \times G_2)^{\mathrm{ab}} = G_1^{\mathrm{ab}} \times G_2^{\mathrm{ab}}$ we see that for any quadratic sections $s_i : G_i^{\mathrm{ab}} \to G_i$ of the natural projections $G_i \twoheadrightarrow G_i^{\mathrm{ab}}$, $i = 1, 2$, the composite $s \circ (s_1 \times s_2) : (G_1 \vee G_2)^{\mathrm{ab}} \to G_1 \vee G_2$ is a section. Since s, s_1, s_2 are q-maps, $s \circ (s_1 \times s_2)$ is also a q-map and the result follows. □

7.4. **Example.** It follows that the dihedral group $D_4 \cong \mathbf{Z}/2\mathbf{Z} \vee \mathbf{Z}/2\mathbf{Z}$ of order 8 is q-split. Let us show that the quaternion group $Q_8 = \langle \tau, \omega \mid 2\tau = 2\omega, \omega + \tau - \omega = -\tau \rangle$ of order 8 is also q-split. Observe that τ and ω are of order 4 and $[\omega, \tau] = 2\tau$. So one has $Q_8^{\mathrm{ab}} \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \cong D_4^{\mathrm{ab}}$ and $[Q_8, Q_8] \cong \mathbf{Z}/2\mathbf{Z} \cong [D_4, D_4]$. One easily checks that the map $s : Q_8^{\mathrm{ab}} \to Q_8$ given by $s(0) = 0$, $s(\hat{\omega}) = \omega$, $s(\hat{\tau}) = \tau$, $s(\hat{\omega} + \hat{\tau}) = \omega + \tau$ is a quadratic section of $Q_8 \to Q_8^{\mathrm{ab}}$. It follows from Corollary 7.6 below that Q_8 and D_4 are isomorphic in $\mathbf{Niq}$.

7.5. **Lemma.** *A nil$_2$-group G is q-split iff the class $\mathsf{e}(G) \in H^2(G^{\mathrm{ab}}, [G, G])$ belongs to the subgroup $H_b^2(G^{\mathrm{ab}}, [G, G])$.*

Proof. Let $u : G^{\mathrm{ab}} \to G$ be a quadratic section. Then the class $\mathsf{e}(G)$ can be represented by the cocycle $(a_1, a_2) \mapsto u(a_1) + u(a_2) - u(a_1 + a_2)$ which is bilinear and therefore lies in $H^2_b(G^{\mathrm{ab}}, [G,G])$. Conversely, if the class $\mathsf{e}(G) \in H^2(G^{\mathrm{ab}}, [G,G])$ is represented by a bilinear map $f : G^{\mathrm{ab}} \times G^{\mathrm{ab}} \to [G,G]$, then G is isomorphic to the set $G^{\mathrm{ab}} \times [G,G]$ with group structure defined by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + f(a_1, a_2))$, and the projection of the latter to G^{ab} has a quadratic section given by $a \mapsto (a, 0)$. □

We denote by $\mathbf{Spl}(\mathbf{Niq})$ and $\mathbf{Spl}(\mathbf{Niq}^{\approx})$ the full subcategories of, respectively, $\mathbf{Niq}$ and $\mathbf{Niq}^{\approx}$ with objects all q-split groups. They are related via the following linear extension:

$$0 \to D^{\approx} \to \mathbf{Spl}(\mathbf{Niq}) \to \mathbf{Spl}(\mathbf{Niq}^{\approx}) \to 0$$

and in particular they have the same isoclasses of objects. According to Theorem 6.6 and Lemma 7.5 the category $\mathbf{Spl}(\mathbf{Niq}^{\approx})$ is equivalent to the category $\mathbf{Spl}^{\mathrm{ab}}$, which is the full subcategory of the category $\mathbf{Nil}^{\mathrm{ab}}$ on those objects (A, B, e) of $\mathbf{Nil}^{\mathrm{ab}}$ satisfying $e \in H^2_b(A, B)$. Let us observe that

$$\mathsf{Hom}_{\mathbf{Nil}^{\mathrm{ab}}}((A, B, e), (A', B', e')) = \mathsf{Hom}(A, A') \times \mathsf{Hom}(B, B')$$

because the compatibility condition required in the definition of morphisms in $\mathbf{Nil}^{\mathrm{ab}}$ holds authomatically in $\mathbf{Spl}^{\mathrm{ab}}$.

