The Noncommutative Residue for Manifolds with Boundary

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Abstract

We construct a trace on the algebra of all classical elements in Boutet de Monvel's calculus on a compact manifold with boundary of dimension $n > 1$. This trace coincides with Wodzicki's noncommutative residue in case the boundary is empty. Moreover, we show that it is the unique continuous trace on this algebra up to a constant.

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Introduction

Let M be a closed compact manifold. Denote by Ψ^∞ the algebra of all classical pseudodifferential operators on M with integral order and by $\Psi^{-\infty}$ its ideal of all smoothing pseudodifferential operators. The noncommutative residue is a trace on the algebra $\mathcal{A} = \Psi^\infty/\Psi^{-\infty}$, i.e. a surjective linear map

$$\mathrm{res} : \mathcal{A} \to \mathbf{C},$$

which vanishes on all commutators: $\mathrm{res}\,(PQ - QP) = 0$ for all $P, Q \in \mathcal{A}$. In fact, this trace, res, turns out to be the unique trace on this algebra, up to a multiplicative constant, provided that the manifold is connected and of dimension higher than 1. It is given as an integral over a local density, res_x, on M.
For one-dimensional manifolds, the noncommutative residue was discovered by Manin [10] and Adler [1] in connection with geometric aspects of nonlinear partial differential equations. In this situation, the algebra $\mathcal{A}$ can be viewed as the algebra of formal Laurent series in the covariable $\xi \in \mathbf{R}$, and the local density res_x indeed takes the form of a classical residue: it is the coefficient of ξ^{-1}. For arbitrary closed compact n-dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [14] using the theory of zeta functions of elliptic pseudodifferential operators. Later, Wodzicki gave a more geometric account based on the theory of homogeneous forms on symplectic cones [15]. In this framework of symplectic cones, Guillemin had independently discovered the noncommutative residue as an important ingredient of his so-called 'soft' proof of Weyl's formula on the asymptotic distribution of eigenvalues [6].
Meanwhile, the noncommutative residue has found many applications in both mathematics and mathematical physics. A detailed introduction to the noncommutative residue

together with its mathematical consequences was given by Kassel in [8]. For applications in physics cf. e.g. Connes [4], Radul [11], Kravchenko and Khesin [9].
In the present paper, we introduce a noncommutative residue for the operators in Boutet de Monvel's algebra on manifolds with boundary. More precisely, let M be a compact connected manifold with boundary of dimension $n > 1$. Denote by $\mathcal{B}^\infty$ the algebra of all operators in Boutet de Monvel's calculus (with integral order) and by $\mathcal{B}^{-\infty}$ the ideal of the smoothing operators. We then construct a trace on the quotient $\mathcal{B} = \mathcal{B}^\infty/\mathcal{B}^{-\infty}$. Moreover, we show that this trace is the only continuous trace on $\mathcal{B}$. As in [15] we work directly at the symbol level. We avoid zeta function techniques, primarily, because the known results are not sufficiently good for our purposes. Computations of the Hochschild and cyclic homologies of $\mathcal{B}^\infty/\mathcal{B}^{-\infty}$ will be the subject of a forthcoming paper in the spirit of Brylinski-Getzler [3].
The paper is organized as follows. In Section 1 we give a simplified proof (see Theorem 1.4 and formula (1.8)) for the existence and uniqueness of the noncommutative residue on a compact manifold without boundary. Section 2 starts with a short review of Boutet de Monvel's algebra $\mathcal{B}$. We then consider two natural subalgebras of $\mathcal{B}$: the algebra $\mathcal{B}_0$ of all operators with vanishing interior pseudodifferential symbol, and the subalgebra $\mathcal{B}_s$ of all operators whose interior pseudodifferential symbol stabilizes in a neighborhood of the boundary. We define (see formula (2.13)) an analog of the noncommutative residue on $\mathcal{B}_0$ and show that it is the unique continuous trace on $\mathcal{B}_0$, cf. Proposition 2.3. We can extend this trace to $\mathcal{B}_s$; however, this extension will no longer be the only trace on $\mathcal{B}_s$. In Section 3 we finally treat the general case. We define the noncommutative residue on $\mathcal{B}$ (see formula (2.23)). We prove in Theorem 3.1 that this noncommutative residue is a trace on $\mathcal{B}$. In Theorem 3.2, we prove that it is the unique continuous trace up to a multiplicative constant. This is achieved by using the uniqueness property in $\mathcal{B}_0$ and by proving that there is no trace on $\mathcal{B}/\mathcal{B}_0$.

1 Wodzicki's Residue on a Closed Compact Manifold

In this section we recall the construction of the noncommutative residue for a closed compact manifold. We will need some lemmata on homogeneous functions on $\mathbf{R}^n \setminus 0$.
Let $\mathbf{R}^n$ be the standard oriented Euclidian space, $n > 1$, with coordinates $\xi_1, \xi_2, \ldots, \xi_n$. A smooth function $p(\xi)$ on $\mathbf{R}^n \setminus 0$ is homogeneous of degree $\lambda \in \mathbf{R}$ if for any $t > 0$

$$p(t\xi) = t^\lambda p(\xi). \tag{1.1}$$

Euler's theorem for homogeneous functions states that then

$$\sum_{j=1}^n \xi_j \frac{\partial p}{\partial \xi_j} = \lambda p; \tag{1.2}$$

this follows by differentiating (1.1) with respect to t and setting $t = 1$.
Consider the $n-1$-form

$$\sigma = \sum_{j=1}^n (-1)^{j+1} \xi_j d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n,$$

where the hat indicates that the corresponding factor has been omitted. Clearly, $d\sigma = n\, d\xi_1 \wedge d\xi_2 \wedge \ldots \wedge d\xi_n$. Restricted to the unit sphere $S = S^{n-1}$, σ gives the volume form on S^{n-1}.

Lemma 1.1 *For any function $p_{-n}(\xi)$ which is homogeneous of degree $-n$ the form $p_{-n}\sigma$ is closed.*

Proof. We have

$$\begin{aligned} d(p_{-n}\sigma) &= \sum_{j=1}^{n} \frac{\partial p_{-n}}{\partial \xi_j} d\xi_j \wedge \sigma + p_{-n} d\sigma \\ &= \sum_{j=1}^{n} \frac{\partial p_{-n}}{\partial \xi_j} \xi_j d\xi_1 \wedge d\xi_2 \wedge \ldots \wedge d\xi_n + n p_{-n} d\xi_1 \wedge \ldots \wedge d\xi_n = 0 \end{aligned}$$

by Euler's theorem. ♯

We will consider the integral

$$\int_S p_{-n}\sigma \tag{1.3}$$

over the unit sphere oriented by the outer normal. For a bounded domain $D \subset \mathbf{R}^n$ containing the origin,

$$\int_{\partial D} p_{-n}\sigma = \int_S p_{-n}\sigma \tag{1.4}$$

since the form $p_{-n}\sigma$ is closed. Here we suppose that ∂D is also oriented by the outer normal, otherwise we have to change the sign in (1.4).

Consider the behavior of (1.3) under a linear change of variables. Let g be a linear map, and let $\eta = g\xi$. Using (1.4) with the proper sign we get

$$\begin{aligned} \int_S p_{-n}(\eta)\sigma_\eta &= \pm \int_{gS} p_{-n}(\eta)\sigma_\eta = \pm \int_S g^*(p_{-n}(\eta)\sigma_\eta) \\ &= \pm \int_S p_{-n}(g\xi)(g^*\sigma)_{g\xi} = |\det g| \int_S p_{-n}(g\xi)\sigma_{g\xi}, \end{aligned} \tag{1.5}$$

since under the linear change we have

$$(g^*\sigma)_\xi = \det g\, \sigma_{g\xi}.$$

Equality (1.4) also holds for some unbounded domains D. We will need the case when D is the cylinder $\{\xi \in \mathbf{R}^n : |\xi'| < 1,\ \ \xi_n \in \mathbf{R}\}$. Here, $\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbf{R}^{n-1}$. Denoting by S' and σ' the $n-2$-dimensional unit sphere and the corresponding $n-2$-form we obtain from (1.4)

$$\int_S p_{-n}\sigma = \int_{S'} \left(\int_{-\infty}^{\infty} p_{-n}(\xi', \xi_n) d\xi_n \right) \sigma'. \tag{1.6}$$

The orientation of S' is completely defined by this equality.

Lemma 1.2 *Let p_{-n} be a derivative*

$$p_{-n} = \frac{\partial}{\partial \xi_k} p_{-(n-1)},$$

where $p_{-(n-1)}$ is a smooth homogeneous function on $\mathbf{R}^n \setminus 0$ of degree $-(n-1)$. Then

$$\int_S p_{-n}\sigma = 0.$$

Proof. Without loss of generality take $k = n$. Then by (1.6)

$$\int_S p_{-n}\sigma = \int_{S'} \left(\int_{-\infty}^{\infty} \frac{\partial p_{-(n-1)}}{\partial \xi_n} d\xi_n \right) \sigma' = 0,$$

since the inner integral is equal to

$$p_{-(n-1)}(\xi', \infty) - p_{-(n-1)}(\xi', -\infty)$$

and $p_{-(n-1)}$ vanishes at infinity. ♯

Lemma 1.2 raises the question as to whether a homogeneous function may be represented as a sum of derivatives.

Lemma 1.3 *Let p be a homogeneous function on $\mathbf{R}^n \setminus 0$. Each of the following conditions is sufficient for p to be a sum of derivatives:*

(i) $\deg p \neq -n$.

(ii) $\deg p = -n$ *and* $\int_S p\sigma = 0$.

(iii) $p = \xi^\alpha \partial^\beta q$ *where q is a homogeneous function and* $|\beta| > |\alpha|$.

