Non-coherence of some lattices in Isom($\mathbb{H}^n$)

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To the memory of Heiner Zieschang

1 Introduction.

The aim of this paper is to prove non-coherence of certain families of lattices in the isometry group Isom($\mathbb{H}^n$) of the hyperbolic n-space $\mathbb{H}^n$ ($n > 3$). We recall that a group G is called *coherent* if every finitely generated subgroup of G is finitely presented. It is well known that all lattices in Isom($\mathbb{H}^2$) and Isom($\mathbb{H}^3$) are coherent. Indeed, it is easy to prove that every finitely generated Fuchsian group is finitely presented. The coherence of 3-manifold groups was proved by P. Scott [Sc2]. First examples of geometrically finite non-coherent discrete subgroups of Isom($\mathbb{H}^4$) were constructed in [KP1], [P1], [P2]. An example of non-coherent uniform lattice in Isom($\mathbb{H}^4$) was given in [BM].

In what follows we will identify $\mathbb{H}^n$ with a connected component of the hyperboloid

$$\{x : f(x) = -1\} \subset \mathbb{R}^{n+1},$$

where f is a real quadratic form of signature $(n, 1)$ in $n+1$ variables. Then the group Isom($\mathbb{H}^n$) is identified with the index 2 subgroup $O'(f, \mathbb{R}) \subset O(f, \mathbb{R})$ preserving $\mathbb{H}^n$.

Let f and g be quadratic forms on finite-dimensional vector spaces V and W over $\mathbb{Q}$. It is said that f *represents* g if the vector space V admits an orthogonal decomposition (with respect to f)

$$V = V' \oplus V''$$

so that $f|V'$ is isometric to g. In other words, after a change of coordinates, the form f can be written as

$$f(x_1, ..., x_n) = g(x_1, ..., x_k) + h(x_{k+1}, ..., x_n)$$

where $n = \dim(V)$ and $k = \dim(W)$. Whenever f represents g, a finite index subgroup of $O(g, \mathbb{Z})$ is naturally embedded into $O(f, \mathbb{Z})$.

The main result of this paper is:

Theorem A. *For every $n \geq 4$ and every rational quadratic form f of signature $(n, 1)$ which represents the form*

$$q_3 = -x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

the lattice $O(f, \mathbb{Z})$ is non-coherent.

Corollary 1.1. *For every $n \geq 4$ there are infinitely many commensurability classes of non-uniform non-coherent lattices in* $\mathrm{Isom}(\mathbb{H}^n)$.

We refer the reader to Section 3 for the discussion of uniform lattices. By combining Theorem A with some standard facts on rational quadratic forms, we prove

Theorem B. For every $n \geq 6$, every non-uniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ is non-coherent.

As a by-product of the proof, in Section 2.2, we obtain a simple proof of the following result of independent interest (which was proven in [ALR] in the case $n = 3$). Recall that a subgroup of a group Γ is called *separable* if it can be represented as the intersection of a family of finite index subgroups of Γ. For instance, separability of the trivial subgroup is nothing else than residual finiteness of Γ.

Theorem C. In every non-uniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ $(n \leq 5)$, every geometrically finite subgroup is separable.

We refer the reader to [K] for the definition of geometrically finite discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. Recall only that every discrete group which admits a finitely-sided convex fundamental polyhedron is geometrically finite.

In Section 4 we adopt the method of [GP] to obtain examples of non-arithmetic non-coherent lattices:

Theorem D. For each $n \geq 4$ there exist both uniform and non-uniform non-coherent non-arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^n)$.

The above results provide a strong evidence for the negative answer to the following question in the case of non-uniform lattices:

Question 1.2 (D. Wise). *Does there exist a coherent lattice in* $\mathrm{Isom}(\mathbb{H}^n)$ *for any $n > 3$?*

In Section 5 we provide some tentative evidence for the negative answer to this question in the uniform case as well.

Our proof of the non-coherence in the non-uniform case is different from the one in [BM]: The finitely generated infinitely presented subgroup that we construct is generated by four subgroups stabilizing 4 distinct hyperplanes in $\mathbb{H}^n$, while in the construction used in [BM] two hyperplanes were enough. Direct repetition of the arguments used in [BM] does not seem to work in the non-uniform case.

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2 Preliminaries.

We refer the reader to [K, Ma] for the basics of discrete groups of isometries of the hyperbolic spaces $\mathbb{H}^n$.

Notation. Given a convex polyhedron $Q \subset \mathbb{H}^n$ let $G(Q)$ denote the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ generated by the reflections in the walls of Q.

We will frequently use the quadratic forms

$$q_n = -x_0^2 + x_1^2 + ... + x_n^2.$$

Let f be a quadratic form

$$f = \sum_{i,j} a_{ij} x_i x_j$$

defined over a number field $K \subset \mathbb{R}$, and σ be an embedding $K \to \mathbb{R}$. Then f^σ will denote the form

$$\sum_{i,j} \sigma(a_{ij}) x_i x_j.$$

2.1 Arithmetic groups.

Let f be a quadratic form of signature $(n, 1)$ in $n+1$ variables with coefficients in a totally real algebraic number field $K \subset \mathbb{R}$ satisfying the following condition:

(*) For every nontrivial (i.e., different from the identity) embedding $\sigma : K \to \mathbb{R}$ the quadratic form f^σ is positive definite.

