# The Length of the Shortest Closed Geodesics on a Positively Curved Manifold


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**Abstract**

We give a metric characterization of the Euclidean sphere in terms of the lower bound of the sectional curvature and the length of the shortest closed geodesics.

## 1 Introduction

Let $M$ be a complete connected Riemannian manifold of dimension $d$ and class $C^\infty$. The study of global structure of closed geodesics on $M$ *vis a vis* certain quantitative restrictions on the sectional curvature $K$ of $M$ has attracted considerable interest. Henceforth, we assume $k$ to be a positive constant. It is well-known that if $K \geq k^2$ on all tangent 2-planes of $M$, then there must exist on $M$ a closed geodesic whose length is $\leq 2\pi/k$. The purpose of the present paper is to describe a rigidity phenomenon observed when this length is extremal on $M$. More precisely, we prove

**Main Theorem.** *If $M$ satisfies $K \geq k^2$ and if the shortest closed geodesics on $M$ have the length $= 2\pi/k$, then $M$ is isometric to $S^d_k$, the Euclidean sphere of radius $1/k$ in $\boldsymbol{R}^{d+1}$.*

Note that we make no assumption about the geodesics' having no self-intersections. There exists an example of a 2-dimensional smooth surface all of whose shortest closed geodesics have self-intersections (see Appendix). These examples have some regions where the curvature is negative. It is reported that E. Calabi has proved that on a positively curved surface, at least one of the shortest closed geodesics is always without self-intersections.

We now mention some related rigidity phenomena. Previously, Sugimoto

Figure 1: $S^2$ like surface with highly curved "equator"

[Su], improving on an earlier work of Tsukamoto [Ts], proved

**Theorem A.** *Suppose that $M$ satisfies $4k^2 \geq K \geq k^2$. If $d$ is odd, assume that $M$ is simply connected. Then, if $M$ has a closed geodesic of length $2\pi/k$, it is isometric with $S^d_k$.*

Recall that under the curvature assumption of Theorem A, if $M$ is simply connected, the celebrated Injectivity radius theorem, which is primarily due to Klingenberg (see [CE (§§5.9,10)], [GKM, §§7.5,7] and also [CG], [KS] and [Sa2]) states that all closed geodesics on $M$ have length $\geq \pi/k$.

However, we point out that, in general, an assumption on the length of the shortest closed geodesic is a nontrivially weaker condition than an upper bound on the sectional curvature. In fact, Suppose we are given a general curvature bound $\alpha k^2 \geq K \geq k^2$. Although we cannot use Klingenberg's injectivity radius theorem, we obtain a lower estimate for the volume of $M$ in terms of the supremum of the sectional curvature. Then the Cheeger-type estimate (sharpened by Heintze-Karcher) implies a lower bound on the length of the shortest closed geodesics on $M$ (see [C], [CE], [HK] and [Sa2]). It is also possible to construct, for any given $k$ and $\delta$, a Riemannian metric on $S^2$ with $K \geq k^2$ and the length of the shortest closed geodesic $\delta$-close to $2\pi/k$ but whose curvature grows arbitrarily large. This construction means that, from the point of view of rigidity theorems in Riemannian geometry, imposing an upper bound on the curvature is nat natural in characterizing a Euclidean sphere among complete Riemannian manifolds with $K \geq k^2$ having a shortest closed geodesic of length just $2\pi/k$. See Figure 1. In the spacial case of dimension 2, we have

**Theorem B** (*Toponogov* [T]). *Suppose that $M$ is an abstract surface with Gauss curvature $K \geq k^2$. If there exists on $M$ a closed geodesic without self-intersections whose length $= 2\pi/k$, then $M$ is isometric to $S_k^2$.*

However, in higher dimensions, there are lens spaces of constant sectional curvature $k^2$ so that all geodesics are closed, the prime ones have no self-intersections, and they are either

(a) homotopic to 0 and have length $= 2\pi/k$, or

(b) homotopically nontrivial and can be arbitrarily short.

See [Sa1]. Of course, it follows from our Main Theorem that

**Corollary.** *If $K \geq k^2$ and the shortest closed geodesics that are homotopic to 0 in $M$ have the length $2\pi/k$, then the universal covering of $M$ must be isometric to $S_k^d$.*

Note also that Theorem B is false without the assumption of the closed geodesics' not having self-intersections. In fact, for any $k$, one can construct an ellipsoid in $\boldsymbol{R}^3$ which possess a prime clised geodesic of length $= 2\pi/k$ and whose curvature is $> k^2$.

Finally, we mention a previous related result of the first author which gives another rigidity solution for the non simply connected case.

**Theorem C** (*Itokawa* [I1,2]). *If the Ricci curvature of $M$ is $\geq (d-1)k^2$ and if the shortest closed geodesics on $M$ have the length $\geq \pi/k$, then either $M$ is simply connected or else $M$ is isometric with the real projective space all of whose prime closed geodesics have length $= \pi/k$.*

It is not yet known if our Main Theorem remains true with the weaker assumption on the Ricci curvature. However, we point out that examples were shown in [I1,2] so that for the Ricci curvature assumption, the shortest closed geodesics may have length arbitrarily close to $2\pi/k$ without the manifold's even being homeomorphic to $S^d$. This indicates how delicate the Ricci curvarure assumption could be.

*Acknowledgement.* This paper is based on the first author's preprint [I3] which was distributed but not published (since the proof of the main result in [I3] contained a gap). In the academic year 1990-1991, the second author visited the Johns Hopkins University. The authors took this opportunity to collaborate on the present paper and filled the gap in [I3]. The first author wishes to acknowledge his gratitude to D. Gromoll and G. Thorbergsson for providing him with valuable suggestions and informations at the early stage of this work. The second author is grateful to the Johns Hopkins University for its support and hospitality. Last but not least, the authors owe much to the anonymous referee to [I3] for patient corrections and encouragement.

