ON L^2-COHOMOLOGY AND PROPERTY (T) FOR AUTOMORPHISM GROUPS OF POLYHEDRAL CELL COMPLEXES

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ABSTRACT. We present an update of Garland's work on the cohomology of certain groups, construct a class of groups many of which satisfy Kazhdan's Property (T) and show that properly discontinuous and cocompact groups of automorphisms of (4, 4) or (6, 3)-complexes do not satisfy Property (T).

For a simplicial complex X, the link X_v of X at a vertex v of X is defined to be the subcomplex consisting of all simplices τ of X which do not contain v, but whose union with v is a simplex of X. If $\dim X = 2$, then X_v is a graph for any vertex v of X.

For a finite graph L with set of vertices $\mathcal{V}_L$, consider the *Laplacian* Δ on the space of real valued functions on $\mathcal{V}_L$ defined by

$$\Delta f(v) = f(v) - Af(v),$$

where $Af(v)$ is the mean value of f on the vertices adjacent to v. Clearly, Δ is a self adjoint operator. We denote by $\kappa = \kappa(L)$ its smallest positive eigenvalue.

Theorem 1. *Let X be a locally finite 2-dimensional simplicial complex such that*

(1) *for any vertex v of X the link X_v is connected;*
(2) *there is an $\varepsilon > 0$ such that $\kappa(X_v) > \frac{1}{2} + \varepsilon$ for each vertex v of X.*

Let Γ be a properly discontinuous group of automorphisms of X and ρ be a unitary representation of Γ. Then $L^2H^1(X, \rho) = 0$, where $L^2H(X, \rho)$ denotes the cohomology of the complex of $\operatorname{mod} \Gamma$ *square integrable cochains on X which are twisted by ρ.*

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Here the square integrability of a cochain refers to a natural norm, see (1.4) in Section 1.

Theorem 1 is a special case of Theorem 2.5 in the text, where we also consider higher dimensional complexes and higher cohomology. However, the formulation of Theorem 2.5 needs more preparation and we refer the reader to Section 2 below for the exact statement. The proof of Theorem 1 and its extension to higher dimension is based on arguments from Garland's paper [Ga] (see also [Bo]) and a recent paper by P. Pansu [Pa].

The class of spaces satisfying the assumptions of Theorem 1 is extremely rich as can be seen from the construction in Section 1 of [BB] (see also Proposition 4.1 below). It is more difficult to obtain such spaces admitting properly discontinuous and cocompact groups of automorphisms. We adress this problem in Theorem 2. Note however that Theorem 1 applies also in the particular case where the group Γ is trivial, hence $L^2H^1(X,\mathbb{R}) = 0$ for all such spaces X.

For the convenience of the reader we recall below the definition of Property (T). Before we do this, a word about the language: groups will always be assumed to be topological groups. Representations will always be assumed to be continuous in the strong topology. The most interesting groups occuring in this paper are countable with the discrete topology and these two conventions will be of no relevance.

Definitions. Let Γ be a locally compact group. Then we say that

(1) a unitary representation ρ of Γ on a Hilbert space H *has almost invariant vectors* if for any compact set $K \subset \Gamma$ and any $\varepsilon > 0$ there is a unit vector $v \in H$ such that $|\rho(g)v - v| < \varepsilon$ for all $g \in K$.

(2) Γ *satisfies Property (T)* if any unitary representation which has almost invariant vectors has an invariant unit vector.

The most prominent class of groups satisfying Property (T) are the isometry groups of symmetric spaces of noncompact type and rank ≥ 2, of the quaternionic hyperbolic spaces and of the Cayley hyperbolic plane. The isometry groups of the real and complex hyperbolic spaces do not satisfy Property (T). Note also that a lattice in a group satisfies Property (T) iff the group itself satisfies Property (T).

It is known that a locally compact group Γ with a countable base for the topology satisfies Property (T) iff $H^1(\Gamma,\rho) = 0$ for any unitary representation ρ of Γ, see Theorem 4.7 in [HV]. On the other hand, if Γ is a properly discontinuous group of automorphisms of a contractible simplicial complex X, then $H^1(\Gamma,\rho) = H^1(X,\rho)$, where $H^1(X,\rho)$ is the cohomology of the complex of cochains on X twisted by ρ. If the action is also cocompact, then $H^1(X,\rho) = L^2H^1(X,\rho)$. Therefore Theorem 1 has the following consequence.

Corollary 1. *Let X be a contractible 2-dimensional simplicial complex and Γ be a properly discontinuous and cocompact group of automorphisms of X. Assume that for any vertex v of X (1) the link X_v is connected and (2) $\kappa(X_v) > 1/2$. Then Γ satisfies Property (T).*

The class of graphs satisfying $\kappa > 1/2$ includes all thick spherical buildings of type $A_1 \times A_1$ and A_2 and those of type B_2 and G_2 if their valence is sufficiently large, see [FH] (see also Subsection 3.1 below). Hence locally finite thick Euclidean buildings of type $\tilde{A}_2$ satisfy the assumptions of the above results and those of type $\tilde{B}_2$ or $\tilde{G}_2$ do if they are sufficiently thick. Recall that cocompact lattices in simple algebraic groups of rank at least two over non-Archimedean fields satisfy Property (T). Such lattices act properly discontinuously and cocompactly on locally finite thick Euclidean buildings of the corresponding type. If the group has rank two, the building has dimension two and Property (T) also follows from Corollary 1 above, at least when the building is sufficiently thick.

In dimension two, there are numerous examples of Euclidean buildings and non-arithmetic groups acting on them. D. Cartwright, W. Młotkowski and T. Steger proved Property (T) for a certain class of such groups which act on Euclidean buildings of type $\tilde{A}_2$, see [CMS] (the examples were constructed earlier in [CMSZ]). Property (T) in these examples also follows from Corollary 1 above.

Theorem 1 and Corollary 1 in the special case where X is a Euclidean building of type $\tilde{A}_2$ with constant thickness > 2 have been proved by P. Pansu, see [Pa]. The results of Pansu apply in particular to examples of non-arithmetic groups acting on $\tilde{A}_2$-buildings constructed by J. Tits (see [Ti], Section 3.1) and M. Ronan (see [Ro]). Another version of Theorem 1 and a somewhat refined version of Corollary 1 was proved independently by Żuk [Zu].

It seems to us that the examples mentioned above are the only ones known so far where Corollary 1 applies. It is possible that there is some kind of rigidity regarding Property (T). However, such a rigidity phenomenon, if it exists, does not involve Tits buildings only: we will construct some new examples where the underlying space X is not a Tits building. We make use of specific Ramanujan graphs. In Chapter 3 of [Sa], P. Sarnak describes some explicit examples of such graphs. They are Cayley graphs of finite groups and most of them satisfy the assumptions of Theorem 2 and Corollary 2 below.

Theorem 2. *Let H be a finite group, $S \subset H \setminus \{e\}$ a set of generators of H and $\langle S, R\rangle$ a presentation of H. Assume that the girth of the Cayley graph $L = C(H, S)$ is at least 6. Then the group Γ given by the presentation*

$$(*) \qquad \langle S \cup \{\tau\} \mid R \cup \{\tau^2\} \cup \{(s\tau)^3 \mid s \in S\}\rangle$$

acts properly discontinuously and cocompactly on a contractible simplicial 2-complex X such that the links of all vertices of X are isomorphic to L.

Here we recall that the *girth* $g = g(L)$ of a finite graph L is the minimal number of edges in closed circuits of L.

Corollary 2. *Let H be a finite group, $S \subset H \setminus \{e\}$ a set of generators of H and $\langle S, R\rangle$ a presentation of H. Assume that the Cayley graph $L = C(H, S)$ satisfies (1) $g(L) \geq 6$ and (2) $\kappa(L) > 1/2$. Then the group Γ given by the presentation $(*)$ satisfies Property (T).*

In combination with the Ramanujan graphs of Sarnak mentioned above, this gives rise to an infinite class of new groups satisfying Property (T). In most of the examples of Sarnak, the girth is > 6 and therefore the corresponding groups Γ are hyperbolic in the sense of Gromov.

A *polygonal complex* is a 2-dimensional polyhedral cell complex. The class of infinite and contractible polygonal complexes is very rich and is studied from various points of view. If X is a polygonal complex and $v \in X$ is a vertex, the link of X at v is the graph X_v whose vertices represent the edges of X adjacent to v, where two vertices are connetced by an edge if the correspoding two edges in X are adjacent to a common face. This generalizes the corresponding notion for simplicial complexes. Say that a polygonal complex is a (k,l)-*complex* if each face of X has at least k sides and if the links at the vertices of X have girth $\geq l$. Simplicial complexes are special $(3,3)$-complexes since each face has exactly 3 sides.

Under our assumptions we always have $(k,l) \geq (3,3)$. The condition $kl \geq 2(k+l)$, which has its origins in small cancellation theory, is very natural from the point of view of geometry and crucial in the construction of polygonal complexes in [BB]. If X is a (k,l)-complex with $kl \geq 2(k+l)$, then the length metric on X, which turns every edge of X into a geodesic segment of length 1 and each face of X into a regular Euclidean polygon, is a complete metric of nonpositive curvature (in the triangle comparison sense of Alexandrov). The minimal solutions of the inequality $kl \geq 2(k+l)$ are $(k,l) = (6,3)$, $(4,4)$ or $(3,6)$. They correspond to the tesselations of the Euclidean plane into regular hexagons, squares and equilateral triangles respectively.

It is natural to ask which groups of automorphisms of polygonal complexes satisfy Property (T). To state our results in that respect we need a further definition.

Definition. Let Γ be a group and $f : \Gamma \to \mathbb{C}$ a continuous function. We say that f is of *negative type* if for all $m \geq 1$, all $g_1, \ldots, g_m \in \Gamma$ and all $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ with $\lambda_1 + \cdots + \lambda_m = 0$ we have

$$\sum_{i,j} \overline{\lambda_i} \lambda_j f(g_i^{-1} g_j) \leq 0.$$

It is known that a locally compact group Γ with a countable base for the topology satisfies Property (T) iff any continuous function of negative type on Γ is bounded, see Theorem 5.20 in [HV].