Below we discuss discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ defined using the form f. Let A denote the ring of integers of K. We define the group $\Gamma := O(f, A)$ consisting of matrices with entries in A preserving the form f. Then Γ is a discrete subgroup of $O(f, \mathbb{R})$. Moreover, it is a *lattice*, i.e., its index 2 subgroup

$$\Gamma' = O'(f, A) := O(f, A) \cap O'(f, \mathbb{R})$$

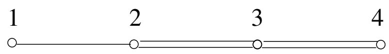
acts on $\mathbb{H}^n$ so that $\mathbb{H}^n/\Gamma'$ has finite volume. Such groups Γ (and subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ commensurable to them) are called *arithmetic subgroups of the simplest type* in $O(n, 1)$, see [VS].

Remark 2.1. *If $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is an arithmetic lattice so that either Γ is non-uniform or n is even, then it follows from the classification of rational structures on $\mathrm{Isom}(\mathbb{H}^n)$ that Γ is commensurable to an arithmetic lattice of the simplest type. For odd n there is another family of arithmetic lattices given as the groups of units of appropriate skew-Hermitian forms over quaternionic algebras. Yet other families of arithmetic lattices exist for $n = 3$ and $n = 7$. See e.g. [VS, LM].*

A lattice $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is called *uniform* if $\mathbb{H}^n/\Gamma$ is compact and *non-uniform* otherwise. An arithmetic lattice $O(f, A)$ of the simplest type is non-uniform if and only if $K = \mathbb{Q}$ and f is *isotropic*, i.e., there exists a non-zero vector $v \in \mathbb{Q}^{n+1}$ such that $f(v) = 0$.

Meyer's theorem (which follows from the Hasse–Minkowski principle, see [BS, pp. 61–62] or [Ca, Corollary 1, page 75]) states that every indefinite rational quadratic form of rank ≥ 5 is isotropic. Thus, for each rational quadratic form f of signature $(n, 1)$, $n \geq 4$, the lattice $O(f, \mathbb{Z})$ is non-uniform. Conversely, every non-uniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ is commensurable to $O(f, \mathbb{Z})$, where f is a rational quadratic form.

In particular, the groups $O'(q_n, \mathbb{Z}) \subset \mathrm{Isom}(\mathbb{H}^n)$ are non-uniform arithmetic lattices. The group $O'(q_3, \mathbb{Z})$ coincides with the group $G(\Delta)$, where $\Delta \subset \mathbb{H}^3$ is the simplex with the Coxeter diagram



(see [VS, Chapter 6, 2.1] and references therein).

Lemma 2.2. *The group $G(\Delta)$ contains a finite index subgroup Γ such that $\mathbb{H}^3/\Gamma$ fibers over the circle.*

Proof: Let $v_4 \in \Delta$ denote the (finite) vertex of Δ disjoint from the 4-th face. Consider the union O of the images of Δ under the stabilizer of v_4 in $G(\Delta)$. Then O is a regular right-angled ideal hyperbolic octahedron in $\mathbb{H}^3$, [PV, VS]. The group $G(\Delta)$ contains $G(O)$ as a finite index subgroup. It is well known that $G(O)$ is commensurable with the fundamental group of the Borromean rings complement which fibers over the circle, [Th]. The property of being the fundamental group of a surface bundle over the circle is hereditary with respect to subgroups of finite index. Thus $G(\Delta)$ contains a subgroup Γ of finite index so that $\mathbb{H}^3/\Gamma$ fibers over the circle. *QED.*

2.2 Rational quadratic forms.

The following proposition is well–known in the theory of rational quadratic forms, see [Ca, Excercise 8, Page 101]. We present a proof for the sake of completeness.

Proposition 2.3. *Let f and g be non-singular rational quadratic forms having respectively the signatures (r, s) and (p, q) such that $r \geq p$ and $s \geq q$. If $\mathrm{rank}(f) - \mathrm{rank}(g) \geq 3$ then f represents g.*

Proof: Recall that a rational quadratic form f on a rational vector space V *represents* $b \in \mathbb{Q}$ if there exists a vector $v \in V \setminus \{0\}$ such that $f(v) = b$.

Lemma 2.4. *Suppose that f is a non-singular rational quadratic form in $n \geq 4$ variables and b is an arbitrary non-zero rational number. Then*

(a) If f is positive definite and $b > 0$ then f represents b.

(b) If f is indefinite then f represents b.

