Homogeneous flows and geometry of numbers

Gregory Margulis

Yale Mathematics Department

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This talk is based on the joint work with J. Athreya on the logarithm laws for unipotent flows.

Two important classes of dynamical systems on non-compact manifolds are the geodesic and horocycle flows on the unit tangent bundle of a finite volume non-compact hyperbolic surface. Both of these flows are known to be ergodic, and thus, generic points are dense. A natural question is to understand the behavior of excursion of the trajectories into the cusp(s).

For geodesic flows, statistical properties of these excursions were first studied by D. Sullivan (in the context of finite volume hyperbolic manifolds) and later, in the more general context of finite volume locally symmetric spaces, by D. Kleinbock and myself. About ten years ago we proved the following result:

Let S be a noncompact irreducible locally symmetric space of noncompact type and finite volume. Let $S = K\backslash G/\Gamma$ for some semisimple Lie group G, and irreducible non-uniform lattice $\Gamma \subset G$. Here K denotes the maximal compact subgroup of G. For $x \in S$, let $T^1_x(S)$ denote the unit tangent space at x, let ν denote the Lebesgue measure on $T^1_x(S)$. For $\theta \in T^1_x(S)$, and $t \in \mathbb{R}$ let $g_t(x, \theta)$ denote the image of (x, θ) under geodesic flow for time t, let d_S denote the distance function on S.

Theorem 1. There exists $k = k(S) > 0$ such that the following holds: for all $x, y \in S$, and any sequence of times t, and almost every $\theta \in T^1_x(S)$,

$$\limsup_{t\to\infty} \frac{d(g_t(x,\theta), y)}{\log t} = 1/k.$$

The proof of Theorem 1 is mostly based on the exponential rate of mixing for the geodesic flow on the unit tangent bundle of S.

Recently J. Athreya and myself considered the question of excursion into the cusp for unipotent flows and horospherical actions on finite volume homogeneous spaces, and in particular, on the spaces of unimodular lattices in $\mathbb{R}^n$. We used probabilistic techniques (generalized Borel-Cantelli lemmas), estimates on decay of matrix coefficients for unitary representations of semisimple groups, dynamics of horospherical actions and some other methods. Here I will concentrate on results related to the geometry of numbers. These results apply to unipotent flows on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$.

Let $\Omega_n = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ denote the space of unimodular lattices in $\mathbb{R}^n$. Define $\alpha_1 : \Omega_n \mapsto \mathbb{R}^+$ by

$$\alpha_1(\Lambda) := \sup_{0\neq\nu\in\Lambda} \frac{1}{||\nu||}$$

The logarithm of $\alpha_1(\Lambda)$ characterizes, up to a bounded function, one of the natural distances (in Ω_n) from Λ to $\mathbb{Z}^n$.

Let $\{u_t\}_{t\in\mathbb{R}}$ denote a unipotent one-parameter subgroup of $SL(n,\mathbb{R})$.

Theorem 2. For almost every $\Lambda \in \Omega_n$ (with respect to the Haar measure $\mu = \mu_n$ on Ω_n),

$$\limsup_{t\to\infty} \frac{\log \alpha_1(u_t\Lambda)}{\log t} = \frac{1}{n}.$$

We split the proof of Theorem 2 into two halves:

(1)Upper bounds: $\limsup_{t\to\infty} \frac{\log \alpha_1(u_t\Lambda)}{\log t} \le \frac{1}{n}$.

(2)Lower bounds: $\limsup_{t\to\infty} \frac{\log \alpha_1(u_t\Lambda)}{\log t} \ge \frac{1}{n}$.

For the upper bound, we use the Siegel integral formula, presented below, to estimate the measure of the set of lattices that have a short vector, and apply the easy (convergence) half of the Borel-Cantelli Lemma).

For the lower bound, we construct a sequence of times t_k so that $u_{t_k}\Lambda$ has an appropriately short vector. In order to do this, we construct a sequence of regions $A_k \subset \mathbb{R}^n$ such that if $\Lambda \cap A_k \neq \{0\}$, then $u_{t_k}\Lambda$ has a short vector. For $n = 2$ the regions A_k that we construct are centrally symmetric rectangles of area 4, and by Minkowski's theorem, we can find non-zero lattice points in these regions for *any* lattice (and thus, our lower bound holds for every lattice Λ whose $\{u_t\}$ orbit is not closed). For $n > 2$, not all lattices intersect our A_k's non-trivially. Instead, we estimate the measure of the set of lattices who do not intersect A_k non-trivially.

