LINEAR REPRESENTATION OF QUASISPLIT TORI AND PROBLEM OF RATIONALITY

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Abstract

We study faithful representations of quasisplit algebraic tori and prove the old-standing conjecture on rationality of a stably rational torus.

It is an important problem in the theory of linear algebraic groups defined over a nonclosed field to classify the varieties of these groups up to birational equivalence. This problem is still far from its solution. At present time we can only classify algebraic tori up to stable equivalence, and have a few partial results on their birational classification. In this paper we prove the old-standing conjecture on the rationality of stably rational tori. Consequently we get a very effective criterion allowing us to obtain definitive results in some of the classification problems, and to simplify some of the proofs.

Let us recall some facts and definitions. Let X be an irreducible algebraic variety defined over a field k. The variety X is called *rational* if the function field $k(X)$ is purely transcendental over k, and it is called *unirational* if its function field is contained in a purely transcendental extension of k. The affine space A^n_k is rational over k, hence we will denote the purely transcendental extension of k of dimension n by $k(A^n)$. The unirational varieties X naturally arise in the problems of the parametrisation of objects by rational functions of several variables. If $\dim X = 1$, then X is rational over k (Lüroth's theorem); if $\dim X = 2$ and k is algebraically closed, then the variety X is also rational (Castelnuovo's theorem). In any other case, an example of a non-rational unirational variety has been constructed. Since the variety of a connected linear group defined over a perfect field is known to be unirational over the field of definition [1], the classification problem of unirational varieties arises quite naturally in the theory of linear algebraic groups. Moreover, many of the important parameters of linear groups turn out to be of birational nature [2, 3]. The notions of stable rationality and stable equivalence appear to be rather useful in the classification problems of the varieties of connected linear groups. The k–varieties X_1 and X_2 are said to be stably equivalent if the products $X_1 \times_k A^m$ and $X_2 \times_k A^p$ are birationally isomorphic over k for some m and p. If there is a birational k–isomorphism $X \times_k A^m \cong A^p$, then the variety X is said to be *stably rational*. Clearly, a stably rational variety is unirational. In the paper [4] in 1984, examples of stably rational varieties, which are not rational, have been constructed. If the field k is not algebraically closed, then such "monsters" exist in any dimension bigger than 1; over the complex number field, there are such examples in any dimension bigger than 2. In the same paper [4], one can find the detailed discussion of this question and the exact bibliographical references.

Let us now turn to the variety of a connected linear algebraic k-group G. If k is algebraically closed, then G is rational, that is the field $k(G)$ is purely transcendental. If k is perfect, then the variety G is unirational (Chevalley and Grothendieck, [1]). In the case of algebraic tori, we have the following more detailed classification. Let $D_k(n)$ be a decomposable k–torus, that is a connected diagonal group defined over k; thus $D_k(n) = Spec(k[M])$, where M is a commutative multiplicative torsion free group of rank n, and $k[M]$ stands for its group ring. The algebra $k[M]$ is isomorphic to the Laurent ring $k[M] = k[\chi_1, \ldots, \chi_n, \chi_1^{-1}, \ldots, \chi_n^{-1}]$, where χ_i are the generators of the group M. Let R be a commutative k–algebra; the set $D_k(n)(R)$ of the R-points of the scheme $D_k(n)$ can be easily described in terms of the group M, if one makes use of the functorial properties of the group ring

$$D_k(n)(R) = Hom_{k-ring}(k[M], R) = Hom_{gr}(M, R^*) \cong (R^*)^n,$$

where R^* stands for the multiplicative group of invertable elements of a ring R. Hence any R-point of the scheme $D_k(n)$ can be represented by a diagonal matrix $diag(a_1, \ldots, a_n)$ with $a_i \in R^*$. The product of matrices defines the group structure on the set $D_k(n)(R)$, and the functor $R \to D_k(n)(R)$