Proof: The form

$$F(y_1, ..., y_n, y_{n+1}) := f(y_1, ..., y_n) - by_{n+1}^2$$

is an indefinite non-singular form of rank ≥ 5. By Meyer's theorem the form F represents 0. Hence by [BS, Theorem 6, p. 393], the form f represents b. *QED.*

Proof of Proposition 2.3. Let $n := r+s, k := p+q$ be the ranks of f and g respectively. After changing coordinates in $\mathbb{Q}^k$ we may assume that g has the diagonal form

$$g = b_1x_1^2 + ... + b_kx_k^2,$$

where $b_i \in \mathbb{Q}_+$, if $i \leq p$ and $b_i \in \mathbb{Q}_-$, if $i > p$.

Since f represents b_1 by Lemma 2.4, the form f is isomorphic to $b_1y_1^2+f_1(y_2, ..., y_n)$. By applying the same procedure to f_1 and arguing inductively we obtain, after k steps,

$$f = b_1y_1^2 + ... + b_ky_k^2 + f_k$$

where f_k is a form in $n-k$ variables. Note that the argument works as long as $n-k \geq 3$. Indeed, if $n = k+3$ we will have

$$f = b_1y_1^2 + ... + b_{k-1}y_{k-1}^2 + f_{k-1}(y_k, y_{k+1}, y_{k+2}, y_{k+3})$$

and therefore we can apply the above argument the last time to f_{k-1}. *QED.*

We now use the above proposition to prove Theorem C stated in Introduction.

Proof: We will use the following result proven by P. Scott in [Sc1] for the convex–cocompact subgroups and in [ALR] for the geometrically finite subgroups:

Suppose that $P \subset \mathbb{H}^n$ is a right–angled polyhedron of finite volume. Then every geometrically finite subgroup of $G(P)$ is separable.

Let Γ be a non-uniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^k)$, $k \leq 5$. Then Γ is commensurable to $O(g, \mathbb{Z})$ where g is a non-singular rational quadratic form of signature $(k, 1)$.

According to [PV] there exists a right-angled noncompact convex polyhedron of finite volume $P^8 \subset \mathbb{H}^8$. Moreover, the group $G(P^8)$ is a finite index subgroup in $O'(q_8, \mathbb{Z})$, see [VS]. Since $\mathrm{rank}(q_8) - \mathrm{rank}(g) \geq 3$, it follows that q_8 represents g, see Proposition 2.3. Hence we have a natural embedding of a finite index subgroup of Γ into $G(P^8)$. As P^8 is right-angled, every geometrically finite subgroup of $G(P^8)$ is separable. Since subgroup separability is hereditary with respect to passing to a subgroup, we conclude that every geometrically finite subgroup of Γ is separable. *QED.*

2.3 Hyperplane separability.

In Section 4 we will need the following variation on subgroup separability. Suppose that $\Gamma = O'(f, \mathbb{Z})$ is an arithmetic subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, where f is a rational quadratic form of signature $(n, 1)$. Let $V_i \subset \mathbb{R}^{n+1}$, $i = 0, 1, ..., k$ be rational vector subspaces of codimension 1, so that $V_i \otimes \mathbb{R}$ intersects $\mathbb{H}^n$ along the hyperplane H_i, $i = 0, 1, ..., k$. We assume that

$$H_0 \cap H_i = \emptyset, \quad i = 1, ..., k. \tag{1}$$

The following proposition is a generalization of [L]; its proof follows the lines of the proof of [MV, Lemma 10].

Proposition 2.5. *There exists a finite index subgroup* $\Gamma' \subset \Gamma$ *so that for every* $\gamma \in \Gamma'$ *either* $\gamma(H_0) = H_0$ *or*

$$\gamma(H_0) \cap (H_0 \cup H_1 \cup ... \cup H_k) = \emptyset.$$

Proof: Let $(\cdot, \cdot)$ denote the symmetric bilinear form on $\mathbb{R}^{n+1}$ corresponding to f. Suppose that $V, V' \subset \mathbb{R}^{n+1}$ are codimension 1 vector subspaces which intersect $\mathbb{H}^n$ along hyperplanes H, H'. Let $e, e' \in \mathbb{R}^{n+1}$ be non-zero vectors orthogonal to V, V' respectively. Then H intersects H' transversally iff

$$|(e, e')| < \sqrt{(e, e)(e', e')}.$$

For each V_i ($i = 0, 1, ..., k$) choose an orthogonal primitive integer vector e_i. Then (1) implies that

$$|(e_i, e_0)| \geq \sqrt{(e_i, e_i)(e_0, e_0)}, \quad i = 1, ..., k.$$

Choose a natural number N which is greater than

$$2 \max_{i=0,1,...,k} |(e_0, e_i)|.$$

Let $\Gamma' = \Gamma(N)$ denote the level N congruence subgroup in Γ, i.e., the kernel of the natural homomorphism

$$\Gamma \to GL(n + 1, \mathbb{Z}/N\mathbb{Z}).$$

Then for every $\gamma \in \Gamma'$, $i = 0, 1, ..., k$,

$$(\gamma(e_i), e_0) \equiv (e_i, e_0) \pmod{N}$$

and therefore either

$$|(\gamma(e_i), e_0)| = |(e_i, e_0)|$$

or

$$|(\gamma(e_i), e_0)| > |(e_i, e_0)| \geq \sqrt{(e_i, e_i)(e_0, e_0)} = \sqrt{(\gamma(e_i), \gamma(e_i))(e_0, e_0)},$$

hence either $\gamma(H_0) = H_i$ or $\gamma(H_0) \cap H_i = \emptyset$.