Theorem 3. *Let X be a simply connected $(4,4)$- or $(6,3)$-complex and let Γ be a group of automorphisms of X. If Γ does not have a fixed point, then Γ admits an unbounded function of negative type.*

Note that we do not assume any kind of regularity conditions on X. We do not need that X is locally finite or that all edges of X bound a face or two faces. In fact, our proof of Theorem 3 also applies in the case when X is a tree.

Corollary 3. *Let X be a locally finite simply connected $(4,4)$- or $(6,3)$-complex and let Γ be a properly discontinuous group of automorphisms of X. If Γ is infinite, then Γ does not satisfy Property (T).*

A group does not satisfy Property (T) if it acts without fixed points on a tree. Hence the following result is a refinement of Theorem 3 in the case where the stronger assumptions on X are satisfied.

Theorem 4. *Let X be a simply connected polygonal complex. Assume that the links of vertices of X are connected and bipartite and that at least one of the following conditions is satisfied:*

(1) *for some $k \geq 4$ each face of X is a k-gon;*
(2) *each face of X has an even number of sides.*

Let Γ be a group of automorphisms of X. If Γ does not have a fixed point, then Γ has a subgroup of finite index which admits a fixed point free action on a tree.

Let X be a noncompact simply connected (k,l)-complex with $kl \geq 2(k+l)$ and let Γ be a properly discontinuous and cocompact group of automorphisms of X. So far it remains unclear what the precise conditions on X for Γ to have Property (T) are and whether there are such. Corollary 3 gives a necessary condition, namely $(k,l) = (3,6)$. Corollary 1 gives a sufficient condition. Of course one needs regularity assumptions on X — such as (1) in Corollary 1 — if one wants to tie Property (T) to the structure of X.

Structure of the paper. In the first two sections we develop the ideas of Garland from [Ga]. There are several technical improvements, and we also include Pansu's improvements to L^2-cohomolgy and infinite dimensional unitary representations in our more general framework. Thus Sections 1 and 2 can be viewed as an update of the main part of Garland's work. In Section 3 we discuss applications to groups acting on Tits buildings and to groups constructed in Section 4. In Section 4 we prove Theorem 2 and in Section 5 Theorems 3 and 4. Sections 4 and 5 can be read independently of the rest of the paper each.

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1. Localization

Let X be a locally finite simplicial complex of dimension n. For a simplex σ in X, denote by $m(\sigma)$ the number of n-simplices containing σ. Throughout we assume $m(\sigma) \geq 1$, that is, every simplex of X is contained in an n-simplex.

Denote by $\Sigma(k)$ the set of *ordered* k-simplices of X. For $\sigma = (v_0, ..., v_k) \in \Sigma(k)$ define

$$m(\sigma) := m(\{v_0, ..., v_k\}), \tag{1.1}$$

where $\{v_0, ..., v_k\}$ is the k-simplex (without ordering) underlying σ. Clearly, for $\tau \in \Sigma(k)$,

$$\sum_{\substack{\sigma \in \Sigma(k+1) \\ \sigma \supset \tau}} m(\sigma) = (n-k)(k+2)! m(\tau), \tag{1.2}$$

where the factor $(k+2)!$ corresponds to the number of possible reorderings of an ordered $(k+1)$-simplex and where $\tau \subset \sigma$ means that the vertices of τ are vertices of σ.

Let Γ be a group acting properly discontinuously and by automorphisms on X. For an ordered simplex σ of X, denote by Γ_σ the stabilizer of σ. We choose a set $\Sigma(k,\Gamma) \subset \Sigma(k)$ of representatives of Γ-orbits.

1.3 Lemma. *For $0 \le l < k \le n$, let $f = f(\tau,\sigma)$ be a Γ-invariant function on the set of pairs (τ,σ), where τ is an ordered l-simplex and σ an ordered k-simplex with $\tau \subset \sigma$, that is, the vertices of τ are vertices of σ. Then*

$$\sum_{\sigma\in\Sigma(k,\Gamma)} \sum_{\substack{\tau\in\Sigma(l)\\ \tau\subset\sigma}} \frac{f(\tau,\sigma)}{|\Gamma_\sigma|} = \sum_{\tau\in\Sigma(l,\Gamma)} \sum_{\substack{\sigma\in\Sigma(k)\\ \tau\subset\sigma}} \frac{f(\tau,\sigma)}{|\Gamma_\tau|}$$

whenever one of the sums on the left hand side or right hand side is absolutely convergent.

Proof. Since f is Γ-invariant,

$$\begin{aligned}\sum_{\sigma\in\Sigma(k,\Gamma)} \sum_{\substack{\tau\in\Sigma(l)\\ \tau\subset\sigma}} \frac{f(\tau,\sigma)}{|\Gamma_\sigma|} &= \sum_{\substack{\sigma\in\Sigma(k,\Gamma)\\ \tau\in\Sigma(l,\Gamma)}} \sum_{\substack{\gamma\in\Gamma\\ \gamma\tau\subset\sigma}} \frac{f(\tau,\sigma)}{|\Gamma_\tau|\cdot|\Gamma_\sigma|}\\ &= \sum_{\substack{\tau\in\Sigma(l,\Gamma)\\ \sigma\in\Sigma(k,\Gamma)}} \sum_{\substack{\gamma\in\Gamma\\ \tau\subset\gamma\sigma}} \frac{f(\tau,\sigma)}{|\Gamma_\tau|\cdot|\Gamma_\sigma|} = \sum_{\tau\in\Sigma(l,\Gamma)} \sum_{\substack{\sigma\in\Sigma(k)\\ \tau\subset\sigma}} \frac{f(\tau,\sigma)}{|\Gamma_\tau|}. \quad \square\end{aligned}$$

Let ρ be a representation of Γ on a complex, possibly infinite dimensional Hilbert space H. Let $C^k(X,\rho)$ be the space of simplicial k-cochains of X, which are *twisted* by ρ, that is, $\phi \in C^k(X,\rho)$ is an alternating map on the ordered k-simplices of X with values in H such that

$$\phi(\gamma\cdot\sigma) = \rho(\gamma)\cdot\phi(\sigma) \quad \text{for all } \gamma\in\Gamma,\ \sigma\in\Sigma(k).$$

We say that $\phi \in C^k(X,\rho)$ is *square integrable* mod Γ if

$$\|\phi\|^2 := \sum_{\sigma\in\Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!|\Gamma_\sigma|}(\phi(\sigma),\phi(\sigma)) < \infty. \tag{1.4}$$

We denote by $L^k(X,\rho)$ the space of mod Γ square integrable cochains in $C^k(X,\rho)$. If X is finite or, more generally, if Γ acts cocompactly on X, we have $L^k(X,\rho) = C^k(X,\rho)$. On $L^k(X,\rho)$, we have the Hermitian form

$$(\phi,\psi) := \sum_{\sigma\in\Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!|\Gamma_\sigma|}(\phi(\sigma),\psi(\sigma)), \quad \phi,\psi\in L^k(X,\rho).$$

For $\sigma = (v_0,\dots,v_k) \in \Sigma(k)$ and $0 \le i \le k$ let $\sigma_i := (v_0,\dots,\hat{v}_i,\dots v_k)$. The differential $d : C^k(X,\rho) \to C^{k+1}(X,\rho)$ is given by

$$d\phi(\sigma) = \sum_{i=0}^{k}(-1)^i\phi(\sigma_i).$$

1.5 Proposition. *For $\phi \in L^k(X,\rho)$ we have $\|d\phi\|^2 \leq (n-k)(k+2)\|\phi\|^2$. In particular, $d : L^k(X,\rho) \to L^{k+1}(X,\rho)$ is a bounded operator.*

The point of this proposition is that we do not assume that X is uniformly locally finite.

Proof of Proposition 1.5. We have

$$\begin{aligned}
\|d\phi\|^2 &= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!|\Gamma_\sigma|}|d\phi(\sigma)|^2 \\
&= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!|\Gamma_\sigma|}|\sum_i(-1)^i\phi(\sigma_i)|^2 \\
&\leq \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!|\Gamma_\sigma|}(k+2)\sum_i|\phi(\sigma_i)|^2 \\
&= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{m(\sigma)(k+2)}{(k+2)!|\Gamma_\sigma|(k+1)!}\sum_{\substack{\tau\in\Sigma(k)\\ \tau\subset\sigma}}|\phi(\tau)|^2 \\
&\overset{(1.3)}{=} \sum_{\tau\in\Sigma(k,\Gamma)}\sum_{\substack{\sigma\in\Sigma(k+1)\\ \sigma\supset\tau}} \frac{m(\sigma)(k+2)}{(k+2)!(k+1)!|\Gamma_\tau|}|\phi(\tau)|^2 \\
&\overset{(1.2)}{=} \sum_{\tau\in\Sigma(k,\Gamma)} \frac{(n-k)(k+2)!m(\tau)(k+2)}{(k+2)!(k+1)!|\Gamma_\tau|}|\phi(\tau)|^2 \\
&= (n-k)(k+2)\sum_{\tau\in\Sigma(k,\Gamma)} \frac{m(\tau)}{(k+1)!|\Gamma_\tau|}|\phi(\tau)|^2 \\
&= (n-k)(k+2)\|\phi\|^2
\end{aligned}$$

and hence $\|d\phi\|^2 \leq (n-k)(k+2)\|\phi\|^2$. $\square$

Denote by $\delta : L^{k+1}(X,\rho) \to L^k(X,\rho)$ the adjoint operator of d,

$$(d\phi,\psi) = (\phi,\delta\psi) \quad \text{for all } \phi\in L^k(X,\rho),\ \psi\in L^{k+1}(X,\rho).$$

Recall that δ is a bounded operator with the same norm as d.

1.6 Proposition. *For $\phi \in L^k(X,\rho)$ and $\tau\in\Sigma(k-1)$,*

$$\delta\phi(\tau) = \sum_{\substack{v\in\Sigma(0)\\ v\tau\in\Sigma(k)}} \frac{m(v\tau)}{m(\tau)}\phi(v\tau),$$

where $v\tau$ is the ordered k-simplex obtained by juxtaposition of the vertex v. Furthermore, $\|\delta\phi\|^2 \leq (n-k+1)(k+1)\|\phi\|^2$.

