ON FREE ROTA–BAXTER ALGEBRAS

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Abstract. A Rota–Baxter algebra, also known as a Baxter algebra, is an algebra with a linear operator satisfying a relation, called the Rota–Baxter relation, that generalizes the integration by parts formula. Most of the studies on Rota–Baxter algebras have been for commutative algebras. Free commutative Rota–Baxter algebras were constructed by Rota and Cartier in the 1970s. A later construction was obtained by Keigher and one of the authors in terms of mixable shuffles. Recently, noncommutative Rota–Baxter algebras have appeared both in physics in connection with the work of Connes and Kreimer on renormalization in perturbative quantum field theory, and in mathematics related to the work of Loday and Ronco on dendriform dialgebras and trialgebras.

We give explicit constructions of free noncommutative Rota–Baxter algebras in various contexts. Our strategy is to obtain these free objects from Rota–Baxter algebra structures on classes of trees and forests. Elements of free Rota–Baxter algebras are expressed in terms of angularly decorated planar rooted forests. This furthers our understanding of free Rota–Baxter algebras and facilitates their further study. Such a forest interpretation then translates naturally into one in terms of bracketed words, which has already appeared in special cases in our recent study of dendriform algebras, thereby making the relation between Rota–Baxter algebras and dendriform algebras more transparent.

Contents

1. Introduction

This paper studies the constructions of free Rota–Baxter algebras with emphasis on their underlying rooted forest structure.

1.1. **Rota–Baxter algebras.** A Rota–Baxter algebra (also known as a Baxter algebra) is an associative algebra R with a linear endomorphism P satisfying the **Rota–Baxter relation**:

$$P(x)P(y) = P\big(P(x)y + xP(y) + \lambda xy\big),\ \forall x, y \in R. \tag{1}$$

Here λ is a fixed element in the base ring and is sometimes denoted by $-\theta$. The relation was introduced by the mathematician Glen E. Baxter [8] in his probability study, and was popularized mainly by the work of G.-C. Rota [48, 49, 50] and his school.

Note that the Rota–Baxter relation (1) is defined even if the binary operation is not associative. In fact, for Lie algebras such a relation was introduced independently by Belavin and Drinfeld [9], and Semenov-Tian-Shansky [52] in the 1980s. In this context they were related to solutions, called r-matrices, of the (modified) classical Yang–Baxter equation, named after the physicists Chen-ning Yang and Rodney Baxter. Recently, there have been several interesting developments of Rota–Baxter algebras in theoretical physics and mathematics, including quantum field theory [13, 14, 39, 40], Yang–Baxter equations [1, 2, 3], shuffle products [20, 34, 35], operads [5, 16, 21, 41, 42], Hopf algebras [7, 20], combinatorics [32] and number theory [25, 33]. The most prominent of these is the work of Connes and Kreimer in their Hopf algebraic approach to renormalization theory in perturbative quantum field theory [13, 14], continued in [18, 19, 24, 26].

1.2. **Free commutative Rota–Baxter algebras.** As pointed out by Cartier [11] thirty years ago, "The existence of free (Rota–)Baxter algebras follows from well-known arguments in universal algebra but remains quite immaterial as long as the corresponding word problem is not solved in an explicit way as Rota was the first to do." Both Rota's aforementioned construction [48] and the construction of Cartier himself in the above cited paper dealt with free commutative Rota–Baxter algebras. Later, a third construction was obtained by the second named author and Keigher [34, 35] later as a generalization of shuffle product algebras.

Other than the theoretical significance of these constructions of free commutative Rota–Baxter algebras, they have important applications. For example, Rota [49, 50] applied his construction to give a proof of the celebrated Spitzer identity [51, 19] by relating it to Waring's identity, another fundamental formula in combinatorics. The product that Cartier used to obtain his construction is readily seen to be the same as the one given by Ehrenborg [27] for the product of monomial quasi-symmetric functions and more recently the one given by Bradley [10] to explicitly describe stuffles and q-stuffles for multiple zeta values. Furthermore, the mixable shuffle product in the construction in [34] appeared also in the work of Goncharov [29] to study motivic shuffle relations. In [20], the mixable shuffle product is shown to be the same as Hoffman's quasi-shuffle product [38] which has played a fundamental role in the study of algebraic relations among multiple zeta values. There is also a description [4, 44] of quasi-shuffle in terms of piecewise linear paths (Delannoy paths).

1.3. **Motivations for going noncommutative.** Our goal in this paper is to give explicit constructions of free *noncommutative* Rota–Baxter algebras. There are several motivations for this study beyond a simple pursuit of generalizations. From a broad perspective, the crucial role played by noncommutative Rota–Baxter algebras in the seminal work of Connes and Kreimer [13, 14, 18, 19] on renormalization of quantum field theory helps to move the theoretical study of noncommutative Rota–Baxter algebras, including the free objects, to the forefront.

In a more theoretical context, there have been quite strong interests lately in possible noncommutative generalizations of shuffles and quasi-shuffles (that is, mixable shuffles). From the connection of these shuffles with free commutative Rota–Baxter algebras mentioned above, such noncommutative generalizations should be related to free noncommutative Rota–Baxter algebras. Indeed one such generalization is the Hopf algebra of planar rooted trees of Loday and Ronco [45] and we have shown in [23] that this algebra naturally embeds into a free noncommutative Rota–Baxter algebra. More generally, Our consideration of noncommutative free Rota–Baxter algebras is also motivated by the connection [1, 16] between Rota–Baxter algebras and dendriform algebras of Loday and Ronco [43]. In an earlier paper [23], we apply such a free object to obtain adjoint functors of the functors from Rota–Baxter algebras to dendriform dialgebras and trialgebras.