Proof. (i) If $\deg p = \lambda \neq -n$, then

$$\sum_{j=1}^{n} \frac{\partial}{\partial \xi_j}(\xi_j p) = \sum_{j=1}^{n} \xi_j \frac{\partial p}{\partial \xi_j} + np = (n + \lambda)p$$

by Euler's theorem.
(ii) On the unit sphere S consider the equation

$$\Delta_S q = p|_S$$

where Δ_S is the Laplace-Beltrami operator, and $p|_S$ denotes the restriction of p to S. This equation has a solution since $p|_S$ is orthogonal to $\operatorname{Ker} \Delta_S$ which consists of the constants. Denoting $|\xi|$ by r and applying the Laplace operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial \xi_j^2} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_S$$

to the function $\frac{1}{r^{n-2}} q$ we obtain

$$\Delta \left(\frac{1}{r^{n-2}} q \right) = \frac{1}{r^n} \Delta_S q = \frac{1}{r^n}\, p|_S = p,$$

from which the second statement follows.
(iii) Let $\partial^\beta = \frac{\partial}{\partial \xi_j} \partial^\gamma$. Then

$$p = \xi^\alpha \frac{\partial}{\partial \xi_j} \partial^\gamma q = \frac{\partial}{\partial \xi_j} (\xi^\alpha \partial^\gamma q) - \frac{\partial \xi^\alpha}{\partial \xi_j} \partial^\gamma q,$$

and the case of multi-indices α, β is reduced to the case $\alpha - \{j\}$, γ with $|\gamma| = |\beta| - 1$. Hence, the third statement follows by induction. ♯

Now let M be a closed compact manifold of dimension $n > 1$. Consider the algebra $\mathcal{A} = \Psi^{\infty}/\Psi^{-\infty}$, where Ψ^{∞} denotes the algebra of all classical scalar-valued pseudodifferential operators on M and $\Psi^{-\infty}$ its ideal of smoothing elements. We assume all orders to be integers. Let U be an arbitrary local coordinate chart. An operator $P \in \Psi^{\infty}$ of order $m \in \mathbf{Z}$ on $U \subset M$ is defined up to a smoothing operator by its 'symbol'

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi); \tag{1.7}$$

this is an infinite formal sum of functions $p_k(x,\xi)$ on $U \times (\mathbf{R}^n \setminus 0)$, which are homogeneous in ξ of degree $k, k = m, m-1, \dots$. The present version of symbol differs from the usual notion of the complete symbol only by a term of order $-\infty$, but it is sufficiently precise for our purposes.

The form $dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$ defines an orientation of U and induces the orientation of $\mathbf{R}^n_\xi$ given by $d\xi_1 \wedge d\xi_2 \wedge \dots \wedge d\xi_n$. For $P \in \mathcal{A}$ with symbol p we define the local density $\operatorname{res}_x P, x \in M$, by

$$\operatorname{res}_x P = \left(\int_S p_{-n}(x,\xi)\sigma \right) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n. \tag{1.8}$$

The definition of the noncommutative residue and its properties are given in the following theorem.

Theorem 1.4 (Wodzicki) *Expression* (1.8) *is a density on M not depending on the local representation of the symbol, so that*

$$\operatorname{res} P = \int_M \operatorname{res}_x P \tag{1.9}$$

is well-defined.
For any $P, Q \in \mathcal{A}$

$$\operatorname{res}[P,Q] = 0, \tag{1.10}$$

hence the noncommutative residue is a trace on the algebra $\mathcal{A}$.
Any trace defined on the algebra $\mathcal{A}$ coincides with the trace res *up to multiplication by a constant.*

The case of pseudodifferential operators with values in sections of vector bundles over M is an easy consequence, cf. Remark 1, below.

Proof. Under a change of variables $x = f(y)$ the symbol $p(x,\xi)$ transforms to a symbol $\bar{p}$ according to the formula

$$\bar{p}(y, {}^t f'(y)\xi) \sim \sum_{|\alpha| \geq 0} \partial_\xi^\alpha p(f(y),\xi)\varphi_\alpha(y,\xi), \tag{1.11}$$

where $\varphi_\alpha(y,\xi)$ are polynomials in ξ of degree $\leq |\alpha|/2$ and $\varphi_0 = 1$ (see Hörmander [7], formula (18.1.30)). Using (1.5) and (1.11) we get

$$\begin{aligned} \int_S \bar{p}_{-n}(y,\eta)\sigma_\eta &= |\det f'(y)| \int_S \bar{p}_{-n}(y, {}^t f'(y)\xi)\sigma_\xi \qquad (1.12) \\ &= |\det f'(y)| \sum_{|\alpha| \geq 0} \int_S (\partial_\xi^\alpha p(f(y),\xi)\varphi_\alpha(y,\xi))_{-n}\sigma_\xi \\ &= |\det f'(y)| \int_S p_{-n}(f(y),\xi)\sigma_\xi \end{aligned}$$

since the terms with $|\alpha| > 0$ do not contribute to the integral in virtue of Lemma 1.3(iii). The transformation law (1.12) shows that expression (1.8) is indeed a density on M, so that (1.9) is well-defined.
We may proceed considering the operators whose symbols have supports in a fixed coordinate chart. The general case may be reduced to this special one using a partition of unity since the density $\mathrm{res}_x P$ does not depend on the choice of local coordinates.
To prove (1.10), consider two operators P, Q with symbols p and q supported in a coordinate chart U. Without loss of generality, we shall assume that U is diffeomorphic to an open ball of $\mathbf{R}^n$. The symbol of $[P, Q]$ is given by

$$\sum_{|\alpha|\geq 0} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha p\, \partial_x^\alpha q - \partial_\xi^\alpha q\, \partial_x^\alpha p). \tag{1.13}$$

This expression may be represented as a sum of derivatives

$$\sum_{j=1}^{n} \frac{\partial}{\partial \xi_j} A_j + \frac{\partial}{\partial x_j} B_j \tag{1.14}$$

where A_j and B_j are bilinear expressions in p and q and their derivatives. In particular, they have compact supports contained in U. Thus, the integrals over S of $\left(\frac{\partial}{\partial \xi_j} A_j\right)_{-n}$ vanish by Lemma 1.2, while the integrals of $\left(\frac{\partial}{\partial x_j} B_j\right)_{-n}$ over U vanish, since all B_j have compact support in U. This proves (1.10).
We will need the explicit expressions of A_j and B_j for the terms in (1.13) with $|\alpha| = 1$, that is

$$-i\sum_{k=1}^{n} \frac{\partial p}{\partial \xi_k}\frac{\partial q}{\partial x_k} - \frac{\partial q}{\partial \xi_k}\frac{\partial p}{\partial x_k} = -i\sum_{k=1}^{n} \frac{\partial}{\partial \xi_k}\left(p\frac{\partial q}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(p\frac{\partial q}{\partial \xi_k}\right). \tag{1.15}$$

Finally, to prove uniqueness, consider an operator P with symbol p supported in a coordinate chart U and let $\widehat{x}_j$ and $\widehat{\xi}_j$ denote any symbols with supports in U coinciding with x_j and ξ_j on the support of p. Then, taking $q = \widehat{x}_j$ or $q = \widehat{\xi}_j$ in (1.15), we obtain

$$[p, \widehat{x}_j] = -i\frac{\partial p}{\partial \xi_j}; \quad [p, \widehat{\xi}_j] = i\frac{\partial p}{\partial x_j}. \tag{1.16}$$

Given a trace τ on the whole algebra of complete symbols the equalities (1.16) imply that

$$\tau\left(\frac{\partial p}{\partial \xi_j}\right) = \tau\left(\frac{\partial p}{\partial x_j}\right) = 0 \tag{1.17}$$

since the trace must vanish on commutators. Let $p \sim \sum_{k\leq m} p_k \in \Psi^\infty/\Psi^{-\infty}$ and define $\overline{p_{-n}}(x) = \frac{1}{\mathrm{vol}\, S}\int_S p_{-n}(x,\xi)\sigma_\xi$. Applying Lemma 1.3 (i) to p_k for all $k \neq -n$, there exist n functions $q_k^{(j)}(x,\xi)$, $1 \leq j \leq n$, homogeneous of degree $k+1$ in ξ such that $p_k = \sum_{1\leq j\leq n} \partial_{\xi_j} q_k^{(j)}$. Define, for all $1 \leq j \leq n$, $b_j(x,\xi) \sim \sum_{k\leq m,\, k\neq -n} q_k^{(j)}$. One has

$$p(x,\xi) - \overline{p_{-n}}(x)|\xi|^{-n} = \sum_{j=1}^{n} \partial_{\xi_j} b_j(x,\xi) + \left(p_{-n}(x,\xi) - \overline{p_{-n}}(x)|\xi|^{-n}\right).$$

Since

$$\int_S (p_{-n}(x,\xi) - \overline{p_{-n}}(x)|\xi|^{-n})\sigma_\xi = 0$$

Lemma 1.3 (ii) shows that the expression $(p_{-n}(x,\xi) - \overline{p_{-n}}(x)|\xi|^{-n})$ is a (finite) sum of derivatives in the variable ξ. Putting this together, it follows that

$$\tau(p) = \tau(\overline{p_{-n}}(x)|\xi|^{-n}).$$

Now, the map $C_0^\infty(U) \ni f \mapsto \mu(f) = \tau(f|\xi|^{-n})$ defines a $\mathbf{C}$-linear form on $C_0^\infty(U)$; it follows from (1.17) above that $\mu(\partial_{x_j} f) = 0$ for all $1 \le j \le n$ and $f \in C_0^\infty(U)$. Hence, since U is diffeomorphic to an open ball of $\mathbf{R}^n$, there exists $c \in \mathbf{C}$ such that $\mu(f) = c\int_U f(x)dx$ for all $f \in C_0^\infty(U)$. ♯

Remark 1. The theorem remains valid for pseudodifferential operators acting on sections of vector bundles over M, if we replace p_{-n} by the matrix trace $\operatorname{Tr} p_{-n}$. Thus, in the general case, the definition of the noncommutative residue will read

$$\operatorname{res} P = \int_M \int_S \operatorname{Tr} p_{-n}(x,\xi)\sigma_\xi \;\; dx_1 \wedge \ldots \wedge dx_n. \tag{1.18}$$

Remark 2. As may be seen from the proof, no continuity condition is required for the uniqueness of the noncommutative residue.