Lastly, we have to ensure that $\gamma(e_0) \neq \pm e_i$ for $i = 1, ..., k$ and all $\gamma \in \Gamma'$. This is achieved by taking N which does not divide some non-zero entries of $e_0 + e_i$ and of $e_0 - e_i$ for all $i = 1, ..., k$. *QED.*

2.4 A construction of non-coherent groups.

Let $L \subset \mathrm{Isom}(\mathbb{H}^3)$ be a subgroup commensurable to the reflection group $G(\Delta)$ defined in Section 2.1. We embed $\mathbb{H}^3$ in $\mathbb{H}^4$ as a hyperplane H and naturally extend the action of L from H to $\mathbb{H}^4$. Let $p_1, p_2 \in \partial H$ be distinct parabolic points of L. Let Π_1, Π_2 be perpendicular hyperplanes in $\mathbb{H}^4$ which are parallel to H and asymptotic to p_1, p_2, respectively. Let τ_i denote the (commuting) reflections in $\Pi_i, i = 1, 2$. Set $\tau_3 := \tau_1\tau_2$. Let G denote the subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ generated by L, τ_1, τ_2.

Theorem 2.6 ([KP1]). *For every choice of the group L, hyperplane H, points p_1, p_2 and hyperplanes Π_1, Π_2 as above, the group G is non-coherent.*

We will need the following:

Corollary 2.7. *Suppose that L_0, L_1, L_2, L_3 are arbitrary finite index subgroups in*

$$L, \quad \tau_1 L \tau_1, \quad \tau_2 L \tau_2, \quad \tau_3 L \tau_3, \quad \textit{respectively.}$$

Then the subgroup S of G generated by L_0, L_1, L_2, L_3 is non-coherent.

Proof: The intersection

$$L' := L_0 \cap \tau_1 L_1 \tau_1 \cap \tau_2 L_2 \tau_2 \cap \tau_3 L_3 \tau_3$$

is a finite index subgroup in L. Let S' denote the subgroup of S generated by

$$L', \quad \tau_1 L' \tau_1, \quad \tau_2 L' \tau_2, \quad \tau_3 L' \tau_3. \tag{2}$$

It is clear that S' has index 4 in the group generated by L', τ_1, τ_2. Since the latter is non-coherent by Theorem 2.6, it follows that S', and thus S, is non-coherent as well. *QED.*

Remark 2.8. *Note that the groups in (2) have the invariant hyperplanes H, $\tau_1(H)$, $\tau_2(H)$, $\tau_3(H)$, respectively. See Figure 1, where we use the projective model of $\mathbb{H}^4$.*

3 Construction of non-coherent arithmetic lattices.

Proof of Theorem A. Our strategy is to embed a non-coherent group G (of the type described in Section 2.4) into the lattice $O(f, \mathbb{Z})$. Then it would follow that $O(f, \mathbb{Z})$ is non-coherent.

Let q_3 be the quadratic form of rank 4 on the rational vector space U as in Section 2. Then $O'(q_3, \mathbb{Z}) = G(\Delta)$, see Section 2.1. We can change the coordinates in U to y_i ($i = 1, 2, 3, 4$) so that q_3 takes the form:

$$g = 2y_1y_2 + y_3^2 + y_4^2.$$

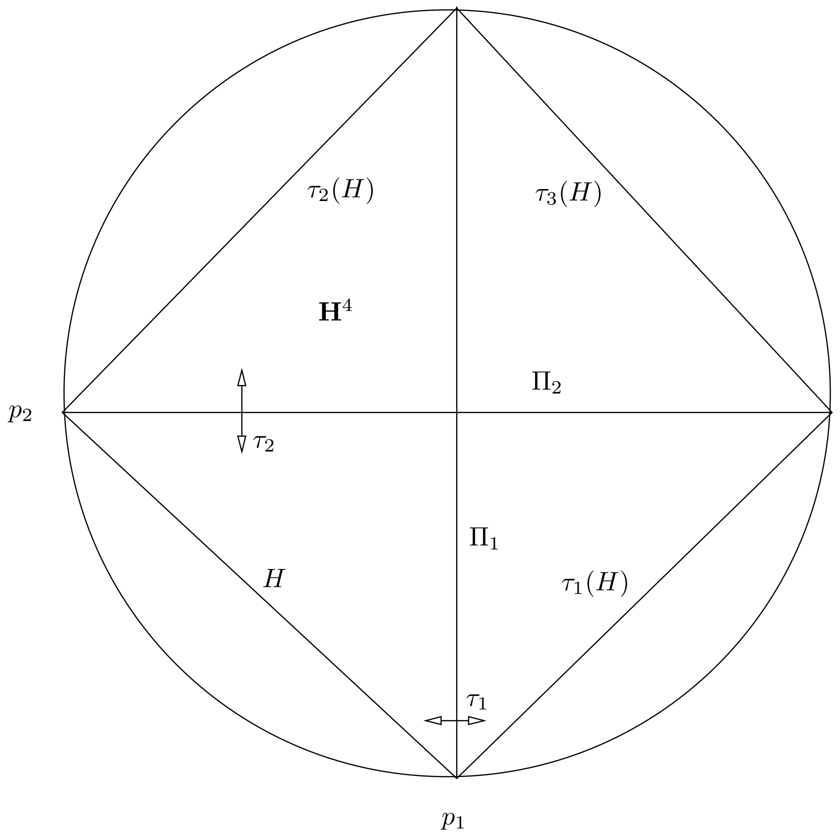


Figure 1:

Let $\{e_1, e_2, e_3, e_4\}$ be the corresponding basis of U. Note that the group $O(g, \mathbb{Z})$ is commensurable to $O(q_3, \mathbb{Z})$.

Let $(U, g) \to (V, f)$ be a rational embedding. Pick a non-zero vector $e_5 \in V$ orthogonal to U. Then

$$a := f(e_5) > 0.$$

Define a 5-dimensional vector space W spanned by the vector e_5 and U. Let h be the restriction of the form f to W; hence we have $(U, g) \subset (W, h) \subset (V, f)$. It therefore suffices to embed some non-coherent group G (as in Section 2.4) into the group $O'(h, \mathbb{Z})$.

We let $(\cdot, \cdot)$ denote the bilinear form on W corresponding to h. The space W splits as the orthogonal direct sum $U \oplus \mathbb{Q}e_5$. We will consider $\mathbb{H}^4$ canonically embedded in $W \otimes \mathbb{R}$ and identify $\mathbb{H}^3$ with the hyperplane $H := U \otimes \mathbb{R} \cap \mathbb{H}^4 \subset \mathbb{H}^4$.

After replacing e_2 with ae_2 we obtain $(e_1, e_2) = a$. Set

$$u_1 := e_1 + e_5, u_2 := -e_2 + e_5.$$

Thus

$$(u_1, u_1) = (u_2, u_2) = a, \quad (u_1, u_2) = (u_1, e_1) = (u_2, e_2) = 0,$$

$$(u_1, e_2) = -(u_2, e_1) = a.$$

Let $U_i \subset W$ $(i = 1, 2)$ be the 4-dimensional vector subspace orthogonal to u_i. Since $a > 0$, it follows that each $U_i \otimes \mathbb{R}$ $(i = 1, 2)$ intersects $\mathbb{H}^4$ along a hyperplane Π_i. The reflection

$$\tau_i : w \mapsto w - 2\frac{(w, u_i)}{(u_i, u_i)} u_i$$

in the subspace U_i is represented by a matrix with integer coefficients in the basis $\{e_1, ..., e_5\}$. Thus $\tau_i \in O'(h, \mathbb{Z})$, $i = 1, 2$.

Because $g(e_i) = 0$, the vector e_i corresponds to a parabolic point $p_i \in \partial\mathbb{H}^4$ of the group $O(g, \mathbb{Z})$, $i = 1, 2$. Since $(u_1, u_2) = 0$, it follows that Π_1 is perpendicular to Π_2. Moreover, since $e_i \in U_i$, we conclude that $\partial\Pi_i$ contains p_i, $i = 1, 2$. Since

$$(u_i, e_5) = \sqrt{(u_i, u_i)(e_5, e_5)},$$

the hyperplane Π_i is parallel to H, see the proof of Proposition 2.5.

Let L be a finite index subgroup of $O'(g, \mathbb{Z})$ contained in $O'(h, \mathbb{Z})$. The group G, generated by L, τ_1, τ_2, is contained in $O'(h, \mathbb{Z})$. Theorem 2.6 then implies that the lattice $O'(h, \mathbb{Z})$ is non-coherent. Hence $O(f, \mathbb{Z})$ is non-coherent as well. Theorem A follows. *QED.*

We next prove Corollary 1.1. For any number $a \in \mathbb{N}$ consider the quadratic form

$$f_a(x_0, x_1, ..., x_n) = q_3(x_0, x_1, x_2, x_3) + ax_4^2 + x_5^2 + ... + x_n^2.$$

Each f_a defines a non-uniform arithmetic lattice $O'(f_a, \mathbb{Z}) \subset \mathrm{Isom}(\mathbb{H}^n)$. Moreover, for infinitely many appropriately chosen primes a these lattices are not commensurable. Since each form f_a represents q_3, Corollary 1.1 follows from Theorem A. *QED.*

Theorem 3.1. *For each $n \geq 4$ there exist uniform non-coherent arithmetic lattices in* $\mathrm{Isom}(\mathbb{H}^n)$. *Moreover, for each $n \geq 5$ there are infinitely many commensurability classes of such lattices.*

Proof: The assertion is a rather direct corollary of the result of Bowditch and Mess [BM], but we present a proof for the sake of completeness. We start with a review of the example of Bowditch and Mess [BM] which is a non-coherent uniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^4)$.