It is also our hope that the explicit constructions of the free Rota–Baxter algebra in this paper lay the foundation for further studies of Rota–Baxter algebra. Indeed, some of such studies [6, 36] have already been carried out concurrently with the writing of this paper. See below for further details. These constructions should also help the understanding of the operadic aspect of Rota–Baxter algebras and classification of Rota–Baxter type operators.

1.4. **Types of noncommutative Rota–Baxter algebras.** We now introduce the main subjects of this paper. We will give several constructions of free Rota–Baxter algebras. These constructions differ mainly in two aspects. One is the kinds of generators of the free Rota–Baxter algebras. The second aspect is that of the objects used in the constructions. To be more precise on the kinds of generators of free Rota–Baxter algebras, note that by leaving out various components of a Rota–Baxter algebra, we obtain forgetful functors from the category of (unitary) Rota–Baxter algebras to the categories of sets, modules, and algebras. The adjoint functors of these forgetful functors give rise to free Rota–Baxter algebras generated by (or over) a set, a module or an algebra. Further, by replacing unitary algebras by nonunitary algebras, we get another class of forgetful functors and their adjoint functors. Therefore there are overall six possible free Rota–Baxter algebras. Let us now give their precise definitions for later references.

Definition 1.1. *A* **free Rota–Baxter algebra over a k-module** *M is a Rota–Baxter algebra $F(M)$ with a Rota–Baxter operator P_M and a* $\mathbf{k}$*-module map $j_M : M \to F(M)$ such that, for any Rota–Baxter algebra R and any* $\mathbf{k}$*-module map $f : M \to R$, there is a unique Rota–Baxter algebra homomorphism $\bar{f} : F(M) \to R$ such that $\bar{f} \circ j_M = f$. That is, such that the following diagram commutes.*

$$\begin{tikzcd} M \arrow[r, "j_M"] \arrow[dr, "f"'] & F(M) \arrow[d, "\bar{f}"] \\ & R \end{tikzcd}$$

There are several variations of this definition.

(a) When the $\mathbf{k}$-module M is replaced by a set X and module maps are replaced by set maps in the definition, we obtain the concept of a **free Rota–Baxter algebra over the set** X. As in the case of free associative algebras, this concept is a specialization of the concept of free Rota–Baxter algebra over a module when the module is the free module generated by the set.
(b) When the $\mathbf{k}$-module M is replaced by an $\mathbf{k}$-algebra A and module maps are replaced by algebra homomorphisms in the definition, we obtain the concept of a **free Rota–Baxter algebra over the algebra** A. This concept is a generalization of the concept of free Rota–Baxter algebra over a module with the later being the special case when the algebra A is the tensor algebra over the module.
(c) In either the definition or its variations above, when all algebras are replaced by nonunitary algebras, we obtain the definition of a **free nonunitary Rota–Baxter algebra** over a module, over a set, or over an algebra.

The free Rota–Baxter algebras over a set and over a module are the analogs of the free associative algebra over a set, realized as the algebra of noncommutative polynomials, and over a module, realized as the tensor algebra. In our case, we describe the free Rota–Baxter algebras as rooted trees and more generally rooted forests with angles decorated by elements from the module or set. The concept of free Rota–Baxter algebra over another algebra has no analog for free associative algebra. But these algebras have properties similar to the corresponding algebras over a set and have already been studied in the papers [6, 36, 23] while the current paper is under writing. We will comment on these work later. Let us also point out that in the commutative case, free Rota–Baxter algebras over another commutative algebra were constructed in [34, 35] and it is in this context that the connection with quasi-shuffles of Hoffman [38] was made [20].

The distinction between unitary and nonunitary for associative algebra is often regarded as negligible since any nonunitary algebra A has a simple unitarization $\mathbf{k} \oplus A$. This is far from being the case for Rota–Baxter algebras with the involvement of the Rota–Baxter operator. Actually only with the help of our constructions of unitary and nonunitary free Rota–Baxter algebras, we are able to prove the existence of unitarization of Rota–Baxter algebras in Section 7. From the application point of view, unitary or nonunitary Rota–Baxter algebras are the preferred objects to study depending on the subject matter. For example, it is more convenient to use free nonunitary Rota–Baxter algebra to study the adjoint functors of the functors from Rota–Baxter algebra to dendriform dialgebras and trialgebras [23]. This is somehow analog to but more significant than the often use of nonunitary associative algebras in the study of the adjoint functor of the functor from associative algebras to Lie algebras.

