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THE SLICING PROBLEM FOR SECTIONS OF PROPORTIONAL DIMENSIONS

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Abstract. We consider the following problem. Does there exist an absolute constant C so that for every $n \in \mathbb{N}$, every integer $1 \leq k < n$, every origin-symmetric convex body L in $\mathbb{R}^n$, and every measure μ with non-negative even continuous density in $\mathbb{R}^n$,

$$\mu(L) \ \leq \ C^k \max_{H \in Gr_{n-k}} \mu(L \cap H) \ |L|^{k/n}, \tag{1}$$

where Gr_{n-k} is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $|L|$ stands for volume? This question is an extension to arbitrary measures (in place of volume) and to sections of arbitrary codimension k of the slicing problem of Bourgain, a major open problem in convex geometry.

It was proved in [K4, K5] that (1) holds for arbitrary origin-symmetric convex bodies, all k and all μ with $C \leq O(\sqrt{n})$. In this article, we prove inequality (1) with an absolute constant C for unconditional convex bodies and for duals of bodies with bounded volume ratio. We also prove that for every $\lambda \in (0,1)$ there exists a constant $C = C(\lambda)$ so that inequality (1) holds for every $n \in \mathbb{N}$, every origin-symmetric convex body L in $\mathbb{R}^n$, every measure μ with continuous density and the codimension of sections $k \geq \lambda n$. The proofs are based on a stability result for generalized intersection bodies and on estimates of the outer volume ratio distance from an arbitrary convex body to the classes of generalized intersection bodies. In the last section, we show that for some measures the behavior of minimal sections may be very different from the case of volume.

1. Introduction

The slicing problem [Bo1, Bo2, Ba1, MP], a major open problem in convex geometry, asks whether there exists an absolute constant C so that for any origin-symmetric convex body K in $\mathbb{R}^n$ of volume 1 there is a hyperplane section of K whose $(n-1)$-dimensional volume is greater than $1/C$. In other words, does there exist a constant C so that for any $n \in \mathbb{N}$ and any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^{\perp}|, \tag{2}$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to ξ, and $|K|$ stands for volume of proper dimension? The best current result $C \leq O(n^{1/4})$ is due to Klartag [Kl], who removed the logarithmic term from an earlier estimate of Bourgain [Bo3]. We refer the reader to [BGVV] for the history and partial results.

For certain classes of bodies the question has been answered in affirmative. These classes include unconditional convex bodies (as initially observed by Bourgain; see also [MP, J2, BN, BGVV]), unit balls of subspaces of L_p [Ba2, J1, M1], intersection bodies [G, Theorem 9.4.11], zonoids, duals of bodies with bounded volume ratio [MP], the Schatten classes [KMP], k-intersection bodies [KPY, K6].

Iterating (2) one gets the lower dimensional slicing problem asking whether the inequality

$$|K|^{\frac{n-k}{n}} \leq C^k \max_{H\in Gr_{n-k}} |K\cap H| \tag{3}$$

holds with an absolute constant C where $1 \leq k \leq n-1$ and Gr_{n-k} is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$.

In this note we prove (3) in the case where $k \geq \lambda n,\ 0<\lambda<1$, with the constant $C = C(\lambda)$ dependent only on λ. Moreover, we prove this result in a more general setting of arbitrary measures in place of volume. We consider the following generalization of the slicing problem.

Problem 1. *Does there exist an absolute constant C so that for every $n \in \mathbb{N}$, every integer $1 \leq k < n$, every origin-symmetric convex body L in $\mathbb{R}^n$, and every measure μ with non-negative even continuous density f in $\mathbb{R}^n$,*

$$\mu(L) \ \leq\ C^k \max_{H\in Gr_{n-k}} \mu(L\cap H)\ |L|^{k/n}. \tag{4}$$

Here $\mu(B) = \int_B f$ for every compact set B in $\mathbb{R}^n$, and $\mu(B\cap H) = \int_{B\cap H} f$ is the result of integration of the restriction of f to H with respect to Lebesgue measure in H.

In many cases we will write (4) in an equivalent form

$$\mu(L) \ \leq\ C^k \frac{n}{n-k}\, c_{n,k} \max_{H\in Gr_{n-k}} \mu(L\cap H)\ |L|^{k/n}, \tag{5}$$

where $c_{n,k} = |B_2|^{\frac{n-k}{n}}/|B_2^{n-k}|$, and B_2^n is the unit Euclidean ball in $\mathbb{R}^n$. Note that $c_{n,k} \in (e^{-k/2}, 1)$ (see for example [KL, Lemma 2.1]), and

$$1 \leq \frac{n}{n-k} \leq e^{\frac{k}{n-k}} \leq e^k,$$

so these constants can be incorporated in the constant C.