[1] Partially supported by a grant of RFBR (project No. 99-01-00014).

defines the group structure on the scheme $D_k(n)$. The Hopf operators of the Hopf algebra $k[M]$ can be defined as follows

$$m^*(\chi_i) = \chi_i \otimes \chi_i \text{ (coproduct)},$$
$$i^*(\chi_i) = \chi_i^{-1} \text{ (coinversion)},$$
$$e^*(\chi_i) = 1 \text{ (coidentity)}.$$

An element χ of the group M can be identified with a homomorphism of the group $D_k(n)$ to the group $D_k(1) = G_{m,k}$:

$$M = \hat{D_k(n)} = Hom_{gr}(D_k(n), G_{m,k});$$

such an element χ is called a character of the group $D_k(n)$. On the R-points, the basic characters χ_i act naturally as projections: $\chi_i(a_1, \ldots, a_n) = a_i$. There are some reasons to write the group of characters $\hat{D_k(n)}$ additively, the multiplicative notation is used only to avoid possible misunderstanding; for example, the latter notation is used in description of the group ring. In the additive notation, the group $\hat{D_k(n)}$ coincides with $\mathbf{Z}^n$. A decomposable torus $D_k(n)$ is a special case of a general diagonal group $D_k(M) = Spec\ k[M]$, where M is an arbitrary commutative group. If M is a finitely generated, then the k-scheme $D_k(M)$ is of finite type. The category of diagonal k-groups of finite type is known to be an abelian category, dual to the category of finitely generated commutative groups [5]. The duality is established by means of the correspondence $M \to D_k(M)$, or the inverse correspondence $D_k(M) \to \hat{D_k(M)}$. An exact sequence of k–groups

$$1 \to D_k(M'') \to D_k(M) \to D_k(M') \to 1$$

uniquely determines the exact sequence of the abelian groups

$$0 \to M' \to M \to M'' \to 0,$$

and conversely. Hence a curious fact: a homomorphic image of a diagonal k–torus $D_k(n)$ is a diagonal k-torus again. In particular, the image of a diagonal torus under an isogeny is isomorphic to that torus.

Let us now recall a few results from the general theory of k-tori. Denote by $\bar{k}$ the algebraic closure of the field k.

Definition. *An algebraic k–group T is called a k–torus if the k-group $T \otimes_k \bar{k}$ is isomorphic to a connected diagonal group $D_k(n) \otimes_k \bar{k} = D_{\bar{k}}(n)$, $n := \dim T$.*