# 2 Preliminaries

The purpose of this section is to gather all the well-known results which will be used in proving Main Theorem as well as to set straight our notational conventions and normalizations. In this paper, we agree that by the term *curve* we mean an absolutely continuous mapping $c : \boldsymbol{R} \to M$ whose derivative $\boldsymbol{R} \to TM$ is an $L_2$ map on each closed interval. We refer to the restriction of a curve to any interval as an *arc*. If $c$ is a curve and $a < b$ are reals, we write $c_{a,b}$ to denote the arc $c\,|_{[a,b]}$. If $c$ happens to be differentiable, the normal bundle; respectively, the unit normal bundle of $c$; which are in fact bundles over $\boldsymbol{R}$, are denoted $\perp c$; respectively, $U \perp c$. We shall call a curve $c$ closed if $c(s+1) = c(s)$ for all $s$. We denote the set of all closed curves on $M$ by $\Omega$.

For fixed $a$, $b$, let $\mathcal{C}_{a,b}$ denote the set of all arcs $[a,b] \to M$. It is known that $\mathcal{C}_{a,b}$ has the structure of a Riemannian Hilbert manifold where the inner product is given by the natural $L_2$ inner product of variation vector fields along a curve. The restriction $c \mapsto c_{0,1}$ embeds $\Omega$ in $\mathcal{C}_{0,1}$ as a closed submanifold. Henceforth, this is the structure we shall always assume on these spaces.

For $\gamma \in \mathcal{C}_{a,b}$, we define the space $\mathcal{B}'_\gamma$ of all square integrable vector fields $v \in T_\gamma \mathcal{C}_{a,b}$ along $\gamma$ such that $V(a) = 0$, $v(b) = 0$, and $v(s) \in \perp_s \gamma$ for all $s$. If $c \in \Omega$, we also define the space $\mathcal{B}_c$ of all $v \in T_c\Omega$ with $v(s) \in \perp_s c$ almost everywhere. Then, $\mathcal{B}'_{c_{0,1}}$ is canonically embedded in $\mathcal{B}_c$.

We normalize the energy of $\gamma \in \mathcal{C}_{a,b}$ by

$$E(\gamma) := \int_a^b \mid \gamma'(s) \mid^2 ds. \tag{1}$$

Also, we denote by $L(\gamma)$ the length of $\gamma$ in the usual sense. Thus, in our convention, $L(\gamma)^2 \leq E(\gamma)$ with equality if and only if $\gamma$ is parametrized proportinal to arclength. The term *geodesic* is always understood to mean a nonconstant geodesic. For $u \in UTM$, the unit tangent bundle, we denote by $c_u$ the geodesic $s \mapsto \exp su$. Recall that the critical points of $E$ on $\Omega$ are closed geodesics and the constant curves.

Let $c$ be a geodesic and $a < b \in \boldsymbol{R}$. The Hessian of $E$ at $c_{a,b}$, here regarded as a symmetric bilinear form on $T\mathcal{C}_{a,b}$, is denoted $H_a^b$. We remind that if $v \in \mathcal{B}'_{c_{a,b}}$ and is differentiable or if $c \in \Omega$, $a = 0$, $b = 1$, and $v \in \mathcal{B}_c$ is differentiable, then $H_a^b$ equals the so called integral

$$-2\int_a^b < v''(s), v(s) > + \mid v(s) \mid^2 \mid c'(s) \mid^2 K(v(s) \wedge c'(s)) ds \tag{2}$$

(for the complete expression, see [BTZ1] where the normalization used is 1/2 that of ours.) We write $\iota'(c_{a,b})$ to denote the index of $H_a^b \mid_{\mathcal{B}'_{c_{a,b}}}$. If $c$ is closed, we put $H := H_0^1 \mid_{\mathcal{B}_c}$ and $\iota(c)$ its index. We recall the basic inequality

$$\iota(c) \geq \iota'(c) = \sum_{0<s<1} \nu'(c_{0,1}) \tag{3}$$

where $\nu'(c_{0,1})$ is the dimension of the space of Jacobi fields in $\mathcal{B}'_{c_{0,s}}$. In this notation, we start the following well-known theorem, which is primarily due to Fet [F].

**Theorem D.** *Assume that $M$ satisfies $K \geq k^2$. Then there exists a closed geodesic $c$ on $M$ such that $L(c) \leq 2\pi/k$ and $\iota(c) \leq d-1$.*

For each $r \in \boldsymbol{R}$, we denote by $\Omega^r$; respectively, $\Omega^{=r}$ and $\Omega^{<r}$, the subspaces $\{c \in \Omega : E(c) \leq r\}$; respectively, $\{c \in \Omega : E(c) = r\}$ and $\{c \in \Omega; E(c) < r\}$. However, $\Omega^0 = \Omega^{=0}$ is identified with $M$ itself and so denoted also by $M$. It is well-known that each $\Omega^r$; $r > 0$, contains a submanifold $'\Omega_r$ which is diffeomorphic to an open set in some finite product $M \times \cdots \times M$ and homotopy equivalent to $\Omega^{<r}$. The functional $E$ becomes a proper function on $'\Omega_r$. The space $'\Omega_r$ contains all the critical points in $\Omega^{<r}$ and the Hessian of $E\,|_{'\Omega_r}$ retains the same index as $E$ at each critical point. For details, see [Mi (§16)] and [Bo]. If $a < r$, we put $'\Omega_r^a := {'\Omega_r} \cap \Omega^a$.

We must laler consider a more general functional $F$ on a finite dimensional Riemannian or separable Hilbert manifold $X$. For our purpose, $X$ will be either $\Omega$ or $'\Omega :=' \Omega_r$ for some fixed $r$. Let $c \in X$ be a critical point of $F$. Then $T_cX$ decomposes into a direct sum

$$T_cX = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{Z}$$

where $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{Z}$ are the spaces on which the Hessian $H_F$ of $F$ at $c$ is positive definite, negative definite and zero respectively. We write $\|\cdot\|$ for the norm in $T_cX$. Then, we can state the following important fact due to Gromoll and Meyer [GM1]:

**Theorem E** (Generalized Morse Lemma). *In the setting described above, there exists a neighborhood $U$ of $c$, a coordinate chart*

$$\xi_c : U \longrightarrow T_cX,$$

*with respect to which $F$ takes the form*

$$F \circ \xi_c^{-1}(v) - F(c) = \|x\|^2 - \|y\|^2 + f(z)$$

*where $x$, $y$ and $z$ are the projections of $v \in \xi^{-1}(U)$ on $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{Z}$ respectively, and the Taylor expansion of $f$ at $z = 0$ starts with a polynomial of degree at least 3 in $z$. For this decomposition, $c$ needs not be an isolated critical point of $F$, but if $F$ has other critical points in $U$, they are contained in the zero eigenspace of $H_F$.*

We use the notation $U_c^{-0} := \xi_c^{-1}(\mathcal{N} \oplus \mathcal{Z})$ and $U_c^- := \xi_c^{-1}(\mathcal{N})$ and call them the *unstable*; respectively, the *strong unstable*; submanifolds of $F$ at $c$, even though we make no assunption on $\dim \mathcal{Z}$.