We now consider another category $\mathbf{Spl}$, which is a full subcategory of the product category $\mathbf{Ab} \times \mathbf{Ab}$. Objects of the category $\mathbf{Spl}$ are pairs of abelian groups (A, B) for which there exists a homomorphism $f : A \otimes A \to B$ such that $f^a : \Lambda^2(A) \to B$ is an epimorphism, where $f^a(x, y) := f(x, y) - f(y, x)$.

7.6. **Theorem.** *The categories* $\mathbf{Spl}$ *and* $\mathbf{Nil}^{\mathrm{ab}}$ *are equivalent. Thus, two q-split groups are isomorphic in* $\mathbf{Niq}$ *iff they are similar.*

Proof. Take any object (A, B) of $\mathbf{Spl}$ and choose $f : A \otimes A \to B$ for which f^a is an epimorphism. Let $e_f \in H^2_b(A, B)$ be the class correspobding to f. Then $(A, B, e_f) \in \mathbf{Nil}^{\mathrm{ab}}$. Then $(A, B, f) \mapsto (A, B, e_f)$ yields expected equivalence of categories. □

7.7. **Remark.** One easily sees that the class S of abelian groups for which the natural short exact sequence $0 \to \Lambda^2(A) \to A \otimes A \to \mathrm{S}^2(A) \to 0$ splits contains all cyclic groups, all uniquely 2-divisible groups and is closed under direct sums. In particular any finitely generated abelian group lies in S. If $A \in \mathsf{S}$ then for any homomorphism $g : \Lambda^2(A) \to B$ there exists a homomorphism $f : A \otimes A \to B$ such that $f^a = g$. It follows that a pair of abelian groups (A, B) with $A \in \mathsf{S}$ belongs to $\mathbf{Spl}$ iff there exists an epimorphism $\Lambda^2(A) \to B$.

7.8. **Proposition.** *For abelian groups A, B there is a commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Hom}(\mathrm{S}^2 A, B) \arrow[r] \arrow[d, dashed] & \mathsf{Hom}(A \otimes A, B) \arrow[r] \arrow[d, "\alpha"] & \mathsf{Hom}(\Lambda^2 A, B) \arrow[d, equal] & \\ 0 \arrow[r] & \mathsf{Ext}(A, B) \arrow[r] & H^2(A; B) \arrow[r, "c"] & \mathsf{Hom}(\Lambda^2 A, B) \arrow[r] & 0 \end{tikzcd} \tag{6}$$

where the image of the homomorphism α is equal to the subgroup $H^2_b(A, B)$ from 6.5.

Proof. For any abelian group A one has a short exact sequence

$$0 \to \Lambda^2 A \to A \otimes A \to \mathrm{S}^2 A \to 0,$$

We place in the upper row of (6) the sequence induced by this short exact sequence. The lower row is the universal coefficient exact sequence, and the map α is given by considering a bilinear map as a 2-cocycle. The rest is obvious. □

7.9. **Remark.** The arrow on the upper right of (6) is surjective if A is either uniquely 2-divisible or is a direct sum of cyclic groups. Indeed in these cases the aforementioned short exact sequence splits.

7.10. **Proposition.** *If 2 is invertible in B then the above homomorphism $\mathsf{Hom}(\mathrm{S}^2 A, B) \to \mathsf{Ext}(A, B)$ is zero.*

Proof. For a homomorphism $f : \mathrm{S}^2 A \to B$ the class $\alpha(f)$ is represented by the cocycle $(x, y) \mapsto f(xy)$. This cocycle is the coboundary of the cochain $g : A \to B$ given by $a \mapsto \frac{1}{2} f(a^2)$. □

7.11. **Lemma.** *Let A be any abelian group and let B be a uniquely 2-divisible group. Then for any $x \in H^2_b(A; B)$ and any $0 \neq a \in \mathsf{Ext}(A, B)$ one has $x + a \notin H^2_b(A, B)$.*

Proof. Otherwise one would have $a \in \mathsf{Im}\,\alpha$, which contradicts the previous lemma. □