Proof. The straightforward computation of $(\phi, d\psi) = (\delta\phi,\psi)$ — to avoid summability problems restricted to those $\psi\in L^{k-1}(X,\rho)$ which are nonzero on finitely many Γ-orbits in $\Sigma(k-1)$ only — shows that $\delta\phi(\tau)$ is as claimed. Since δ has the same norm as d, we have $\|\delta\phi\|^2 \leq (n-k+1)(k+1)\|\phi\|^2$. $\square$

1.7 Corollary. *For $\phi \in L^k(X,\rho)$ and $\sigma \in \Sigma(k)$,*

$$\delta d\phi(\sigma) = (n-k)\phi(\sigma) - \sum_{\substack{v\in\Sigma(0)\\ v\sigma\in\Sigma(k+1)}} \sum_{0\le i\le k} (-1)^i \frac{m(v\sigma)}{m(\sigma)} \phi(v\sigma_i). \quad \square$$

Let $\tau = (v_0, ..., v_j)$ be an ordered j-simplex of X. Denote by X_τ the *link* of τ in X, that is, the subcomplex of dimension $n-j-1$ consisting of all simplices $\{w_0, ..., w_l\}$ of X which are disjoint from $\{v_0, ..., v_j\}$ such that the union $\{v_0, ..., v_j\} \cup \{w_0, ..., w_l\}$ is a simplex of X. For an ordered simplex $\eta = (w_0, ..., w_l)$ of X_τ, let $\tau\eta = (v_0, ..., v_j, w_0, ..., w_l) \in \Sigma(j+l+1)$ and denote by $m_\tau(\eta)$ the number of $(n-j-1)$-simplices in X_τ containing η. Then

$$m_\tau(\eta) = m(\tau\eta). \tag{1.8}$$

The set of ordered l-simplices of X_τ is denoted $\Sigma_\tau(l)$. The isotropy group Γ_τ acts by automorphisms on X_τ. Since Γ acts properly discontinuously, Γ_τ is finite. Note that $\Gamma_{\tau\eta} = \Gamma_\tau \cap \Gamma_\eta$ is the stabilizer in Γ_τ of the ordered simplex η of X_τ. We let ρ_τ be the restriction of ρ to Γ_τ and $C^l(X_\tau, \rho_\tau)$ be the space of simplicial l-cochains of X_τ which are twisted by ρ_τ. Thus X_τ, the action of Γ_τ on X_τ and ρ_τ is a triple as X, the action of Γ on X and ρ, except that we are in the special case that X_τ is finite. Our notational guideline is to use the same symbols as in the general case, but to indicate by a subscript τ that we are considering X_τ. For example, the differential on $C^l(X_\tau, \rho_\tau)$ is denoted d_τ.

Let $\Sigma_\tau(l, \Gamma_\tau) \subset \Sigma_\tau(l)$ be a set of representatives of Γ_τ-orbits. For $\psi \in C^l(X_\tau, \rho_\tau)$ we have

$$\begin{aligned} \|\psi\|^2 &= \sum_{\eta\in\Sigma_\tau(l,\Gamma_\tau)} \frac{m_\tau(\eta)}{(l+1)!|\Gamma_{\tau\eta}|} |\psi(\eta)|^2 \\ &= \frac{1}{(l+1)!|\Gamma_\tau|} \sum_{\eta\in\Sigma_\tau(l)} m(\tau\eta)|\psi(\eta)|^2. \end{aligned} \tag{1.9}$$

Juxtaposition of τ to an ordered simplex η of X_τ, denoted $\tau\eta$ as above, defines a *localization map*

$$C^k(X,\rho) \to C^{k-j-1}(X_\tau, \rho_\tau), \quad \phi \mapsto \phi_\tau,$$

where ϕ_τ is given by

$$\phi_\tau(\eta) := \phi(\tau\eta).$$

1.10 Lemma. *Let $\phi \in L^k(X,\rho)$ and $0 \le j < k$. Then*

$$(k+1)! \cdot \|\phi\|^2 = (k-j)! \sum_{\tau\in\Sigma(j,\Gamma)} \|\phi_\tau\|^2.$$

Proof. We compute

$$\begin{aligned}
\sum_{\tau\in\Sigma(j,\Gamma)} \|\phi_\tau\|^2 &\overset{(1.9)}{=} \sum_{\tau\in\Sigma(j,\Gamma)} \sum_{\eta\in\Sigma_\tau(k-j-1)} \frac{m(\tau\eta)}{(k-j)!|\Gamma_\tau|} |\phi_\tau(\eta)|^2 \\
&= \sum_{\tau\in\Sigma(j,\Gamma)} \sum_{\substack{\sigma\in\Sigma(k) \\ \sigma=\tau\eta}} \frac{m(\sigma)}{(k-j)!|\Gamma_\tau|} |\phi(\sigma)|^2 \\
&= \sum_{\tau\in\Sigma(j,\Gamma)} \sum_{\substack{\sigma\in\Sigma(k) \\ \tau\subset\sigma}} \frac{m(\sigma)}{(k+1)!|\Gamma_\tau|} |\phi(\sigma)|^2 \\
&= \sum_{\sigma\in\Sigma(k,\Gamma)} \sum_{\substack{\tau\in\Sigma(j) \\ \tau\subset\sigma}} \frac{m(\sigma)}{(k+1)!|\Gamma_\sigma|} |\phi(\sigma)|^2 \\
&= \frac{(k+1)!}{(k-j)!} \cdot \|\phi\|^2. \quad \square
\end{aligned}$$

The relation between localization and δ is straightforward, at least if $j < k-1$.

1.11 Lemma. *Let $0 \le j < k \le n$, $\tau \in \Sigma(j)$ and $\phi \in L^k(X,\rho)$.*

(1) *If $j < k-1$, then $\delta_\tau\phi_\tau = (-1)^{j+1}(\delta\phi)_\tau$.*

(2) *If $j = k-1$ and ϕ_τ^0 denotes the component of ϕ_τ in the subspace of constant maps in $C^0(X_\tau,\rho_\tau)$, then $(-1)^k(n-k+1)\phi_\tau^0 \equiv \delta\phi(\tau)$. In particular,*

$$\|\phi_\tau^0\|^2 = \frac{m(\tau)}{(n-k+1)|\Gamma_\tau|}|\delta\phi(\tau)|^2.$$

Proof. The proof of (1) is straightforward. As for (2) we have by (1.9) that ϕ_τ^0 is the constant function with value equal to

$$\Big(\sum_{v\in\Sigma_\tau(0)} m_\tau(v)\Big)^{-1} \cdot \sum_{v\in\Sigma_\tau(0)} m_\tau(v)\phi_\tau(v).$$

On the other hand,

$$\delta\phi(\tau) = \frac{1}{m(\tau)} \sum_{v\in\Sigma_\tau(0)} m(v\tau)\phi(v\tau) = \frac{(-1)^k}{m(\tau)} \sum_{v\in\Sigma_\tau(0)} m_\tau(v)\phi_\tau(v)$$

and therefore

$$\delta\phi(\tau) \equiv \frac{(-1)^k}{m(\tau)} \cdot \Big(\sum_{v\in\Sigma_\tau(0)} m_\tau(v)\Big) \cdot \phi_\tau^0 = (-1)^k(n-k+1)\phi_\tau^0.$$

In particular,

$$\begin{aligned}
\|\phi_\tau^0\|^2 &= \sum_{v\in\Sigma_\tau(0)} \frac{m_\tau(v)}{|\Gamma_\tau|}|\phi_\tau^0(v)|^2 \\
&= \sum_{v\in\Sigma_\tau(0)} \frac{m_\tau(v)}{(n-k+1)^2|\Gamma_\tau|}|\delta\phi(\tau)|^2 = \frac{m(\tau)}{(n-k+1)|\Gamma_\tau|}|\delta\phi(\tau)|^2. \quad \square
\end{aligned}$$

In the special case of Tits buildings, a formula similar to the one below is contained in the proof of Lemma 8.1 of [Ga].

1.12 Theorem. *Let $0 \le j < k < n$ and $\phi \in L^k(X,\rho)$. Then*

$$k! \cdot \left(\|d\phi\|^2 - (n-k)\|\phi\|^2\right) = (k-j-1)! \sum_{\tau \in \Sigma(j,\Gamma)} \left(\|d_\tau \phi_\tau\|^2 - (n-k)\|\phi_\tau\|^2\right).$$

Proof. By Lemma 1.7

$$\begin{aligned}
&\|d\phi\|^2 - (n-k)\|\phi\|^2 = (\delta d\phi - (n-k)\phi, \phi) \\
&= - \sum_{\sigma \in \Sigma(k,\Gamma)} \sum_{\substack{v \in \Sigma(0) \\ v\sigma \in \Sigma(k+1)}} \sum_{0 \le i \le k} (-1)^i \frac{m(v\sigma)}{(k+1)!|\Gamma_\sigma|} \left(\phi(v\sigma_i), \phi(\sigma)\right) \\
&= - \sum_{\sigma \in \Sigma(k,\Gamma)} \sum_{\substack{\eta \in \Sigma(k-1) \\ \eta \subset \sigma}} \sum_{\substack{v \in \Sigma(0) \\ v\sigma \in \Sigma(k+1)}} [\sigma : \eta] \frac{m(v\sigma)}{k!(k+1)!|\Gamma_\sigma|} \left(\phi(v\eta), \phi(\sigma)\right),
\end{aligned}$$

where $[\sigma : \eta]$ denotes the incidence coeeficient of η with respect to σ,

$$\begin{aligned}
&= - \sum_{\eta \in \Sigma(k-1,\Gamma)} \sum_{\substack{\sigma \in \Sigma(k) \\ \sigma \supset \eta}} \sum_{\substack{v \in \Sigma(0) \\ v\sigma \in \Sigma(k+1)}} [\sigma : \eta] \frac{m(v\sigma)}{k!(k+1)!|\Gamma_\eta|} \left(\phi(v\eta), \phi(\sigma)\right) \\
&= - \sum_{\eta \in \Sigma(k-1,\Gamma)} \sum_{\substack{vw \in \Sigma(1) \\ vw\eta \in \Sigma(k+1)}} \frac{m(vw\eta)}{k!|\Gamma_\eta|} \left(\phi(v\eta), \phi(w\eta)\right),
\end{aligned}$$