Suppose that $a \in \boldsymbol{R}$. We set $\Omega_F^a := \{c \in \Omega \text{ or } '\Omega;\ F(c) \leq a\}$. Let $I$

be the interval $[-1,1]$. Suppose $c$ is a critical point of $F$ with $a := F(c)$ and $\iota :=$ index $H_F |_c = \dim \mathcal{N}$. Let $U$ be a neighborhood of $c$ as defined in Theorem E. A differentiable embedding $\sigma : (I^\iota, \partial I^\iota) \to (\Omega, \Omega_F^a - U)$ will be called an *weak unstable simplex* of $F$ at $c$ if $\sigma(0) = c$ and $F |_\sigma \leq a$. If $\sigma$ is a weak unstable simplex of $F$ at $c$, then $\sigma \cap U$ must be contained in the topological cone

$$\{\gamma \in U\,;\, H_F(\xi(\gamma)) \leq 0\}.$$

In particular, if $\sigma(I^\iota) \cap \xi^{-1}(\mathcal{N})$ contains an open neifhborhood of $\xi^{-1}(0)$ in $\xi^{-1}(\mathcal{N})$, we call $\sigma$ a *strong unstable simplex*. We note that

**Proposition 1.** *Any embedded differentiable simplex* $\sigma : (I_\iota) \to (X, X - U)$ *with* $\sigma(0) = c$ *and* $H_F(\xi(\sigma(t))) < 0$ *for all* $t \in I^\iota - \{0\}$ *is homotopic to one of the strong unstable simplexes modulo* $X - U$.

We shall say that a critical point $c$ of $F$ is a *nondegenerate* critical point if $\mathcal{Z} = \{0\}$. Note that this agreement is different from the often-used convention of calling a closed geodesic nondegenerate if $\mathcal{Z}$ is the $S^1$ orbit of the geodesic. With our convention, a closed geodesic is never a nondegenerate critical point and for $E$. We put $a := F(c)$ and write $X^r := \{x \in X\,;\, F(x) \leq r\}$. If $c$ is a nondegenerate critical point of $F$, then of course $c$ is an isolated critical point and, for some $\varepsilon > 0$, the strong unstable simplexes at $c$ represent a nontrivial class in $\pi_\iota(X^{a+\varepsilon}, X^{a-\varepsilon})$.

Let $t > r$. The set $\Sigma^\iota$ of all absolutely continuous $\sigma : (I, \partial I) \to (X^t, X^r)$ can be given the compact-open topology. We will need the following topological

**Proposition 2.** *Suppose* $M$ *is compact. Let* $\sigma \in \Sigma^\iota$. *Then there is a neighborhood* $\mathcal{O}$ *of* $\sigma$ *in* $\Sigma^\iota$ *so that any* $\sigma' \in \mathcal{O}$ *is homotopic to* $\sigma$ *modulo* $X^r$ *up to orientation.*

# 3 Proof of Main Theorem

It is clear that, in order to prove Theorem 1, we need consider only one specific $k$. So, hereafter we assume that $M$ satisfies $K \geq k^2$ where $k := 2\pi$.

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In the present section, we further assume that $M$ contains no closed geodesic of length $< 1$, or equivalently that there are no critical points of $E$ in $\Omega^{<1} - M$. It now remains for us to prove that then $M$ is isometric to $S^d_{2\pi}$.

We set

$$\mathcal{C} := \{c \in \Omega\,;\ c \text{ is a closed geodesic and } \iota(c) = d-1\}$$

and

$$\mathcal{C}^* := \{c \in \mathcal{C}\,;\ \text{an unstable simplex of } E \text{ at } c \text{ represents a nontrivial element in } \pi_{d-1}(\Omega, M)\}.$$

Theorem D and Theorem E assert that $\mathcal{C} \neq \emptyset$. If we can asume that each $c \in \mathcal{C}$ has an isolated critical $S^1$-orbit, the technique of Gromoll and Meyer [GM2] fairly readily shows that $\mathcal{C}^*$ too is non-empty. In our case, it will be precisely one of our points that no $c \in \mathcal{C}^*$ has an isolated critical orbit. Under the stronger hypothesis of $4k^2 \geq K \geq k^2$, Ballman [Ba] showed that all closed geodesics have nontrivial unstable simplexes. However, he makes essential use of the upper bound for $K$ which is not available to us. Nonetheless, we shall still prove in the next section

**Lemma 1.** *Under the assumption of this section, $\mathcal{C}^*$ is nonempty and is a closed set in $\Omega$.*

In this section, we assume Lemma 1 for the time being, and prove

**Lemma 2.** *For each $c \in \mathcal{C}^*$, there is a neighborhood $\mathcal{U}$ of $c'(0)$ in $UT_{c(0)}M$ such that whenever $u \in \mathcal{U}$ and $\tau$ is any tangent 2-plane containing $c_u'(s)$ for some $s \in \boldsymbol{R}$, then $K(\tau) = k^2$.*

The idea for proving this is to construct, for each $c \in \mathcal{C}$, a specific unstable simplex $\sigma$ and its deformation so that, unless the conclusion of the lemma is met, $\sigma$ is deformed into $M$, which is a contradiction if $c \in \mathcal{C}^*$. First, we show

**Lemma 3.** *If $c \in \mathcal{C}$, then for any $s \in \boldsymbol{R}$ and any $v \in \perp_s c$, $K(c'(s) \wedge v) = k^2$.*