7.12. **Remark.** It follows that for any object (A, B, x) of $\mathbf{Niq}^{\mathrm{ab}}$ and any $a \in \mathsf{Ext}(A, B)$, also $(A, B, x + a)$ gives an object of $\mathbf{Niq}^{\mathrm{ab}}$, since in the universal coefficient exact sequence in (6) one has $c(x + a) = c(x)$. In particular, if (A, B, x) with uniquely 2-divisible B lies in the subcategory $\mathbf{Spl}^{\mathrm{ab}}$ and $a \neq 0$, then $(A, B, x + a) \in \mathbf{Niq}^{\mathrm{ab}}$ cannot belong to $\mathbf{Spl}^{\mathrm{ab}}$, since by Lemma 7.11, $x + a \notin H^2_b(A; B)$. Thus not all nil$_2$-groups are q-split. Some explicit examples of non-q-split groups follow.

7.13. **Example.** For each prime p consider the semidirect product $\mathbf{Z}/p^2\mathbf{Z} \ltimes \mathbf{Z}/p\mathbf{Z}$, where the generator of $\mathbf{Z}/p\mathbf{Z}$ acts on $\mathbf{Z}/p^2\mathbf{Z}$ via multiplication by $p + 1$. This group is similar in the sense of Definition 7.1 to $\mathbf{Z}/p\mathbf{Z} \vee \mathbf{Z}/p\mathbf{Z}$ (both have abelianizations isomorphic to $(\mathbf{Z}/p\mathbf{Z})^2$ and commutator subgroups isomorphic to $\mathbf{Z}/p\mathbf{Z}$). For $p = 2$ these groups are in fact isomorphic; however for odd p they are not, since the former has exponent p^2 and the latter has exponent p. Thus in the diagram (6) for $A = (\mathbf{Z}/p\mathbf{Z})^2$ and $B = \mathbf{Z}/p\mathbf{Z}$, classes of $\mathbf{Z}/p\mathbf{Z} \vee \mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/p^2\mathbf{Z} \ltimes \mathbf{Z}/p\mathbf{Z}$ in $H^2((\mathbf{Z}/p\mathbf{Z})^2; \mathbf{Z}/p\mathbf{Z})$ are not equal. On the other hand one can choose isomorphisms of their commutator subgroups with $\mathbf{Z}/p\mathbf{Z}$ in a way which makes obvious that these classes have the same image under the homomorphism c defined in (6) above, hence they differ by a nonzero element of $\mathsf{Ext}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$. But $\mathbf{Z}/p\mathbf{Z} \vee \mathbf{Z}/p\mathbf{Z}$ is q-split by Lemma 7.3, hence its class is in the image of the homomorphism α from (6). Then by Lemma 7.11 we conclude that $\mathbf{Z}/p^2\mathbf{Z} \ltimes \mathbf{Z}/p\mathbf{Z}$ is not q-split. In particular, the above similar groups are also not isomorphic in $\mathbf{Niq}$.

8. q-MAPS FOR UNIQUELY 2-DIVISIBLE GROUPS

Let us recall the relevant part of the classical Maltsev correspondence between nilpotent groups and Lie algebras. In the nil$_2$ case it amounts to an isomorphism of categories from the category of nil$_2$ Lie algebras over $\mathbf{Z}[\frac{1}{2}]$, i. e. Lie algebras with $[L, [L, L]] = 0$ to the category of uniquely

2-divisible nil_2-groups. In what follows, all Lie algebras are understood to be of the above kind, i. e. class two nilpotent Lie $\mathbf{Z}[\frac{1}{2}]$-algebras. Let us denote by $\mathsf{Nil}(\mathbf{Z}[\frac{1}{2}])$ the category of these algebras and their homomorphisms. Moreover we will denote by $\mathsf{Nil}^{\frac{1}{2}}$ the category of uniquely 2-divisible nil_2-groups.

One defines an isomorphism of categories

$$\exp : \mathsf{Nil}(\mathbf{Z}[\tfrac{1}{2}]) \to \mathsf{Nil}^{\frac{1}{2}}$$

by declaring, for an algebra $L \in \mathsf{Nil}(\mathbf{Z}[\frac{1}{2}])$, $\exp(L)$ to be the set L equipped with the operation

$$a \oplus b = a + b + \tfrac{1}{2}[a, b].$$

This is a group, with zero element 0 and inverse of an element a given by $-a$. Moreover the commutator with respect to this group structure coincides with the Lie bracket, so that for any L one has $[\exp(L), \exp(L)] = [L, L]$ and $\exp(L)^{\mathrm{ab}} = L^{\mathrm{ab}}$, where $L^{\mathrm{ab}} = L/[L, L]$ is the abelianization of the Lie algebra L.