2 Boutet de Monvel's Algebra

Let M be a compact manifold with boundary ∂M and dimension $\dim M = n > 1$. In a neighborhood of the boundary we consider local coordinates x', x_n where $x' = (x_1, \ldots, x_{n-1})$ are coordinates on ∂M and x_n is the geodesic distance to ∂M in some Riemannian metric. So, any boundary coordinate chart U is diffeomorphic to the closed half-space $\overline{\mathbf{R}^n_+} = \{x_n \ge 0\}$, and transition diffeomorphisms change $x' \in \mathbf{R}^{n-1} = \partial\overline{\mathbf{R}^n_+}$ only, while x_n remains unchanged.
For a detailed introduction to Boutet de Monvel's algebra see Boutet de Monvel [2], Grubb [5], Rempel-Schulze [12] or Schrohe-Schulze [13]. In the following we will give a review of some basic facts we need.
By $\mathcal{B}^\infty$ let us denote the algebra of all operators in Boutet de Monvel's calculus of arbitrary (integer) order and type; by $\mathcal{B}^{-\infty}$ denote the ideal of all regularizing elements of arbitrary type in $\mathcal{B}^\infty$. We will be interested in the algebra $\mathcal{B} = \mathcal{B}^\infty/\mathcal{B}^{-\infty}$. This quotient can be viewed as the set of all pairs $\{p_i, p_b\}$, where p_i and p_b are called the *interior* and *boundary* symbol, respectively.

In order to introduce the interior and boundary symbols we need some preparations. By θ denote the characteristic function on the half-line $\mathbf{R}^*_+$ in $\mathbf{R}$: $\theta(t) = 0$ for $t \le 0$, $\theta(t) = 1$ for $t > 0$.

H^+ is the space of all Fourier transforms of functions of the form $\theta u, u \in \mathcal{S}(\mathbf{R})$. It consists precisely of all functions $h \in C^\infty(\mathbf{R})$ which have an analytic extension to the lower complex half-plane $\{\operatorname{Im}\zeta < 0\}$ and an asymptotic expansion

$$h(\zeta) \sim \sum_{k=1}^{\infty} \frac{c_k}{\zeta^k} \tag{2.1}$$

as $|\zeta| \to \infty$, $\operatorname{Im}\zeta \leq 0$, that can be formally differentiated.
Similarly, H_0^- is the space of all Fourier transforms of functions of the form $(1-\theta)u$, with $u \in \mathcal{S}(\mathbf{R})$. It can be characterized as the space of all functions in $C^\infty(\mathbf{R})$ that have an analytic extension into the upper half-plane $\{\operatorname{Im}\zeta > 0\}$ and an asymptotic expansion (2.1) as $|\zeta| \to \infty$, $\operatorname{Im}\zeta \geq 0$, that can be formally differentiated, cf. [12], Section 2.1.1.1.
Finally, H' is the space of all polynomials. We let $H^- = H_0^- \oplus H'$ and $H = H^+ \oplus H^-$. By Π^+ (resp. Π^-) we denote the projections onto $H^\pm$ (resp. H^-) parallel to H^- (resp. H^+).
For calculations it is convenient to think of H as a space of rational functions having no poles on the real axis (these functions form a dense set in the topology of H). On these functions, the projectors $\Pi^\pm$ may be represented by Cauchy integrals

$$(\Pi^\pm h)(\xi_n) = \mp\frac{1}{2\pi i}\int_{\Gamma^+} \frac{h(\eta_n)}{\eta_n - \xi_n \pm i0} d\eta_n,$$

where Γ^+ is a contour consisting of a segment of the real axis and of a half-circle surrounding all the singularities of h in the upper half-plane. Introduce also the functional Π' on H, defined by

$$\Pi' h = \lim_{x_n \to 0^+} (\mathcal{F}^{-1}h)(x_n) = \frac{1}{2\pi}\int_{\Gamma^+} h(\xi_n)d\xi_n.$$

For rational functions the contour Γ^+ may be shifted slightly into the upper (lower) half-plane. Clearly, Π' vanishes on the subspace H^-. For functions $h \in H \cap L^1(\mathbf{R})$ the integral may be taken over the real axis instead of Γ^+. Moreover, if $h \in H^+ \cap L^1(\mathbf{R})$, then $\Pi' h = 0$, because h is holomorphic in the lower half-plane with an estimate $|h(\xi_n)| = O\left(\frac{1}{|\xi_n|^2}\right)$.
Let us now focus first on the interior symbols. The interior symbols are classical pseudodifferential symbols in the sense of Section 1, cf. (1.7), except that the coordinate x varies in $\overline{\mathbf{R}^n_+}$ for boundary charts. Moreover, they have the *transmission property*. For a classical pseudodifferential symbol p with an asymptotic expansion $p \sim \sum p_l$ into homogeneous terms p_l of degree l this means that in every boundary chart we have

$$D^\alpha_{\xi'} D^k_{x_n} p_l(x', 0, 0, +1) = e^{i\pi(l-|\alpha|)} D^\alpha_{\xi'} D^k_{x_n} p_l(x', 0, 0, -1)$$

for every multi-index α and all $k \in \mathbf{N}$, cf. [12], Section 2.2.2.3. In particular, we will then have $D^\alpha_{\xi'} D^k_{x_n} p_l(x', 0, \xi', \xi_n) \in H$ as a function of ξ_n for all fixed $(x', \xi') \in T^*\partial M \setminus 0$.

We make the following observations.
In a boundary chart the composition formula for interior symbols on M takes the form

$$p_1(x', x_n, \xi', \xi_n) \circ p_2(x', x_n, \xi', \xi_n) = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \partial^k_{\xi_n} p_1 \circ' \partial^k_{x_n} p_2, \tag{2.2}$$

where $\circ'$ means the composition of symbols on ∂M and x_n as well as ξ_n are regarded as parameters, that is

$$p_1 \circ' p_2 = \sum_{|\alpha|\geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi'} p_1 \partial^\alpha_{x'} p_2. \tag{2.3}$$

Using representation (2.2) we obtain the following formula for a commutator in a boundary chart.

Lemma 2.1

$$[p_1, p_2] = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} [\partial^k_{x_n} p_1, \partial^k_{\xi_n} p_2]' + \frac{\partial}{\partial \xi_n} A_n + \frac{\partial}{\partial x_n} B_n. \tag{2.4}$$

Here $[\cdot,\cdot]'$ *denotes the commutator with respect to* $\circ'$,

$$A_n = \sum_{m,j\geq 0} \frac{(-i)^{m+j+1}}{(m+j+1)!} \partial^j_{\xi_n} \partial^m_{x_n} p_1 \circ' \partial^m_{\xi_n} \partial^{j+1}_{x_n} p_2 \tag{2.5}$$

and

$$B_n = -\sum_{m,j\geq 0} \frac{(-i)^{m+j+1}}{(m+j+1)!} \partial^j_{\xi_n} \partial^m_{x_n} p_1 \circ' \partial^{m+1}_{\xi_n} \partial^j_{x_n} p_2. \tag{2.6}$$

Proof. Straightforward calculation. ♯

In particular, the terms in (2.5) and (2.6) with $j = m = 0$ give (1.15).

Next we describe the boundary symbols. The boundary symbol is a family of operators parametrized by $T^*\partial M \setminus 0$. In a local chart on ∂M it has the form of a 2×2 matrix

$$\begin{pmatrix} b(x',\xi',D_n) & k(x',\xi',D_n) \\ t(x',\xi',D_n) & q(x',\xi') \end{pmatrix}, \quad (x',\xi') \in T^*\partial M \setminus 0. \tag{2.7}$$

It acts on pairs of the form $\begin{pmatrix} h \\ v \end{pmatrix}$, where $h \in H^+$ is in general vector-valued, and v is a vector in $\mathbf{C}^k$. The entries of the above matrix (2.7) are operators given again by symbols $b = b(x',\xi',\xi_n,\eta_n), k = k(x',\xi',\xi_n), t = t(x',\xi',\xi_n)$, and $q = q(x',\xi')$, respectively. First of all, these symbols are formal sums of jointly homogeneous smooth functions with respect to all the variables except for x'. So,

$$\begin{aligned} b(x',\xi',\xi_n,\eta_n) &\sim \sum_{-\infty<l\leq m} b_l(x',\xi',\xi_n,\eta_n), \\ k(x',\xi',\xi_n) &\sim \sum_{-\infty<l\leq m} k_l(x',\xi',\xi_n), \\ t(x',\xi',\xi_n) &\sim \sum_{-\infty<l\leq m} t_l(x',\xi',\xi_n), \\ q(x',\xi') &\sim \sum_{-\infty<l\leq m} q_l(x',\xi'), \end{aligned}$$

where, for $\lambda > 0$,

$$\begin{aligned} b_l(x', \lambda\xi', \lambda\xi_n, \lambda\eta_n) &= \lambda^l b_l(x', \xi', \xi_n, \eta_n), \\ k_l(x', \lambda\xi', \lambda\xi_n) &= \lambda^l k_l(x', \xi', \xi_n), \\ t_l(x', \lambda\xi', \lambda\xi_n) &= \lambda^l t_l(x', \xi', \xi_n), \\ q_l(x', \lambda\xi') &= \lambda^l q_l(x', \xi'). \end{aligned}$$

Under a change of variables on ∂M they obey the same rule (1.11) as symbols on ∂M, that is with respect to the variables x', ξ', the extra variables ξ_n, η_n can be considered as parameters. In order to state the additional properties of the symbols and the way they act we consider them separately.

(1) The symbol b is called the *singular Green symbol.* For every l and fixed x', ξ'

$$b_l(x', \xi', \xi_n, \eta_n) \in H^+ \hat{\otimes}_\pi H^-$$

(where as usual $\hat{\otimes}_\pi$ denotes Grothendieck's completion of the algebraic tensor product). The operator $b(x', \xi', D_n) : H^+ \to H^+$ is given by

$$[b(x', \xi', D_n)h](\xi_n) = \Pi'_{\eta_n}(b(x', \xi', \xi_n, \eta_n)h(\eta_n)).$$

Singular Green symbols on ∂M form an algebra under composition; this algebra is denoted by $\mathcal{G}$.