*Proof.* Assume that, for some $s_1 \in \boldsymbol{R}$ and $v_1 \in \perp_{s_1} c$, $K(c'(s_1) \wedge v_1) > k^2$. By virtue of the natural $S^1$-action on $\Omega$, it is no loss of generality to assume that $0 < s_1 < 1/2$. Now, we define a real number $\delta$ as follows. If there is a point in $(0, s_1]$ which is conjugate to 0 along $c$, we choose any $\delta$ so that $s_1 < \frac{1}{2} - \delta < \frac{1}{2}$. If, on the other hand, there is no conjugate point in $(0, s_1]$, there is a unique Jacobi field $Y$ along $c$ with $Y(0) = 0$ and $Y(s_1) = v_1$, and by a consequence of the original Rauch comparison theorem [CE (§1.10, Remark, p.35)], there is an $s_2$, $s_1 < s_2 < 1/2$ so that $Y(s_2) = 0$. In this case, we choose $\delta$ so that $s_2 < \frac{1}{2} - \delta < \frac{1}{2}$. In either case, we have $\iota'(c_{0,\frac{1}{2}-\delta}) \geq 1$. On the other hand, by the Morse-Schoenberg index comparison with $S^d_k$, we have $\iota'(c_{\frac{1}{2}-\delta,1}) \geq d - 1$, since $L(c_{\frac{1}{2}-\delta,1}) \geq \frac{1}{2}$. Therefore, we have

$$\iota(c) \geq \iota'(c) \geq \iota'(c_{0,\frac{1}{2}-\delta}) + \iota'(c_{\frac{1}{2}-\delta,1}) \geq 1 + d - 1 = d,$$

which is a contradiction. □

As a consequence, we have

**Lemma 4.** *Jacobi fields in $\mathcal{B}'|_{c_{0,1}}$ are constant multiple of the fields $\sin(ks)V(s)$; $0 \leq s \leq 1$, where $V$ is any parallel vector field of elements in $U \perp c\,|_{[0,1]}$.*

Now, let $V_1, \cdots, V_{d-1}$ be parallel vector fields of orthonormal elements in $U \perp c\,|_{[0,1]}$. By compactness argument, there exists an $\eta > 0$ so that each orthogonally trajecting geodesics $t \mapsto \exp tx$ where $x \in U \perp c$ has no point focal to $c$ in $t < \eta$. Define $2(d-1)$ vector fields $X_i(s)$ and $Y_i(s)$ $(0 \leq s \leq 1)$ along $c$ as follows. These vector fields are not continuous at $s = 0$ and $s = \frac{1}{2}$.

$$X_i(s) = \begin{cases} V_i(s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < s < 1 \end{cases}$$

and

$$Y_i(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq \frac{1}{2} \\ V_i(s) & \text{if } \frac{1}{2} < s < 1. \end{cases}$$

Let $x = (x_1, \cdots, x_{d-1}) \in I \subset \boldsymbol{R}^{d-1}$ and $y = (y_1, \cdots, y_{d-1}) \in I \subset \boldsymbol{R}^{d-1}$, where $I$ is a small interval in $\boldsymbol{R}^{d-1}$. We define a $(2n-2)$-dimensional simplex $\tilde{\sigma}$ in

$\Omega$ as follows:

$$\tilde{\sigma}(x,y)(s) = \exp_{c(s)} \arctan\{\sin(2\pi s)(\sum_{i=1}^{d-1}(x_i X_i(s) + y_i Y_i(s)))\}.$$

The strange impression of "arctan" of a vector will disappear if we "define" $\arctan x$ for $x \in U \perp c$ to be $\eta(\arctan \|x\|)x$. We deform this simplex in the following way. If $x = y$, we make no change on the corresponding loop. If $x \neq y$, then we make a suitable short cut at the non-trivial angle (created by the discrepancy $x \neq y$) at $s = \frac{1}{2}$. For instance, we fix a small positive number $\delta$ and make a short cut between points corresponding to $s = \frac{1}{2} - \delta$ and $s = \frac{1}{2} + \delta$. After performing this modification and reparametrizing the corresponding loops by arc length, we get a $(2d-2)$-dimensional simplex $\sigma$. We note that

(i) the $(d-1)$-dimensional simplex $\tilde{\sigma} \cap \sigma$ consists of those closed curves and is defined by $x = y$, i.e., variations which integrate global Jacobi fields on $c$, and

(ii) the vector fields $\sin(2\pi s)X_i(s)$ and $\sin(2\pi s)Y_i(s)$ are naturally regarded as Jacobi fields along $c\,|_{[0,\frac{1}{2}]}$; respectively, $c\,|_{[\frac{1}{2},1]}$ which vanish at end points.

If $x \neq y$, then, after performing the above modification, we see that $\sigma(x,y)$ is strictly under the level set $\Omega^{=1}$ of $E = 1$. We see this, by applying Rauch type comparison theorem of Berger (Rauch's second comparison; see [CE (§1.10)]) to variations:

$$\exp_{c(s)} \arctan\{\sin(2\pi s) \sum_{i=1}^{d-1} x_i X_i(s)\}$$

for $s \in [0, \frac{1}{2}]$ of $c\,|_{[0,\frac{1}{2}]}$, and

$$\exp_{c(s)} \arctan\{\sin(2\pi s) \sum_{i=1}^{d-1} x_i Y_i(s)\}$$

for $s \in [\frac{1}{2}, 1]$ of $c\,|_{[\frac{1}{2},1]}$ fixing their end points. Summing up, we have

**Lemma 5.** *There exists a neighborhood $W$ of $c \in \mathcal{C}$ so that the $(2d-2)$-dimensional simplex $\sigma \cap W$ is contained in $\Omega^1$.*

We now consider the foliation on $\boldsymbol{R}^{2d-2}$ by affine subspaces orthogonal to the linear subspace defined by the equations $x = y$. It follows from Proposition 1 that the $(n-1)$-simplex $\tau$ defined by

$$\tau(x)(s) = \sigma(x,-x)(s) = \exp_{c(s)} \arctan\{\sin(2\pi s)(\sum_{i=1}^{d-1}(x_i X_i(s) - x_i Y_i(s)))\},$$

i.e., $\tau(x) = \tau(x,y)$, where $(x,y)$ varies over the leaf containing the origin (so $y = -x$), is homotopic to an unstable simplex in the sense of Proposition 1. Indeed, we can easily show that, for any vector field $v$ in $\mathcal{B}'_{c_{0,1}}$ tangent to $\tau$, $H_E(v,v) < 0$ for the Hessian $H_E$ at $c$. This is essentially because of the Taylor expansion $\cos\theta = 1 - \frac{\theta^2}{2} + O(\theta^4)$ as $\theta \to 0$. Thus, from Proposition 1, we have