Now clearly any Lie algebra homomorphism is also a homomorphism with respect to $\oplus$. Moreover we have $a \oplus c = a + c$ for $c \in [L, L]$, so that for any $a, b \in L$

$$a + b = a \oplus b \oplus \tfrac{1}{2}[b, a].$$

It follows that also conversely, a map which is a homomorphism with respect to $\oplus$ is a Lie algebra homomorphism, so that $\exp$ is an isomorphism of categories, with the inverse isomorphism $\log$ defined as follows: for a uniquely 2-divisible nil_2-group G the Lie algebra $\log(G)$ is the set G equipped with the addition as above and with the bracket equal to the commutator map.

Our aim in this section is to prove

8.1. **Theorem.** *Two uniquely 2-divisible nil_2 groups G, G' are isomorphic as objects of* Niq *if and only if there exists an isomorphism of abelian groups $g : \log(G) \to \log(G')$ such that $g[G, G] = [G', G']$.*

For the proof we must define an analog of the category Niq from Section 4 for Lie algebras. For this, we first define

8.2. **Definition.** A map $f : L \to L'$ between Lie algebras is called a q-map if it is a quadratic map between the underlying abelian groups and moreover for any $a, b \in L$ and any $c \in [L, L]$ one has $(a \mid b)_f \in [L', L']$, $f(a + c) = f(a) + f(c)$ and $f(c) \in [L', L']$.

Moreover we consider the following category $\mathsf{Niq}(\mathbf{Z}[\frac{1}{2}]) \supset \mathsf{Nil}(\mathbf{Z}[\frac{1}{2}])$ with the same objects as $\mathsf{Nil}(\mathbf{Z}[\frac{1}{2}])$. A morphism $L \to L'$ in $\mathsf{Niq}(\mathbf{Z}[\frac{1}{2}])$ is a q-map in the sense just defined.

The key observation is then

8.3. **Theorem.** *The functor* $\exp$ *extends to an isomorphism of categories*

$$\mathsf{Niq}^{\frac{1}{2}} \simeq \mathsf{Niq}(\mathbf{Z}[\tfrac{1}{2}]).$$

This theorem follows immediately from the following

8.4. **Proposition.** *Let $f : L \to L'$ be a map between Lie algebras. Then the following assertions are equivalent:*

i) *f is a q-map in the sense of 8.2;*
ii) *f is a q-map when considered as a map $\exp(L) \to \exp(L')$;*
iii) *there exists a linear map $g : L \to L'$ with $g[L, L] \subseteq [L', L']$ and a symmetric bilinear map $h : L^{\mathrm{ab}} \times L^{\mathrm{ab}} \to [L', L']$ such that one has*

$$f(a) = g(a) + \tfrac{1}{2}h(\hat{a}, \hat{a})$$

for all $a \in L$.

Proof. ii) $\iff$ iii):

Let $(a \mid b)^{+}_{f}$, $(a \mid b)^{\oplus}_{f}$ denote the cross-effect of f with respect to the corresponding operations. Thus f is a q-map when considered as a map $\exp(L) \to \exp(L')$ iff $(a \mid b)^{\oplus}_{f}$ is bilinear and lands in $[L', L']$. In that case we have

$$f(a + b) = f(a \oplus b \oplus \tfrac{1}{2}[b, a]) = f(a \oplus b) \oplus f(\tfrac{1}{2}[b, a]) = fa \oplus fb \oplus (a \mid b)^{\oplus}_{f} \oplus \tfrac{1}{2}f[b, a]$$

and

$$fa + fb = fa \oplus fb \oplus \tfrac{1}{2}[fb, fa],$$

hence

$$\begin{aligned}(a \mid b)^{+}_{f} &= -(fa + fb) + f(a + b) = -(fa + fb) \oplus f(a + b) \oplus \tfrac{1}{2}[f(a + b), -(fa + fb)]\\ &=\tfrac{1}{2}[fa, fb] \oplus (a \mid b)^{\oplus}_{f} \oplus \tfrac{1}{2}f[b, a] \oplus \tfrac{1}{2}[fa \oplus fb \oplus (a \mid b)^{\oplus}_{f} \oplus \tfrac{1}{2}f[b, a], -(fa \oplus fb \oplus \tfrac{1}{2}[fb, fa])]\\ &=\tfrac{1}{2}[fa, fb] \oplus (a \mid b)^{\oplus}_{f} \oplus \tfrac{1}{2}f[b, a].\end{aligned}$$