A field L containing k is called a splitting field of the torus T if $T \otimes_k L \cong D_L(n)$. Any k–torus splits over a finite separable extension of the field k [1]. Let $\hat{T} = Hom_{gr}(T_{k_s}, G_{m,k_s})$ be the group of characters of the torus T, where k_s denotes the separable closure of the field k. The Galois group $\mathcal{G}$ of the extension k_s/k acts on the lattice $\hat{T}$ in a natural way. The group $\hat{T}$ is of finite rank, equal to the dimention of T. Hence there is an integral representation $\varphi : \mathcal{G} \to GL(n, \mathbf{Z})$, $n = \dim_k T$. The kernel $Ker(\varphi)$ of this map is a normal subgroup of a finite index in $\mathcal{G}$, and the field of invariants $L = k_s^{Ker(\varphi)}$ is the Galois extension of k of the lowest degree such that the torus T splits over L. The image $Im(\varphi)$ is a finite subgroup of the group $GL(n, \mathbf{Z})$, which determines the symmetries of the lattice $\hat{T}$ and provides important information about the structure of the torus T. Let $\Pi = \mathrm{Gal}\,(L/k)$, then $T = Spec(L(\hat{T})^{\Pi})$. The *permutation* Π-modules $\hat{S}$, that is those modules with a finite $\mathbf{Z}$–basis whose elements are permuted by the group Π, play a special role in the birational geometry of algebraic tori. A torus S defined over k by a permutation Π-module $\hat{S}$ is called a *quasisplit* torus. Only the k–decomposable tori are simpler than the quasisplit ones. On decomposing a basis of the Π-module $\hat{S}$, whose elements are permuted by the group Π, in Π-orbits, one defines a decomposition of the module $\hat{S}$ in a direct sum $\oplus_{i=1}^{s} \mathbf{Z}[\Pi/\pi_i]$, where π_i is a subgroup of the group Π, $1 \leq i \leq s$. The torus S with $\hat{S} = \mathbf{Z}[\Pi/\pi]$ can be easily described. Let $F = L^{\pi}$ be the field of invariants of such a torus, and let $\omega_1, \ldots, \omega_m$ be a basis of the extension L/k. Consider the regular representation of the field F in the algebra $M(m, k)$; this representation defines the group F^* as a subgroup in the group $GL(m, k)$. The torus S may be regarded as the maximal k-torus of the group $GL_k(m)$ satisfying the condition $S(k) = F^*$; it is usually denoted by $S = R_{F/k}(G_m)$. Thus any quasisplit torus L/k-torus is isomorphic to the product of the tori $R_{F_i/k}(G_m)$, where $k \subset F_i \subset L$. If $x_1, x_2, \ldots, x_m$ are the coordinates of a general element of the field F with respect to the basis $\omega_1, \dots, \omega_n$, then the coordinate ring of the torus $R_{F/k}(G_m)$ can be given explicitly: $k[R_{F/k}(G_m)] = k[x_1, \dots, x_m, y^{-1}]$, where $y = N_{F/k}(x_1\omega_1 + \dots + x_m\omega_m)$. The embedding $k[x_1, \dots, x_m] \subset k[x_1, \dots, x_m, y^{-1}]$ determines a faithful action of the group $R_{F/k}(G_m)$ on the space $R_{F/k}(A^1_F) \cong A^m_k$, and embeds the variety $R_{F/k}(G_m)$ in A^m as an open orbit. Making use of the described direct product decomposition of a quasisplit torus, one concludes that any quasisplit k-torus of dimension n can be realised as a maximal k-torus of the group $GL_k(A^n)$, and that the variety S may be regarded as an open orbit in the affine space A^n_k. A quasisplit torus S is k–rational. The following notion is important when discussing the problems of rationality.

Definition. *The Π-modules A and B are said to be similar, if there are permutation Π-modules $\hat{S}_1$ and $\hat{S}_2$ such that the direct sum $A \oplus \hat{S}_1$ is isomorphic to the direct sum $B \oplus \hat{S}_2$.*

Let us denote by $[A]$ the similarity class of a module A. The Π-module $\hat{T}$ of characters of a torus T has a flasque resolvent, that is there is an exact sequence of torsion-free Π-modules of finite rank

$$0 \to \hat{T} \to \hat{S} \to \hat{N} \to 0,$$

such that $\hat{S}$ is a permutation Π-module, and $\hat{N}$ is a *flasque* module (a module N is said to be flasque if the condition $H^{-1}(\pi, \hat{N}) = 0$ holds for any subgroup π of Π). Although a flasque resolvent of $\hat{T}$ is not uniquely defined, the similarity class $[\hat{N}]$ of $\hat{N}$ is determined uniquely by the module $\hat{T}$; we denote this class by $p(\hat{T})$ or $p(T)$. The class $p(T)$ is a birational invariant of the $k-$torus T [2,3]. Let X be a smooth projective completion of the torus T over k, the class $p(T)$ coincides with the class $[PicX_L]$ of the Picard module $PicX_L$. Let $C(L/k)$ be the category of algebraic k-tori decomposable over L, and let $P(\Pi)$ be the semigroup of the similarity classes of Π-modules generated by the flasque modules. The sum of the classes $[\hat{N}_1] + [\hat{N}_2]$ is defined to be equal to the class $[\hat{N}_1 \oplus \hat{N}_2]$. Let us define a semigroup $Z(L/k)$ generated by the classes of stably equivalent tori lying in $C(L/k)$, the product of two classes being given as follows:

$$\{T_1\} \times \{T_2\} = \{T_1 \times_k T_2\}.$$

We recall that the k-varieties X_1 and X_2 are said to be stably equivalent, if the products $X_1 \times_k \mathbf{A}^m$ and $X_2 \times_k \mathbf{A}^p$ are birationally isomorphic over k for some m and p. The following result can be found in [3].