where $w\eta$ takes over the role of σ, more precisely, of $v_i\sigma_i$ for $\sigma = (v_0, \dots, v_k)$. If $j = k-1$, the asserted equality follows immediately. If $j < k-1$, we apply the above formula to ϕ_τ, $\tau \in \Sigma(j,\Gamma)$, and get

$$\begin{aligned}
&(k-j-1)! \sum_{\tau \in \Sigma(j,\Gamma)} \left(\|d_\tau \phi_\tau\|^2 - (n-k)\|\phi_\tau\|^2\right) \\
&= - \sum_{\tau \in \Sigma(j,\Gamma)} \sum_{\rho \in \Sigma_\tau(k-j-2,\Gamma_\tau)} \sum_{\substack{vw \in \Sigma_\tau(1) \\ vw\rho \in \Sigma_\tau(k-j)}} \frac{m_\tau(vw\rho)}{|\Gamma_{\tau\rho}|} \left(\phi_\tau(v\rho), \phi_\tau(w\rho)\right) \\
&= k! \cdot \left(\|d\phi\|^2 - (n-k)\|\phi\|^2\right),
\end{aligned}$$

where we use that the set of $\tau\rho$, for $\tau \in \Sigma(j,\Gamma)$ and $\rho \in \Sigma_\tau(k-j-2,\Gamma_\tau)$, is a set $\Sigma(k-1,\Gamma)$ of representatives of Γ-orbits. □

For $\tau \in \Sigma(j)$, define a quadratic form Q_τ on $C^{k-j-1}(X_\tau, \rho_\tau)$ by

$$Q_\tau(\psi) = \|d_\tau \psi\|^2 - \frac{j+1}{k+1}(n-k)\|\psi\|^2.$$

As an application of Lemma 1.10 and Theorem 1.12 we obtain:

1.13 Corollary. *Let $0 \le j < k < n$ and $\phi \in L^k(X,\rho)$. Then*

$$k! \cdot \|d\phi\|^2 = (k-j-1)! \sum_{\tau \in \Sigma(j,\Gamma)} Q_\tau(\phi_\tau). \quad \square$$

In [Pa], Pansu discusses a related formula for 1-forms in the case where X is a building of type $\tilde{A}_2$.

2. Vanishing of L^2-cohomology

Let X, Γ and ρ be as in the previous section. The L^2-cohomology of X with respect to ρ is defined as

$$L^2H^k(X,\rho) = \ker\big(d|L^k(X,\rho)\big)/\operatorname{im}\big(d|L^{k-1}(X,\rho)\big).$$

If X is finite or, more generally, if Γ acts cocompactly on X, then $L^2H^k(X,\rho) = H^k(X,\rho)$. Let

$$\Delta^+ = \delta d, \quad \Delta^- = d\delta, \quad \Delta = \Delta^+ + \Delta^-.$$

Define the space $L^2\mathcal{H}^k(X,\rho)$ of mod Γ square integrable *harmonic k-forms* twisted by ρ by

$$L^2\mathcal{H}^k(X,\rho) = \ker\big(\Delta|L^k(X,\rho)\big) = \ker\big(\Delta^+|L^k(X,\rho)\big) \cap \ker\big(\Delta^-|L^k(X,\rho)\big).$$

If X is finite or, more generally, if Γ acts cocompactly on X, then $L^2\mathcal{H}^k(X,\rho) = \mathcal{H}^k(X,\rho)$, the space of harmonic forms on X twisted by ρ. We have

$$\begin{aligned}\big(\ker\big(\Delta^+|L^k(X,\rho)\big)\big)^\perp &= \overline{\operatorname{im}\big(\Delta^+|L^k(X,\rho)\big)} = \overline{\operatorname{im}\big(\delta|L^{k+1}(X,\rho)\big)}\\ \big(\ker\big(\Delta^-|L^k(X,\rho)\big)\big)^\perp &= \overline{\operatorname{im}\big(\Delta^-|L^k(X,\rho)\big)} = \overline{\operatorname{im}\big(d|L^{k-1}(X,\rho)\big)}\end{aligned} \tag{2.1}$$

and the orthogonal decompositions

$$\begin{aligned}\ker\big(\Delta^+|L^k(X,\rho)\big) &= L^2\mathcal{H}^k(X,\rho) \oplus \overline{\operatorname{im}\big(\Delta^-|L^k(X,\rho)\big)}\\ \ker\big(\Delta^-|L^k(X,\rho)\big) &= L^2\mathcal{H}^k(X,\rho) \oplus \overline{\operatorname{im}\big(\Delta^+|L^k(X,\rho)\big)}.\end{aligned} \tag{2.2}$$

In particular,

$$L^2\mathcal{H}^k(X,\rho) = \ker\big(d|L^k(X,\rho)\big)/\overline{\operatorname{im}\big(d|L^{k-1}(X,\rho)\big)}.$$

2.3 Lemma. *If X (and Γ) are finite, then $\operatorname{im}\big(\Delta^\pm|C^k(X,\rho)\big)$ is closed. Furthermore, if $\alpha^\pm$ is the minimal positive eigenvalue of $\Delta^\pm$ on $C^k(X,\mathbb{R})$, the space of (untwisted) k-chains on X with values in $\mathbb{R}$, then $\|\Delta^\pm\phi\| \geq \alpha^\pm\|\phi\|$ for all ϕ in $C^k(X,\rho)$ perpendicular to $\ker\Delta^\pm$.*

Proof. We let $\Delta^+_{\mathbb{R}} = \Delta^+|C^k(X,\mathbb{R})$. Since $C^k(X,\mathbb{R})$ is a finite dimensional Euclidean space and $\Delta^+_{\mathbb{R}}$ is a self adjoint operator on $C^k(X;\mathbb{R})$, we have an orthogonal decomposition

$$C^k(X,\mathbb{R}) = \ker\Delta^+_{\mathbb{R}} \oplus \operatorname{im}\Delta^+_{\mathbb{R}}.$$

Denote by $C^k(X,H)$ the space of (untwisted) k-cochains on X with values in the Hilbert space H. Then

$$C^k(X,H) = C^k(X,\mathbb{R}) \otimes H$$

as a Hilbert space. For $\Delta^+_H = \Delta^+|C^k(X,H)$ we have

$$\Delta^+_H(\phi\otimes v) = (\Delta^+_{\mathbb{R}}\phi)\otimes v$$

for $\phi \in C^k(X, \mathbb{R})$ and $v \in H$. In other words, $\Delta_H^+ = \Delta_{\mathbb{R}}^+ \otimes \mathrm{id}$. Hence

$$\ker \Delta_H^+ = (\ker \Delta_{\mathbb{R}}^+) \otimes H, \quad \operatorname{im} \Delta_H^+ = (\operatorname{im} \Delta_{\mathbb{R}}^+) \otimes H.$$

In particular, $\ker \Delta_H^+$ and $\operatorname{im} \Delta_H^+$ are closed and

$$C^k(X, H) = \ker \Delta_H^+ \oplus \operatorname{im} \Delta_H^+$$

is an orthogonal decomposition. This proves the first assertion in the case of Δ^+ on $C^k(X, H)$.

Concerning the second assertion, we let P_0 be the orthogonal projection in $C^k(X, \mathbb{R})$ onto $\ker \Delta_{\mathbb{R}}^+$ and P_1 be the orthogonal projection onto $\operatorname{im} \Delta_{\mathbb{R}}^+$. Then $P_0 + P_1 = \mathrm{id}$ and $P_0 P_1 = P_1 P_0 = 0$. We have

$$\|\Delta_{\mathbb{R}}^+(P_1\phi)\| \geq \alpha^+ \|P_1\phi\|$$

for all $\phi \in C^k(X, \mathbb{R})$. Now $P_0 \otimes \mathrm{id}$ is the orthogonal projection of $C^k(X, H)$ onto $\ker \Delta_H^+$ and $P_1 \otimes \mathrm{id}$ the orthogonal projection onto $\operatorname{im} \Delta_H^+$. Hence the asserted inequality follows in the case of Δ_H^+ on $C^k(X, H)$, and the proof of the corresponding statements for Δ_H^- on $C^k(X, H)$ is similar.

Now the space $C^k(X, \rho)$ of twisted k-cochains is a closed subspace of $C^k(X, H)$, invariant under Δ_H^+ and Δ_H^-. $\square$

Let $0 < k < n$ and consider the case $j = k-1$ in Corollary 1.13. Let $\tau \in \Sigma(k-1)$ and $\psi \in C^0(X_\tau, \rho_\tau)$. Then

$$Q_\tau(\psi) = \|d_\tau \psi\|^2 - \frac{k}{k+1}(n-k)\|\psi\|^2 = (\Delta_\tau \psi - \frac{k}{k+1}(n-k)\psi, \psi),$$

where we note that $\Delta_\tau = \Delta_\tau^+$ on 0-cochains. Let ψ^0 be the component of ψ in $\ker \Delta_\tau$ and set $\psi^1 = \psi - \psi^0$. Then

$$Q_\tau(\psi) = -\frac{k}{k+1}(n-k)\|\psi^0\|^2 + \left(\Delta_\tau \psi^1 - \frac{k}{k+1}(n-k)\psi^1, \psi^1\right).$$

Denote by κ_τ the smallest positive eigenvalue of Δ_τ on $C^0(X_\tau, \mathbb{R})$.