It appears that some results on the original slicing problem can be extended to the case of arbitrary measures. The first result of this kind was established in [K3], namely, when L is an intersection body (see definition below) and $k = 1$, inequality (5) holds with the best possible constant $C = 1$. This result was later proved for arbitrary k in [KM]. For arbitrary origin-symmetric convex bodies, inequality (5) was proved with $C = \sqrt{n}$ in [K4] and [K5], for $k = 1$ and for arbitrary k, respectively. When L is the unit ball of a subspace of $L_p,\ p \geq 2$, the constant C can be improved to $n^{\frac{1}{2}-\frac{1}{p}}$; see [K6]. In [K6], (4) was also proved for the unit balls of normed spaces that embed in $L_p,\ -\infty < p \leq 2$ with C depending only on p. In the case where $k = 1$ and the measure μ is log-concave, (4) holds for any origin-symmetric convex body with $C \leq O(n^{1/4})$, as shown in [KZ] using the estimate of Klartag [Kl] mentioned above and the technique of Ball [Ba1] relating log-concave measures to convex bodies.

In this article, we prove inequality (4) for unconditional convex bodies and duals of bodies with finite volume ratio, with an absolute constant C. We also prove that for every $\lambda \in (0, 1)$ there exists a constant $C = C(\lambda)$ so that inequality (4) holds for every $n \in \mathbb{N}$, arbitrary origin-symmetric convex body L, every measure μ with continuous density and every codimension of sections k satisfying $\lambda n \leq k < n$.

In Section 6, we show that the properties of the minimal measures of sections may be different from the case of volume. We prove that there exist a symmetric convex body L in $\mathbb{R}^n$ and a measure μ with continuous density so that

$$\mu(L) < \frac{n}{n-1} c_{n,1} \min_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n}.$$

Note that in the case of volume

$$\int_{S^{n-1}} |K \cap \xi^\perp| d\sigma(\xi) \leq c_{n,1} |K|^{\frac{n-1}{n}},$$

where σ is the normalized uniform measure on the sphere; see [L1] for more general results.

2. Reduction to intersection bodies

The approach to Problem 1 suggested in this paper is based on the concept of an intersection body. In this section we reduce the problem to computing the outer volume ratio distance from an origin-symmetric convex body to the class of generalized intersection bodies.

We need several definitions and facts. A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \ge 0 : \ x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \qquad x \in \mathbb{R}^n, \ x \neq 0.$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of K in the direction of x.

We use the polar formula for volume of a star body

$$|K| = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{6}$$

The class of intersection bodies was introduced by Lutwak [L2]. Let K, L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^\perp|$$
$$= \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} R\left(\|\cdot\|_L^{-n+1}\right)(\xi),$$

where $R : C(S^{n-1}) \to C(S^{n-1})$ is the *spherical Radon transform*

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(x)dx, \qquad \forall f \in C(S^{n-1}).$$

All bodies K that appear as intersection bodies of different star bodies form *the class of intersection bodies of star bodies*. A more general class of *intersection bodies* is defined as follows. If μ is a finite Borel measure on S^{n-1}, then the spherical Radon transform $R\mu$ of μ is defined as a functional on $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x)d\mu(x), \qquad \forall f \in C(S^{n-1}).$$

A star body K in $\mathbb{R}^n$ is called an *intersection body* if $\|\cdot\|_K^{-1} = R\mu$ for some measure μ, as functionals on $C(S^{n-1})$, i.e.

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x)dx = \int_{S^{n-1}} Rf(x)d\mu(x), \qquad \forall f \in C(S^{n-1}).$$

Intersection bodies played a crucial role in the solution of the Busemann-Petty problem and its generalizations; see [K1, Chapter 5].