The latter expression is then symmetric since it is the cross-effect of some map with respect to the commutative operation $+$. It is bilinear with respect to $\oplus$ and satisfies

$$(a \oplus c \mid b)^{+}_{f} = (a \mid b \oplus c)^{+}_{f} = (a \mid b)^{+}_{f}$$

for any $c \in [L, L]$ and any $a, b \in L$. Hence it is also bilinear with respect to $+$ and defining

$$h(\hat{a}, \hat{b}) = (a \mid b)^{+}_{f}$$

gives a well-defined symmetric bilinear map $h : L^{\mathrm{ab}} \times L^{\mathrm{ab}} \to [L', L']$. Then the map $g : L \to L'$ given by

$$g(a) = f(a) - \tfrac{1}{2}h(\hat{a}, \hat{a}) = f(a) - \tfrac{1}{2}(a \mid a)^{\oplus}_{f}$$

carries $[L, L]$ to $[L', L']$. Moreover this map is linear since $(a \mid a)^{\oplus}_{f} = (a \mid a)^{+}_{f}$ for any $a \in L$, so that

$$\begin{aligned}g(a + b) &=f(a + b) - \tfrac{1}{2}(a + b \mid a + b)^{+}_{f}\\ &=fa + fb + (a \mid b)_{f} - \tfrac{1}{2}(a \mid a)^{+}_{f} - \tfrac{1}{2}(b \mid b)^{+}_{f} - \tfrac{1}{2}(a \mid b)^{+}_{f} - \tfrac{1}{2}(b \mid a)^{+}_{f}\\ &=g(a) + g(b).\end{aligned}$$

Conversely, given g and h as in iii), we compute

$$
\begin{aligned}
&(a \mid b)_f^{\oplus} = -(fa \oplus fb) \oplus f(a \oplus b)\\
=&-(fa + fb + \tfrac{1}{2}[fa, fb]) + f(a + b + \tfrac{1}{2}[a, b]) + \tfrac{1}{2}[fa + fb + \tfrac{1}{2}[fa, fb], f(a + b + \tfrac{1}{2}[a, b])]\\
=&-(ga + \tfrac{1}{2}h(\hat{a}, \hat{a}) + gb + \tfrac{1}{2}h(\hat{b}, \hat{b}) + \tfrac{1}{2}[ga, gb])\\
&+ ga + gb + \tfrac{1}{2}g[a, b] + \tfrac{1}{2}h(\hat{a} + \hat{b}, \hat{a} + \hat{b}) + \tfrac{1}{2}[-(ga + gb), ga + gb]\\
=&-\tfrac{1}{2}[ga, gb] + \tfrac{1}{2}g[a, b] + h(\hat{a}, \hat{b})
\end{aligned}
$$

which lies in $[L', L']$ and is bilinear, so indeed f is a q-map.

i) $\iff$ iii):

Obviously any f satisfying iii) is quadratic. Moreover, a map f between $\mathbf{Z}[\frac{1}{2}]$-modules is quadratic if and only if it has the form

$$f(a) = g(a) + \tfrac{1}{2}h(a, a)$$

for unique linear map g and bilinear symmetric map h. One just takes $g(a) = 2f(a) - \frac{1}{2}f(2a)$ and $h(a, b) = f(a + b) - f(a) - f(b)$. Then it is easy to check that a quadratic map is a q-map of Lie algebras if and only if the corresponding g and h satisfy conditions in iii). $\square$

This enables us to obtain an extension to the q-world of the above classical Maltsev equivalence, by identifying the full subcategory $\mathsf{Niq}^{\frac{1}{2}} \subset \mathsf{Niq}$ on the uniquely 2-divisible nil$_2$-groups with the following category defined in terms of Lie $\mathbf{Z}[\frac{1}{2}]$-algebras.

Moreover in this situation 6.6 admits a strengthening. To formulate it we will need some more categories.