(2) For fixed x', ξ', each component $k_l(x', \xi', \xi_n)$ of the *potential symbol* $k(x', \xi', \xi_n)$ belongs to $H^+ \otimes (\mathbf{C}^k)^*$ with respect to ξ_n. The operator $k(x', \xi', D_n) : \mathbf{C}^k \to H^+$ acts on $v \in \mathbf{C}^k$ by multiplication $v \mapsto k(x', \xi', \xi_n)v \in H^+$.

(3) For fixed x', ξ', each component $t_l(x', \xi', \xi_n)$ of the *trace symbol* $t(x', \xi', \xi_n)$ belongs to $H^- \otimes \mathbf{C}^k$ with respect to ξ_n. The operator $t(x', \xi', D_n) : H^+ \to \mathbf{C}^k$ acts by

$$t(x', \xi', D_n)h = \Pi'(t(x', \xi', \xi_n)h(\xi_n)).$$

(4) The symbol $q = q(x', \xi')$ is simply a classical pseudo-differential symbol on ∂M in the sense of (1.7) with values in $\mathcal{L}(\mathbf{C}^k)$; $q(x', \xi')$ and $q_l(x', \xi')$ act by matrix multiplication on $\mathbf{C}^k$.

Given two operators in $\mathcal{B}$ with symbols (p_{i1}, p_{b1}) and (p_{i2}, p_{b2}) the composition is again an operator in $\mathcal{B}$. It has the symbol (p_i, p_b), where $p_i = p_{i1} \circ p_{i2}$ simply is the composition of the pseudodifferential symbols in the sense of (2.2); it again satisfies the transmission condition. The resulting boundary symbol has the form

$$p_{b1} \circ' p_{b2} + \begin{pmatrix} L(p_{i1}, p_{i2}) + p_{i1}^+(D_n)b_2(D_n) + b_1(D_n)p_{i2}^+(D_n) & p_{i1}^+(D_n)k_2(D_n) \\ t_1(D_n)p_{i2}^+(D_n) & 0 \end{pmatrix}. \tag{2.8}$$

Here, $p_{b1} \circ' p_{b2}$ is the pseudodifferential composition of p_{b1} and p_{b2} with respect to the variables (x', ξ'), cf. (2.3), together with composition of the operator-valued matrices (2.7). The terms in the second summand come from the interior symbols. There, the composition is the pseudo-differential composition for operator-valued symbols with respect to (x', ξ'), cf. (2.3). We have denoted the entries of $p_{bj}, j = 1, 2$ by b_j, k_j, and t_j and omitted the variables (x', ξ') for better legibility.

(i) $L(p_{i1}, p_{i2})$ is the so-called *leftover term.* It is induced by the particular way the action of a pseudodifferential operator P on the manifold ***with boundary*** M is defined: We assume that M is embedded in a manifold without boundary and that P extends to it. Given a function or distribution on M of sufficiently high regularity we first extend it by zero to the full manifold, then apply P and finally restrict to M, in other words we apply the operator $P_M = r_M P e_M$; here e_M denotes extension by zero and r_M restriction to M. Given two pseudodifferential operators P_1 and P_2 with interior symbols p_{i1} and p_{i2} respectively, the difference $[P_1]_M[P_2]_M - [P_1P_2]_M$ turns out to be a singular Green operator with an associated singular Green boundary symbol operator $L(p_{i1}, p_{i2})$. The asymptotic expansion of the associated singular Green symbol can be computed from the knowledge of p_{i1}, p_{i2}, and their derivatives at the boundary. Obviously it is zero if either p_{i1} or p_{i2} is zero.

(ii) Given an interior symbol p_i, the operator $p_i^+(x', \xi', D_n) : H^+ \to H^+$ is induced from the action of the interior symbol in the normal direction for fixed (x', ξ'). More precisely, understanding p_i as a full classical symbol rather than the associated formal sum, one lets

$$p_i^+(x', \xi', D_n)h = \mathcal{F}(\theta u), \quad \text{where} \quad u(x_n) = \frac{1}{2\pi}\int e^{ix_n\xi_n} p_i(x', x_n, \xi', \xi_n)h(\xi_n)d\xi_n.$$

The last integral should be understood as an oscillatory integral. It is a consequence of the transmission property that $u|_{\mathbf{R}_+} \in \mathcal{S}(\mathbf{R}_+)$, so that $\mathcal{F}(\theta u) \in H^+$. This, however, is of minor importance here. We shall mainly be interested in the (nontrivial) fact that for a singular Green boundary symbol operator $b(x', \xi', D_n)$ both compositions

$$p_i^+(x', \xi', D_n)b(x', \xi', D_n) \quad \text{and} \quad b(x', \xi', D_n)p_i^+(x', \xi', D_n)$$

are singular Green boundary symbol operators and that the asymptotic expansion of the corresponding symbols can be computed from the knowledge of p_i and its derivatives at the boundary.

More information will be given later when we need it.
It should be pointed out that the current terminology is not quite the standard terminology. In general, the boundary symbol operator also contains the term $p_i^+(x', \xi', D_n)$ from the interior symbol p_i. Noting that the only operator which is induced by both a classical pseudodifferential symbol and a classical singular Green symbol is 0, both parts can be neatly separated in this context. So it makes the presentation easier to set up things the way we did.

Now we give the following definition:

Definition 2.2 (a) *$\mathcal{B}_0$ is the subalgebra of $\mathcal{B}$ consisting of all elements with zero interior symbol.*
(b) *$\mathcal{B}_s$ is the subalgebra of $\mathcal{B}$ where the interior symbol stabilizes near the boundary: $p_i(x', x_n, \xi) = p_i(x', 0, \xi)$ for small x_n.*

Note. The following facts are well-known but important:

(i) the identity is not a singular Green operator; in particular, $\mathcal{B}_0$ is a nonunital algebra;

(ii) the operator induced by an interior symbol p_i^+ is not a singular Green operator unless it is zero; however composing p_i^+ with a singular Green operator on the right or on the left yields a singular Green operator.

Clearly $\mathcal{B}_0 \subseteq \mathcal{B}_s \subseteq \mathcal{B}$, and $\mathcal{B}_0$ is an ideal in $\mathcal{B}$, since both the resulting pseudodifferential part and the leftover term in any composition will be zero.

For fixed x', ξ', and a singular Green boundary symbol operator

$$b(x', \xi', D_n) : H^+ \to H^+$$

acting by

$$[b(x', \xi', D_n)h](\xi_n) = \Pi'_{\eta_n}(b(x', \xi', \xi_n, \eta_n)h(\eta_n)) = \frac{1}{2\pi}\int_{\Gamma+} b(x', \xi', \xi_n, \eta_n)h(\eta_n)d\eta_n$$

we define a trace similarly to the trace of usual integral operators

$$\operatorname{tr} b(x', \xi', D_n) = \Pi'_{\xi_n} b(x', \xi', \xi_n, \xi_n) = \frac{1}{2\pi}\int_{\Gamma+} b(x', \xi', \xi_n, \xi_n)d\xi_n. \tag{2.9}$$

In case $b(x', \xi', D_n)$ is matrix-valued we additionally take the matrix trace under the integral. We clearly have the trace property:

$$\operatorname{tr}(b_1(x', \xi', D_n)b_2(x', \xi', D_n)) = \operatorname{tr}(b_2(x', \xi', D_n)b_1(x', \xi', D_n)).$$

As indicated by the missing $\circ'$, the composition is with respect to the x_n-action D_n only. Moreover, taking the trace of a singular Green boundary symbol operator $b(x', \xi', D_n)$ yields a symbol $\bar{b}(x', \xi')$ on ∂M:

$$\bar{b}(x', \xi') = \operatorname{tr} b(x', \xi', D_n) = \Pi'_{\xi_n} b(x', \xi', \xi_n, \xi_n) = \frac{1}{2\pi}\int_{\Gamma+} b(x', \xi', \xi_n, \xi_n)d\xi_n \tag{2.10}$$

is a sum of homogeneous components; the component $\bar{b}_k(x', \xi')$ of degree k is obtained from $b_{k-1}(x', \xi', \xi_n, \eta_n)$. Indeed,

$$\begin{aligned}
\bar{b}_k(x', t\xi') &= \frac{1}{2\pi}\int_{\Gamma+} b_{k-1}(x', t\xi', \xi_n, \xi_n)d\xi_n & (2.11)\\
&= \frac{t}{2\pi}\int_{\Gamma+} b_{k-1}(x', t\xi', t\eta_n, t\eta_n)d\eta_n & (2.12)\\
&= \frac{t^k}{2\pi}\int_{\Gamma+} b_{k-1}(x', \xi', \eta_n, \eta_n)d\eta_n\\
&= t^k\bar{b}_k(x', \xi').
\end{aligned}$$

Since the change of variables acts on x', ξ', and does not affect the variables ξ_n, η_n, of the Green symbol $b(x', \xi', \xi_n, \eta_n)$, $\bar{b}(x', \xi')$ is indeed a symbol on ∂M

Proposition 2.3 *Assume that ∂M is connected. For the boundary symbol we use the notation of* (2.7). *Then the functional*

$$\operatorname{res}_{\partial M} p_b = \int_{\partial M} \operatorname{res}_{x'}\{\operatorname{tr} b(x', \xi', D_n) + \operatorname{Tr} q(x', \xi')\} = \int_{\partial M} \operatorname{res}_{x'}(\bar{b} + \operatorname{Tr} q) \tag{2.13}$$

is the unique continuous trace functional on the subalgebra $\mathcal{B}_0 \subset \mathcal{B}$ up to multiplication by a constant.
As before, Tr *denotes the matrix trace on $\mathcal{L}(\mathbb{C}^k)$.*

In the proof of Proposition 2.3, we shall use the following simple argument that we have isolated below in the following