**Lemma 6.** *If $c \in \mathcal{C}^*$, then there is a neighborhood $U$ of $c$ in $\Omega$ so that, for $\varepsilon > 0$ and a subneighborhood $W \subset U$, $\tau$ constructed above represents a nontrivial element in $\pi_{d-1}(W, W \cap \Omega^{1-\varepsilon}.)$*

The proof of Lemma 6 is easy using standard Morse theoretic arguments. If we define a $(d-1)$-dimensional simplex $\overline{\tau}$ by

$$\overline{\tau}(x) := \exp_{c(s)} \arctan\{\sin(\pi s)\sum_{i=1}^{d-1} x_i V_i(s)\},$$

then direct calculation of the Hessian implies that $\overline{\tau}$ also defines an unstable simplex at $c$ and belongs to the same class as $\tau$ in the relative homotopy group. This unstable simplex corresponds to the eigen vector of the index form (2) with negative eigenvalue. In this sense, $\overline{\tau}$ is more natural than $\tau$. Now define a $(2d-2)$-simplex $\overline{\sigma}$, which also contains the $(d-1)$-simplex (defined by $x = y$ in our simplex $\sigma$) corresponding to the global Jacobi field on $c$, by

$$\overline{\sigma}(x,y)(s) := \exp_{c(s)} \arctan\{\sum_{i=1}^{d-1}(x_i \sin(\pi s) + y_i \sin(k\pi s))V_i(s)\}.$$

Although this construction is natural, it turns out to be not clear whether there exists an interval $I$ containing 0 such that $\overline{\sigma}(I \times I)$ is contained in $\Omega^1$. This is the reason why our construction of $(2d-2)$-simplex $\sigma$ is based on

Figure 2: unstable simplex $\sigma$

the short cut argument of broken geodesics in the model space, although the unstable simplex $\tau$ does not directly integrate the negative eigenspace of the Hessian of the energy functional $E$ at $c$.

One of the following two cases is possible. Namely, either

(A) For at least one choice of $x_0 \in I$, there is some $\varepsilon > 0$ such that

$$\exp_{c(s)} \arctan\{\sin(2\pi s)(\textstyle\sum_{i=1}^{d-1} x_{0,i}X_i(s) + x_{0,i}Y_i(s))\}$$

$$= \exp_{c(s)} \arctan\{\sin(2\pi s)(\textstyle\sum_{i=1}^{d-1} x_{0,i}V_i(s))\}$$

is contained in $\Omega^{1-2\varepsilon}$ as is illustrated in Figure 2, in which the variation vector fields $Y$ and $Z$ along $c$ are of the form

$$\begin{aligned} Y &= \sin(2\pi s)\textstyle\sum_{i=1}^{d-1} x_i V_i(s) \quad \text{(Jacobi fields)} \\ Z &= \sin(2\pi s)\textstyle\sum_{i=1}^{d-1}(x_i X_i(s) - x_i Y_i(s)), \end{aligned}$$

or,

(B) There exists an $\alpha$; $0 < \alpha < 1$ so that whenever $| x_1, \cdots, x_{d-1} | \leq \alpha$,

$$\exp_{c(s)} \arctan\{\sin(2\pi s)(\sum_{i=1}^{d-1} x_i V_i(s))\}$$

lies in $\Omega^{=1}$.

If Case (A) prevails, $\sigma$ can be deformed into $U \cap W \cap \Omega^{1-\varepsilon}$ which contradicts the conclusion of Lemma 5. Hence $c \notin \mathcal{C}^*$. If, on the other hand, we start out with a $c \in \mathcal{C}^*$, then Case (B) must really be the case. We thus get a $(d-1)$-dimensional local submanifold $S$ of $\Omega^{=1}$ which is tangent to the 0 eigenspace of the Hessian of $E$ defined on $\mathcal{B}_{c_{0,1}}$ through $c$.

**Lemma 7.** *In the present situation, each member $\tilde{c}$ of $S$ is a closed geodesic in $\mathcal{C}$.*

*Proof.* If $\tilde{c}$ is not a critical point of $E$, there exists at least one $x_0 \in I \subset \boldsymbol{R}^{d-1}$ such that the $(d-1)$-dimensional simplex defined by the $(d-1)$-dimensional affine subspace through $x_0$ orthogonal to the linear subspace defined by $x = y$ contains no critical point. Then, by following the trajectory of $-\text{grad}\, E$, $\tau$ is deformed into $U \cap W \cap \Omega^{1-\varepsilon}$, which contradicts assumption that we strated with $c \in \mathcal{C}^*$. Hence all $\tilde{c} \in S$ are closed geodesics. If some $\tilde{c} \notin \mathcal{C}$, then $\iota(c) > d-1$, so $\tau$ is again deformed into $U \cap W \cap \Omega^{1-\varepsilon}$ via the unstable simplex of $\tilde{c}$. Hence, either way, we get a contradiction. □

By construction, we also see that for any $\tilde{c} \in S$, $\tilde{c}(0) = c(0)$. Translared into $M$, this means that there is a open tube $B$ around the set $c(0, \frac{1}{2}) \cup c(\frac{1}{2}, 1)$ such that for each $q \in B$, a geodesic joining $c(0)$ to $q$ extends to a closed geodesic in $\mathcal{C}$ whose image lies in $B$ except at $s =$ half integers. Applying Lemma 3 to each geodesics proves Lemma 2. □

Even more is true. By Proposition 2, we get

**Lemma 8.** *Let $c \in \mathcal{C}^*$ and let $\mathcal{U} \subset T_{c(0)}M$ be the set in Lemma 2. Then, there exists an open set $\mathcal{U}^*$; $c'(0) \in \mathcal{U}^* \subset \mathcal{U}$, so that, for all $u \in \mathcal{U}^*$, $c_u \in \mathcal{C}^*$.*