Let $n = 2$. Then we can assume that

$$u_t = h_t \overset{\text{def.}}{=} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Let $\Lambda \in \Omega_2$ be a lattice without a horizontal vector (i.e. it is not $\{h_t\}$-periodic). For $k \in \mathbb{N}$ define

$$A_k := \{(x,y) \in \mathbb{R}^2 : |x| \le \sqrt{k}, |y| \le 1/\sqrt{k}\}.$$

A_k is a convex, centrally symmetric region of area 4, so by Minkowski's theorem, there is a non zero point $(x_k, y_k) \in A_k \cap \Lambda$, and moreover, since Λ has no horizontal vectors, $y_k \neq 0$ for all k. Let $t_k = -x_k/y_k$. The sequence $\{(x_k, y_k)\}$ is unbounded: otherwise, our lattice would have an accumulation point. Therefore $t_k \to \infty$. Now, note that

$$\alpha_1(h_{t_k}\Lambda) \ge 1/|y_k|,$$

since $h_{t_k}(x_k, y_k) = (0, y_k)$. Now $|t_k| = |-x_k/y_k| \le \sqrt{k}/y_k \le 1/y_k^2$, so

$$\log \alpha_1(h_{t_k}\Lambda) \ge \log 1/|y_k| \ge \frac{1}{2}\log|t_k|,$$

so we have produced an unbounded sequence of times t_k where we achieve our lower bound (but with maybe $t_k \to -\infty$).

Now, let $n > 2$. As all essential ideas are contained in the case $n = 3$, we describe only this case. Then we can assume that either

(i) $u_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$ or (ii) $u_t = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$.

Fix $\epsilon > 0$. In the case (i) we put $A_k := \{(x, y, z) \in \mathbb{R}^3 : |x| \le |k|^{-1/3}, -k^{2/3} \le y \le -1, k^{-1/3-\epsilon} \le z \le k^{-1/3+\epsilon}\}$ and in the case (ii) we put $A_k := \{(x, y, z) \in \mathbb{R}^3 : |x - y^2/2z| \le |k|^{-1/3}, -k^{2/3} \le y \le -1, k^{-1/3-\epsilon} \le z \le k^{-1/3+\epsilon}\}$.

In both cases $m(A_k) \to \infty$ where m is the Lebesgue measure. Suppose that we proved the following

Lemma 1. Let $\{B_k\}_{k\in\mathbb{N}}$ be a sequence of Borel measurable sets in $\mathbb{R}^n, n \ge 3$, such that $m(B_k) \to \infty$.

Then

$$\lim_{k\to\infty} \mu(\Lambda \in \Omega_n : \Lambda \cap B_k = \emptyset) \to 0.$$

Then, for almost all $\Lambda \in \Omega_3$, by passing to a subsequence of needed, we can produce a sequence of distinct points $\{(x_k, y_k, z_k) \in A_k\}$ with $z_k \ne 0$.

Set $t_k = -y_k/z_k$. Then $t_k \to +\infty$. Also in both cases (i) and (ii)

$$\alpha(u_{t_k}\Lambda) \ge \frac{1}{2k^{-1/3+\epsilon}}$$

and

$$t_k \le k^{1+3\epsilon}.$$

From this we get that

$$\limsup_{t\to\infty} \frac{\log \alpha_1(u_t\Lambda)}{\log t} > \frac{1}{3} - \delta(\epsilon)$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. It gives the desired lower bound for almost all $\Lambda \in \Omega_n$.

The proof lemma 1 is based on classical results from the geometry of numbers. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a bounded, compactly-supported function. We define $\hat{f} : \Omega_n \mapsto \mathbb{R}$ by

$$\hat{f}(\Lambda) = \sum_{0\ne\nu\in\Lambda} f(\nu).$$

We have the Siegel Integral Formula:

$$\int_{\Omega_n} \hat{f} d\mu = \int_{\mathbb{R}^n} f dm$$

where m is the Lebesgue measure on $\mathbb{R}^n$ and μ is the normalized Haar measure on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. As a consequence of this, we got that the L^1-norm of $\hat{f}$ is not greater than the L^1-norm of f.