A generalization of the concept of an intersection body was introduced by Zhang [Z] in connection with the lower dimensional Busemann-Petty problem. For $1 \le k \le n-1$, the *$(n-k)$-dimensional spherical*

Radon transform $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k}g(H) = \int_{S^{n-1}\cap H} g(x)\ dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

We say that an origin symmetric star body K in $\mathbb{R}^n$ is a *generalized k-intersection body*, and write $K \in \mathcal{BP}_k^n$, if there exists a finite Borel non-negative measure μ on Gr_{n-k} so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_K^{-k} g(x)\ dx = \int_{Gr_{n-k}} R_{n-k}g(H)\ d\mu(H). \tag{7}$$

When $k = 1$ we get the class of intersection bodies. It was proved by Grinberg and Zhang [GZ, Lemma 6.1] that every intersection body in $\mathbb{R}^n$ is a generalized k-intersection body for every $k < n$. More generally, as proved later by E.Milman [M2], if m divides k, then every generalized m-intersection body is a generalized k-intersection body. Note that in [Z, GZ] generalized k-intersection bodies are called "i-intersection bodies".

We need a stability result for generalized k-intersection bodies proved in [K5, Theorem 1]. Here we present a slightly simpler version.

Theorem 1. *Suppose that $1 \le k \le n-1$, K is a generalized k-intersection body in $\mathbb{R}^n$, f is an even continuous non-negative function on K, and $\varepsilon > 0$. If*

$$\int_{K\cap H} f \ \le \varepsilon, \qquad \forall H \in Gr_{n-k},$$

then

$$\int_K f \ \le \frac{n}{n-k}\ c_{n,k}\ |K|^{k/n}\varepsilon.$$

Recall that $c_{n,k} \in (e^{-k/2}, 1)$.

Proof : Writing integrals in spherical coordinates we get

$$\int_K f = \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\ dr \right) d\theta,$$

and

$$\int_{K\cap H} f = \int_{S^{n-1}\cap H} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\ dr \right) d\theta$$

$$= R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\ \cdot)\ dr\right)(H),$$

so the condition of the theorem can be written as

$$R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\ \cdot)\ dr\right)(H) \le \varepsilon, \qquad \forall H \in Gr_{n-k}.$$

Integrate both sides with respect to the measure μ on Gr_{n-k} that corresponds to K as a generalized k-intersection body by (7). We get

$$\int_{S^{n-1}} \|\theta\|_K^{-k}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\ dr\right) d\theta \le \varepsilon\mu(Gr_{n-k}).$$

Estimate the integral in the left-hand side from below using $f \ge 0$:

$$\int_{S^{n-1}} \|\theta\|_K^{-k}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\ dr\right) d\theta$$

$$= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\ dr\right) d\theta$$

$$+ \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} (\|\theta\|_K^{-k} - r^k) r^{n-k-1} f(r\theta)\ dr\right) d\theta$$

$$\ge \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\ dr\right) d\theta = \int_K f.$$

Now we estimate $\mu(Gr_{n-k})$ from above. We use $1 = R_{n-k}1(H)/|S^{n-k-1}|$ for every $H \in Gr_{n-k}$, definition (7), Hölder's inequality and the fact that $n|B_2^n| = |S^{n-1}|$:

$$\mu(Gr_{n-k}) = \frac{1}{|S^{n-k-1}|}\int_{Gr_{n-k}} R_{n-k}1(H) d\mu(H)$$

$$= \frac{1}{|S^{n-k-1}|}\int_{S^{n-1}} \|\theta\|_K^{-k}\ d\theta$$

$$\le \frac{1}{|S^{n-k-1}|}\left|S^{n-1}\right|^{\frac{n-k}{n}} \left(\int_{S^{n-1}} \|\theta\|_K^{-n}\ d\theta\right)^{\frac{k}{n}}$$

$$= \frac{1}{|S^{n-k-1}|}\left|S^{n-1}\right|^{\frac{n-k}{n}} n^{k/n}|K|^{k/n} = \frac{n}{n-k} c_{n,k}|K|^{k/n}.$$

Combining the estimates,

$$\int_K f \ \le\ \frac{n}{n-k} c_{n,k}|K|^{k/n}\varepsilon.$$

□

For a convex body L in $\mathbb{R}^n$ and $1 \le k < n$, denote by

$$\text{o.v.r.}(L, \mathcal{BP}_k^n) = \inf \left\{ \left(\frac{|K|}{|L|} \right)^{1/n} : \ L \subset K, \ K \in \mathcal{BP}_k^n \right\}$$

the outer volume ratio distance from a body L to the class $\mathcal{BP}_k^n$.