Consider the right-angled regular 120-cell $D \subset \mathbb{H}^4$: It is a compact regular polyhedron, see for instance [D, VS]. It appears that it was first discovered by Schlegel in 1883 in [Sch], who was interested in classifying *honeycombs* in the spaces of constant curvature, see [Co].

Each facet of D is a right-angled regular dodecahedron. Let $\Gamma = G(D) \subset \mathrm{Isom}(\mathbb{H}^4)$ be the reflection group determined by D. The group Γ is commensurable to $O(q, A)$, where $q(x_0, x_1, x_2, x_3, x_4)$ is the quadratic form given by the matrix

$$\begin{bmatrix} 1 & -\cos(\pi/5) & 0 & 0 & 0 \\ -\cos(\pi/5) & 1 & -1/2 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -\cos(\pi/5) \\ 0 & 0 & 0 & -\cos(\pi/5) & 1 \end{bmatrix} \qquad (3)$$

and A is the ring of integers of the field $K = \mathbb{Q}(\sqrt{5})$. Thus Γ is a (uniform) arithmetic lattice. Consider the facets F_1, F_2 of D which share a common 2-dimensional face F. There is a canonical isomorphism $\varphi : G(F_1) \to G(F_2)$ fixing $G(F)$ elementwise. The reflection group $G(F_1)$ contains a finite index subgroup isomorphic to the fundamental group of a hyperbolic 3-manifold M^3 which fibers over S^1, see [Th]. Let $N_1 \subset \pi_1(M^3)$ be a normal surface subgroup and set $N_2 := \varphi(N_1) \subset G(F_2)$. In particular, both N_1, N_2 are finitely generated. On the other hand, $N_i \cap G(F)$ is a free group E of infinite rank, $i = 1, 2$. One then verifies that the subgroup of Γ generated by N_1 and N_2 is isomorphic to $N_1 *_E N_2$ and therefore is not finitely presentable, [Ne]. Hence Γ is a non-coherent uniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^4)$.

In order to construct lattices in $\mathrm{Isom}(\mathbb{H}^n)$ consider the quadratic forms

$$f_a(x_0, x_1, ..., x_n) = q(x_0, x_1, x_2, x_3, x_4) + ax_5^2 + x_6^2 + ... + x_n^2,$$

where $a \in \mathbb{N}$ are primes. Since q^σ is positive definite for the (unique) nontrivial embedding $\sigma : K \to \mathbb{R}$, it follows that each $O(f_a, A)$ is a uniform arithmetic lattice in $O(f_a, \mathbb{R})$. As in the non-compact case, the groups $O(f_a, \mathbb{R})$ are not commensurable for infinitely many primes a. As $O(q, A) \subset O(f_a, A)$, the assertion follows. *QED.*

Remark 3.2. *Clearly, the subgroup generated by any finite index subgroups of* $G(F_1)$ *and* $G(F_2)$ *is non-coherent as well.*

Remark 3.3. *The above construction produces only one commensurability class of non-coherent lattices in* $\mathrm{Isom}(\mathbb{H}^4)$. *Using noncommensurable arithmetic lattices in* $\mathrm{Isom}(\mathbb{H}^4)$ *containing* $G(F_1)$, *one can construct infinitely many commensurability classes of uniform non-coherent arithmetic lattices in* $\mathrm{Isom}(\mathbb{H}^4)$.

Theorem B. *For every* $n \geq 6$, *every non-uniform arithmetic lattice in* $\mathrm{Isom}(\mathbb{H}^n)$ *is non-coherent.*

Proof: Let Γ be a non-uniform arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ where $n \geq 6$. Then Γ is commensurable to $O(f, \mathbb{Z})$ for some rational form f of signature $(n, 1)$. Since $n + 1 \geq 7$ and q_3 has rank 4, it follows from Proposition 2.3 that f represents q_3. Therefore, by Theorem A, the group $O(f, \mathbb{Z})$ is non-coherent. Thus Γ is non-coherent as well. *QED.*

4 Non-arithmetic non-coherent lattices.

The goal of this section is to prove

Theorem D. *For each $n \geq 4$ there exist both uniform and non-uniform, non-arithmetic, non-coherent lattices in* $\mathrm{Isom}(\mathbb{H}^n)$.

Proof: We produce these non-coherent examples by using the construction of non-arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^n)$ due to Gromov and Piatetski–Shapiro [GP]. We begin with a review of their construction.

Let f be a quadratic form of signature $(n-1, 1)$ in n variables with coefficients in a totally real algebraic number field $K \subset \mathbb{R}$. Let A denote the ring of integers of K. We assume that f satisfies Condition (*) from Section 2.1.