1.5. **Rota–Baxter algebras of forests.** So the first feature of this article is that it will consider all these cases of free Rota–Baxter algebras. The second feature of this article is that we will obtain these free Rota–Baxter algebras from the underlying Rota–Baxter algebras on rooted trees and forests. Other than being of interest on its own right, the later Rota–Baxter algebras highlight as well simplify the various constructions of the free Rota–Baxter algebras. We now elaborate this point a little further with the precise definitions postponed to later sections. We will consider planar rooted forests $\mathcal{F}$ and two of their subsets, $\mathcal{F}^0$ of the ladder-free forests and $\mathcal{F}^r$ of the controlled forests. Together with the intersection $\mathcal{F}^{r,0}$ of the two subsets, we have four sets of forests and corresponding free $\mathbf{k}$-modules $\mathbf{k}\,\mathcal{F}, \mathbf{k}\,\mathcal{F}^0, \mathbf{k}\,\mathcal{F}^r$ and $\mathbf{k}\,\mathcal{F}^{r,0}$. We show in Section 2 and 5 that each of these four modules has a Rota–Baxter algebra structure (Theorem 2.5, Corollary 2.6, Theorem 5.1 and Corollary 5.2) and they together fit into the following commutative diagram.

$$\begin{tikzcd}
\mathbf{k}\,\mathcal{F}^0 \arrow[r] \arrow[d] & \mathbf{k}\,\mathcal{F} \arrow[d] \\
\mathbf{k}\,\mathcal{F}^{r,0} \arrow[r] & \mathbf{k}\,\mathcal{F}^r
\end{tikzcd}$$

Here the horizontal maps are injective Rota–Baxter algebra homomorphisms and vertical maps are surjective Rota–Baxter algebra homomorphisms. The surjections are constructed in Section 5.3.

Using these Rota–Baxter algebras, we construct in Section 3 the free unitary (resp. nonunitary) Rota–Baxter algebra $Ш^{\mathrm{N}C}(M)$ (resp. $Ш^{\mathrm{N}C,0}(M)$) over a module M (Theorem 3.4 and Theorem 3.6), and obtain in Section 6 the free unitary (resp. nonunitary) Rota–Baxter algebra $Ш^{\mathrm{N}C}(A)$ (resp. $Ш^{\mathrm{N}C,0}(A)$) over an algebra A (Theorem 6.3 and Theorem 6.7). When A is taken to be the tensor algebra of M, these free Rota–Baxter algebras likewise fit into in the following commutative diagram.

$$\begin{tikzcd}
Ш^{\mathrm{N}C,0}(M) \arrow[r] \arrow[d] & Ш^{\mathrm{N}C}(M) \arrow[d] \\
Ш^{\mathrm{N}C,0}(A) \arrow[r] & Ш^{\mathrm{N}C}(A)
\end{tikzcd}$$

We consider free Rota–Baxter algebras over a set in Section 4 and display two canonical bases of the free Rota-Baxter algebra, one in the form of angularly decorated forests (Theorem 4.1), the other one in the form of bracketed words formed by the generating set as the alphabet set together with the Rota–Baxter operator as the brackets (Theorem 4.6). More generally, given a free Rota–Baxter algebra over an algebra where the algebra has a choice of basis over the base ring, we can also give two realizations of the free Rota–Baxter algebra with bases in terms of decorated forests and bracketed words. Each of the two bases has its own advantages as is already shown in the papers [6, 36]. In [6] combinatorial properties of free Rota–Baxter algebras are discovered in connection with Schröder, Motzkin paths and Catalan numbers. There the authors worked with variations of our tree form of the basis that the authors derived largely independently. In [36], enumerative and algorithmic properties of free Rota–Baxter algebras are studied again related to Catalan numbers and their generating functions. There it is more convenient to work with the bracketed word form for computational purposes. These articles both display the utility of having an explicit bases available, and indicate that these free Rota–Baxter algebras are interesting objects to study on its own right.

As an application of these free Rota–Baxter algebras, the unitarization of Rota–Baxter algebras is studied in Section 7.

1.6. **Notations.** In this paper, $\mathbf{k}$ is a commutative unitary ring. By a $\mathbf{k}$-algebra we mean a unitary algebra over the base ring $\mathbf{k}$ unless otherwise stated. The same applies to Rota–Baxter algebras. For a set X, let $\mathbf{k}\,X$ be the free $\mathbf{k}$-module $\oplus_{x\in X}\mathbf{k}\,x$ generated by X. If X is a semigroup (resp. monoid), $\mathbf{k}\,X$ is given the natural nonunitary (resp. unitary) $\mathbf{k}$-algebra structure. Let $\mathbf{Alg}$ be the category of unitary $\mathbf{k}$-algebras A whose unit is identified with the unit $\mathbf{1}$ of $\mathbf{k}$ by the structure homomorphism $\mathbf{k} \to A$. Let $\mathbf{Alg}^0$ be the category of nonunitary $\mathbf{k}$-algebras. Similarly let $\mathbf{RB}_\lambda$ (resp. $\mathbf{RB}_\lambda^0$) be the category of unitary (resp. nonunitary) Rota–Baxter $\mathbf{k}$-algebras of weight λ. The subscript λ will be suppressed if there is no danger of confusion.

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2. The Rota–Baxter algebra of planar rooted forests

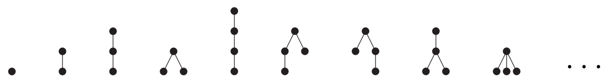
We first obtain a Rota–Baxter algebra structure on planar rooted forests and their various subsets. This allows us to give a uniform construction of free Rota–Baxter algebras in different settings in § 3 and in § 6.

2.1. **Planar rooted forests.** For the convenience of the reader and for fixing notations, we start by recalling basic concepts and facts of planar rooted trees. For references, see [15, 53].