Corollary 1. *Let L be an origin-symmetric star body in $\mathbb{R}^n$. Then for any measure μ with even continuous density on L we have*

$$\mu(L) \le (\text{o.v.r.}(L, \mathcal{BP}_k^n))^k \ \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(L \cap H) \ |L|^{k/n}.$$

Proof : Let $C > \text{o.v.r.}(L, \mathcal{BP}_k^n)$, then there exists a body K in $\mathcal{BP}_k^n$ such that $L \subset K$ and $|K|^{1/n} \le C \ |L|^{1/n}$.

Let g be the density of the measure μ, and define a function f on K by $f = g\chi_L$, where χ_L is the indicator function of L. Clearly, $f \ge 0$ everywhere on K. Put

$$\varepsilon = \max_{H \in Gr_{n-k}} \int_{K \cap H} f = \max_{H \in Gr_{n-k}} \int_{L \cap H} g = \max_{H \in Gr_{n-k}} \mu(L \cap H),$$

and apply Theorem 1 to f, K, ε (f is not continuous, but we can do an easy approximation). We have

$$\begin{aligned} \mu(L) = \int_L g = \int_K f &\le \frac{n}{n-k} c_{n,k} |K|^{k/n} \max_{H \in Gr_{n-k}} \mu(L \cap H) \\ &\le C^k \ \frac{n}{n-k} c_{n,k} |L|^{k/n} \max_{H \in Gr_{n-k}} \mu(L \cap H). \end{aligned}$$

The result follows by sending C to $\text{o.v.r.}(L, \mathcal{BP}_k^n)$.

□

3. Unconditional bodies

Let $e_i, \ 1 \le i \le n$, be the standard basis of $\mathbb{R}^n$. A star body K in $\mathbb{R}^n$ is called unconditional if for every choice of real numbers x_i and $\delta_i = \pm 1, \ 1 \le i \le n$ we have

$$\| \sum_{i=1}^n \delta_i x_i e_i \|_K = \| \sum_{i=1}^n x_i e_i \|_K.$$

Theorem 2. *For every $n \in \mathbb{N}$, every $1 \le k < n$, every unconditional convex body L in $\mathbb{R}^n$ and every measure μ with even continuous non-negative density on L*

$$\mu(L) \ \le \ e^k \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(L \cap H) \ |L|^{k/n}. \tag{8}$$

Proof : By a result of Lozanovskii [Lo] (see the proof in [P, Corollary 3.4]), there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$T(B_\infty^n) \subset L \subset nT(B_1^n),$$

where B_1^n and B_∞^n are the unit balls of the spaces ℓ_1^n and ℓ_∞^n, respectively. Let $K = nT(B_1^n)$. By [K2, Theorem 3] and the fact that a linear transformation of an intersection body is an intersection body (see [L2] or [K2, Theorem 1]), the body K is an intersection body in $\mathbb{R}^n$. By a result of Grinberg and Zhang [GZ, Lemma 6.1], K is a generalized k-intersection body for every $1 \le k < n$.

Since $|B_1^n| = 2^n/n!$ (see for example [K1, Lemma 2.19]), we have $|K|^{1/n} \le 2e|\det T|^{1/n}$. On the other hand, $|T(B_\infty^n)| = 2^n|\det T|$, and $T(B_\infty^n) \subset L$, so $|K|^{1/n} \le e\ |L|^{1/n}$. Therefore, o.v.r$(L, \mathcal{BP}_k^n) \le e$. Now (8) follows from Corollary 1.

□

4. Duals of bodies with bounded volume ratio

The volume ratio of a convex body K in $\mathbb{R}^n$ is defined by

$$\text{v.r.}(K) = \inf_E \left\{ \left(\frac{|K|}{|E|} \right)^{1/n} :\ E \subset K,\ E - \text{ellipsoid} \right\}.$$

The following argument is standard and first appeared in [BM] and [MP]. Let K° and E° be polar bodies of K and E, respectively. If E is an ellipsoid, then

$$|E||E^\circ| = |B_2^n|^2.$$

By the reverse Santalo inequality of Bourgain and Milman [BM], there exists an absolute constant $c > 0$ such that

$$(|K||K^\circ|)^{1/n} \ge \frac{c}{n}.$$

Combining these and using the asymptotics of B_2^n we get that there exists an absolute constant C such that

$$\left(\frac{|E^\circ|}{|K^\circ|} \right)^{1/n} \le C \left(\frac{|K|}{|E|} \right)^{1/n}.$$