For L^2-norms we have the following Lemmas 2 and 3, which are special cases of results obtained by C. A. Rogers more than fifty years ago. We specialize to the case where $f = I_A$, the indicator function of a bounded set A. Define $a := ||I_A||_1 = m(A)$. Let B_a be the ball around the origin 0 with $m(B_a) = a$.

Lemma 2. For $n \geq 3$,

$$\int_{\Omega_n} \hat{I}_A^2 d\mu \leq \int_{\Omega_n} \hat{I}_{B_A}^2 d\mu.$$

Remark. Rogers works with general Borel measurable functions and their *spherical symmetrizations*.

Lemma 2 is a corollary of the following integral formula.

Lemma 3. Let $n \geq 3$. Let f be a non-negative Borel measurable function on $\mathbb{R}^n$. Then

$$\int_{\Omega_n} \hat{f}^2(\Lambda) d\mu = (\int_{\mathbb{R}^n} f d\mu)^2 + \sum_{k,q \in \mathbb{Z}:(k,q)=1} \int_{\mathbb{R}^n} f(kx) f(qx) dm(x)$$

where (k, q) is the greatest common divison of k and q.

In fact, the proof of this result (and its analogs for p-norms, $p > 2$, also given by Rogers) and the proof of the Siegel integral formula (which can be thought of as such a formula for $p = 1$) follow on much the same lines. Define a functional T_p on the space $C_c(\mathbb{R}^n \setminus \{0\})^p$ of continuous, compactly supported functions on $(\mathbb{R}^n \setminus \{0\})^p$ by

$$T_p(h) = \int_{\Omega_n} \sum_{v_1,\dots,v_p \in \Lambda \setminus \{0\}} h(v_1, \dots v_p).$$

This is clearly $SL(n, \mathbb{R})$ invariant, and thus the measure that defines it must be combination of $SL(n, \mathbb{R})$-invariant measures on $SL(n, \mathbb{R})$-orbits in $(\mathbb{R}^n \setminus \{0\})^p$. We perform this decomposition and apply it to functions h on $(\mathbb{R}^n \setminus \{0\})^p$ defined by $h(v_1, \dots, v_p) = f(v_1) \dots f(v_p)$, where $f \in C_c(\mathbb{R}^n \setminus \{0\})$.

For $p = 1$ there is only one orbit, all of $\mathbb{R}^n \setminus \{0\}$. For $p = 2$, there is an orbit consisting of linearly independent paris of vectors, which yields the first term ($||f||_1^2 = ||\hat{f}||_1^2$), and the second term captures the contribution of paris of linearly dependent vectors. For $p > 2$, the formula becomes much more complicated. It should be noted that the p-norm formula only works for dimensions $n > p$ (in fact if $n = p$ then the L^p-norm of $\hat{f}$ is usually infinite).

Using Siegel integral formula, one can easily get the following:

Lemma 4. For $n \geq 3$,

$$\lim_{a \to \infty} \frac{a^2}{||\hat{I}_{B_a}||^2} = 1.$$

Lemma 1 is an easy consequence of Siegel intergral formula, lemmas 2 and 4, and of the Cauchy-Schwartz inequality.

Concluding remarks. (a) If $p < n$ is an integer, Rogers proved that $||\hat{I}_A||_p \leq ||\hat{I}_{B_A}||_p$ in the following cases: (1) $p = 3$, (2) A is convex. I do not know if this inequality is proved for any A and any integer $p < n$. It will be interesting to prove the inequality when $p \geq 1$ is not an integer.

(b) A. Venkatesh informed us that the statement of Lemma 1 is true also for $n = 2$. In this case the proof is much more complicated and is based on the theory of Eisenstein series.

(c) A. Venkatesh also suggested the following question which is in the spirit of Lemma 1. Let $A \subset \mathbb{R}$ be a set of positive Lebesgue measure. Is it true that the complement in $\mathbb{R}$ of the union $\bigcup_{n\in\mathbb{Z}} nA$ has finite Lebesgue measure?

(d) It would be interesting to get an analog of Lemma 1 for the set of saddle connections in a strata if abelian differentials on a surface of genus $g > 1$.