A free tree is an undirected graph that is connected and contains no cycles. Equivalently, a free tree is an undirected graph in which any two vertices of the graph can be connected by a unique simple path. A **rooted tree** is a free tree in which a particular vertex has been distinguished as the **root**. Such a distinguished vertex endows the tree with a directed graph structure when the edges of the tree are given the orientation of pointing away from the root. If two vertices of a rooted tree are connected by such an oriented edge, then the vertex on the side of the root is called the **parent** and the vertex on the opposite side of the root is called a **child**. A vertex with no children is called a **leaf**. By our convention, in a tree with only one vertex, this vertex is a leaf, as well as the root. The number of edges in a path connecting two vertices in a rooted tree is called the **length** of the path.

The **depth** or **height** of a rooted tree is the length of the longest path from its root to its leafs. A **plane rooted tree** is a rooted tree that can be embedded into the plane. (A plane rooted tree can also be defined [12] to be a connected and simply connected set of oriented edges and vertices such that there is only one incoming edge at each vertex, except one vertex, the root, which has only outgoing edges.) A **planar rooted tree** is a plane rooted tree with a fixed embedding into the plane.

There are two ways to draw planar rooted trees. In one drawing all vertices are represented by a dot and the root is usually at the top of the tree. The following list shows the first few of them.



Note that we distinguish the sides of the trees, so the trees are non-symmetric. This drawing is used, for example, in the above reference [15, 53] of trees and in the Hopf algebra of non-planar rooted trees of Connes and Kreimer [12, 13].

In the second drawing the leaf vertices are removed with only the edges leading to them left, and the root, placed at the bottom in opposite to the first drawing, gets an extra edge pointing down. The following list shows the first few of them.

This is used, for example in the Hopf algebra of planar rooted trees of Loday and Ronco [43, 45] and noncommutative variation of the Connes-Kreimer Hopf algebra [37, 28]. In the following we will mostly use the first drawing. The tree $\bullet$ (or $|$ in the second drawing) with only the root is called the **empty tree**.

Let $\mathcal{T}$ be the set of planar rooted trees and let $\mathcal{F}$ be the free semigroup generated by $\mathcal{T}$ in which the product is denoted by $\sqcup$. Thus each element in $\mathcal{F}$ is a noncommutative product of elements in $\mathcal{T}$, called a **planar rooted forest**.

Remark 2.1. *For the rest of this paper, a tree or forest means a planar rooted one unless otherwise specified.*

So a forest is of the form $T_1 \sqcup \cdots \sqcup T_n$ consisting of trees $T_1, \cdots, T_n$. Here $\sqcup$ means putting two trees next to each other, i.e., the concatenation. We also use the abbreviation

$$T^{\sqcup n} = \underbrace{T \sqcup \cdots \sqcup T}_{n \text{ terms}}. \tag{2}$$

We use the (grafting) **brackets** $\lfloor T_1 \sqcup \cdots \sqcup T_n \rfloor$ to denote the tree obtained by **grafting**, that is, by adding a new root together with an edge from the new root to the root of each of the trees $T_1, \cdots, T_n$. In the new tree, the trees $T_1, \cdots, T_n$, now the branches, are in the same order as in $T_1 \sqcup \cdots \sqcup T_n$. This is the B^+ operator in the work of Connes and Kreimer [13]. The operation is also denoted by $T_1 \vee \cdots \vee T_n$ in some other literatures, such as in Loday and Ronco [43, 45]. Note that our operation $\sqcup$ is different from $\vee$. Their relation is

$$\lfloor T_1 \sqcup \cdots \sqcup T_n \rfloor = T_1 \vee \cdots \vee T_n.$$

Note that parentheses for $\sqcup$ is associative and so is nonconsequential:

$$(T_1 \sqcup T_2) \sqcup T_3 = T_1 \sqcup T_2 \sqcup T_3 = T_1 \sqcup (T_2 \sqcup T_3),$$

whereas grafting brackets for $\sqcup$ is highly nonassociative: $\lfloor T_1 \sqcup T_2 \rfloor \sqcup T_3$, $\lfloor T_1 \sqcup T_2 \sqcup T_3 \rfloor$, and $T_1 \sqcup \lfloor T_2 \sqcup T_3 \rfloor$ are all different forests.

The **depth** of a forest F is the maximal depth $\mathrm{d} = \mathrm{d}(F)$ of trees in F. It is clear that the depth of a forest F is increased by one in $\lfloor F \rfloor$. The trees in a forest F are called root branches of $\lfloor F \rfloor$. Furthermore, for a forest $F = T_1 \sqcup \cdots \sqcup T_b$ with trees $T_1, \cdots, T_b$, we define $b = b(F)$ to be the **breadth** of F. The number of leafs of a rooted forest, $F \in \mathcal{F}$, is denoted by $\ell(F)$. For a forest $F = T_1 \sqcup \cdots \sqcup T_b \in \mathcal{F}$ consisting of trees $T_1, \cdots, T_b$, we have the relation

$$\ell(F) := \sum_{i=1}^{b} \ell(T_i). \tag{3}$$

Example 2.1. *For example, the forest* $\lfloor \lfloor \bullet \rfloor \bullet \rfloor \bullet$ *has breadth 2, depth 2 and has 3 leafs.*