Theorem 3. *There exists an absolute constant C such that for every $n \in \mathbb{N}$, every $1 \le k < n$, every origin-symmetric convex body L in $\mathbb{R}^n$ and every measure μ with even continuous non-negative density on L*

$$\mu(L)\ \le\ (C\ \text{v.r}(L^\circ))^k \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(L \cap H)\ |L|^{k/n}.$$

Proof : If E is an ellipsoid, $E \subset L^\circ$, then the ellipsoid E° contains L. Also every ellipsoid is an intersection body as a linear image of the Euclidean ball, so it is also a generalized k-intersection body for every k. By the argument before the statement of the theorem,

$$\text{o.v.r}(L, \mathcal{BP}_k^n) \le C \text{ v.r.}(L^\circ).$$

The result follows from Corollary 1. □

5. Sections of proportional dimensions

The outer volume ratio distance from a general convex body to the class of generalized k-intersection bodies was estimated in [KPZ].

Proposition 1. *([KPZ, Theorem 1.1]) Let L be an origin-symmetric convex body in $\mathbb{R}^n$, and let $1 \le k \le n-1$. Then*

$$\text{o.v.r.}(L, \mathcal{BP}_k^n) \le C_0 \sqrt{\frac{n}{k}} \left(\log\left(\frac{en}{k}\right)\right)^{3/2},$$

where C_0 is an absolute constant.

Remark. In [KPZ, Theorem 1.1], the result was formulated with the logarithmic term raised to the power $1/2$ instead of $3/2$. This happened because the proof in [KPZ, p.2705] uses Corollary 3.2 which holds for $\alpha = 1$. However, the constant α used in the proof is $\alpha = 2 - \frac{1}{\log(en/k)}$, so Corollary 3.2 should have been formulated for this different value of α. We now correct this at the expense of an extra logarithmic term.

We use a result of Pisier [P, Corollary 7.16], generalizing V.Milman's M-position. For two symmetric convex bodies K and L in $\mathbb{R}^n$, the covering number of K by L, denoted by $N(K, L)$, is defined as the minimal number of translates of L, with their centers in K, needed to cover K.

Theorem 4. *([P, p.120]) For every $\alpha \in (0, 2)$ and every origin-symmetric convex body K in $\mathbb{R}^n$, there exists a linear image K_α of K such that*

$$\max\{N(K_\alpha, tB_2^n), N(B_2^n, tK_\alpha)\} \le \exp\left(\frac{cn}{t^\alpha(2-\alpha)}\right),$$

for every $t \ge 1$, where c is an absolute constant.

The constant $c/(2-\alpha)$ is not written precisely in Corollary 7.16 of [P], but it can be established by combining Corollary 7.15 and the proofs of Theorems 7.13 and 7.11 in the same book.

Theorem 4 implies a generalization of V.Milman's reverse Brunn-Minkowski inequality; one can find this in [P] as a combination of several results. We present a proof for the sake of completeness.

Corollary 2. *Let $\alpha \in [1,2)$, let K be an origin-symmetric convex body in $\mathbb{R}^n$, and let K_α be the position of K established in Theorem 4. Then for every $t \geq 1$,*

$$|K_\alpha + tB_2^n|^{1/n} \leq 2e^c\ t|K_\alpha|^{1/n}\frac{1}{2-\alpha}\exp\left(\frac{c}{t^\alpha(2-\alpha)}\right),$$

where c is the same absolute constant as in Theorem 4.

Proof : We first use the part of Theorem 4 estimating $N(B_2^n, tK_\alpha)$. Put $t = (2-\alpha)^{-1/\alpha}$ in Theorem 4. Then

$$|B_2^n|^{1/n} \leq t|K_\alpha|^{1/n}\left(N(B_2^n, tK_\alpha)\right)^{1/n}$$

$$\leq (2-\alpha)^{-1/\alpha}e^c|K_\alpha|^{1/n} \leq \frac{e^c}{2-\alpha}|K_\alpha|^{1/n}.$$

Now for every $t \geq 1$ we use the estimate for $N(K_\alpha, tB_2^n)$ from Theorem 4. We have

$$\frac{|K_\alpha + tB_2^n|^{1/n}}{2t|K_\alpha|^{1/n}} \leq \frac{e^c}{2-\alpha}\frac{|K_\alpha + tB_2^n|^{1/n}}{2t|B_2^n|^{1/n}}$$

$$\leq \frac{e^c}{2-\alpha}\left(N(K_\alpha + tB_2^n, 2tB_2^n)\right)^{1/n}$$

$$\leq \frac{e^c}{2-\alpha}\left(N(K_\alpha, tB_2^n)\right)^{1/n} \leq \frac{e^c}{2-\alpha}\exp\left(\frac{c}{t^\alpha(2-\alpha)}\right). \quad \square$$