2.4 Lemma. *In the above notation, assume $\kappa_\tau \geq \frac{k(n-k)}{k+1} + \varepsilon$ for some $\varepsilon > 0$. Then*

$$\|\psi\|^2 \leq C_\varepsilon \|\psi^0\|^2 + \frac{1}{\varepsilon} Q_\tau(\psi).$$

for all $\psi \in C^0(X_\tau, \rho_\tau)$, where $C_\varepsilon = 1 + \frac{k(n-k)}{\varepsilon(k+1)}$. $\square$

2.5 Theorem. *Let $0 < k < n = \dim X$. Assume that X_τ is connected and that there is an $\varepsilon > 0$ such that $\kappa_\tau \geq \frac{k(n-k)}{k+1} + \varepsilon$ for all $(k-1)$-simplices τ of X, where κ_τ is the smallest positive eigenvalue of Δ_τ on $C^0(X_\tau, \mathbb{R})$. Then $L^2H^k(X, \rho) = 0$ for any unitary representation ρ of Γ.*

Proof. Let τ be a $(k-1)$-simplex of X. Now X_τ is connected and $\ker \Delta_\tau = \ker d_\tau$. Therefore the kernel of Δ_τ on $C^0(X_v, \rho_v)$ consists of the constant maps. Hence by Lemma 1.11(2),

$$\|\phi_\tau^0\|^2 = \frac{m(\tau)}{(n-k+1)|\Gamma_\tau|}|\delta\phi(\tau)|^2.$$

Let $\phi \in \ker\big(\Delta^+|L^k(X,\rho)\big)$. Then $d\phi = 0$ and we have

$$
\begin{aligned}
(k+1)! \cdot \|\phi\|^2 &\overset{(1.10)}{=} \sum_{\tau\in\Sigma(k-1,\Gamma)} \|\phi_\tau\|^2 \\
&\overset{(2.4)}{\leq} C_\varepsilon \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{m(\tau)}{(n-k+1)|\Gamma_\tau|}|\delta\phi(\tau)|^2 + \frac{1}{\varepsilon}\sum_{\tau\in\Sigma(k-1,\Gamma)} Q_\tau(\phi_\tau) \\
&\overset{(1.13)}{=} C_\varepsilon \frac{k!}{n-k+1}\cdot\|\delta\phi\|^2 + \frac{1}{\varepsilon}\cdot 0 = C'_\varepsilon \cdot \|\delta\phi\|^2 = C'_\varepsilon\cdot(\Delta^-\phi,\phi).
\end{aligned}
$$

This shows that $L^2\mathcal{H}^k(X,\rho) = 0$ and that the image of $\Delta^-|L^k(X,\rho)$ is closed in $\ker\big(\Delta^+|L^k(X,\rho)\big)$. Hence $\ker\big(\Delta^+|L^k(X,\rho)\big) = \operatorname{im}\big(\Delta^-|L^k(X,\rho)\big)$ and therefore $L^2H^k(X,\rho) = 0$. □

Note that Theorem 1 of Introduction corresponds to the case $n = 2$ and $k = 1$ in Theorem 2.5.

3. Applications

Our main applications of Theorem 2.5 concern the case $k = 1$ and $\dim X = n = 2$. More generally, if $\dim X = n \geq 2$ and if $k = n-1$, then the link of $\tau \in \Sigma(k-1) = \Sigma(n-2)$ is a graph. Therefore we are interested in estimates for the smallest positive eigenvalue $\kappa = \kappa_L$ of the Laplacian Δ_L, acting on the space $C^0(L,\mathbb{R})$ of real-valued functions on the vertices of a finite graph L. In order to apply Theorem 2.5, we actually need $\kappa > 1/2$ if $n = 2$ and $k = 1$ respectively $\kappa > (n-1)/n$ if $n \geq 2$ and $k = n-1$.

For $\phi \in C^0(L,\mathbb{R})$ and v a vertex of L, we have

$$\Delta_L\phi(v) = \phi(v) - \frac{1}{m_L(v)}\sum_{w\in L_v}\phi(w),$$

where $m_L(v)$ denotes the *valence* of v, that is, the number of edges adjacent to v, and L_v is the link of v in L, consisting of all vertices adjacent to v. Defining the *averaging operator* $A : C^0(L,\mathbb{R}) \to C^0(L,\mathbb{R})$ by

$$A\phi(v) = \frac{1}{m_L(v)}\sum_{w\in L_v}\phi(w),$$

we get

$$\Delta_L\phi = \phi - A\phi.$$

Obviously, the spectrum of A is in $[-1,1]$ and hence the spectrum of Δ is in $[0,2]$.

If all the vertices of L have the same valence m, then the adjacency operator considered in graph theory is m times the averaging operator A considered above. The complete spectrum of the adjacency operator, including the multiplicities of the eigenvalues, has been computed in many cases, see e.g. [BCN], [CDS].

3.1 Tits buildings. We say that a graph L is *bipartite* if the set $\mathcal{V}$ of vertices of L is a disjoint union $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1$, where $v \in \mathcal{V}_0$ is adjacent to vertices from $\mathcal{V}_1$ only and vice versa. If L is bipartite, then we have an orthogonal decomposition

$$C^0(L, \mathbb{R}) = \mathcal{F}_0 \oplus \mathcal{F}_1,$$

where $\mathcal{F}_0$ denotes the space of real valued functions on $\mathcal{V}_0$ and $\mathcal{F}_1$ the space of real valued functions on $\mathcal{V}_1$, and a corresponding decomposition of A,

$$A = \begin{pmatrix} 0 & A_1 \\ A_0 & 0 \end{pmatrix}.$$

Since the decomposition $\mathcal{F}_0 \oplus \mathcal{F}_1$ is orthogonal and A is self adjoint, we have $A_0^* = A_1$. Clearly,

$$\ker A = \ker A_0 \oplus \ker A_1.$$

Nonzero eigenvalues of $A_1 A_0$ are positive. If $\phi \in \mathcal{F}_0$ is an eigenfunction of $A_1 A_0$ with eigenvalue $\lambda > 0$, then $(\sqrt{\lambda}\phi, A_0\phi)$ and $(\sqrt{\lambda}\phi, -A_0\phi)$ are eigenfunctions of A for the eigenvalues $\sqrt{\lambda}$ and $-\sqrt{\lambda}$. In this way, we obtain all nonzero eigenvalues of A and the corresponding eigenfunctions. Observe that the spectrum of A is symmetric about 0.

Assume now that L is a Tits building (also called *generalized polygon*). Then L is bipartite. Assume that there are numbers m_0, m_1 such that the valence of each vertex $v \in \mathcal{V}_0$ is m_0 and the valence of each vertex $v \in \mathcal{V}_1$ is m_1. This holds automatically when L is thick. By what we said above, κ and the biggest eigenvalue μ of $M = m_1 m_0 A_1 A_0$ on the space of functions in $\mathcal{F}_0$ perpendicular to the constant functions are related by

$$\kappa = 1 - \sqrt{\frac{\mu}{m_1 m_0}}.$$

Now the spectrum of M was determined by Feit and Higman [FH, Lemmas 4.1, 5.1, 6.1] (see also [Ga], [Pa]). Their results imply:

(1) if L is of type $A_1 \times A_1$, i.e., if L is a complete bipartite graph, then $\kappa = 1$.

(2) if L is of type A_2, i.e., if L is the flag complex of a projective plane, then $m_0 = m_1 =: m$ and $\kappa = 1 - \dfrac{\sqrt{m-1}}{m}$. Hence $\kappa > 1/2$ for $m \geq 3$.

(3) if L is of type B_2, then $\kappa = 1 - \sqrt{\dfrac{m_0 + m_1 - 2}{m_0 m_1}}$.

(4) if L is of type G_2, then $\kappa = 1 - \sqrt{\dfrac{m_0 + m_1 - 2 + \sqrt{(m_0 - 1)(m_1 - 1)}}{m_0 m_1}}$.

(5) if L has diameter 8, then $\kappa = 1 - \sqrt{\dfrac{m_0 + m_1 - 2 + \sqrt{2(m_0 - 1)(m_1 - 1)}}{m_0 m_1}}$.

The main theorem of Feit and Higman in [FH] says that there are no finite thick Tits buildings of dimension 1 other then of the types mentioned above. In each case, $\kappa > 1/2$ when $m_0, m_1 \geq 13$ and $\kappa \to 1$ as $m_0, m_1 \to \infty$.

Theorem 2.5 together with the above formulas for κ implies: *if X is a locally finite 2-dimensional Tits building such that each edge of X is adjacent to at least 13 faces,* *then $L^2H^1(X,\rho) = 0$ for any properly discontinuous group Γ of automorphisms of X and any unitary representation ρ of Γ.*

In some cases the assumption on the thickness can be relaxed. For example, if X is a Euclidean building (of dimension 2), then the links of vertices of X are of type $A_1 \times A_1$, A_2, B_2 or G_2, and hence we need to assume only that each edge of X is adjacent to at least 11 faces. If X is of type $\tilde{A}_2$, then all links of X are of type A_2 and it suffices to assume that X is thick.

Lemma 6.3 in [Ga] implies that for any $n \geq 3$ and any constant $\alpha > 0$ there is a constant $\beta > 0$ with the following property: if X is a locally finite n-dimensional Tits building such that $\kappa_\tau > 1 - \beta$ for all $\tau \in \Sigma(n-2)$, then $\kappa_\eta > n - k - \alpha$ for all $\eta \in \Sigma(k-1)$, $0 < k < n$. Now the formulas for κ show that $\kappa \to 1$ as $m_0, m_1 \to \infty$. In particular, *there is a constant C_n such that $L^2H^k(X,\rho) = 0$ for any properly discontinuous group Γ of automorphisms of X, any unitary representation ρ of Γ and any $k \in \{1, ..., n-1\}$, provided any simplex of X of codimension 1 bounds at least C_n simplices of dimension n.*

3.2 Ramanujan graphs. Let L be a finite and connected graph with constant valence m. We say that L is *Ramanujan* if the spectrum of the averaging operator A of L, except for the eigenvalue 1, is contained in $[-2\frac{\sqrt{m-1}}{m}, 2\frac{\sqrt{m-1}}{m}]$. If L is bipartite, we say that L is *bipartite Ramanujan* if the spectrum of A, except for the eigenvalues ± 1, is contained in $[-2\frac{\sqrt{m-1}}{m}, 2\frac{\sqrt{m-1}}{m}]$. If L is Ramanujan or bipartite Ramanujan, then $\kappa > \frac{1}{2}$ if $m \geq 15$. In [Sa], Sarnak constructs families of Ramanujan and bipartite Ramanujan graphs with arbitrarily large valence m and girth g. The graphs constructed in [Sa] are Cayley graphs for groups $PGL(2, \mathbb{Z}/q\mathbb{Z})$ with respect to certain sets of generators.