Lemma 2.4 *Let $\tau : \mathcal{B}_0 \to \mathbf{C}$ be a* **C***-linear form. Then τ is a trace on $\mathcal{B}_0$ if and only if there exists a trace τ_1 on $\mathcal{G}$ (the algebra of singular Green symbols) and a trace τ_2 on $\Psi^\infty(\partial M)/\Psi^{-\infty}(\partial M)$ such that for all trace operators t and all potential operators k*

$$\tau_1(kt) = \tau_2(tk) \tag{2.14}$$

and

$$\tau\left\{\begin{pmatrix} b & k \\ t & q \end{pmatrix}\right\} = \tau_1(b) + \tau_2(q)\,. \tag{2.15}$$

Sketch of proof. Let $\tilde{\mathcal{B}}_0$ be the algebra consisting of matrices of the form

$$\begin{pmatrix} b \oplus cI & k \\ t & q \end{pmatrix},\ c \in \mathbf{C},\quad \begin{pmatrix} b & k \\ t & q \end{pmatrix} \in \mathcal{B}_0\,, \tag{2.16}$$

with the multiplication formula

$$(b \oplus cI)(b' \oplus c'I) = (bb' + cb' + c'b) \oplus cc'I\,. \tag{2.17}$$

$\tilde{\mathcal{B}}_0$ is a unital algebra by construction. Let τ be a trace on $\mathcal{B}_0$; then the C-linear form on $\tilde{\mathcal{B}}_0$ defined by

$$\tilde{\tau}\left\{\begin{pmatrix} b \oplus cI & k \\ t & q \end{pmatrix}\right\} = \tau\left\{\begin{pmatrix} b & k \\ t & q - cI \end{pmatrix}\right\} \tag{2.18}$$

is a trace on $\tilde{\mathcal{B}}_0$. Using the equalities

$$\left[\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \tag{2.19}$$

and

$$\left[\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \tag{2.20}$$

the lemma easily follows for $(\tilde{\mathcal{B}}_0, \tilde{\tau})$ and, according to (2.18), for $(\mathcal{B}_0, \tau)$. ♯

Proof of Proposition 2.3. Correctness and the trace property (1.10) follow from Theorem 1.4 applied to $\mathcal{B}_0$ considered as an operator-valued symbol algebra on ∂M whose coefficient trace is

$$\operatorname{tr} b(x', \xi', D_n) + \operatorname{Tr} q(x', \xi'), \quad (x', \xi') \text{ fixed.} \tag{2.21}$$

Conversely, let τ be a trace on $\mathcal{B}_0$. Applying Lemma 2.4, there exist traces τ_1 and τ_2 such that

$$\tau_1(b) = \tau\left\{\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right\},\quad \tau_2(q) = \tau\left\{\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}\right\}. \tag{2.22}$$

The functional τ_2 is a trace on $\Psi^\infty(\partial M)/\Psi^{-\infty}(\partial M)$; hence, Wodzicki's result shows that there exists a constant $c \in \mathbf{C}$ such that $\tau_2 = c\,\mathrm{res}$. Since by assumption τ is a trace on $\mathcal{B}_0$, (2.14) holds and shows that, for all trace symbols t and potential symbols k, $\tau_1(kt) = c\,\mathrm{res}\,(tk)$. Hence the functional $\tau - c\,\mathrm{res}_{\partial M}$ vanishes on all elements of the form

$$\begin{pmatrix} \sum_{i=1}^m k_i t_i & k \\ t & q \end{pmatrix} \tag{2.23}$$

for all potential symbols k_i, k, all trace symbols t_i, t and all pseudo-differential symbol q. Since the set of such elements is dense in $\mathcal{B}_0$ (see properties (1)-(2)-(3) after Lemma 2.1) and since the trace τ is continuous, one has $\tau = c\,\mathrm{res}_{\partial M}$. ♯

Remark. In case ∂M consists of finitely many components, the preceding proof shows that we may pick a constant factor for each component.

Next we consider the subalgebra $\mathcal{B}_s \subseteq \mathcal{B}$.
We use the notation of (2.7) and set for a pair $P = \{p_i(x,\xi), p_b(x',\xi')\} \in \mathcal{B}_s$

$$\mathrm{res}\, P = \int_M \mathrm{res}_x p_i(x,\xi) - 2\pi \int_{\partial M} \mathrm{res}_{x'}(\mathrm{tr}\, b(x',\xi',D_n) + \mathrm{Tr}\, q(x',\xi')). \tag{2.24}$$

Theorem 2.5 *For any $P, Q \in \mathcal{B}_s$*

$$\mathrm{res}\,[P,Q] = 0.$$

Proof. Note that, in general, neither of the two terms in (2.24) vanishes on commutators. First let us compute the contribution of the interior symbols. In view of Wodzicki's theorem it is sufficient to prove Theorem 2.5 for symbols supported on a boundary chart. Using Lemma 2.1 and formula (1.6) we obtain after a straightforward computation

$$\int_M \mathrm{res}_x[p_i,q_i] = -\int_{\partial M}\int_{S'}\left(\int_{-\infty}^{\infty} (B_n)_{-n}\, d\xi_n\right)\sigma' dx_1 \wedge \ldots \wedge dx_{n-1}.$$

where, as in Section 1, S' is the $(n-1)$-sphere with volume form σ'. Since the symbols p_i, q_i do not depend on x_n the only nonvanishing term of (2.6) is that with $m = j = 0$, so

$$\int_M \mathrm{res}_x[p_i,q_i] = -i\int_{\partial M}\left(\int_{-\infty}^{\infty} d\xi_n \int_{S'}\left(p_i \circ' \frac{\partial q_i}{\partial \xi_n}\right)_{-n}\Big|_{x_n=0}\sigma'\right) dx_1 \wedge \ldots \wedge dx_{n-1}. \tag{2.25}$$

Now for the boundary symbol part. Denote the entries of the matrix p_b by b_1, k_1, t_1, q_1, those of q_b by b_2, k_2, t_2, q_2. Considering the representation (2.8) for the composition of the boundary symbol operators and the fact that res is a trace on $\mathcal{B}_0$, we conclude that the contribution by $p_b \circ' q_b - q_b \circ' p_b$ is zero. The only terms that may contribute are those in the difference $g_1(x',\xi',D_n) - g_2(x',\xi',D_n)$ where

$$\begin{aligned} g_1(D_n) &= L(p_i,q_i) + p_i^+(D_n)b_2(D_n) + b_1(D_n)q_i^+(D_n) \quad \text{and} \\ g_2(D_n) &= L(q_i,p_i) + q_i^+(D_n)b_1(D_n) + b_2(D_n)p_i^+(D_n). \end{aligned}$$

We will show the following identities:

$$\int \mathrm{res}_{x'}\{p_i^+(D_n)b_2(D_n) - b_2(D_n)p_i^+(D_n)\} = 0, \tag{2.26}$$

$$\int \mathrm{res}_{x'}\{b_1(D_n)q_i^+(D_n) - q_i^+(D_n)b_1(D_n)\} = 0, \tag{2.27}$$

$$\int \mathrm{res}_{x'}\{L(p_i,q_i) - L(q_i,p_i)\} = \frac{1}{2\pi}\int_M \mathrm{res}_x[p_i,q_i]. \tag{2.28}$$

Let us start with (2.26). We have the following asymptotic expansion formulas for the symbols c_1 and c_2 of the compositions $p_i^+(x',\xi',D_n)\circ' b_2(x',\xi',D_n)$ and $b_2(x',\xi',D_n)\circ' p_i^+(x',\xi',D_n)$, cf. [5, Theorem 2.7.4, $2^0, 3^0$].

$$c_1(x',\xi',\xi_n,\eta_n) \sim \sum_{j=0}^{\infty}\frac{i^j}{j!}\Pi^+_{\xi_n}\{\partial^j_{\xi_n}\{\partial^j_{x_n}p_i(x',0,\xi',\xi_n)\circ' b_2(x',\xi',\xi_n,\eta_n)\}\}; \quad (2.29)$$

$$c_2(x',\xi',\xi_n,\eta_n) \sim \sum_{j=0}^{\infty}\frac{(-i)^j}{j!}\Pi^-_{\eta_n}\{\partial^j_{\eta_n}b_1(x',\xi',\xi_n,\eta_n)\circ'\partial^j_{x_n}p_i(x',0,\xi',\eta_n)\}. \quad (2.30)$$

Since p_i is independent of x_n close to the boundary this reduces to

$$c_1(x',\xi',\xi_n,\eta_n)\sim \Pi^+_{\xi_n}\{p_i(x',0,\xi',\xi_n)\circ' b_2(x',\xi',\xi_n,\eta_n)\}$$

and

$$c_2(x',\xi',\xi_n,\eta_n)\sim \Pi^-_{\eta_n}\{b_2(x',\xi',\xi_n,\eta_n)\circ' p_i(x',0,\xi',\eta_n)\},$$

respectively. For the associated symbols $\bar c_1$ and $\bar c_2$, cf. (2.10), we then have

$$\begin{aligned}\bar c_1(x',\xi') &= \Pi'_{\xi_n}(\Pi^+_{\xi_n}\{p_i(x',0,\xi',\xi_n)\circ' b_2(x',\xi',\xi_n,\eta_n)\}|_{\eta_n=\xi_n})\\ &= \Pi'_{\xi_n}(\{p_i(x',0,\xi',\xi_n)\circ' b_2(x',\xi',\xi_n,\eta_n)|_{\eta_n=\xi_n}),\end{aligned}$$

since $\Pi^-_{\xi_n}\{p_i(x',0,\xi',\xi_n)\circ' b_2(x',\xi',\xi_n,\eta_n)|_{\eta_n=\xi_n}\}\in H^-$, where Π' vanishes. In particular, we have, according to (2.11),

$$(\bar c_1(x',\xi'))_{-(n-1)} = \Pi'_{\xi_n}(p_i(x',0,\xi',\xi_n)\circ' b_2(x',\xi',\xi_n,\eta_n)|_{\eta_n=\xi_n})_{-n}. \quad (2.31)$$

Similarly,

$$\begin{aligned}(\bar c_2(x',\xi'))_{-(n-1)} &= \Pi'_{\xi_n}(\Pi^-_{\eta_n}\{b_2(x',\xi',\xi_n,\eta_n)\circ' p_i(x',0,\xi',\eta_n)\}|_{\eta_n=\xi_n})_{-n}\\ &= \Pi'_{\xi_n}(b_2(x',\xi',\xi_n,\eta_n)\circ' p_i(x',0,\xi',\eta_n)|_{\eta_n=\xi_n})_{-n}. \quad (2.32)\end{aligned}$$

This time the reason is that $(\Pi^+_{\eta_n}\{b_2(x',\xi',\xi_n,\eta_n)\circ' p_i(x',0,\xi',\eta_n)\}|_{\eta_n=\xi_n})_{-n}$ is a function in $H^+\cap L_1(\mathbf{R})$, where Π' also vanishes (recall that $n>1$ and that the subscript $-n$ denotes the component of homogeneity $-n$). We therefore have

$$\begin{aligned}&\int_{\partial M}\operatorname{res}_{x'}\{\operatorname{tr} c_1(x',\xi',D_n)-\operatorname{tr} c_2(x',\xi',D_n)\}\\ =\ &\Pi'_{\xi_n}\int_{\partial M}\operatorname{res}_{x'}\{p_i(x',0,\xi',\xi_n)\circ' b_2(x',\xi',\xi_n,\xi_n)-b_2(x',\xi',\xi_n,\xi_n)\circ' p_i(x,0,\xi',\xi_n)\}=0\end{aligned}$$

by Theorem 1.4. This proves (2.26). It also proves (2.27), since the situation there is completely analogous.