We now show that planar rooted forests are characterized by a recursive structure. For any subset X of $\mathcal{F}$, let $\langle X \rangle$ be the sub-semigroup of $\mathcal{F}$ generated by X. Let $\mathcal{F}_0 = \langle \bullet \rangle$, consisting of forests generated by $\bullet$. These are also the forests of depth zero. Then recursively define

$$\mathcal{F}_n = \langle \{\bullet\} \cup \lfloor \mathcal{F}_{n-1} \rfloor \rangle. \tag{4}$$

It is clear that $\mathcal{F}_n$ is the set of forests with depth less or equal to n. From this observation, we see that $\mathcal{F}_n$ form a linear direct system: $\mathcal{F}_n \supseteq \mathcal{F}_{n-1}$, and

$$\mathcal{F} = \cup_{n \geq 0} \mathcal{F}_n = \varinjlim \mathcal{F}_n. \tag{5}$$

Then we have

$$\lfloor \mathcal{F} \rfloor \subseteq \mathcal{F}.$$

In fact, $\mathcal{F}$ satisfies a universal property with respect to the product $\sqcup$ and the operator $\lfloor\ \rfloor$. Recall that a pointed set is a set together with a particular element of the set. We similarly define

Definition 2.2. *A* **mapped semigroup** *is a semigroup U together with a map $\alpha : U \to U$. A morphism from a mapped semigroup (U, α) to a mapped semigroup (V, β) is a semigroup homomorphism $f : U \to V$ such that $f \circ \alpha = \beta \circ f$.*

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "f"] & U \arrow[d, "f"'] \\ V \arrow[r, "\beta"] & V \end{tikzcd}$$

When semigroup is replaced by monoid, $\mathbf{k}$-algebra or nonunitary $\mathbf{k}$-algebra in the above definition, we obtain the concept of **mapped monoid**, **mapped k-algebras** or **mapped nonunitary k-algebra**. For example, the semigroup $\mathcal{F}$ of forests with the map $\lfloor\ \rfloor$ and the product $\sqcup$ is a mapped semigroup. The $\mathbf{k}$-module $\mathbf{k}\,\mathcal{F}$ generated by $\mathcal{F}$ is a mapped nonunitary $\mathbf{k}$-algebra.

The adjoint functor of the forgetful functor from the category of mapped semigroups to the category of sets gives the free mapped semigroups in the usual way. More precisely, a **free mapped semigroup** on a set X is a mapped semigroup (U_X, α_X) together with a map $j_X : X \to U_X$ with the property that, for any mapped semigroup (V, β) together with a map $f : X \to V$, there is a unique morphism $\bar{f} : (U_X, \alpha_X) \to (V, \beta)$ of mapped semigroups such that $f = \bar{f} \circ j_X$.

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[dr, "f"'] & U_X \arrow[d, "\bar{f}"] \\ & V \end{tikzcd}$$

We similarly define the concept of free mapped (nonunitary) $\mathbf{k}$-algebras.

Proposition 2.3. *Let $j_\bullet : \{\bullet\} \to \mathcal{F}$ be the natural embedding $j_\bullet(\bullet) = \bullet \in \mathcal{T}$.*

(a) *The triple $(\mathcal{F}, \lfloor\ \rfloor, j_\bullet)$ is the free mapped semigroup on the generator $\bullet$.*

(b) *The triple $(\mathbf{k}\,\mathcal{F}, \lfloor\ \rfloor, j_\bullet)$ is the free mapped nonunitary algebra on one generator $\bullet$.*

We will see later that this free property underlies the construction of the free Rota–Baxter algebras.

Proof. (a) Let (V, β) be a mapped semigroup with its product denoted by $*$, and let $f : \{\bullet\} \to V$ be a set map. The homomorphism $\bar{f} : \mathcal{F} \to V$ is defined recursively as follows.

First define $\bar{f} : \mathcal{F}_0 = \langle \bullet \rangle \to V$ by multiplicity:

$$\bar{f}(\bullet \sqcup \cdots \sqcup \bullet) = f(\bullet) * \cdots * f(\bullet).$$

Suppose $\bar{f} : \mathcal{F}_n \to V$ is defined. Define

$$\bar{f} : \lfloor \mathcal{F}_n \rfloor \to V, \quad \bar{f}(\lfloor F \rfloor) = \beta(\bar{f}(F)),\ F \in \mathcal{F}_n$$

which is defined by the induction hypothesis. We then extend the map to

$$\bar{f} : \mathcal{F}_{n+1} := \langle \mathcal{F}_0 \cup \lfloor \mathcal{F}_n \rfloor \rangle \to V$$

by multiplicity. The resulting $\bar{f} : \mathcal{F} \to V$ is easily checked to be a homomorphism of mapped semigroups, and the unique one such that $\bar{f}(\bullet) = f(\bullet)$.

The proof of (b) is similar. □

2.2. **Rota-Baxter operator on rooted forests.** We note that $\mathbf{k}\,\mathcal{F}$ with the product $\sqcup$ is also the free noncommutative nonunitary $\mathbf{k}$-algebra over $\mathcal{T}$. We are going to define, for each fixed $\lambda \in \mathbf{k}$, another product $\diamond = \diamond_\lambda$ on $\mathbf{k}\,\mathcal{F}$, making it into a unitary Rota–Baxter algebra (of weight λ). To ease notation, we will fix λ throughout and suppress it unless there is a danger of confusion.