8.5. **Definition.** Let $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}]) \subset \mathsf{Niq}(\mathbf{Z}[\frac{1}{2}])$ be the subcategory with the same objects and those morphisms which are actually linear. That is, a morphism from L to L' in $\mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ is an abelian group homomorphism $g : L \to L'$ with $g[L, L] \subseteq [L', L']$.

Moreover let $\widetilde{\mathsf{Niq}}_0(k)$ be the quotient category of $\mathsf{Niq}_0(k)$ obtained by identifying those $g_1, g_2 : L \to L'$ for which $g_1|_{[L,L]} = g_2|_{[L,L]}$ and $g_1^{\mathrm{ab}} = g_2^{\mathrm{ab}} : L^{\mathrm{ab}} \to L'^{\mathrm{ab}}$.

We then have

8.6. **Proposition.** *There are linear extensions*

$$0 \to \Phi \to \mathsf{Niq}(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{q} \mathsf{Niq}_0(\mathbf{Z}[\tfrac{1}{2}]) \to 0$$

and

$$0 \to \tilde{\Phi} \to \mathsf{Niq}_0(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{\tilde{q}} \widetilde{\mathsf{Niq}}_0(\mathbf{Z}[\tfrac{1}{2}]) \to 0$$

defined as follows. The functor $q : \mathsf{Niq}(\mathbf{Z}[\frac{1}{2}]) \to \mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ is identity on objects and given on morphisms via

$$q(f)(a) = 2f(a) - \tfrac{1}{2}f(2a).$$

The bifunctor $\Phi : \mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}])^{\mathrm{op}} \times \mathsf{Niq}_0(\mathbf{Z}[\frac{1}{2}]) \to \mathsf{Ab}$ is given by

$$\Phi(L, L') = \mathsf{Hom}(\mathrm{S}^2(L^{\mathrm{ab}}), [L', L']).$$

The functor $\tilde{q}$ is the canonical quotient functor, and $\tilde{\Phi}$ is given by

$$\tilde{\Phi}(L, L') = \mathsf{Hom}(L^{\mathrm{ab}}, [L', L']).$$

Moreover the categories $\mathbf{Niq}_0(k)$ and $\widetilde{\mathbf{Niq}}_0(k)$ are both additive, and the functor q has a section given by the embedding.

Proof. Additivity of $\mathbf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ follows from the obvious fact that for any morphisms $g_1, g_2 : L \to L'$ in $\mathbf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ the maps $g_1 \pm g_2$ are morphisms of $\mathbf{Niq}_0(\mathbf{Z}[\frac{1}{2}])$ too.

The rest is clear in view of the above considerations. Indeed we can replace a morphism $f : L \to L'$ in $\mathbf{Niq}(\mathbf{Z}[\frac{1}{2}])$ by a pair (g, h) as in iii) of Proposition 8.4. Under this identification the functor q becomes the projection sending (g, h) to g and the first linear extension becomes obvious. The second one is straightforward. $\square$

8.7. **Definition.** Let $\mathbf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ denote the following category. Objects of $\mathbf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ are short exact sequences

$$0 \to B \to E \to A \to 0$$

of $\mathbf{Z}[\frac{1}{2}]$-modules such that there exists a surjective homomorphism $\pi : \Lambda^2(A) \twoheadrightarrow B$. A morphism from $0 \to B \to E \to A \to 0$ to $0 \to B' \to E' \to A' \to 0$ is a pair $(\alpha : A \to A', \beta : B \to B')$ of homomorphisms such that there exists $\varepsilon : E \to E'$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\beta"'] & E \arrow[r] \arrow[d, dashed, "\varepsilon"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & B' \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

commute. We do not make π or ε part of the structure, in particular π is not required to be compatible with α and β in any way.

Note that $\mathbf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ is an additive category, since for any A, A' there are surjective homomorphisms $\Lambda^2(A \oplus A') \twoheadrightarrow \Lambda^2(A) \oplus \Lambda^2(A')$ and moreover for any morphism (α, β) in $\mathbf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ the pair $(-\alpha, -\beta)$ is also a morphism.