Theorem 1. *The correspondence $T \to p(T)$ defines an isomorphism of the semigroups $Z(L/k)$ and $P(\Pi)$.*

This isomorphism may be regarded as a first approximation to the problem of birational classification of tori lying in $C(L/k)$. As one may expect, tori T with $p(T) = [0]$ have some special features. Such a torus T can be described as the following factor-group:

$$1 \to S_1 \overset{\alpha}{\to} S_2 \overset{\beta}{\to} T \to 1, \tag{1}$$

where S_1 and S_2 are quasisplit k-tori. Since the map β is surjective, it follows that the field $k(T)$ is contained in $k(S_2)$; therefore any k-torus is an unirational variety. A quasisplit torus S satisfies the following important condition: $H^1(M/M_1, S(M)) = 0$ for any Galois extension M/M_1 with $M_1 \supset k$. Consequently the map β of the sequence (1) has a k-rational section $\gamma : T \to S_2$ with $\beta\gamma = 1$. Thus there is a k-rational map

$$\varphi = (\alpha, \gamma) : S_1 \times_k T \to S_2, \tag{2}$$

given by

$$\varphi(s,t) = \alpha(s)\gamma(t), \quad s \in S_1(k_s), \quad t \in T(k_s).$$

The map φ is birational. For let $\alpha(s)\gamma(t) = \alpha(s_1)\gamma(t_1)$, then $\beta[\alpha(s)\gamma(t)] = \beta[\alpha(s_1)\gamma(t_1)]$; hence $t = t_1$, and therefore $s = s_1$. The torus T with the property $p(T) = [0]$ is stably rational over k. The tori S_1 and S_2 can be realised as maximal k-tori in the groups $GL(V_1)$ and $GL(V_2)$, $\dim V_i = \dim S_i$. Thus we see that the function field $k(T)$ of the torus T is isomorphic to the field of invariants $k(V_2)^{S_1}$ of the group S_1, which acts linearly and faithfully on V_2 by means of the representation α. We can rewrite the birational map φ defined by (2) as follows:

$$\varphi \ : \ V_1 \times_k T \to V_2. \tag{3}$$

Let now S be any maximal k-torus in $GL(V_1)$. We can define an action of S on $V_1 \times_k T$ by letting S act linearly and faithfully on V_1, and trivially on T (it is a reflection with respect to the the space T, in a generalised sense). The group S acts then birationally on V_2 in the obvious way: $g \to \varphi g \varphi^{-1}$. Curiously enough, the field $k(T)$ can be regarded as the field of invariants $k(V_2)^S$ for any maximal k-torus S of the group $GL(V_1)$, assuming S acts on V_2 in the way we have just described. One can now take S to be equal to a maximal diagonalizable subgroup $D \subset GL(V_1)$. It suffices to show that the group D acts linearly on the field $k(V_2)$, that is there is a basis $\{f_1, ..., f_n\}$ of the field $k(V_2)$ such that D acts linearly on the k-vector space $U =< f_1, ..., f_n >$, generated by this basis. For then D is contained in the maximal k-decomposable torus D_2 of the group $GL(U)$, and therefore the field $k(T) = k(V_2)^D = k(D_2/D)$ is a purely transcendental extension of k. Let us study the representation α in more detail. The representation α of the group S_1 in the group $S_2 \subset GL(V_2)$ defines on V_2 a structure of an S_1-module. Let us decompose the module V_2 in a direct sum of indecomposible S_1-modules. Without loss of generality, it can be assumed that this decomposition contains no trivial S_1-component, for if $V_2 = U \oplus W$ and the group S_1 acts trivially on the k-vector space W, then U is a faithful S_1-module and therefore the field of invariants $k(V_2)^{S_1}$ coincides with the purely transcendental extension $k(U)^{S_1}(W)$ of the field $k(U)^{S_1}$, so that the field $k(V_2)^{S_1}$ is rational as soon as $k(U)^{S_1}$ is. Similarly, one can assume that the chosen decomposition of the S_1-module V_2 contains no multiple components. Thus we have reduced the problem of the k-rationality of a stably rational k-torus to the problem of rationality of the field of invariants $k(V_2)^{S_1}$ of a quasisplit k-torus S_1, acting linearly and faithfully on a k-vector space V_2; moreover, it may be assumed that the space V_2 contains no S_1-subspace U with a faithful action of S_1. Let us call such a representation minimal and describe its structure. Consider at first the simplest case when $S_1 = R_{F/k}(G_m)$. Let L be the normal closure of F in k_s, and let

$$\Pi = \mathrm{Gal}\,(L/k),\ \Pi_1 = \mathrm{Gal}\,(L/F),\ \hat{S} = \mathbf{Z}[\Pi/\Pi_1] = \mathbf{Z}[\Pi] \otimes_{\Pi_1} \mathbf{Z}.$$

Let us choose a basis $\chi_1, .., \chi_d$ of the group $\hat{S}_1$ transitively permuted by the group Π, and let Π_1 denote the stabilizer of the character χ_1. There is an one-dimensional faithful linear representation of the group $G_{m,F}$ on the vector space $\mathbf{A}^1_F$. Then $V_1 = R_{F/k}(\mathbf{A}^1_F)$ is a space of the regular representation of the group $S_1 = R_{F/k}(\mathbf{G}_m)$. The trace of this representation is equal to the sum $\chi_1 + ... + \chi_d$. Let $e_1, ..., e_d$ be a basis of the space V_1 over k, let $\Theta_1 = \alpha_1 e_1 + ... + \alpha_d e_d$ be the eigenvector corresponding to the character χ_1, $\alpha_i \in F$, $\Theta_1 \in F \otimes_k V_1$. The other characters χ_i and their eigenvectors Θ_i can be obtained acting on χ_1, and Θ_1 by the Galois group. The module V_2 can be decomposed in a direct sum of indecomposable S_1-modules. Let V be one of the indecomposable components of the S_1-module V_2. Let us describe its structure. The group S_1 is diagonalizable over L, therefore the L-vector space $V \otimes_k L$ is isomorphic to a direct sum of one-dimensional components V_ξ transitively permuted by the Galois group $\Pi :\ \sigma V_\xi = V_{\sigma(\xi)}$. Each of the components V_ξ affords an one-dimensional representation of the group $S_{1,L}$ belonging to the character ξ. It follows therefore that the trace of restriction on V of the representation α is equal to

$$\mathrm{tr}\,(\alpha \mid V) = \xi_1 + ... + \xi_r,\ r = \dim V.$$

We call the characters ξ_i the weights of the representation of the group $S_{1,L}$. Let $\xi_1 = \chi_1^{a_1} \cdots \chi_d^{a_d}$ be one of the characters. Then any of the weights ξ_i can be obtained from ξ_1 by the permuting the exponents; these permutations are given by the action of the Galois group on the element ξ_1. The one-dimensional component V_{ξ_1} is generated by the eigenvector $\Theta_1^{a_1} \cdots \Theta_d^{a_d}$ belonging to the character ξ_1; here $\Theta_1^{a_1} \cdots \Theta_d^{a_d}$ is regarded as an element of the field of fractions $L(V_1)$ of the symmetric algebra $L[V_1]$ on the space $V_1 \otimes_k L$. Let π be the stabiliser of the character ξ_1 in the Galois group Π, and let $F_1 = L^\pi$ be its field of invariants. The space V of the linear representation of S_1 is generated over k by the k-rational functions $P_i(e_1, ..., e_d)$, $1 \le i \le r$, defined by the decomposition