We let K_+ denote the set of $a \in K$ such that for each embedding $\sigma : K \to \mathbb{R}$ we have $\sigma(a) > 0$. For $a \in K_+$ we consider the quadratic form

$$h_a(x_0, x_1, ..., x_n) = f(x_0, x_1, ..., x_{n-1}) + a x_n^2.$$

It has signature $(n, 1)$ and satisfies Condition (*). Then $\Gamma_a := O'(h_a, A)$ is a lattice in $\mathrm{Isom}(\mathbb{H}^n)$. Similarly, $\Gamma_0 := O'(f, A)$ is a lattice in $\mathrm{Isom}(\mathbb{H}^{n-1})$.

In what follows we will consider pairs of groups Γ_a, Γ_1, where $a \in \mathbb{N}$. Observe that both groups contain the subgroup Γ_0. Let $\Gamma'_a \subset \Gamma_a, \Gamma'_1 \subset \Gamma_1$ be torsion-free finite index subgroups such that

$$\Gamma'_1 \cap \Gamma_0 = \Gamma'_a \cap \Gamma_0.$$

We let Γ'_0 denote this intersection and set $M_1 := \mathbb{H}^n/\Gamma'_1, M_a := \mathbb{H}^n/\Gamma'_a$.

Without loss of generality (after passing to deeper finite index subgroups), we may assume that $\mathbb{H}^{n-1}/\Gamma'_0$ isometrically embeds into M_1 and M_a as a non-separating totally geodesic hypersurface, see [Mi]. Cut M_1 and M_a open along these hypersurfaces. The resulting manifolds M_1^+, M_a^+ both have totally geodesic boundaries isometric to the disjoint union of two copies of $M_0 = \mathbb{H}^{n-1}/\Gamma'_0$.

Let M be the connected hyperbolic manifold obtained by gluing M_1^+, M_a^+ via the isometry of their boundaries. It is easy to see that M is complete. Then there exists a lattice $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ such that $M = \mathbb{H}^n/\Gamma$. Note that M is compact iff both M_1, M_a are. It is proven in [GP] that

Γ *is arithmetic if and only if a is not a square in K.*

Note that there exist infinitely numbers a which are not squares in K. Indeed, it is well known that square roots of prime numbers are linearly independent over $\mathbb{Q}$. Therefore only finitely many of them belong to K.

We now prove Theorem D by working with specific examples.

1. Compact case. For $n \geq 5$ take $K = \mathbb{Q}(\sqrt{5})$ and consider the quadratic form

$$f = q(x_0, x_1, x_2, x_3, x_4) + x_5^2 + ... + x_{n-1}^2,$$

where the form q is given by the matrix (3). The quadratic form q yields a uniform arithmetic lattice $O'(q, A)$ in $\mathrm{Isom}(\mathbb{H}^4)$ which is commensurable to the reflection group $G(D)$ defined in the proof of Theorem 3.1. The group $O'(q, A)$ is non-coherent according to Theorem 3.1. On the other hand, by applying the Gromov– Piatetski–Shapiro construction to f and taking any prime number $a \neq 5$, we obtain a uniform non-arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ which contains $O'(q, A)$ and, hence, is non-coherent.

It remains to analyze the case $n = 4$. Take facets F_1, F_2, F_3 of D so that F_1 and F_2 intersect along a 2-dimensional face and

$$F_3 \cap F_1 = F_3 \cap F_2 = \emptyset.$$

Then the group generated by the reflections in the facets of F_1 and F_2 is non-coherent, see the proof of Theorem 3.1.

By taking an appropriate finite index subgroup $\Gamma_1 \subset G(D)$, we obtain a hyperbolic 4-manifold $M_1 = \mathbb{H}^4/\Gamma_1$ which contains embedded totally geodesic hypersurfaces S_i corresponding to the facets F_i, $i = 1, 2, 3$, so that

$$S_3 \cap S_1 = S_3 \cap S_2 = \emptyset, \quad S_1 \cap S_2 \neq \emptyset.$$

Now cut M_1 open along S_3 and apply the gluing construction of Gromov and Piatetski–Shapiro. In this way one can obtain a non-arithmetic compact hyperbolic manifold M whose fundamental group contains the subgroup of Γ_1 generated by some finite index subgroups of $G(F_1)$ and $G(F_2)$ and, hence, is non-coherent (see Remark 3.2).

2. Non-compact case. For $n \geq 5$ take $K = \mathbb{Q}$ and consider the quadratic form $f = q_{n-1}$. Taking any prime number for a, apply the same argument as in the compact case.

Consider $n = 4$. We will imitate the proof in the compact case. However we will appeal to the results of Section 2.3 instead of using a particular fundamental domain.

Let $\Gamma := O'(q_4, \mathbb{Z})$. Clearly, q_4 represents the form q_3. Set $L := O'(q_3, \mathbb{Z}) \subset \Gamma$, and let $\tau_1, \tau_2 \in O'(q_4, \mathbb{Z})$ be the commuting reflections constructed in the proof of Theorem A. Set $\tau_3 := \tau_1\tau_2$ and $L_0 := L, L_i := \tau_i L \tau_i$, $i = 1, 2, 3$.