Given a Cayley graph L of girth ≥ 6 we construct in the section below an infinite and contractible simplicial 2-complex X such that the link of any vertex of X is isomorphic to L and such that X admits a properly discontinuous and cocompact group Γ of automorphisms. Applying this construction to the examples of Sarnak mentioned above, we obtain an infinite family of groups satisfying the assumptions of Theorem 2.5. Since the spaces X are contractible, the first cohomology of these groups Γ with coefficients in any unitary representation vanishes. Hence these groups satisfy Property (T).

4. A FAMILY OF GROUPS

In this section we prove Theorem 2 of the Introduction. We first explain a general method of constructing contractible 2-complexes. Then we apply this method carefully to obtain a group of automorphisms as asserted in Theorem 2.

Let L be a connected finite graph, and assume that it is *univalent*, i.e. there is a number $m \geq 2$ such that each vertex of L is adjacent to m edges. An *L-complex* is a simplicial 2-complex X such that for each vertex v of X the link at v is isomorphic to L.

4.1 Proposition. *Let L be a finite, connected and univalent graph. If the girth of L is at least 6, then there are contractible L-complexes.*

Proof. Proposition 4.1 follows from the more general result of [BB]. We briefly sketch the construction in [BB], emphasizing the features that are important for our purposes.

For a simplicial 2-complex B, the *boundary* ∂B of B is the subcomplex consisting of the (closed) edges that are adjacent to exactly one face of B. Consider the following conditions on B:

(a) all the non-boundary edges of B are contained in exactly m triangles, where m is the valence of L;

(b) the links of B at non-boundary vertices are isomorphic to L;

(c) the links of B at boundary vertices are isomorphic to finite connected trees with diameter 2 or 3 and with valences of vertices equal to 1 or m.

Step 1. Let B_1 be the simplicial cone over L. Clearly $B = B_1$ satisfies conditions (a)–(c).

Step 2. Assume inductively that $B = B_n$ is constructed and that it satisfies conditions (a)–(c). To construct B_{n+1}, first glue $m-1$ new triangles to each boundary edge of B_n, thus getting a complex $\widetilde{B}_n$. Note that by property (c) above, the links of $\widetilde{B}_n$ at all boundary vertices of B_n are isomorphic to finite connected trees with diameter 4 or 5 and with valences of vertices equal to 1 or m. This means that we can embed each such a link (in many different ways) into the graph L since the latter has constant valence m and girth $g(L) \geq 6$. We then use a collection of such embeddings, one for each vertex, as a pattern for extending $\widetilde{B}_n$ by glueing to it new triangles around boundary vertices of B_n. This we do independently (and disjointly) for all boundary vertices of B_n, and the result is B_{n+1}. Observe that $B = B_n$ satisfies conditions (a)–(c) and that

(d) the boundary vertices of B_n are no longer on the boundary of B_{n+1};

(e) B_{n+1} can be contracted to B_n.

Step 3. By repeating Step 2 we obtain an infinite sequence $B_1 \subset B_2 \subset \ldots$ of complexes, and we define $X := \cup_{n=1}^{\infty} B_n$. Using properties (a)–(e), it is easy to check that X is a contractible L-complex. □

4.2 Remark. There is the following addition to Proposition 4.1. Let X be a contractible L-complex, where L is as in Proposition 4.1, and let v be a vertex in X. Let B_1 be the star of v and B_{n+1} be the star of B_n, $n \geq 1$. Then the sequence (B_n) is obtained by the construction in the proof of Proposition 4.1.

As we have seen, the construction of L-complexes depends on many choices. This is the reason why for some graphs L uncountably many non-isomorphic L-complexes can be constructed (see [BB] or [Sw]), only few of them admitting nontrivial automorphisms. To construct L-complexes with a large group of automorphisms, one has to perform the construction more carefully, for example by using specific graphs L and some additional structure given by a system of labels as described below. In this section we deal with the case when L is a Cayley graph of a finite group.

Let H be a group and S a generating set of H with $e \notin S$. The *Cayley graph* $C(H,S)$ of H with respect to S is the graph with vertex set H, where two vertices h_1, h_2 are joined by an edge (not oriented) iff $h_1 s = h_2$ or $h_2 s = h_1$ for some $s \in S$. Note that $C(H,S)$ is a connected graph and that H acts (on the left) on it by automorphisms. Under our assumptions, $C(H,S)$ is a simplicial graph.

Let H be a finite group with a generating set S. Assume that H and S satisfy the assumption of Theorem 2 and consider $L = C(H, S)$. Equip the set of oriented edges of L with labels from the set $S \cup S^{-1}$ according to the following rule: label (h_1, h_2) by s if $h_1 s = h_2$. This rule determines labels uniquely and the label of an oppositely oriented edge is the inverse of the original one. Moreover, the group of label preserving automorphisms of L coincides with H.

Let X be an L-complex. Consider the set $\mathcal{F}$ of all *flags* in X, i.e. incident triples (vertex, edge, triangle) in X. Given a vertex v of X, the flags containing v are in 1–1 correspondence with the oriented edges in the link X_v of X at v. Let Λ be a labelling of $\mathcal{F}$ by the elements of $S \cup S^{-1}$. We say that Λ is *modelled on L*, if for any vertex v there is an isomorphism of X_v and L such that the labels of the flags containing v coincide with the labels of the oriented edges in L via this isomorphism and the above 1–1 correspondence. A *labelled L-complex* is a pair (X, Λ), where X is an L-complex and Λ is a labelling of the flags in X modelled on L.

Consider the following conditions on labellings of $\mathcal{F}$ by elements of $S \cup S^{-1}$:

(C1) the labels of any two flags containing the same edge and triangle, but different vertices, are inverse to each other;

(C2) the labels of any two flags containing the same vertex and triangle, but different edges, are inverse to each other;

(C3) the labels of all flags containing the same vertex and edge are pairwise different.

Note that conditions (C2) and (C3) are satisfied automatically for labellings modelled on L. We will use these conditions in the construction of labelled L-complexes below.

4.3 Proposition. *Let H be a finite group and S a generating set of H. Suppose H and S satisfy the assumption of Theorem 2 and let $L = C(H, S)$. Then there exists a unique (up to label preserving isomorphism) contractible labelled L-complex (X, Λ), where Λ satisfies Condition (C1). Moreover, the group* $\mathrm{Aut}(X, \Lambda)$ *of all label preserving automorphisms of X acts transitively on vertices of X with stabilizers isomorphic to H. In particular,* $\mathrm{Aut}(X, \Lambda)$ *is simply transitive on oriented edges of X.*

Proof. We repeat the inductive construction of the proof of Proposition 4.1, taking labels into account.

Step 1. Identify the link at the center of B_1 with L. This induces a labelling of the flags of B_1 with vertex at the center. Extend this labelling to all flags in B_1 using Condition (C1). Observe that such an extension exists, is unique and satisfies (C2) and (C3).

Step 2. Suppose inductively that the flags in B_n are labelled such that Conditions (C1)–(C3) are satisfied. Consider the complex $\widetilde{B}_n$ as in the proof of Proposition 4.1 and extend the labelling of the flags of B_n to all flags of $\widetilde{B}_n$ so that (C1)–(C3) are satisfied. Note that such an extension exists, and it is unique up to permutations inside the sets of $m - 1$ new triangles glued to the boundary edges of B_n when constructing $\widetilde{B}_n$.

Now consider the second part of Step 2. For each boundary vertex v of B_n, the labelling of the flags in $\widetilde{B}_n$ containing v induces a labelling of the oriented edges of the link of $\widetilde{B}_n$ at v. These labelled links can be embedded into L so that the labels are preserved. Such embeddings are unique up to translations of L by elements of H. These embeddings determine uniquely the extension from $\widetilde{B}_n$ to B_{n+1} and the extension of the labelling of the flags in $\widetilde{B}_n$ to all flags in B_{n+1} so that Conditions (C1)–(C3) are satisfied. Moreover, a label preserving automorphism of B_{n+1} is uniquely determined by its restriction to B_n and, by induction, by its restriction to the link of the center of B_1.

Step 3. Let $X = \cup_{n=1}^{\infty} B_n$. Note that the labelling Λ of the flags in X constructed inductively in Steps 1 and 2 is modelled on L and satisfies Condition (C1).

It is clear that any labelled L-complex satisfying the assertions of Proposition 4.3 is the result of a construction as above, cf. Remark 4.2. Since the extensions from B_n to B_{n+1} are unique up to label preserving isomorphism, it follows that the resulting labelled complex (X, Λ) is unique.

Finally observe that the only label preserving automorphism of X which extends id_{B_1} is id_X. Moreover, by the uniqueness of the construction up to a label preserving isomorphism, any label preserving automorphism of 1-balls in X extends uniquely to a label preserving automorphism of all of X. This implies the assertion about $\mathrm{Aut}(X, \Lambda)$. $\square$

Proof of Theorem 2 of Introduction. Consider the group $\mathrm{Aut}(X, \Lambda)$ of label preserving automorphisms of the complex X in Proposition 4.3. Since $\mathrm{Aut}(X, \Lambda)$ acts transitively on vertices of X with stabilizers isomorphic to the finite group H, the action is properly discontinuous and cocompact. It remains to prove that $\mathrm{Aut}(X, \Lambda)$ has a presentation as claimed in Theorem 2.

Fix a vertex $v \in X$ and identify the isotropy group $\mathrm{Stab}_v(\mathrm{Aut}(X, \Lambda))$ with H, see Proposition 4.3. Choose an edge e having v as one of its vertices and denote by τ the unique automorphism of $\mathrm{Aut}(X, \Lambda)$ reversing v with the other vertex w of e; the uniqueness of τ follows from the fact that $\mathrm{Aut}(X, \Lambda)$ is simply transitive on oriented edges of X. By the same reason τ is an involution, i.e. $\tau^2 = \mathrm{id}_X$. Since the group H (identified with $\mathrm{Stab}_v(\mathrm{Aut}(X, \Lambda))$) acts transitively on vertices adjacent to v, it follows that H and τ generate $\mathrm{Aut}(X, \Lambda)$.