$$\Theta_1^{a_1} \cdots \Theta_d^{a_d} = P_1 \omega_1 + \cdots + P_r \omega_r, \qquad (4),$$

where $\omega_1, ..., \omega_r$ is a basis of the extension F_1/k, and

$$\sigma(\Theta_1^{a_1} \cdots \Theta_d^{a_d}) = P_1 \sigma(\omega_1) + \cdots + P_r \sigma(\omega_r),\ \sigma \in \Pi.$$

Thus the S_1-module V_2 can be realized on functions from the field of fractions $k(V_1)$. Let $P_1, ..., P_n$ be the sequence of functions obtained from (4) for all the components of the k-vector space V_2; it follows from the minimality of the S_1-module V_2 that these functions are linearly independent over k. Thus the sequence of functions $P_1, ..., P_n$ is a k-basis of the vector space V_2. Since $\alpha : S_1 \to S_2$ is a monomorphism, the characters of the torus S_1, entering into the sum for $tr(\alpha \mid V_2)$ generate the group of characters of the torus S_1. But then the system of functions $\{P_1, ..., P_n\}$ generates the field $k(V_1)$, and in particular, each of the elements $e_1, ..., e_d$ can be written as a rational function of $P_1, ..., P_n$. We associate to each Laurent monomial $\Theta_1^{b_1} \cdots \Theta_d^{b_d}$, coinciding with the eigenvector of a character ξ entering into the sum for $tr(\alpha)$, the monomial $e_1^{b_1} \cdots e_d^{b_d}$ (forgetting about the action of the Galois group). There are exactly n such monomials, and they are pairwise distinct; let us list these monomials in some order: $f_1, ..., f_n$. The above argument shows also that the system of functions $f_1, ..., f_n$ generates the field $k(V_1)$. Therefore each of the functions $f_1, ..., f_n$ can be rationally expressed in terms of the functions $P_1, ..., P_n$ and vice versa: $f_i = R_i(P_1, ..., P_n)$, $P_i = Q_i(f_1, ..., f_n)$; here R_i and Q_i are rational functions of n independent variables with coefficients in k. Let D_1 be a maximal diagonal k-torus in the group $GL(V_1)$ acting on V_1 as follows: $g(e_i) = \lambda_i e_i$, $g \in D_1(k_s), \lambda_i \in k_s^*$. Then this action can be naturally extended to a linear diagonal action on the space $< f_1, ..., f_n >$ because the basic monomials are mapped to the proportional ones. The vector space V_2 is generated by the basis $\{P_1, ..., P_n\}$; let $k[V_2]$ be the symmetric algebra on the vector space V_2, and let $k(V_2)$ be its field of fractions. One remarks that the field $k(V_2)$ can be also described as the field of fractions of the symmetric algebra $k[f_1, \dots, f_n]$, and that the group D_1 acts diagonally on the elements of the basis $\{f_1, \dots, f_n\}$. This has as a consequence that there is an $n-$dimensional subtorus D_2 of the group of all the birational k-transformations of the vector space V_2, which contains the torus D_1 and splits over k. As we have already remarked, this implies that the variety T is birationally equivalent over k to the variety D_2/D_1, which is $k-$rational. Thus if $S_1 = R_{F/k}(G_m)$ and V_2 is a faithful S_1-module, then the field $k(V_2)^{S_1}$ is a purely transcendental extension of k. A general quasisplit $k-$torus S_1 is isomorphic to the product of the tori of the form $R_{F_i/k}(G_m)$, and V_1 is equal to a direct sum of the invariant subspaces of the form $R_{F_i/k}(\mathbf{A}^1) = V_{1,i}$. In each of the subspaces $V_{1,i}$, one chooses a $k-$basis as above, and a permutation basis $L-$basis $\Theta_1^{(i)}, ..., \Theta_d^{(i)}$ of the $L[\Pi]-$module $L \otimes_k V_{1,i}$. Then again the S_1-module V_2 can be generated by a system of functions from the field $k(S_1) = k(V_1)$:

$$V_2 =< P_1, ..., P_n >, \; n = dim_k V_2,$$

with $P_1, ..., P_n$ linearly independent over k. The rest of argument remains the same, as in the special case considered above. We have just obtained the main result of this article.

Theorem 2 (Main Theorem). *Any stably rational k-torus is rational over k.*

Thus the condition $p(T) = [0]$ characterises the algebraic k-tori rational over k.

Corollary 1. *Let $\alpha : S \to GL(V)$ be a faithful linear representation of a quasisplit torus S. There is a birational transformation of the vector space V into a $k-$vector space W, such that the action of the group S on W is quasiregular, that is the S-module W is isomorphic to the direct sum $W = W_1 \oplus W_2$ such that W_1 supports the regular representation of the torus S, while S acts trivially on W_2.*

Suppose now that L/k is a cyclic extension of degree $(L : k) = n$, and let $\Pi = \mathrm{Gal}\,(L/k)$. The ring of integers $\mathbf{Z}[\zeta_n]$ of the cyclotomic field $\mathbf{Q}(\zeta_n)$ can be regarded as a Π-module on letting $\sigma(\alpha) = \zeta_n \alpha$ for $\alpha \in \mathbf{Z}[\zeta_n]$, where ζ_n is a primitive $n-$th root of 1 and σ is a generator of Π. Let T_n be the k-torus, having the Π-module $\hat{T}_n = \mathbf{Z}[\zeta_n]$ as its module of rational characters. Since such a torus T_n is stably rational over k [3], we have the following result.

Corollary 2. *The torus T_n is rational over its field of definition, for any n.*

A direct proof of the rationality of the tori T_n has been given in the author's lecture at the Journées Arithméques in Limoges [6].

The problem of the reduction of a pair matrices to a canonical form gives us another series of stably rational varieties. Let $M(n) = M(n, \mathbf{C})$ be the ring of square matrices of order n with complex coefficients, and let $V_{m,n} = M(n) \oplus ... \oplus M(n)$ be the direct sum of m copies of the vector space $M(n)$. The group $GL(n, \mathbf{C})$ acts on $V_{m,n}$ by the conjugations. Let $K_{m,n} = \mathbf{C}(V_{m,n})^{GL_n}$ be its field of invariants. It is comparatively easy to show that the field $K_{m,n}$ is a purely transcendental extension of the field $K_{2,n}$, $m \geq 2$. The problem of rationality of the field $K_{2,n}$ is an old-standing problem. Sylvester had already proved rationality of the field $K_{2,2}$. Formanek [7,8] has proved that the fields $K_{2,3}$ and $K_{2,4}$ are rational. He has also shown that the field $K_{2,n}$ can be realized as a function field of an explicitly described algebraic torus. This line of thought is followed in [9], where it has been proved that the tori corresponding to the fields $K_{2,5}$ and $K_{2,7}$ are stably rational over $\mathbf{C}$. Since stable rationality of the fields $K_{2,a}$ and $K_{2,b}$ implies stable rationality of the field $K_{2,ab}$ if the integers a and b are relatively prime, it follows that the field $K_{2,d}$ is stably rational for any d dividing 420. We have a stronger result.

Corollary 3. *If* $d \mid 420$, *then the field* $K_{2,d}$ *is rational over* $\mathbf{C}$.

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Translated from the Russian by S.Yu.Popov and B.Z.Moroz