In the proof of Theorem 1.1. in [KPZ, p.2705], we have $\alpha = 2 - \frac{1}{\log e\frac{n}{k}}$ and $t^\alpha(2-\alpha) = \frac{n}{k}$, so $t \sim \sqrt{\frac{n}{k}\log(\frac{en}{k})}$. Then Corollary 2 implies

$$|K_\alpha + tB_2^n|^{1/n} \leq c'\sqrt{\frac{n}{k}}\left(\log\left(\frac{en}{k}\right)\right)^{3/2}|K_\alpha|^{1/n},$$

where c' is an absolute constant. Using this estimate in place of Corollary 3.2 in [KPZ, p.2705], we get Proposition 1.

Proposition 1 in conjunction with Corollary 1 implies the following slicing inequality.

Theorem 5. *There exists an absolute constant C_0 such that for every $n \in \mathbb{N}$, every $1 \leq k < n$, every origin-symmetric convex body L in $\mathbb{R}^n$ and every measure μ with even continuous non-negative density on L*

$$\mu(L)\ \leq\ C_0^k\left(\sqrt{\frac{n}{k}}\left(\log\left(\frac{en}{k}\right)\right)^{3/2}\right)^k\frac{n}{n-k}c_{n,k}\max_{H\in Gr_{n-k}}\mu(L\cap H)\ |L|^{k/n}.$$

Corollary 3. *If the codimension of sections k satisfies $\lambda n \le k < n$, for some $\lambda \in (0,1)$, then for every origin-symmetric convex body L in $\mathbb{R}^n$ and every measure μ with continuous non-negative density in $\mathbb{R}^n$,*

$$\mu(L) \ \le \ C_0^k \left(\sqrt{\frac{(1-\log\lambda)^3}{\lambda}}\right)^k \frac{n}{n-k} c_{n,k} \max_{H\in Gr_{n-k}} \mu(L\cap H)\ |L|^{k/n},$$

where C_0 is an absolute constant.

6. Minimal sections

We consider Schwartz distributions, i.e. continuous functionals on the space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$. For any even distribution f, we have $(\hat{f})^\wedge = (2\pi)^n f$.

If K is a convex body and $0<p<n$, then $\|\cdot\|_K^{-p}$ is a locally integrable function on $\mathbb{R}^n$ and represents a distribution acting by integration. Suppose that K is infinitely smooth, i.e. $\|\cdot\|_K \in C^\infty(S^{n-1})$ is an infinitely differentiable function on the sphere. Then by [K1, Lemma 3.16], the Fourier transform of $\|\cdot\|_K^{-p}$ is an extension of some function $g\in C^\infty(S^{n-1})$ to a homogeneous function of degree $-n+p$ on $\mathbb{R}^n$. When we write $\left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)$, we mean $g(\xi),\ \xi\in S^{n-1}$.

For $f\in C^\infty(S^{n-1})$ and $0<p<n$, we denote by

$$(f\cdot r^{-p})(x) = f(x/|x|_2)|x|_2^{-p}$$

the extension of f to a homogeneous function of degree $-p$ on $\mathbb{R}^n$. Again by [K1, Lemma 3.16], there exists $g\in C^\infty(S^{n-1})$ such that

$$(f\cdot r^{-p})^\wedge = g\cdot r^{-n+p}.$$

If K, L are infinitely smooth convex bodies, the following spherical version of Parseval's formula was proved in [K4] (see also [K1, Lemma 3.22]): for any $p\in(-n,0)$

$$\int_{S^{n-1}} \left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)\left(\|\cdot\|_L^{-n+p}\right)^\wedge(\xi) = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}\,dx. \tag{9}$$

It was proved in [K2, Theorem 1] that an origin-symmetric convex body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution. If K is infinitely smooth, this means that the function $(\|\cdot\|_K^{-1})^\wedge$ is non-negative on the sphere.