Hence consider (2.28). The leftover terms depend only on the behavior of the interior symbols near the boundary. In view of the fact that those stabilize near the boundary by assumption, we may as well assume that both p_i and q_i are *independent of* x_n. Then the action of $p_i^+(x',\xi',D_n): H^+\to H^+$ is simply given by

$$[p_i^+(x',\xi',D_n)h](\xi_n)=\Pi^+_{\xi_n}\{p_i(x',0,\xi',\xi_n)h(\xi_n)\}, \quad (2.33)$$

similarly for q_i. Moreover,

$$L(p_i,q_i)=p_i^+(D_n)\circ' q_i^+(D_n)-(p_i\circ q_i)^+(D_n)=-p_i^+(D_n)\circ' q_i^-(D_n).$$

Again, we have omitted x', ξ', and denoted by $q_i^-(D_n): H^+ \to H^-$ the operator given by $q_i^-(D_n)h = \Pi^-(qh)$, analogously to (2.33). The standard expression for the symbol of this leftover term, cf. [12] or [2] is not helpful for our purposes. We therefore shall now derive a new representation. Keeping (x', ξ') fixed, we let $p^\pm = p^\pm(\xi_n) = \Pi^\pm_{\xi_n} p_i(x', 0, \xi', \xi_n)$ and $q^\pm = q^\pm(\xi_n) = \Pi^\pm_{\xi_n} q_i(x', 0, \xi', \xi_n)$ be the projections of the interior symbols on H^+ and H^-, respectively, not to be confused with the operators $p_i^\pm(D_n)$ and $q_i^\pm(D_n)$.
Applying the operator $q_i^-(D_n)$ to a function $h \in H^+$ and observing that $\Pi^-(q^+h) = 0$ we obtain

$$\begin{aligned}\Pi^-_{\xi_n}(q_i(x',0,\xi',\xi_n)h(\xi_n)) &= \Pi^-(q^-h)(\xi_n) = \frac{1}{2\pi i}\int_{\Gamma^+} \frac{q^-(\zeta)}{\zeta - \xi_n - i0} h(\zeta) d\zeta \\ &= \frac{1}{2\pi i}\int_{\Gamma^+} \frac{q^-(\zeta) - q^-(\xi_n)}{\zeta - \xi_n} h(\zeta) d\zeta,\end{aligned}$$

since

$$\frac{q^-(\xi_n)}{2\pi i}\int_{\Gamma^+} \frac{1}{\zeta - \xi_n - i0} h(\zeta) d\zeta = 0,$$

noting that $h \in H^+$. A similar argument applied to the function $v = \Pi^-(q^-h) \in H^-$ yields

$$\begin{aligned}\Pi^+_{\xi_n}(p_i(x',0,\xi',\xi_n)v(\xi_n)) &= \Pi^+(p^+v)(\xi_n) = -\frac{1}{2\pi i}\int_{\Gamma^+} \frac{p^+(\eta_n)}{\eta_n - \xi_n - i0} v(\eta_n) d\eta_n \\ &= -\frac{1}{2\pi i}\int_{\Gamma^+} \frac{p^+(\eta_n) - p^+(\xi_n)}{\eta_n - \xi_n} v(\eta_n) d\eta_n.\end{aligned}$$

Thus, the singular Green boundary symbol operator $L(p_i, q_i)$ is given by the symbol

$$b_{p_iq_i}(\xi_n, \eta_n) = \frac{1}{2\pi}\int_{\Gamma^+} \frac{p^+(\zeta) - p^+(\xi_n)}{\zeta - \xi_n} \circ' \frac{q^-(\eta_n) - q^-(\zeta)}{\eta_n - \zeta} d\zeta, \tag{2.34}$$

and

$$\bar{b}_{p_iq_i} = \frac{1}{(2\pi)^2}\int_{\Gamma^+} d\xi_n \int_{\Gamma^+} \frac{(p^+(\zeta) - p^+(\xi_n)) \circ' (q^-(\zeta) - q^-(\xi_n))}{(\zeta - \xi_n)^2} d\zeta. \tag{2.35}$$

Considering the inner integral shift the contour Γ^+ to a contour Γ_1^+ in the upper half-plane so that Γ_1^+ is inside Γ^+. Then the inner integral is equal to

$$\int_{\Gamma_1^+} \frac{p^+(\zeta) \circ' (q^-(\zeta) - q^-(\xi_n))}{(\zeta - \xi_n)^2} d\zeta - p^+(\xi_n) \circ' \int_{\Gamma_1^+} \frac{q^-(\zeta) - q^-(\xi_n)}{(\zeta - \xi_n)^2} d\zeta$$

and the second term vanishes being analytic inside Γ_1^+. Integrating the first term over $\xi_n \in \Gamma^+$ we may calculate the integral by means of the residue at the pole $\xi_n = \zeta$ yielding

$$\begin{aligned}\bar{b}_{p_iq_i} &= \frac{1}{(2\pi)^2}\int_{\Gamma_1^+} p^+(\zeta) d\zeta \circ' \int_{\Gamma^+} \frac{q^-(\zeta) - q^-(\xi_n)}{(\zeta - \xi_n)^2} d\xi_n \\ &= \frac{1}{2\pi i}\int_{\Gamma_1^+} p^+(\zeta) \circ' \frac{\partial q^-(\zeta)}{\partial \zeta} d\zeta = -i\Pi'_{\xi_n}\left\{p^+(\xi_n) \circ' \frac{\partial q^-(\xi_n)}{\partial \xi_n}\right\}.\end{aligned}$$

Similarly an integration by parts gives

$$\bar{b}_{q_ip_i} = -i\Pi'_{\xi_n}\left\{q^+(\xi_n) \circ' \frac{\partial p^-(\xi_n)}{\partial \xi_n}\right\} = i\Pi'_{\xi_n}\left\{\frac{\partial q^+(\xi_n)}{\partial \xi_n} \circ' p^-(\xi_n)\right\}.$$

Thus,

$$\begin{aligned}
&\int_{\partial M} \mathrm{res}_{x'}(\bar{b}_{p_i q_i} - \bar{b}_{q_i p_i})_{-(n-1)} \\
&= -\frac{i}{2\pi}\int_{\partial M} \mathrm{res}_{x'} \int_{-\infty}^{\infty} \left(p^+(\xi_n) \circ' \frac{\partial q^-(\xi_n)}{\partial \xi_n} + p^-(\xi_n) \circ' \frac{\partial q^+(\xi_n)}{\partial \xi_n}\right)_{-n} d\xi_n = \\
&= -\frac{i}{2\pi}\int_{\partial M} \mathrm{res}_{x'} \int_{-\infty}^{\infty} \left(p_i(x',0,\xi',\xi_n) \circ' \frac{\partial q_i(x',0,\xi',\xi_n)}{\partial \xi_n}\right)_{-n} d\xi_n
\end{aligned} \tag{2.36}$$

since

$$\Pi'\left(p^+ \circ' \frac{\partial q^+}{\partial \xi^n}\right)_{-n} = \Pi'\left(p^- \circ' \frac{\partial q^-}{\partial \xi^n}\right)_{-n} = 0.$$

Notice that we could interchange the order of p^- and $\partial_{\xi_n} q^+$ as a consequence of Theorem 1.4. Using formulas (2.24), (1.6) and (1.8), one obtains at once (2.28). This completes the proof of Theorem 2.5. ♯

For the algebra $\mathcal{B}_s$ there is *no uniqueness property* of the noncommutative residue. Examples of trace functionals not coinciding with $\mathrm{res}\, P$ may be constructed as follows. For $P = \{p_i, p_b\}$ take $p_i|_{x_n=0} = p_i(x', \xi', \xi_n)$. The variable ξ_n is globally defined so that for any $k = 0, 1, 2, \ldots$ we have a symbol on ∂M defined by

$$a_k = \left\{\frac{\partial^k}{\partial \xi_n^k}\left(p_i|_{x_n=0}\right)\right\}\bigg|_{\xi_n=0}$$

and we may define

$$\mathrm{Tr}_k P = \mathrm{res}_{\partial M} a_k \tag{2.37}$$

taking the noncommutative residue of the symbols a_k on ∂M. It is easy to verify that these functionals are really traces on $\mathcal{B}_s$. The reason is that the restriction map

$$\{p_i, p_b\} \mapsto p_i|_{x_n=0}$$

is a an algebra homomorphism from $\mathcal{B}_s$ to the algebra of classical pseudodifferential operators on ∂M (under the assumption that p_i does not depend on x_n near the boundary), and any trace on the restricted algebra will serve as a trace for the whole algebra $\mathcal{B}_s$.