That is to say, the set

$$\mathcal{U}^* = \{u \in UT_{c(0)}M\,;\, c_u \in \mathcal{C}^*\}$$

is an open set in $UT_{c(0)}M$. On the other hand, by Lemma 1 and the continuous dependence of geodesics on their initial values, the set $\mathcal{U}^*$ is also a closed set. Since $UT_{c(0)}M$ is connected, $\mathcal{U}^*$ must in fact be all of $UT_{c(0)}M$. Together with Lemma 2, we summariae our result as

**Lemma 9.** *Let $M$ be assumed in this section. Then, there exists a point*

*$p \in M$ such that for all $u \in T_pM$, $c_u$ is a closed geodesic of prime length* 1 *and $K(\tau) = k^2$ for all 2-planes $\tau$ tangent to the radial direction from $p$.*

*Proof.* Take a $c \in \mathcal{C}^*$ and let $p := c(0)$. □

Now it is a standard technique to construct an explicit isometry from $M$ onto $S^d_k$ exactly as in Toponogov's maximum diameter theorem (see, for example, [CE (§6.5)] or [GKM (§7.3)]). Thus, Main Theorem is proven as soon as Lemma 1 is established.

# 4 Proof of Lemma 1

In this section, we continue to assume $K \geq 4\pi^2$. The following proposition is essentially contained in earlier works of Berger and is easy to prove by Morse-Schoenberg index comparison with $S^d_k$ and the tautological isomorphism $\pi_i(\Omega) \cong \pi_{i+1}(M)$.

**Proposition 3.** *If $M$ contains no closed geodesic of length $\leq 1/2$, then $M$ has the homotopy type of a sphere. In particular, we have*

$$\pi_i(\Omega, M) \cong \begin{cases} Z & \text{if } i = d-1 \\ 0 & \text{for } 0 \leq i \leq d-2 \end{cases}$$

for the relative homotopy groups $\pi_i(\Omega, M)$ up to $i \leq d-1$.

We now return to the assumption that the length of the shortest closed geodesic on $M$ is 1. Let $\mathcal{C}$ and $\mathcal{C}^*$ be as defined in §3. We wish to prove that a strong unstable simplex at at least one $c \in \mathcal{C}$ represents a nontrivial class of $\pi_{d-1}(\Omega, M)$. Our technique will be to approximate $E$ with other functionals that are guaranteed to have nontrivial unstable simplexes. Although all our arguments carry through in all of $\Omega$ in an $S^1$-invariant fashon, essentially because the functional $E$ satisfies the Condition (C) of Palais and Smale and because an $S^1$-invariant formulation of Theorem E is available [GM2], we find it a little easier to work in a finite dimensional space.

More precisely, choose $r$ sufficiently large, say $r > 2$. Then, all closed geodesics not in $\Omega^r$ will have index $> 2(d-1)$. Let $'\Omega := {'\Omega_r}$. Then

$$\pi_i('\Omega, M) \cong \pi_i(\Omega, M)$$

for all $i$; $0 \leq i \leq 2d-3$, and $d-1 \leq 2d-3$ if $d \geq 2$. Using Theorem E and a partition of unity on $'\Omega$, we can approximate $E$ with a sequence $\{E_n\}_{n=1}^{\infty}$ of functionals on $'\Omega$ with the following properties.

(i) $\lim_{n\to\infty} E_n = E$ in the $C^2$ topology.
(ii) For some $\varepsilon > 0$, all critical points of $E_n$ in the closure of the set $L := {}'\Omega_{1+\varepsilon} - {}'\Omega_{1-\varepsilon}$ either belong in $\Omega_{=1}$ or have index $\geq 2d-2$, and outside $L$, each $E_n$ agree with $E$.
(iii) Each $E_n$ has only nondegenerate critical points in the set $L$, all of which have index $\geq d-1$.

Let $C$ be the set of all closed geodesics in $'\Omega^{=1}$ and let $C_n$ be the set of all critical points of $E_n$ that lie in $L$.

**Lemma 10.** *For each $n$, there exists in $C_n$, at least one critical point of $E_n$ that possesses a strong unstable simplex that represents a nontrivial element in $\pi_{d-1}('\Omega, M)$.*

*Proof.* From the topology described in Proposition 3, there must exist a nontrivial element $\rho$ of $\pi_{d-1}('\Omega, M)$. We first deform $\rho$ so that the only points of $C_n - (M \cap C_n)$ that lies on the image of $\rho$ are the relative maxima of $E_n \circ \rho$. In fact, since there are no critical poionts of index $< d-1$ except in $M$, at every critical point of $E_n$ lying on $\rho$, say $c$, other than relative maxima, the unstable dimension of $E_n$ in $'\Omega$ is strictly greater than the unstable dimension of $E_n \circ \rho$ in the image of $\rho$. Therefore, in some neighborhood of $c$ in which a chart of the form described in Theorem E is valid, we can deform $\rho$ in a direction transversal to itself and which decreases $E_n$. Since the critical points of $E_n$ are isolated and $\rho$ is contained in a compact region, by repeating this deformation a finite number of times and by deforming $\rho$ along the trajectory of $-\text{grad } E_n$, we can deform $\rho$ until it is expressed as a sum of disjoint simplexes, each summand of which is a simplex in $('\Omega, M)$, hanging from a single critical point of index $= d-1$. Such critical points must be in $C_n$, and at least one summand must be nontrivial itself. □

Of course, it is not necessarily true that a sequence of critical points $\{c_n\}$ of $C_n$ converges to a closed geodesic. However, that $\lim_{n\to\infty} C_n \subset C$ in the following weaker sense is clear.