We define $\diamond$ by giving a set map

$$\diamond : \mathcal{F} \times \mathcal{F} \to \mathbf{k}\,\mathcal{F}$$

and then extending it bilinearly. For this, we use the depth filtration $\mathcal{F} = \cup_{n\geq 0} \mathcal{F}_n$ and apply induction on $i + j$ to define

$$\diamond : \mathcal{F}_i \times \mathcal{F}_j \to \mathbf{k}\,\mathcal{F}.$$

When $i + j = 0$, we have $\mathcal{F}_i = \mathcal{F}_j = \langle \bullet \rangle$, the forests consisting of $\bullet$. With the notation in Eq. (2), we define

$$\diamond : \mathcal{F}_0 \times \mathcal{F}_0 \to \mathbf{k}\,\mathcal{F},\ \bullet^{\sqcup m} \diamond \bullet^{\sqcup n} := \bullet^{\sqcup (m+n-1)}. \tag{6}$$

For any $k \geq 0$, suppose that

$$\diamond : \mathcal{F}_i \times \mathcal{F}_j \to \mathbf{k}\,\mathcal{F}$$

is defined for $i+j\leq k$. Consider forests F,F' with $\mathrm{d}(F)+\mathrm{d}(F')=k+1$. First assume that F and F' are trees. Note that a tree is either $\bullet$ or is of the form $\lfloor \overline{F}\rfloor$ for a forest $\overline{F}$ of smaller depth. Thus we can define

$$F\diamond F'=\begin{cases} F, & \text{if } F'=\bullet,\\ F', & \text{if } F=\bullet,\\ \lfloor\lfloor\overline{F}\rfloor\diamond\overline{F}'\rfloor+\lfloor\overline{F}\diamond\lfloor\overline{F}'\rfloor\rfloor+\lambda\lfloor\overline{F}\diamond\overline{F}'\rfloor, & \text{if } F=\lfloor\overline{F}\rfloor, F'=\lfloor\overline{F}'\rfloor,\end{cases}\tag{7}$$

since for the three products on the right hand of the third equation, we have

$$\begin{aligned}\mathrm{d}(\lfloor\overline{F}\rfloor)+\mathrm{d}(\overline{F}') &= \mathrm{d}(\lfloor\overline{F}\rfloor)+\mathrm{d}(\lfloor\overline{F}'\rfloor)-1=\mathrm{d}(F)+\mathrm{d}(F')-1,\\ \mathrm{d}(\overline{F})+\mathrm{d}(\lfloor\overline{F}'\rfloor) &= \mathrm{d}(\lfloor\overline{F}\rfloor)+\mathrm{d}(\lfloor\overline{F}'\rfloor)-1=\mathrm{d}(F)+\mathrm{d}(F')-1,\\ \mathrm{d}(\overline{F})+\mathrm{d}(\overline{F}') &= \mathrm{d}(\lfloor\overline{F}\rfloor)-1+\mathrm{d}(\lfloor\overline{F}'\rfloor)-1=\mathrm{d}(F)+\mathrm{d}(F')-2\end{aligned}\tag{8}$$

which are all less than or equal to k. Note that in either case, $F\diamond F'$ is a tree or a sum of trees.

Now consider arbitrary forests $F=T_1\sqcup\cdots\sqcup T_b$ and $F'=T'_1\sqcup\cdots\sqcup T'_{b'}$ with $\mathrm{d}(F)+\mathrm{d}(F')=k+1$. We then define

$$F\diamond F'=T_1\sqcup\cdots\sqcup T_{b-1}\sqcup(T_b\diamond T'_1)\sqcup T'_2\cdots\sqcup T_{b'}\tag{9}$$

where $T_b\diamond T'_1$ is defined by Eq. (7). By the remark after Eq. (8), $F\diamond F'$ is in $\mathbf{k}\,\mathcal{F}$. This completes the definition of the binary operation $\diamond$ of $Ш^{\mathrm{NC}}(\mathcal{T})$.

We record the following simple properties for later applications.

Lemma 2.4. (a) *For any forests F,F',F'', we have*

$$(F\sqcup F')\diamond F''=F\sqcup(F'\diamond F''),\quad F''\diamond(F\sqcup F')=(F''\diamond F)\sqcup F'.$$

(b) *For any forests F and F', we have*

$$\ell(F\diamond F')=\ell(F)+\ell(F')-1.$$

So $\mathcal{F}$ with the operations $\sqcup$ and $\diamond$ forms a 2-associative algebra in the sense of [46, 47].

Proof. (a). Let $F=T_1\sqcup\cdots\sqcup T_b$, $F'=T'_1\sqcup\cdots\sqcup T'_{b'}$ and $F''=T''_1\sqcup\cdots\sqcup T''_{b''}$ be the decomposition of the forests into trees. Recall that parentheses for $\sqcup$ is associative. Then by Eq. (9),

$$\begin{aligned}(F\sqcup F')\diamond F'' &= (T_1\sqcup\cdots\sqcup T_b\sqcup T'_1\sqcup\cdots\sqcup T'_{b'})\diamond(T''_1\sqcup T''_2\sqcup\cdots\sqcup T''_{b''})\\ &= T_1\sqcup\cdots\sqcup T_b\sqcup T'_1\sqcup\cdots\sqcup T'_{b'-1}\sqcup(T'_{b'}\diamond T''_1)\sqcup T''_2\sqcup\cdots\sqcup T''_{b''}\\ &= (T_1\sqcup\cdots\sqcup T_b)\sqcup(T'_1\sqcup\cdots\sqcup T'_{b'-1}\sqcup(T'_{b'}\diamond T''_1)\sqcup T''_2\sqcup\cdots\sqcup T''_{b''})\\ &= F\sqcup(F'\diamond F'').\end{aligned}$$

The proof of the second equation is the same.