There is a functor $r : \widetilde{\mathbf{Niq}}_0(\mathbf{Z}[\frac{1}{2}]) \to \mathbf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ sending a Lie algebra L to the short exact sequence

$$0 \to [L, L] \to L \to L^{\mathrm{ab}} \to 0$$

and the morphism $[g] : L \to L'$ to the pair $(g^{\mathrm{ab}}, g|_{[L,L]})$, where $[g]$ denotes the equivalence class of g and $g^{\mathrm{ab}} : L^{\mathrm{ab}} \to L'^{\mathrm{ab}}$ is the homomorphism induced by g which exists since $g[L, L] \subseteq [L', L']$.

8.8. **Proposition.** *The above functor r yields an equivalence of categories*

$$\widetilde{\mathbf{Niq}}_0(\mathbf{Z}[\tfrac{1}{2}]) \simeq \mathbf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\tfrac{1}{2}]).$$

Proof. First, r is surjective on objects, since for any object $0 \to B \to E \to A \to 0$ of $\mathbf{Niq}^{\mathrm{ab}}(k)$ any surjective homomorphism $\Lambda^2(A) \twoheadrightarrow B$ determines a bracket

$$[,] : \Lambda^2(E) \twoheadrightarrow \Lambda^2(A) \twoheadrightarrow B \rightarrowtail E$$

on E which turns it into a nil_2 Lie algebra with $[E, E] = B$ and $E^{\mathrm{ab}} = A$.

Next, r is full since by definition a morphism from the object $0 \to [L,L] \to L \to L^{\mathrm{ab}} \to 0$ to the object $0 \to [L',L'] \to L' \to L'^{\mathrm{ab}} \to 0$ in $\mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\frac{1}{2}])$ is by definition a pair of linear maps $\beta : [L,L] \to [L',L']$, $\alpha : L^{\mathrm{ab}} \to L'^{\mathrm{ab}}$ for which there exists a linear map $g : L \to L'$ fitting in the appropriate diagram, which means that $\beta = g|_{[L,L]}$ and $\alpha = g^{\mathrm{ab}}$.

Finally r is faithful since for $g_1, g_2 : L \to L'$ one has $r[g_1] = r[g_2]$ if and only if g_1 and g_2 are equivalent in the sense of 8.5, i. e. if and only if $[g_1] = [g_2]$. □

We can now finish the proof of our theorem.

Proof of 8.1. There is a chain of functors

$$\mathsf{Niq}^{\frac{1}{2}} \xrightarrow{(8.3)} \mathsf{Niq}(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{(8.6)} \mathsf{Niq}_0(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{(8.6)} \widetilde{\mathsf{Niq}}_0(\mathbf{Z}[\tfrac{1}{2}]) \xrightarrow{(8.8)} \mathsf{Niq}^{\mathrm{ab}}(\mathbf{Z}[\tfrac{1}{2}])$$

each of which is either an equivalence or a linear extension. The statement of 8.1 is that objects on the left are isomorphic if and only if their images under the composite functor are. This is clear since any linear extension reflects isomorphy of objects. □

9. A COHOMOLOGICAL OBSTRUCTION TO q-SPLITTING

We start with recalling the definition of the nonabelian cohomology. Let G^* be a cosimplicial group. One denotes by $\pi^0(G^*)$ the subgroup of G^0 consisting of elements $x \in G^0$ such that $d^0(x) = d^1(x)$. Moreover, one defines the pointed set $\pi^1(G^*)$ as the quotient of the pointed set

$$Z^1(G^*) = \left\{ y \in G^1 \mid d^1(y) = d^0(y) + d^2(y) \right\}$$

by the following equivalence relation: $y \sim z$, $y, z \in Z^1(G^*)$ iff there exists $x \in G^0$ such that $z = -d^0x + y + d^1x$. If G^* is abelian then one defines $\pi^*(G^*)$ in all dimensions using the homology of the associated cochain complex $(G^*, d = \sum(-1)^i d^i)$. In particular $\pi^i(G^*)$ is an abelian group, $i \geq 0$, provided G^* is abelian cosimplicial group. The following result is well known.

9.1. **Lemma.** *Let*

$$0 \to A^* \to G^* \to B^* \to 0$$

be a short exact sequence of cosimplicial groups. Then one has the exact sequence of pointed sets:

$$0 \to \pi^0(A^*) \to \pi^0(G^*) \to \pi^0(B^*) \to \pi^1(A^*) \to \pi^1(G^*) \to \pi^1(B^*)$$

Moreover, if A^* is abelian, then the connecting map $\pi^0(B^*) \to \pi^1(A^*)$ is a homomorphism.