3 The Noncommutative Residue on Boutet de Monvel's Algebra

Now we consider the full algebra $\mathcal{B}$. An operator $P \in \mathcal{B}$ will be identified with its symbol, a pair $\{p_i, p_b\}$, consisting of the interior symbol p_i and the boundary symbol p_b as introduced in Section 2. We define the noncommutative residue by the same formula (2.24) as in the case of $\mathcal{B}_s$. As before $\mathrm{res}_x p_i$ and $\mathrm{res}_{x'} p_b$ are densities on M and ∂M, respectively, since the change of variables on a boundary chart does not affect the variables x_n and ξ_n.

Theorem 3.1 *The residue* (2.24) *is a trace on the algebra* $\mathcal{B}$.

Proof. Let P and Q be operators in $\mathcal{B}$ with symbols $\{p_i, p_b\}$ and $\{q_i, q_b\}$, respectively. In order to show that $\operatorname{res}[P,Q]$ vanishes we will use the same set-up and notation as in the proof of Theorem 2.5; we suppose that all symbols are supported in a boundary chart and that the boundary symbols p_b and q_b are given by matrices as in (2.7) whose entries we denote by b_1, k_1, t_1, q_1 and b_2, k_2, t_2, q_2, respectively.
Of course we can rely on what we showed in the proof of 2.5. Since now p_i and q_i may depend on x_n, we will have to revise (2.25). On the other hand we know that the noncommutative residue is a trace on the ideal $\mathcal{B}_0 \subset \mathcal{B}$ of all operators with zero interior symbol. So the contribution of $p_b \circ' q_b - q_b \circ' p_b$ will vanish again, and it will suffice to show the identities (2.26), (2.27), and (2.28) using the composition formulas (2.29) and (2.30), plus an asymptotic expansion formula for the symbol of the leftover term in the x_n-dependent case, cf. (3.1), below.
In analogy to (2.31) we get

$$\begin{aligned}(\bar{c}_1(x',\xi'))_{-(n-1)} &\sim \sum_{j=0}^{\infty} \frac{i^j}{j!}\Pi'_{\xi_n}\left\{ \{\partial^j_{\xi_n}[\partial^j_{x_n} p_i(x',0,\xi',\xi_n) \circ' b_2(x',\xi',\xi_n,\eta_n)]\}\big|_{\eta_n=\xi_n}\right\}_{-n} \\ &\sim \sum_{j=0}^{\infty} \frac{(-i)^j}{j!}\Pi'_{\xi_n}\left\{ \{\partial^j_{x_n} p_i(x',0,\xi',\xi_n) \circ' \partial^j_{\eta_n} b_2(x',\xi',\xi_n,\eta_n)\}\big|_{\eta_n=\xi_n}\right\}_{-n}.\end{aligned}$$

Here we have made use of an induction on j and the following identity

$$\frac{\partial}{\partial \xi_n} a(\xi_n,\xi_n) = \frac{\partial}{\partial \xi_n} a(\xi_n,\eta_n)\bigg|_{\eta_n=\xi_n} + \frac{\partial}{\partial \eta_n} a(\xi_n,\eta_n)\bigg|_{\eta_n=\xi_n}$$

together with the fact that $\Pi'_{\xi_n}(\frac{\partial}{\partial \xi_n} a(\xi_n,\xi_n)) = 0$.
Similarly,

$$(\bar{c}_2(x',\xi'))_{-(n-1)} \sim \sum_{j=0}^{\infty} \frac{(-i)^j}{j!}\Pi'_{\xi_n}\left\{ \{\partial^j_{\eta_n} b_2(x',\xi',\xi_n,\eta_n) \circ' \partial^j_{x_n} p_i(x',0,\xi',\xi_n)\}\big|_{\eta_n=\xi_n}\right\}_{-n}$$

so that

$$\int_{\partial M} (\operatorname{res}_{x'}\bar{c}_1 - \operatorname{res}_{x'}\bar{c}_2) = 0$$

by Theorem 1.4 for ∂M.

Now for the leftover terms. In the x_n-dependent case we will have to replace (2.34) according to [5, Theorem 2.7.7] by

$$b_{p_i q_i}(\xi_n,\eta_n) \sim \sum_{j,l,m=0}^{\infty} \frac{(-1)^m i^{j+l+m}}{j!\,l!\,m!} \partial^j_{\xi_n}\partial^m_{\eta_n} b_{\partial^j_{x_n} p_i|_{x_n=0},\, \partial^l_{\eta_n}\partial^{l+m}_{x_n} q_i|_{x_n=0}}(\xi_n,\eta_n). \qquad (3.1)$$

Abbreviating $a = \partial^j_{x_n} p_i|_{x_n=0}$, and $b = \partial^l_{\eta_n}\partial^{l+m}_{x_n} q_i|_{x_n=0}$, each term b_{ab} under the summation in (3.1) denotes the singular Green symbol obtained from the x_n-independent symbols a and b by (2.34). Writing $a^\pm = \Pi^\pm a, b^\pm = \Pi^\pm b$ we have

$$\partial^j_{\xi_n}\partial^m_{\eta_n} b_{ab}(\xi_n,\eta_n) = \frac{1}{2\pi}\int_{\Gamma^+} \partial^j_{\xi_n} \frac{a^+(\zeta) - a^+(\xi_n)}{\zeta - \xi_n} \circ' \partial^m_{\eta_n} \frac{b^-(\eta_n) - b^-(\zeta)}{\eta_n - \zeta} d\zeta.$$

With the aim of eventually computing $\bar{b}_{p_iq_i}$ we put $\eta_n = \xi_n$, multiply by $(2\pi)^{-1}$, and integrate over $\xi_n \in \Gamma^+$. We obtain, using integration by parts,

$$\begin{aligned}\bar{b}_{ab} &= \frac{1}{(2\pi)^2}\int_{\Gamma^+} d\xi_n \int_{\Gamma^+} \partial^j_{\xi_n}\frac{a^+(\zeta)-a^+(\xi_n)}{\zeta-\xi_n} \circ' \partial^m_{\xi_n}\frac{b^-(\xi_n)-b^-(\zeta)}{\xi_n-\zeta}d\zeta \\ &= \frac{(-1)^j}{(2\pi)^2}\int_{\Gamma^+} d\xi_n \int_{\Gamma^+} \frac{a^+(\zeta)-a^+(\xi_n)}{\zeta-\xi_n} \circ' \partial^{m+j}_{\xi_n}\frac{b^-(\xi_n)-b^-(\zeta)}{\xi_n-\zeta}d\zeta.\end{aligned}$$

This integral may be simplified similarly to (2.35). For the integration over ζ shift the contour Γ^+ to a contour Γ_1^+ inside Γ^+ and note that for fixed ξ_n, the function $\partial^{m+j}_{\xi_n}\frac{b^-(\xi_n)-b^-(\zeta)}{\xi_n-\zeta}$ is analytic in the upper half plane $\{\operatorname{Im}\zeta > 0\}$. We get

$$\begin{aligned}\bar{b}_{ab} &= \frac{(-1)^j}{(2\pi)^2}\int_{\Gamma^+} d\xi_n \int_{\Gamma_1^+} \frac{a^+(\zeta)}{\zeta-\xi_n} \circ' \partial^{m+j}_{\xi_n}\frac{b^-(\xi_n)-b^-(\zeta)}{\xi_n-\zeta}d\zeta \\ &= \frac{(-1)^{j+1}i}{2\pi}\int_{\Gamma_1^+} a^+(\zeta)\circ' \frac{\partial^{m+j+1}_\zeta b^-(\zeta)d\zeta}{(m+j+1)}d\zeta.\end{aligned}$$

For the second equality we have interchanged the order of integration and applied Cauchy's theorem for fixed ζ. The identity is most easily checked using that

$$\partial^{m+j}_{\xi_n}\frac{b^-(\xi_n)-b^-(\zeta)}{\xi_n-\zeta} = \partial^{m+j}_{\xi_n}\int_0^1 (\partial_{\xi_n}b^-)(\zeta+\theta(\xi_n-\zeta))d\theta.$$

Thus,

$$\begin{aligned}\bar{b}_{p_iq_i} &= \frac{1}{2\pi}\int_{\Gamma^+} b_{p_iq_i}(\xi_n,\xi_n)d\xi_n \\ &\sim \sum_{j,l,m=0}^{\infty}\frac{i^{j+l+m+1}(-1)^{m+j+1}}{m!\,l!\,j!\,(m+j+1)}\frac{1}{2\pi}\int_{\Gamma^+}\partial^j_{x_n}p_i^+(x',0,\xi',\xi_n)\circ'\partial^{l+m+j+1}_{\xi_n}\partial^{m+l}_{x_n}q_i^-(x',0,\xi',\xi_n)d\xi_n.\end{aligned}$$

The notation should be obvious: we let $p_i^{\pm}(x',0,\xi',\xi_n) = \Pi^{\pm}_{\xi_n}p_i(x',0,\xi',\xi_n)$ and $q_i^{\pm}(x',0,\xi',\xi_n) = \Pi^{\pm}_{\xi_n}q_i(x',0,\xi',\xi_n)$.
We need to calculate the sum

$$\sum_{m+l=k}\frac{(-1)^m}{m!\,l!\,(m+j+1)}.$$

To this end consider binomial formula

$$(1-t)^k = k!\sum_{m+l=k}\frac{(-1)^m}{m!\,l!}t^m.$$

Multiplying by t^j and integrating over $[0,1]$ we obtain

$$k!\sum_{m+l=k}\frac{(-1)^m}{m!l!(m+j+1)} = \int_0^1 t^j(1-t)^k dt = B(j+1,k+1) = \frac{k!\,j!}{(k+j+1)!}.$$