According to Proposition 2.1 of [BB], to get a presentation of $\mathrm{Aut}(X, \Lambda)$ one needs to choose representatives of orbits of the action of H on triangles adjacent to v and consider relations corresponding to those triangles. Let $T = \{t : (v, e, t) \in \mathcal{F}\}$, where e is the edge corresponding to τ as above. Label each triangle $t \in T$ by the corresponding label of the flag (v, e, t). This gives a 1–1 correspondence of T with $S \cup S^{-1}$. Let T_S be the subset of T consisting of triangles with label in S. Then T_S contains a set of representatives of triangles as required.

It is easy to see that the automorphism τs preserves the triangle in T_S labelled by $s \in S$ and that its restriction to this triangle is a rotation. Therefore we have $(\tau s)^3 = \mathrm{id}_X$, and this is a relation corresponding to our representative triangle, as required by Proposition 2.1 of [BB]. Thus we get the presentation

$$< S \cup \{\tau\} \mid R \cup \{\tau^2\} \cup \{(\tau s)^3 : s \in S\} >$$

for $\mathrm{Aut}(X, \Lambda)$. This finishes the proof of Theorem 2. $\square$

5. Functions of negative type and actions on trees

The aim of this section is to prove Theorems 3 and 4 of Introduction. In the proof of Theorem 3, we restrict to the $(4,4)$-case and indicate the changes of the argument in the $(6,3)$-case.

Let X be a simply connected $(4,4)$-complex. Let X' be the subdivision of X obtained by adding the barycenters of the faces of X as new vertices and as new edges the line segments in the faces connecting their barycenters with the vertices on their boundary. Then X' is a simplicial 2-complex. Let d be the length metric on X which turns each edge of X into a geodesic segment of length 1, each new edge of X' into a geodesic segment of length $1/\sqrt{2}$ and the 2-simplices of X' into corresponding isosceles Euclidean triangles with interior angles $\pi/2, \pi/4, \pi/4$. If a face of X is a 4-gon, then it is a Euclidean unit square with respect to d, if it has more then four sides, then d has a singularity at its barycenter. It is clear that d is complete. By our assumptions on the links and faces, X has nonpositive curvature with respect to d, see [Gr,B1]. Since X is simply connected, X is a *Hadamard space*, that is, (X,d) is a complete metric space such that

(5.1) for any two points $x, y \in X$ there is a unique geodesic connecting them and this geodesic has length $d(x,y)$;

(5.2) for any geodesic triangle with vertices $x, y, z \in X$ the distance of x to the midpoint m on the geodesic from y to z is at most the corresponding distance in a Euclidean triangle with the same side lengths.

One of the important properties of Hadamard spaces is the existence of the circumcenter of bounded subsets, see Proposition 5.10 in Chapter I of [B2]. More precisely,

(5.3) if $B \subset X$ is a bounded subset, then there is a unique closed metric ball in X of smallest possible radius containing B; the center of this ball is called the *circumcenter* of B.

Let f be a face of X and e be an edge adjacent to f. Let x be the point on e of distance $1/4$ to one of the end points and let γ be the geodesic segment of length 1 in f which contains x and is perpendicular to e at x. Then γ meets the boundary in another point x' which is similarly located on an edge e' adjacent to f and is perpendicular to e' at x', see Figure 1. Any geodesic extension of γ in X consists (of parts of) geodesic segments contained in faces of X and similarly located in these faces. Any such geodesic extension misses the vertices of X' and intersects edges of X perpendicularly. It follows that the set T of all points in X which lie on such extensions is a convex subset of X and a tree.

5.4 Lemma. *For a tree T as above, $X \setminus T$ has two connected components.*

Proof. Note that the $1/4$-neighborhood U of T does not contain vertices of X. Since T is a tree, we conclude that $U \setminus T$ consists of two connected components, U_0, U_1. Hence $X \setminus T$ has at most two connected components.

If $X \setminus T$ is connected, then there is a continuous curve $c : [0,1] \to X \setminus T$ connecting $c(0) = y_0 \in U_0$ with $c(1) = y_1 \in U_1$, where y_0, y_1 are points in f on different sides of $\gamma = T \cap f$. Now X is simply connected and hence there is a homotopy $H : [0,1] \times [0,1] \to X$ from c to the line segment in f connecting y_0 with y_1. This homotopy can be chosen to be generic. Then the number of intersections

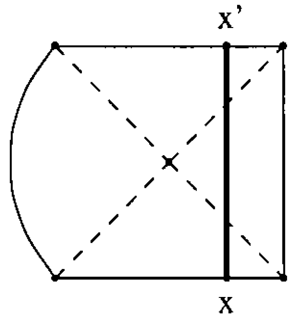

FIGURE 1. The geodesic segment γ in f.

with T does not change mod 2. This is a contradiction. $\square$

We let $\mathcal{T}$ be the set of trees in X obtained in the above way, for all choices of face f, edge e adjacent to f and point x on e of distance $1/4$ from one of the endpoints of e. Each $T \in \mathcal{T}$ defines two *half spaces* in X, namely the connected components of $X \setminus T$. The set of half spaces obtained in this way is denoted $\mathcal{H}_\mathcal{T}$.

Now let e be an edge in X which is not adjacent to any face. Let x be the point on e of distance $1/4$ to one of the endpoints of e. Then $X \setminus \{x\}$ has two connected components since X is simply connected. We denote by $\mathcal{S} \subset X$ the set of such points x, for all choices of edge e of X not adjacent to any face. Thus each $x \in \mathcal{S}$ also defines two *half spaces* in X, namely the connected components of $X \setminus \{x\}$. The set of half spaces obtained in this way is denoted $\mathcal{H}_\mathcal{S}$. We set $\mathcal{H} := \mathcal{H}_\mathcal{T} \cup \mathcal{H}_\mathcal{S}$.

Let $\mathcal{B}$ be the set of barycenters of faces of X and edges of X which are not adjacent to faces. For $p, q \in \mathcal{B}$ set

$$N(p,q) = |\{H \in \mathcal{H} : p \in H, q \notin H\}|.$$

Since $\mathcal{H}$ is invariant under the natural action of automorphisms of X we have $N(gp, gq) = N(p,q)$ for any automorphism g of X.

5.5 Lemma. *There exist constants $c_0, c_1 > 0$ such that*

$$c_0\, d(p,q) \le N(p,q) \le c_1\, d(p,q) \quad \textit{for all } p,q \in \mathcal{B}.$$

Proof. Let $p, q \in \mathcal{B}$ and $\delta := d(p,q)$. Let $\sigma : [0,\delta] \to X$ be the unit speed geodesic from p to q. Now p and q are not on any tree $T \in \mathcal{T}$ and hence σ is transversal to any $T \in \mathcal{T}$. It follows that σ intersects precisely those $T \in \mathcal{T}$ for which one of the corresponding half planes in $\mathcal{H}_\mathcal{T}$ is counted by N. Similarly, σ meets precisely those points $x \in \mathcal{S}$ for which one of the corresponding half planes in $\mathcal{H}_\mathcal{S}$ is counted by N. Hence $N(p,q)$ is equal to the number of intersections of these two types. Now any subsegment of σ of length $1/\sqrt{2}$ meets at least one $T \in \mathcal{T}$ or $x \in \mathcal{S}$ and a subsegment of length $< 1/2$ at most two. Hence the lemma. $\square$

5.6 Lemma. *The function $N : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ has the following properties:*

(i) N vanishes on the diagonal;

(ii) N is symmetric;

(iii) for all $m \geq 1$, all points $p_1, \dots, p_m \in \mathcal{B}$ and all numbers $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ with $\lambda_1 + \dots + \lambda_m = 0$ we have

$$\sum_{i,j} \overline{\lambda_i} \lambda_j N(p_i, p_j) \leq 0.$$

Remarks. (1) Lemma 5.3 says that N is a *kernel of negative type* in the sense of Definitions 5.12 and 5.17 in [HV]. Continuity of N is not an issue here since $\mathcal{B}$ is discrete.

(2) The proof of (iii) below uses the argument in the proof of Proposition 6.14 in [HV], see also [BJS].

Proof of Lemma 5.6. Assertion (i) is clear. Assertion (ii) follows since for any $H \in \mathcal{H}$ containing p but not q, the opposite half space contains q but not p. For any half space $H \in \mathcal{H}$, denote by χ_H the characteristic function of H. Then

$$N(p,q) = \sum_{H \in \mathcal{H}} \chi_H(p)(1 - \chi_H(q)).$$

Therefore

$$\begin{aligned} \sum_{i,j} \overline{\lambda_i} \lambda_j N(p_i, p_j) &= \sum_{i,j} \overline{\lambda_i} \lambda_j \sum_{H \in \mathcal{H}} \chi_H(p_i)(1 - \chi_H(p_j)) \\ &= \sum_{H \in \mathcal{H}} \big(\sum_i \overline{\lambda_i} \chi_H(p_i) \big) \big(\sum_j \lambda_j (1 - \chi_H(p_j)) \big). \end{aligned}$$

Since $\sum \lambda_i = 0$ we have

$$\sum_i \lambda_i \chi_H(p_i) = - \sum_j \lambda_j (1 - \chi_H(p_j))$$

for each $H \in \mathcal{H}$, hence (iii). $\square$

Proof of Theorem 3 in the $(4,4)$*-case.* Choose $p_0 \in \mathcal{B}$ and define $f : \Gamma \to \mathbb{R}$ by $f(g) = N(p_0, gp_0)$. By Lemma 5.6 and since N is invariant under automorphisms of X, f is a continuous function of negative type on Γ. By Lemma 5.5, f is bounded if and only if the orbit $\Gamma(p_0)$ is bounded. If $\Gamma(p_0)$ is bounded, then its circumcenter is a fixed point of Γ, see (5.3). Therefore f is not bounded if Γ does not have a fixed point. $\square$

Sketch of changes in the $(6,3)$*-case.* Let X' be the same subdivision of X as above and let d be the length metric on X which turns each edge of X' into a geodesic segment of length 1 and each 2-simplex of X' into an equilateral Euclidean triangle. Then d is complete and X is a Hadamard space with respect to d. The rest of the proof is the same, except that the edges of the corresponding trees have length $\sqrt{3}$,

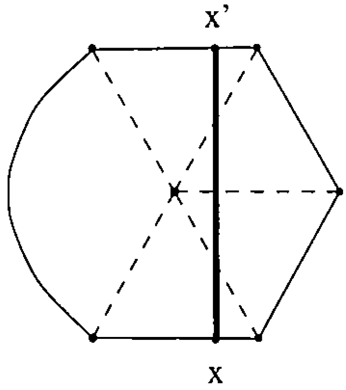

FIGURE 2. The geodesic segment in a face in case (6,3).

see Figure 2, and that the explicit numbers appearing in the proof of Lemma 5.5 change slightly. □

This finishes the proof of Theorem 3 and we come to the proof of Theorem 4. From now on, let X be a simply connected polygonal complex satisfying the assumptions of Theorem 4(1) in Introduction. Then X is a $(4,4)$-complex and we may and will use the above constructions. Since the links of the vertices of X are connected, any edge of X is adjacent to at least one face.