We also need the following

9.2. **Lemma.** *Let G and H be nil$_2$-groups. Then one has the following exact sequence:*

$$0 \to \mathsf{Quad}(G, [H,H]) \to \mathsf{Q}(G,H) \to \mathsf{Hom}(G, H^{\mathrm{ab}}).$$

If additionally G is free in Nil, then the last map is surjective.

Proof. Assume $f : G \to H$ is a q-map. Then the composite of f with the quotient map $H \to H^{\mathrm{ab}}$ is a homomorphism, which is zero provided the image of f lies in $[H, H]$. Then the resulting map is quadratic. Conversely any quadratic map $G \to [H, H]$ considered as a map $G \to H$ is a q-map. If G is free then any homomorphism $G \to H^{\mathrm{ab}}$ has a lifting to a homomorphism $G \to H$ and the result follows. □

For any $G \in \mathbf{Nil}$ and any $A \in \mathbf{Ab}$ define the groups $\mathsf{Quad}^*(G, A)$ as the simplicial derived functors of the functor $\mathsf{Quad}(-, A)$. More precisely, let G_* be a free simplicial resolution of G. Thus G_* is a simplicial object in $\mathbf{Nil}$ such that for each $n \geqslant 0$ the group G_n is free in $\mathbf{Nil}$ and $\pi_i(G_*) = 0$ for $i > 0$ and $\pi_0(G_*) = G$. Then one can consider the cosimplicial abelian group $\mathsf{Quad}(G_*, A)$. It is well known that the groups $\pi^*(G_*, A)$ do not depend on the choice of a free simplicial resolution and they are denoted by $\mathsf{Quad}^*(G, A)$. Actually $\mathsf{Quad}^0(G, A) = \mathsf{Quad}(G, A)$.

For $N \in \mathbf{Nil}$ the sets $\pi^i(\mathsf{Q}(G_*, N))$ for $i = 0, 1$ also do not depend on the choice of a free simplicial resolution of G. We will denote them by $\mathsf{Q}^i(G, H)$, $i = 0, 1$. Actually $\mathsf{Q}^0(G, H) = \mathsf{Q}(G, H)$.

9.3. **Proposition.** *Let G and H be nil$_2$-groups. Then one has the following exact sequence:*

$$0 \to \mathsf{Quad}(G, [H, H]) \to \mathsf{Q}(G, H) \to \mathsf{Hom}(G, H^{\mathrm{ab}}) \\ \to \mathsf{Quad}^1(G, [H, H]) \to \mathsf{Q}^1(G, H) \to H^2_{\mathbf{Nil}}(G, H^{\mathrm{ab}}),$$

where all terms are groups except for $\mathsf{Q}^1(G, H)$ and all maps are homomorphisms except for the last two maps.

Proof. By Lemma 9.2 we have a short exact sequence of cosimplicial groups

$$0 \to \mathsf{Quad}(G_*, [H, H]) \to \mathsf{Q}(G_*, H) \to \mathsf{Hom}(G_*, H^{\mathrm{ab}}) \to 0,$$

where G_* is a free simplicial resolution of G. The rest follows from Lemma 9.1. □

9.4. **Corollary.** *For nil$_2$-groups G and H and a homomorphism $f : G \to H^{\mathrm{ab}}$, there is a well-defined element $o(f) \in \mathsf{Quad}^1(G, [H, H])$ which vanishes if and only if f lifts to a q-map $G \to H$.*

□

In particular, taking above H to be arbitrary, $G = H^{\mathrm{ab}}$ and f the identity map, denote the corresponding element $o(f)$ by $o(H)$; this is thus an element in $\mathsf{Quad}^1(H^{\mathrm{ab}}, [H, H])$. Then we have

9.5. **Corollary.** *For a nil$_2$-group G there is a well-defined element $o(G) \in \mathsf{Quad}^1(G^{\mathrm{ab}}, [G, G])$ which vanishes if and only if G is q-split.*

□

Acknowledgements

The paper was written during visits of the authors to the University of Bielefeld and the Max-Planck-Institut für Mathematik in Bonn. The authors gratefully acknowledge hospitality of these institutions.

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