We also need a result from [K7] (see also [K1, Theorem 3.8]) expressing volume of central hyperplane sections in terms of the Fourier transform. For any origin-symmetric star body K in $\mathbb{R}^n$, the distribution $(\|\cdot\|_K^{-n+1})^\wedge$ is a continuous function on the sphere extended to a homogeneous function of degree -1 on the whole of $\mathbb{R}^n$, and for every $\xi \in S^{n-1}$,

$$|K \cap \xi^\perp| = \frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\xi). \tag{10}$$

In particular, if $K = B_2^n$ and $|\cdot|_2$ is the Euclidean norm, then for every $\xi \in S^{n-1}$

$$(|\cdot|_2^{-n+1})^\wedge(\xi) = \pi(n-1)|B_2^{n-1}|. \tag{11}$$

Lemma 1. *Let K be an origin-symmetric infinitely smooth convex body in $\mathbb{R}^n$. Then*

$$\int_{S^{n-1}} \left(\|\cdot\|_K^{-1}\right)^\wedge(\xi)d\xi \le \frac{(2\pi)^n n}{\pi(n-1)} c_{n,1}|K|^{1/n},$$

Proof : By (11), Parseval's formula, Hölder's inequality, polar formula for volume (6) and $|S^{n-1}| = n|B_2^n|$, we get

$$\int_{S^{n-1}} \left(\|\cdot\|_K^{-1}\right)^\wedge(\xi)d\xi$$

$$= \frac{1}{\pi(n-1)\left|B_2^{n-1}\right|}\int_{S^{n-1}} \left(\|\cdot\|_K^{-1}\right)^\wedge(\xi)\left(|\cdot|_2^{-n+1}\right)^\wedge(\xi)$$

$$= \frac{(2\pi)^n}{\pi(n-1)\left|B_2^{n-1}\right|}\int_{S^{n-1}} \|\theta\|_K^{-1}\, d\theta$$

$$\le \frac{(2\pi)^n}{\pi(n-1)\left|B_2^{n-1}\right|}\left|S^{n-1}\right|^{\frac{n-1}{n}}\left(\int_{S^{n-1}} \|\theta\|_K^{-n}\, d\theta\right)^{\frac{1}{n}}$$

$$= \frac{(2\pi)^n}{\pi(n-1)\left|B_2^{n-1}\right|}\left|S^{n-1}\right|^{\frac{n-1}{n}} n^{1/n}|K|^{1/n} = \frac{(2\pi)^n n}{\pi(n-1)} c_{n,1}|K|^{1/n}. \quad \square$$

The following theorem provides examples where the minimal measure of sections behaves in a different way from the case of volume. Note that every non-intersection body can be approximated in the radial metric by infinitely smooth non-intersection bodies with strictly positive curvature; see [K1, Lemma 4.10]. Different examples of convex bodies that are not intersection bodies (in dimensions five and higher, as in dimensions up to four such examples do not exist) can be found in [K1, Chapter 4]. In particular, the unit balls of the spaces $\ell_q^n,\ q > 2,\ n \ge 5$ are not intersection bodies.

Theorem 6. *Suppose that L is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$ with strictly positive curvature that is not an intersection body. Then for small enough $\varepsilon > 0$ there exists an origin-symmetric convex body K in $\mathbb{R}^n$, $K \subset L$, such that*

$$|K \cap \xi^\perp| \le |L \cap \xi^\perp| - \varepsilon, \qquad \forall \xi \in S^{n-1},$$

but

$$|K|^{\frac{n-1}{n}} > |L|^{\frac{n-1}{n}} - c_{n,1}\,\varepsilon.$$

Note that $c_{n,1} \in (\frac{1}{\sqrt{e}}, 1)$.

Proof : Since L is infinitely smooth, the Fourier transform of $\|\cdot\|_L^{-1}$ is a continuous function on the sphere S^{n-1}. Also, L is not an intersection body, so $\left(\|\cdot\|_L^{-1}\right)^\wedge < 0$ on an open set $\Omega \subset S^{n-1}$. Let $\phi \in C^\infty(S^{n-1})$ be an even non-negative, not identically zero, infinitely smooth function on S^{n-1} with support in $\Omega \cup -\Omega$. Extend ϕ to an even homogeneous of degree -1 function $\phi \cdot r^{-1}$ on $\mathbb{R}^n \setminus \{0\}$. The Fourier transform of this function in the sense of distributions is $\psi \cdot r^{-n+1}$ where ψ is an infinitely smooth function on the sphere.