**Lemma 11.** *Given any open neighborhood $\mathcal{U}$ of $C$ in $'\Omega$, whenever $n$ is large enough, $C_n \subset \mathcal{U}$.*

In fact, since the convergence is specified in the $C^2$ topology, we can state the even stronger

**Lemma 12.** *Let $\{\mathcal{U}_c^- \subset \mathcal{U}_c^{-0}\}_{c\in C}$ be a family of pairs of open sets in $'\Omega$ so that, for each $c \in C$, $\mathcal{U}_c^-$ is a neighborhood of the strong unstable submanifold $U_c^-$ of $E$ at $c$ and $\mathcal{U}_c^{-0}$ is a neighborhood of the unstable submanifold $U_c^{-0}$. Then, for $n$ sufficiently large, for each $c_n \in C_n$, there exists some $c \in C$, so that $U_{c_n}^-$, the strong unstable manifold of $E_n$ at $c_n$ is contained in $\mathcal{U}_c^{-0}$. Moreover, for such $c_n$ and $c$, a strong unstable simplex $\tau_n$ of $c_n$ contains a subsimplex $\tau_n'$ with $\dim \tau_n' = \dim U_c^- = \iota(c)$ which is actually contained in $\mathcal{U}_c^-$.*

To see the above, we can take a local coordinate expression around each $c \in C$ as described in Theorem E and look at the partial derivatives. By taking $n$ large, if $c_n \in C_n$ is close to $c \in C$, the corresponding second derivatives respectively of $E_n$ at $c_n$, $E$ at $c_n$ and $E$ at $c$ can all be made arbitratily close to each other by the property (i). But, in $U$, the strong unstable submanifolds and unstable submanufolds are determined by the second partial derivatives.

Now, for each $n$, let $c_n$ be the critical point in Lemma 11 which has a strong unstable simplex $\tau_n$ that is nontrivial in $\pi_{d-1}(\Omega, M)$. For such a $c_n$, $\tau_n \cap U$ must itself be contained in a neighborhood $\mathcal{U}_c^-$ of the strong unstable submanifold at some $c \in C$ by index comparison and the dimensional consideration. From the construction of $\tau_n$, this $c$ must be $\in \mathcal{C}$. Let $\tau$ be a strong unstable simplex at $c$ with $\tau(\partial I^{d-1}) \subset M$. By repeating the standard Morse theoretic arguments in the proof of Lemma 6, $\tau$ is seen to represent a nontrivial element in $\pi_{d-1}(\Omega, M)$. Hence $c \in \mathcal{C}^*$. Then, that $\mathcal{C}^*$ is closed follows from Proposition 2. This completes the proof of Lemma 1 and thus of Main Theorem. □

Figure 3: triangle with "cusps"

# Appendix

Yoe Itokawa

This is a reproduction of Appendix in Itokawa [I3] (1985), which was distributed but never published because the proof of the main theorem in [I3] contained a gap.

The purpose of this Appendix is to show

**Theorem (A).** *There exists a compact embedded surface in* $\boldsymbol{R}^3$ *all of whose shortest closed geodesics have self-intersections.*

This result is due independently and simultaneously to E. Calabi [Private communication], but his construction does not appear in print. We appreciate Calabi for encouraging us to publish our construction. Although our construction is essentially similar to his, it allows slightly more explicit computations. Both of these constructions are based on Calabi's example of a sequence of convex surfaces with closed geodesics with self-intersections whose length approaches the shortest lengths arbitrarily close.

*Proof.* Take three circles of unit radius in the $(x, y)$-plane arranged so that each is tangent to the other two. Let $S$ be the bounded component of the intersection of the exteriors of the three circles (Figure 3). For any $\delta$, let $M_\delta := \partial(S \times [0, \delta])$, where the second factor is taken in the $z$-axis. Let $\pi : M_\delta \longrightarrow S$ be the perpendicular projection. Now, while $M_\delta$ os of course not smooth, it has the structure of a Hölder continuous 2-manifold. If we interpret a geodesic on $M_\delta$ to be a Hölder continuous piecewise smooth curve that monimizes length on sufficiently short segments, it is easy to calculate that the only closed geodesics on $M_\delta$ without self-intersections are of the type $c_1$ whose image is the set $z$ = constant and the type $c_2$ such that $\pi(c_2)$ is the axis in $S$ of the symmetry that interchanges two of the three circles. Upon calculation, we obtain

$$L(c_1) = \pi \approx 3.1416$$

and

$$L(c_2) = 2(\sqrt{3} - 1) \approx 1.4641 + 2\delta.$$

On the other hand, consider the nonsimply closed curve which starts at the center of the three circular arcs of $S$, extending as a straight line segment perpendicular to an adjacent circular arc, climbing vertically up to $z = \delta$, returning straight to the point at $z = \delta$ with the same $(x, y)$ coordinates as the initial point, dropping vertically down to the initial point, and then repeating the same procedure on the morror image of the reflection that interchanges the two arcs not containing the initial point. The length of this curve is

$$4(\sqrt{5 - 2\sqrt{3}} - 1) + 4\delta \approx 0.9573 + 4\delta.$$

By following the negative gradient trajectory of the length functional in the loop space of $M_\delta$, whose direction near the present curve can be easily identified on $M_\delta$ itself, we reach a local minimum $c_3$ of $L$, which is nontrivial, has a self-intersection at $z = \frac{1}{2}\delta$, and whose image is invariant under the reflection that interchanges two of the circular arcs. For $\delta$ sufficiently small, $c_3$ has a shorter length than any of the closed geodesics without self-intersections. The non-simple closed curve $c_3$ is illustrated in Figure 4. Now we approximate $M_\delta$ with smooth surface. We can do this so that the lengths of geodesics on the smooth surface converge to the length of the limiting geodesics on $M_\delta$. Moreover, below a set length, no new closed geodesics are introduced. This is a consequence of the fact that $M_\delta$ is $C^\infty$ outside a set of lower dimension and Hölder continuous along the singularity. We omit the details, since the argument will be technical, while geometrically (or visually), the argument is clear. □