(b). We prove by induction on the sum $m:=\mathrm{d}(F)+\mathrm{d}(F')$. When $m=0$, it follows from Eq. (6). Assume that the equation holds for all F and F' with $m\leq k$ and consider F and F' with $\mathrm{d}(F)+\mathrm{d}(F')=k+1$. If F and F' are trees, then the equation holds by Eq. (7), the induction hypothesis and the fact that $\ell(\lfloor\overline{F}\rfloor)=\ell(\overline{F})$ for a forest $\overline{F}$. Then for forests F and F', the equation follows from Eq. (9) and Eq. (3) $\square$

Extending $\diamond$ bilinearly, we obtain a binary operation

$$\diamond : \mathbf{k}\,\mathcal{F} \otimes \mathbf{k}\,\mathcal{F} \to \mathbf{k}\,\mathcal{F}.$$

For $F \in \mathcal{F}$, we use the grafting operation to define

$$P_{\mathcal{F}}(F) = \lfloor F \rfloor. \tag{10}$$

Then $P_{\mathcal{F}}$ extends to a linear operator on $\mathbf{k}\,\mathcal{F}$.

The following is our first main result and will be proved in the next subsection.

Theorem 2.5. (a) *The pair $(\mathbf{k}\,\mathcal{F}, \diamond)$ is a unitary associative algebra.*
(b) *The triple $(\mathbf{k}\,\mathcal{F}, \diamond, P_{\mathcal{F}})$ is a unitary Rota–Baxter algebra of weight λ.*

We will construct a nonunitary sub-Rota–Baxter algebra in $\mathbf{k}\,\mathcal{F}$.

Corollary 2.6. *Let $\mathcal{F}^0$ be the subset of $\mathcal{F}$ consisting of forests that are not $\bullet$ and do not contain any subtree $\lfloor \bullet \rfloor$. The submodule $\mathbf{k}\,\mathcal{F}^0$ of $\mathbf{k}\,\mathcal{F}$ is a nonunitary Rota–Baxter subalgebra of $\mathbf{k}\,\mathcal{F}$ under the product $\diamond$.*

Forests in $\mathcal{F}^0$ will be called the **ladder-free forests**. A more intuitive interpretation of a forest containing a $\lfloor \bullet \rfloor$ is that the forest has a leaf that is the only child of its parent vertex and therefore gives a ladder tree branch . Our proof does not depend on this interpretation.

Proof. We only need to check that $\mathbf{k}\mathcal{F}^0$ is closed under $\diamond$ and $P_{\mathcal{F}} = \lfloor\ \rfloor$. We achieve this by two lemmas.

Lemma 2.7. *If F is in $\mathcal{F}^0$, then $\lfloor F \rfloor$ does not contain $\lfloor \bullet \rfloor$ and hence is in $\mathcal{F}^0$.*

Proof. Let F be in $\mathcal{F}^0$. Then F does not contain $\lfloor \bullet \rfloor$. In other words, none of the brackets $\lfloor B \rfloor$ in F is of the form $\lfloor \bullet \rfloor$. The only other brackets in $\lfloor F \rfloor$ is $\lfloor F \rfloor$ itself. So suppose $\lfloor F \rfloor$ contains a $\lfloor \bullet \rfloor$, then we must have $\lfloor F \rfloor = \lfloor \bullet \rfloor$, implying $F = \bullet$. This is a contradiction. So we have $\lfloor F \rfloor \in \mathcal{F}^0$. $\square$

By Lemma 2.7, $\mathbf{k}\,\mathcal{F}^0$ is closed under the Rota–Baxter operator $P_{\mathcal{F}}$.

To prove that $\mathbf{k}\,\mathcal{F}^0$ is closed under the multiplication $\diamond$, consider F and F' in $\mathcal{F}^0$. Since none of F or F' is $\bullet$, we have $F \diamond F' \neq \bullet$. So we only need to prove the following lemma.

Lemma 2.8. *If F and F' are in $\mathcal{F}^0$, then $F \diamond F'$ is either a forest that does not contain $\lfloor \bullet \rfloor$ or is a linear combination of forests that do not contain $\lfloor \bullet \rfloor$.*

Proof. Let $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$. We will prove the lemma using induction on $n := \mathrm{d}(T_b) + \mathrm{d}(T'_1)$.

When $n = 0$, we have $T_b = T'_1 = \bullet$. Since none of F or F' is $\bullet$, we have $b > 1$ and $b' > 1$. So by Eq. (7),

$$F \diamond F' = T_1 \sqcup \cdots \sqcup T_{b-1} \sqcup \bullet \sqcup T'_2 \sqcup \cdots \sqcup T'_{b'}.$$

Since neither F nor F' contains $\lfloor \bullet \rfloor$, none of T_i or T'_j contains $\lfloor \bullet \rfloor$. Then none of the trees in the right hand side contains $\lfloor \bullet \rfloor$. So the right hand side does not contain $\lfloor \bullet \rfloor$, as needed.