Substituting this result we get

$$\begin{aligned}\bar{b}_{p_iq_i} &= \sum_{j,k=0}^{\infty}\frac{(-1)^{j+1}i^{j+k+1}}{(j+k+1)!}\Pi'_{\xi_n}\left(\partial^j_{x_n}p_i^+(x',0,\xi',\xi_n)\circ'\partial^{j+k+1}_{\xi_n}\partial^k_{x_n}q_i^-(x',0,\xi',\xi_n)\right) \\ &= \sum_{j,k=0}^{\infty}\frac{(-i)^{j+k+1}}{(j+k+1)!}\Pi'_{\xi_n}\left(\partial^j_{x_n}\partial^k_{\xi_n}p_i^+(x',0,\xi',\xi_n)\circ'\partial^{j+1}_{\xi_n}\partial^k_{x_n}q_i^-(x',0,\xi',\xi_n)\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{b}_{q_ip_i} &= \sum_{j,k=0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!}\Pi'_{\xi_n}\left(\partial^j_{x_n}\partial^k_{\xi_n} q_i^+(x',0,\xi',\xi_n) \circ' \partial^{j+1}_{\xi_n}\partial^k_{x_n} p_i^-(x',0,\xi',\xi_n)\right)\\ &= -\sum_{j,k=0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!}\Pi'_{\xi_n}\left(\partial^{k+1}_{\xi_n}\partial^j_{x_n} q_i^+(x',0,\xi',\xi_n) \circ' \partial^j_{\xi_n}\partial^k_{x_n} p_i^-(x',0,\xi',\xi_n)\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\partial M}(\mathrm{res}_{x'}\,\bar{b}_{p_iq_i} - \mathrm{res}_{x'}\,\bar{b}_{q_ip_i})\\ =& \frac{1}{2\pi}\int_{\partial M}\int_{S'}\int_{-\infty}^{\infty}\sum_{j,k=0}^{\infty}\frac{(-i)^{j+k+1}}{(j+k+1)!}\left(\partial^j_{x_n}\partial^k_{\xi_n}p_i^+(x',0,\xi',\xi_n)\circ'\partial^{j+1}_{\xi_n}\partial^k_{x_n}q_i^-(x',0,\xi',\xi_n)\right.\\ &\left.+\ \partial^{k+1}_{\xi_n}\partial^j_{x_n}q_i^+(x',0,\xi',\xi_n)\circ'\partial^j_{\xi_n}\partial^k_{x_n}p_i^-(x',0,\xi',\xi_n)\right)_{-n}d\xi_n\,\sigma'\,dx_1\wedge\ldots\wedge dx_{n-1}\end{aligned}$$

In the last expression we may interchange the order of $\partial^{k+1}_{\xi_n}\partial^j_{x_n}q_i^+(x',0,\xi',\xi_n)$ and $\partial^j_{\xi_n}\partial^k_{x_n}p_i^-(x',0,\xi',\xi_n)$ as a consequence of Theorem 1.4. With the considerations justifying (2.36) we then conclude that

$$\begin{aligned}&\int_{\partial M}(\mathrm{res}_{x'}\,\bar{b}_{p_iq_i} - \mathrm{res}_{x'}\,\bar{b}_{q_ip_i}) \qquad\qquad (3.2)\\ =& \frac{1}{2\pi}\int_{\partial M}\int_{S'}\int_{-\infty}^{\infty}\sum_{j,k=0}^{\infty}\frac{(-i)^{j+k+1}}{(j+k+1)!}\\ &\left(\partial^j_{x_n}\partial^k_{\xi_n}p_i(x',0,\xi',\xi_n)\circ'\partial^{j+1}_{\xi_n}\partial^k_{x_n}q_i(x',0,\xi',\xi_n)\right)_{-n}d\xi_n\,\sigma'\,dx_1\wedge\ldots\wedge dx_{n-1}\end{aligned}$$

since the $++$ and $--$ parts vanish after integration with respect to ξ_n.
Now the representation of the commutator $p_i\circ q_i - q_i\circ p_i$ in (2.6) together with (1.6) shows that (3.2) coincides precisely with

$$-\frac{1}{2\pi}\int_{\partial M}\int_{S'}\int_{-\infty}^{\infty}(B_n)_{-n}|_{x_n=0}d\xi_n\,\sigma'\,dx_1\wedge\ldots\wedge dx_{n-1} = \frac{1}{2\pi}\int_M \mathrm{res}_x[p_i,q_i].$$

Hence the sum of both is zero, and we have proven the theorem. ♯

Unlike in the case of $\mathcal{B}_s$, the noncommutative residue is the unique continuous trace on the full algebra $\mathcal{B}$.

Theorem 3.2 *Denote by $\mathcal{B}$ Boutet de Monvel's algebra on M as introduced in Section 2. Then any continuous trace on $\mathcal{B}$ coincides with the noncommutative residue* res *up to a constant factor.*

Proof. Let Tr be a continuous trace functional on $\mathcal{B}$. Choose a boundary chart U that intersects only one component of ∂M. Denote by $\mathcal{B}^U\subseteq\mathcal{B}$ the ideal of those elements whose interior symbol has support in U and whose boundary symbol has support in $U\cap\partial M$. By $\mathcal{B}_0^U$ denote the subset of those elements with zero interior symbol. Restricted to $\mathcal{B}_0^U$ the trace Tr must coincide with c_U res for a suitable constant c_U. This is a consequence of

the considerations for the uniqueness part in the proof of Theorem 1.4 together with the fact that there is only one continuous trace on the algebra of boundary symbol operators, established in the proof of Proposition 2.3. Then $\mathrm{Tr}' = \mathrm{Tr} - c_U \,\mathrm{res}$ is a trace functional on $\mathcal{B}^U$ vanishing on the subalgebra $\mathcal{B}^U_0$.
Clearly, $\mathcal{B}^U_0$ is a two-sided ideal in $\mathcal{B}^U$, so Tr' is actually defined on the algebra $\mathcal{B}^U/\mathcal{B}^U_0$. This quotient is understood purely algebraically (no topology on $\mathcal{B}^U/\mathcal{B}^U_0$ is required); moreover, it obviously can be identified with the algebra of all interior symbols supported in U. Without loss of generality we may assume that U is an interval in $\overline{\mathbf{R}}^n_+$. It therefore follows from the lemma, below, that any trace functional on this algebra is trivial. This yields the assertion of the theorem. ♯

Lemma 3.3 *Let $U =]-1,1[^{n-1} \times [0,1[\subset \overline{\mathbf{R}}^n_+$. Denote by $\mathcal{C}$ the algebra of all classical pseudodifferential symbols with x-support in U that satisfy the transmission condition at $x_n = 0$. Then any trace on $\mathcal{C}$ vanishes as a consequence of the following three assertions:*

(a) *We have $\mathcal{C} = [\mathcal{C},\mathcal{C}] + \mathcal{C}_0$, where $\mathcal{C}_0$ denotes the subalgebra of all elements of $\mathcal{C}$ vanishing identically in a neighborhood of $\{x_n = 0\}$.*

(b) *Let Tr' be a trace on $\mathcal{C}$. Then $\mathrm{Tr}' = c\,\mathrm{res}$ for a suitable constant c.*

(c) *The constant in (b) is necessarily 0.*

Proof. (a) Let p be an arbitrary classical symbol with the transmission property. We may confine ourselves to the case where $p(x,\xi)$ vanishes for x outside $[-1/4,1/4]^{n-1} \times [0,1/4]$. Choose a smooth function $\alpha \geq 0$ on $[0,\infty[$ with $\alpha(t) \equiv 1$ for $0 \leq t \leq 1/3$ and $\alpha(t) \equiv 0$ for $t \geq 1/2$. By $\widehat{\xi_n}$ denote a symbol with the transmission property which is equal to ξ_n for $x \in [-1/2,1/2]^{n-1} \times [0,1/2]$ and vanishes for x outside a compact set in U.
Let $q(x,\xi) = \alpha(x_n) \int_0^{x_n} p(x',t,\xi)\,dt$. Then q is a classical symbol with the transmission property. The symbol of the commutator $[i\widehat{\xi_n}, q]$ is

$$\partial_{x_n} q(x,\xi) = p(x,\xi) + \partial_{x_n}\alpha(x_n) \int_0^\infty p(x',t,\xi)\,dt \tag{3.3}$$

This gives the desired decomposition.
(b) Let Tr' be a trace on $\mathcal{C}$. The restriction of Tr' to $\mathcal{C}_0$ is a trace, and according to the considerations in the proof of Theorem 1.4 it coincides with $c\,\mathrm{res}$ for a suitable constant c. We conclude from (3.3) and the fact that $\mathrm{Tr}'\partial_{x_n} q = 0$ that

$$\begin{aligned} \mathrm{Tr}'p &= -c\,\mathrm{res}\left(\partial_{x_n}\alpha(x_n)\int_0^\infty p(x',t,\xi)\,dt\right) \\ &= -c\int_0^\infty \partial_{x_n}\alpha(x_n)\,dx_n \int_{\mathbf{R}^{n-1}} \left(\int_S\int_0^\infty p_{-n}(x',t,\xi)\,dt\sigma_\xi\right) dx_1 \ldots dx_{n-1} \\ &= c\,\mathrm{res}\,p. \end{aligned}$$

(c) In order to see that c vanishes, choose a homogeneous function $h(\xi)$ of degree $-n$ with $\int_S h(\xi)\sigma_\xi \neq 0$ that satifies the transmission condition; it is well-known that such functions exist, cf. [12, Section 2.3.2.4]. Then pick $\beta \in C_0^\infty(]-1/4,1/4[^{n-1},[0,1])$, not identically zero, and let $p(x',x_n,\xi) = \alpha(x_n)\beta(x')h(\xi)$ with the function α introduced in (a). Define

$$q(x,\xi) = -\int_{x_n}^\infty p(x',t,\xi)\,dt.$$

Clearly, the symbol q satisfies the transmission condition and, in the notation of (a), $[i\widehat{\xi}_n, q] = \partial_{x_n} q = p$. This implies that $\mathrm{Tr}'p = 0$, while, by construction, $\mathrm{res}\, p \neq 0$. Hence $c = 0$. ♯

Remark. What we have implicitly used in the proof of the Theorem is, of course, the fact that the first Cech cohomology group with compact support $H^1_{compact}([0,\infty), \mathbf{R}) = \{0\}$.

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