Figure 4: $c_1$, $c_2$ and $c_3$ in $M_5$

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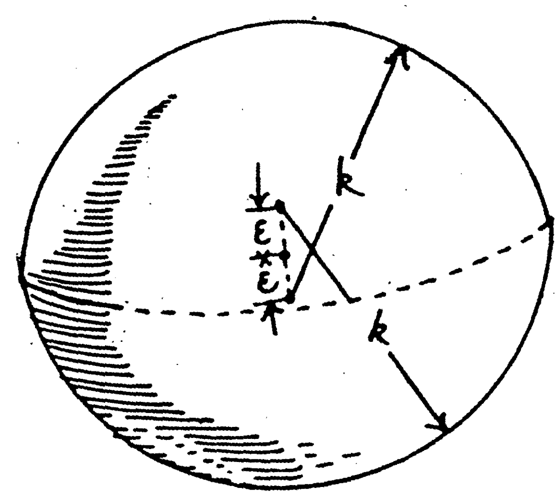


fig. 1

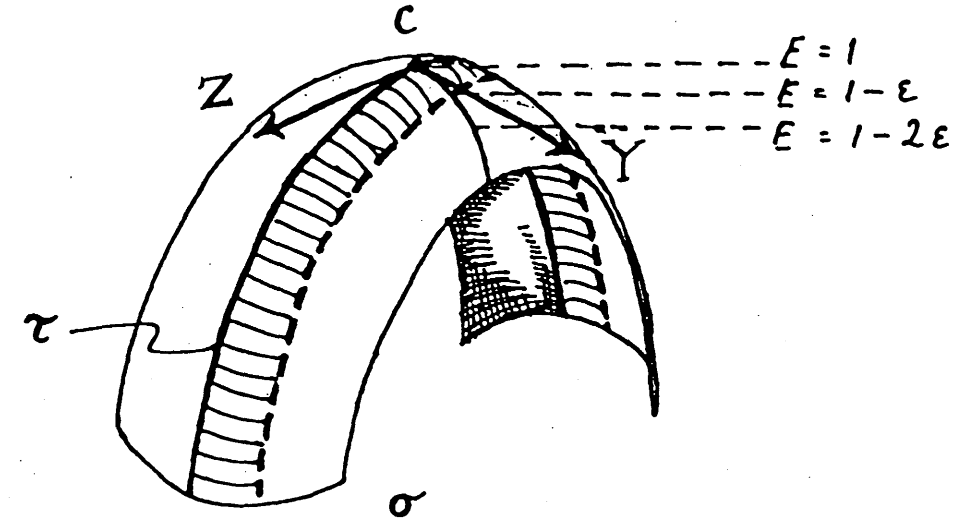


fig. 2

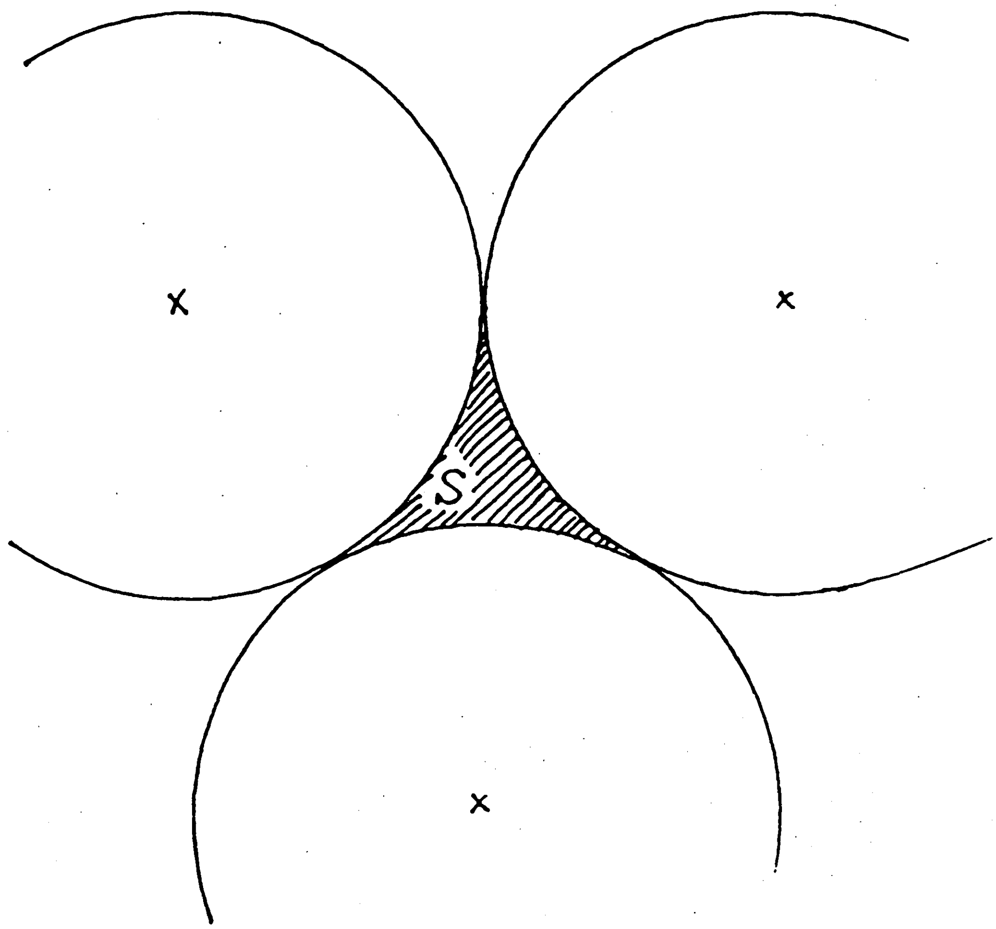


fig. 3

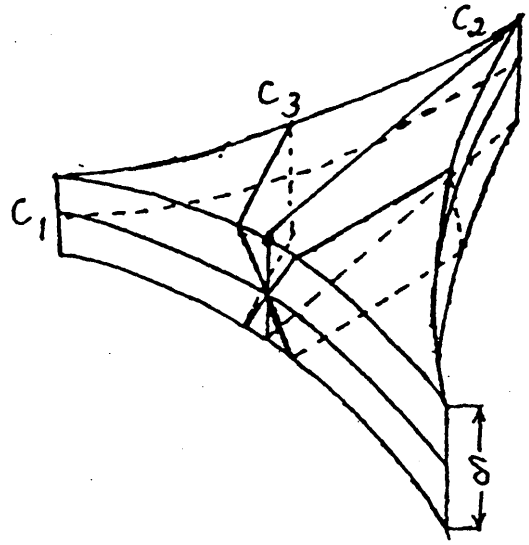


fig. 4