5.7 Lemma. *There exists a labelling of the edges of X by $1, \dots, k$ such that*

(i) for any face f of X, the edges of X adjacent to f are consecutively labelled by the numbers 1 to k;

(ii) for any edge e and faces f_1, f_2 adjacent to e, the labels of the edges adjacent to f_1 and f_2 coincide with respect to the combinatorial isomorphism of f_1 with f_2 which fixes e pointwise.

Furthermore, any automorphism of X induces a permutation of the labels. In particular, any group of automorphisms of X has a subgroup of finite index consisting of label preserving automorphisms.

Figure 3 illustrates the above properties of the labelling in the case of pentagonal faces.

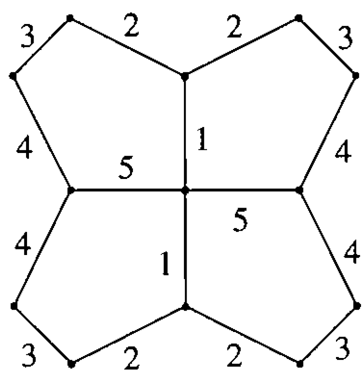

FIGURE 3. Labelling of edges.

Proof. A *gallery* in X is a sequence $\Omega = (f_0, \ldots, f_n)$ of faces in X such that any two consecutive faces f_{i-1} and f_i in it have a common edge and do not coincide. Given such a gallery, consider the map

$$\phi_\Omega := \phi_{n-1,n} \circ \phi_{n-1,n-2} \circ \cdots \circ \phi_{12} \circ \phi_{01},$$

where $\phi_{i-1,i} : f_{i-1} \to f_i$ is the combinatorial isomorphism of f_{i-1} with f_i fixing the common edge $f_{i-1} \cap f_i$ pointwise, $1 \leq i \leq n$. Then ϕ_Ω is a well defined combinatorial isomorphism between f_0 and f_n.

We say that a gallery $\Omega = (f_0, \ldots, f_n)$ is *closed* if $f_0 = f_n$.

Sublemma. *For any closed gallery $\Omega = (f_0, \ldots, f_n = f_0)$, the induced mapping $\phi_\Omega : f_0 \to f_0$ is the identity.*

Proof. Since the links of X are connected, generic closed curves in X miss vertices and cross edges transversally. Hence a generic closed curve in X determines a unique closed gallery and each closed gallery arises in this way. Consider a closed gallery $\Omega = (f_0, \ldots, f_n = f_0)$ and choose a generic closed curve γ which induces Ω such that the base point p of γ is inside f_0. Since X is simply connected, there exists a contraction γ_t of γ to the constant curve p. We may choose this contraction such that the curves γ_t are generic for all $t \in [0, 1]$ except for a finite number $t_1, \ldots, t_m$ of times at which γ_{t_i} crosses a single vertex of X (but still is transversal to all the edges whose interior it crosses). Passing through those t_i's results in modifications of the galleries Ω_t determined by the curves γ_t, but any such modification is performed inside the star of the vertex crossed by the corresponding γ_{t_i}. Since the links of the vertices of X are bipartite, the corresponding isomorphisms ϕ_{Ω_t} do not change under such modifications. After the last modification we are left with the trivial gallery (f_0), hence $\phi_\Omega = \mathrm{id}$. □

To conclude the proof of Lemma 5.7, choose a face f_0 of X and label the edges adjacent to it consecutively by $1, \ldots, k$. Since the links of X are connected, there is a gallery from f_0 to any other face of X. Given such a face f, choose a gallery Ω from f_0 to f and label the edges of f so that their labels coincide with the labels of the edges adjacent to f_0 with respect to the isomorphism ϕ_Ω. By the above sublemma, the labelling of f does not depend on the chosen gallery. The labelling obtained in this way clearly satisfies properties (i) and (ii) of the lemma. The other assertions follow easily. □

Fix a labelling of the edges of X as in Lemma 5.7. For a tree T as defined before Lemma 5.4 there is an $i \in \{1, \ldots, k\}$ such that T only intersects faces of X' adjacent to edges with labels i, $i+1$ and $i+2$ (mod k). Thus we obtain a partition $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_k$ and a corresponding partition of half spaces $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_k$. Fix an $i \in \{1, \ldots, k\}$ and let $|\mathcal{T}_i| \subset X$ be the union of all the trees in $\mathcal{T}_i$.

5.8 Lemma. *Let $X_i = X \setminus |\mathcal{T}_i|$. Then each $T \in \mathcal{T}_i$ intersects the closures of exactly two connected components of X_i and is contained in any of those two closures. The closures of two different connected components of X_i are either disjoint or intersect in exactly one $T \in \mathcal{T}_i$.*

Proof. For $T \in \mathcal{T}_i$, the 1/4-neighborhood U of T is disjoint from the 1/4-neighborhood of any other $T' \in \mathcal{T}_i$. Furthermore, $U \setminus T$ has two connected components. □

Now define the *dual graph* $\mathcal{T}_i^*$ of $\mathcal{T}_i$ as follows: the vertices of $\mathcal{T}_i^*$ correspond to the connected components of X_i and two vertices are connected by an edge if the closures of the corresponding components of X_i intersect in a tree $T \in \mathcal{T}_i$.

5.9 Lemma. *$\mathcal{T}_i^*$ is a tree.*

Proof. By Lemma 5.4, the removal of an edge disconnects $\mathcal{T}_i^*$. □

Proof of Theorem 4(1) of Introduction. Let Γ be a group of automorphisms of X and suppose that Γ does not have a fixed point. Let Γ' be the sugroup of Γ consisting of label preserving automorphisms. Then Γ' is a normal subgroup of Γ of finite index and leaves X_i invariant. Hence there is an induced action of Γ' on $\mathcal{T}_i^*$, $1 \le i \le k$.

Suppose $x_0 \in X$ is a fixed point of Γ'. Then the Γ-orbit $\Gamma(x_0)$ of x_0 is finite and hence bounded. In particular, the circumcenter of $\Gamma(x_0)$ is a fixed point of Γ, a contradiction. Hence Γ' does not have fixed points.

The function N can be written as $N = N_1 + \cdots + N_k$, where

$$N_i(p,q) = |\{H \in \mathcal{H}_i : p \in H, q \notin H\}|, \quad 1 \le i \le k.$$

Let $p_0 \in \mathcal{B}$. Since Γ' does not have fixed points, the Γ'-orbit $\Gamma'(p_0)$ is unbounded. Hence there is an $i \in \{1, \ldots, k\}$ such that the function

$$f_i : \Gamma \to \mathbb{R}, \quad f_i(g) = N_i(p_0, gp_0),$$

is unbounded.

Let $v_0 \in \mathcal{T}_i^*$ be the vertex corresponding to the component of X_i containing p_0. Let d_i be a length metric on $\mathcal{T}_i^*$ such that all edges of $\mathcal{T}_i^*$ have length 1. Then $d_i(v_0, gv_0) = f_i(g)$ for all $g \in \Gamma'$. In particular, the orbit $\Gamma'(v_0)$ is unbounded. Now $\mathcal{T}_i^*$ is a Hadamard space with respect to d_i, hence the action of Γ' on $\mathcal{T}_i^*$ does not have a fixed point. □

Proof of Theorem 4(2) of Introduction. Theorem 4(2) reduces to the case 4(1) in the following way. Consider the subdivision X' of X obtained by adding the barycenters of the faces and edges of X as new vertices and as new edges the line segments in the faces connecting the barycenters of them with the barycenters of the adjacent edges, see Figure 4.

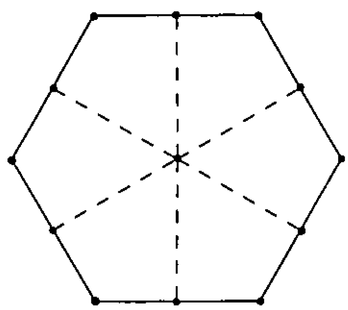

FIGURE 4. Subdivision of a hexagonal face.

Observe that all faces of the subdivided complex X' are 4-gons. Note also that the links at the vertices of X do not change after subdivision, while those at the new vertices are connected and bipartite. Hence the subdivided complex X' satisfies the assumptions of Theorem 4(1) and hence Theorem 4(2) follows. $\square$

5.7 Remarks. (1) By construction, the subgroup Γ' is normal in Γ and Γ/Γ' is isomorphic to a subgroup of the symmetric group S_k in Theorem 4(1) respectively S_4 in Theorem 4(2). In particular, the index of Γ' is at most $k!$ respectively 24.

(2) After a slight modification of the above arguments one gets the conclusion of Theorem 4(1) under the following more general conditions on X:

(i) all links of X are bipartite;

(ii) there is a natural number $k \geq 3$ such that for each face f of X the number of edges of f is km for some natural number m with $km \geq 4$.

Observe that Theorem 4(2) handles the case $k = 2$ in (ii).

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