Let ε be a number such that $|B_2^{n-1}|\|\theta\|_L^{-n+1} > \varepsilon > 0$ for every $\theta \in S^{n-1}$. Define a star body K by

$$\|\theta\|_K^{-n+1} = \|\theta\|_L^{-n+1} - \delta\psi(\theta) - \frac{\varepsilon}{|B_2^{n-1}|}, \qquad \forall \theta \in S^{n-1}, \tag{12}$$

where $\delta > 0$ is small enough so that for every θ

$$|\delta\psi(\theta)| < \min\left\{\|\theta\|_L^{-n+1} - \frac{\varepsilon}{|B_2^{n-1}|},\ \frac{\varepsilon}{|B_2^{n-1}|}\right\}.$$

The latter condition implies that $K \subset L$. Since L has strictly positive curvature, by an argument from [K1, p. 96], we can make ε, δ smaller (if necessary) to ensure that the body K is convex.

Now we extend the functions in (12) from the sphere to $\mathbb{R}^n \setminus \{0\}$ as homogeneous functions of degree $-n+1$ and apply the Fourier transform. We get that for every $\xi \in S^{n-1}$

$$\left(\|\cdot\|_K^{-n+1}\right)^\wedge(\xi) = \left(\|\cdot\|_L^{-n+1}\right)^\wedge(\xi) - (2\pi)^n\delta\phi(\xi) - \pi(n-1)\varepsilon. \tag{13}$$

Here, we used (11) to compute the last term. By (13), (10) and the fact that the function ϕ is non-negative,

$$|K \cap \xi^\perp| = |L \cap \xi^\perp| - \frac{(2\pi)^n}{\pi(n-1)}\delta\phi(\xi) - \varepsilon \le |L \cap \xi^\perp| - \varepsilon. \tag{14}$$

Multiplying both sides of (13) by $\left(\|\cdot\|_L^{-1}\right)^\wedge(\xi)$, integrating over S^{n-1} and using Parseval's formula on the sphere, we get

$$(2\pi)^n \int_{S^{n-1}} \|\theta\|_L^{-1}\ \|\theta\|_K^{-n+1} d\theta$$

$$= (2\pi)^n n|L| - (2\pi)^n \delta \int_{S^{n-1}} \phi(\theta)\left(\|\cdot\|_L^{-1}\right)^\wedge(\theta)d\theta$$

$$-\pi(n-1)\varepsilon \int_{S^{n-1}} \left(\|\cdot\|_L^{-1}\right)^\wedge(\theta)d\theta.$$

Since ϕ is a non-negative function supported in Ω, where $\left(\|\cdot\|_L^{-1}\right)^\wedge$ is negative, the latter equality implies

$$(2\pi)^n n|L| - \pi(n-1)\varepsilon \int_{S^{n-1}} \left(\|\cdot\|_L^{-1}\right)^\wedge(\theta)d\theta$$

$$< (2\pi)^n \int_{S^{n-1}} \|\theta\|_L^{-1}\ \|\theta\|_K^{-n+1} d\theta$$

$$\leq (2\pi)^n \left(\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta\right)^{\frac{n-1}{n}} \left(\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta\right)^{\frac{1}{n}}$$

$$= (2\pi)^n n|L|^{\frac{1}{n}}|K|^{\frac{n-1}{n}}.$$

Combining the latter inequality with the estimate of Lemma 1, we get the result. □

Corollary 4. *Suppose that L is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$ with strictly positive curvature that is not an intersection body. Then there exists an even continuous function $g \geq 0$ on L so that*

$$\int_L g < \frac{n}{n-1} c_{n,1}|L|^{1/n} \min_{\xi\in S^{n-1}} \int_{L\cap\xi^\perp} g. \tag{15}$$

Proof : By Theorem 6 there exist $\varepsilon > 0$ and an origin-symmetric convex body $K \subset L$ such that

$$\varepsilon = \min_{\xi\in S^{n-1}} \left(|L\cap\xi^\perp| - |K\cap\xi^\perp|\right),$$

but

$$|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} < c_{n,1}\varepsilon.$$

Combining these and applying the Mean Value Theorem to the function $t \to t^{\frac{n-1}{n}}$

$$c_{n,1} \min_{\xi\in S^{n-1}} \left(|L\cap\xi^\perp| - |K\cap\xi^\perp|\right) > |L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}$$

$$\geq \frac{n-1}{n}|L|^{-1/n}\left(|L|-|K|\right).$$

The latter shows that $g_0 = \chi_{L\setminus K}$, the indicator function of the set $L \setminus K$, satisfies (15). By simple approximation one can get (15) with a continuous function g. □

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