By passing to any finite index subgroups $L_i' \subset L_i$, we obtain a non-coherent subgroup G' in Γ generated by $L_i', i = 0, 1, 2, 3$, see Corollary 2.7. Since Γ is a linear group, we can assume without loss of generality that Γ is torsion-free. Let $H_0 = H \subset \mathbb{H}^4$ be the L-invariant hyperplane. Then $H_i := \tau_i(H)$ $(i = 1, 2, 3)$ is the L_i-invariant hyperplane.

Lemma 4.1. *There exists a finite index subgroup $\Gamma' \subset \Gamma$ so that for the groups $L_i' := L_i \cap \Gamma'$ we have:*

1. H_0/L_0' embeds as a hypersurface S_0 into $\mathbb{H}^4/\Gamma'$.

2. Let M^+ denote the manifold obtained by cutting $\mathbb{H}^4/\Gamma'$ along S_0. Then G' embeds into $\pi_1(M^+)$.

Proof: We have to find a subgroup Γ' so that:

a. For all $\gamma \in \Gamma'$ either $\gamma(H_0) = H_0$ or

$$\gamma(H_0) \cap (H_0 \cup H_1 \cup H_2 \cup H_3) = \emptyset.$$

b. For all $\gamma \in \Gamma'$, the hyperplane $\gamma(H_0)$ does not separate the above hyperplanes from each other.

This is achieved by applying Proposition 2.5 to the hyperplanes H_0, H_1, H_2, H_3 and $H_4 := \Pi_1$, $H_5 := \Pi_2$. *QED.*

We now glue an appropriately chosen manifold M_a^+ along the boundary of M^+. Let M be the resulting complete hyperbolic manifold. Then, as in the case $n \geq 5$, the fundamental group of M is non-arithmetic. On the other hand,

$$G' \subset \pi_1(M^+) \subset \pi_1(M).$$

Therefore $\pi_1(M)$ is non-coherent. *QED.*

5 Non-coherence and Thurston's conjecture.

We recall the following conjecture:

Conjecture 5.1 (Thurston's virtual fibration conjecture). *Suppose that M is a hyperbolic 3-manifold of finite volume. Then there exists a finite cover over M which fibers over the circle.*

We expect that all lattices in $\mathrm{Isom}(\mathbb{H}^n)$ are non-coherent for $n \geq 4$. Proving this for non-arithmetic lattices is clearly beyond our reach. Therefore we restrict to the arithmetic case. Even in this case our discussion will be rather speculative. We restrict to the arithmetic groups of the simplest type $\Gamma = O(f, A)$, where f is a quadratic form on $V = K^{n+1}$ and $K \subset \mathbb{R}$ is a totally real algebraic number field (see Section 2.1). Choose a basis $\{e_0, e_1, ..., e_n\}$ in which the form f is diagonal:

$$f = a_0 x_0^2 + a_1 x_1^2 + ... + a_n x_n^2.$$

Here $a_0 < 0$, $a_1, ..., a_n > 0$ and for all nontrivial embeddings $\sigma : K \to \mathbb{R}$ we have $\sigma(a_i) > 0$, for all $i = 0, 1, ..., n$. To simplify the discussion, we will assume that Γ is uniform (the non-uniform lattices were discussed in Theorems A and B).

For a 4-element subset $I = \{0, i, j, k\} \subset \{0, 1, ..., n\}$ let $V_I \subset V$ denote the linear span of the basis vectors $e_l, l \in I$. Set $H_I := V_I \otimes \mathbb{R} \cap \mathbb{H}^n$. Then $f|V_I$ determines a lattice Γ_I in $\mathrm{Isom}(H_I)$, which is naturally embedded into Γ. Assuming Thurston's conjecture, up to taking finite index subgroups, each Γ_I contains an (infinite) normal finitely generated surface subgroup N_I. Moreover, by taking I and J such that $I \cap J$ consists of 3 elements, we obtain subgroups Γ_I, Γ_J whose intersection is a Fuchsian group F. It now follows from the separability of F in Γ (see [BM], [L] or Proposition

2.5 of this paper) that, after passing to certain finite index subgroups $\Gamma_I' \subset \Gamma_I, \Gamma_J' \subset \Gamma_J$, we get the inclusion

$$\Gamma_I' *_F \Gamma_J' \subset \Gamma. \tag{4}$$

Set $N_I' := \Gamma_I' \cap N_I, N_J' := \Gamma_J' \cap N_J$. Then $E := N_I' \cap N_J'$ is a free group of infinite rank. Now (4) implies that Γ is non-coherent since the subgroup

$$N_I' *_E N_J' \subset \Gamma$$

is finitely generated but not finitely presented, [Ne]. Therefore we obtain:

Suppose that Thurston's conjecture holds for all compact arithmetic 3-manifolds. Then all uniform arithmetic lattices of the simplest type in $\mathrm{Isom}(\mathbb{H}^n)$*,* $n \geq 4$*, are non-coherent.*

Thus we expect the negative answer to the question of Dani Wise stated in Introduction.

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