Let $k \geq 0$. Assume that the claim has been proved for $n \leq k$ and let F and F' be in $\mathcal{F}^0$ with $n = k + 1$. Then $n \geq 1$. So at least one of $\mathrm{d}(T_b)$ and $\mathrm{d}(T'_1)$ is not zero. If one of them is zero, then the same argument as in the $n = 0$ case works using the first two cases of Eq. (7). If none of them is zero, then by the third case of Eq. (7), we have $T_b = \lfloor \overline{F}_b \rfloor$, $T'_1 = \lfloor \overline{F}'_1 \rfloor$ and

$$T_b \diamond T'_{b'} = \lfloor \lfloor \overline{F}_b \rfloor \diamond \overline{F}'_1 \rfloor + \lfloor \overline{F}_b \diamond \lfloor \overline{F}'_1 \rfloor \rfloor + \lambda \lfloor \overline{F}_b \diamond \overline{F}'_1 \rfloor.$$

Since T_b does not contain $\lfloor \bullet \rfloor$, $\overline{F}_b$ is not $\bullet$ and does not contain $\lfloor \bullet \rfloor$. So $\overline{F}_b$ is in $\mathcal{F}^0$. Similarly, $\overline{F}'_1$ is in $\mathcal{F}^0$. By the induction hypothesis, none of the terms

$$\lfloor \overline{F}_b \rfloor \diamond \overline{F}'_1, \quad \overline{F}_b \diamond \lfloor \overline{F}'_1 \rfloor, \quad \overline{F}_b \diamond \overline{F}'_1$$

contains $\lfloor \bullet \rfloor$. Thus they are in $\mathbf{k}\mathcal{F}^0$. By Lemma 2.7, the terms on the right hand side themselves do not contain $\lfloor \bullet \rfloor$. Therefore $T_b \diamond T'_1$ is a linear combination of terms that do not contain $\lfloor \bullet \rfloor$. Since F and F' do not contain $\lfloor \bullet \rfloor$, none of T_i and T'_j contains $\lfloor \bullet \rfloor$. By Eq. (9), we have

$$F \diamond F' = T_1 \sqcup \cdots \sqcup T_{b-1} \sqcup (T_b \diamond T'_1) \sqcup T'_2 \cdots \sqcup T_{b'}.$$

Then $F \diamond F'$ is a linear combination of terms that do not contain $\lfloor \bullet \rfloor$. This completes the induction. □

Now the proof of Corollary 2.6 is completed. □

2.3. **The proof of Theorem 2.5.**

Proof. (a). By Definition (7), $\bullet$ is the identity under the product $\diamond$. So we just need to verify the associativity. For this we only need to verify

$$(F \diamond F') \diamond F'' = F \diamond (F' \diamond F'') \tag{11}$$

for forests $F, F', F'' \in \mathcal{F}$. We will accomplish this by induction on the sum of the depths $n := \mathrm{d}(F) + \mathrm{d}(F') + \mathrm{d}(F'')$. If $n = 0$, then all of F, F', F'' have depth zero and so are in $\mathcal{F}_0 = \langle \bullet \rangle$, the sub-semigroup of $\mathcal{F}$ generated by $\bullet$. Then we have $F = \bullet^{\sqcup i}$, $F' = \bullet^{\sqcup i'}$ and $F'' = \bullet^{\sqcup i''}$, for $i, i', i'' \geq 1$. Then the associativity follows from Eq. (6) since both sides of Eq. (11) is $\bullet^{\sqcup(i+i'+i''-2)}$ in this case.

Let $k \geq 0$. Assume Eq. (11) holds for $n \leq k$ and assume that $F, F', F'' \in \mathcal{F}$ satisfy $n = \mathrm{d}(F) + \mathrm{d}(F') + \mathrm{d}(F'') = k + 1$. We next reduce the breadths of the forests.

Lemma 2.9. *If the associativity*

$$(F \diamond F') \diamond F'' = F \diamond (F' \diamond F'')$$

holds when F, F' and F'' are trees, then it holds when they are forests.

Proof. We use induction on the sum of breadths $m := b(F) + b(F') + b(F'')$. Then $m \geq 3$. The case when $m = 3$ is the assumption of the lemma. Assume the associativity holds for $3 \leq m \leq j$ and take $F, F', F'' \in \mathcal{F}$ with $m = j + 1$. Then $j + 1 \geq 4$. So at least one of F, F', F'' has breadth greater than or equal to 2.

First assume $b(F) \geq 2$. Then $F = F_1 \sqcup F_2$ with F_1, $F_2 \in \mathcal{F}$. Thus by Lemma 2.4, we have

$$(F \diamond F') \diamond F'' = ((F_1 \sqcup F_2) \diamond F') \diamond F'' = (F_1 \sqcup (F_2 \diamond F')) \diamond F'' = F_1 \sqcup ((F_2 \diamond F') \diamond F'').$$

Similarly,

$$F \diamond (F' \diamond F'') = (F_1 \sqcup F_2) \diamond (F' \diamond F'') = F_1 \sqcup (F_2 \diamond (F' \diamond F'')).$$

Thus

$$(F \diamond F') \diamond F'' = F \diamond (F' \diamond F'')$$

whenever

$$(F_2 \diamond F') \diamond F'' = F_2 \diamond (F' \diamond F'')$$

which follows from the induction hypothesis.

A similar proof works if $b(F'') \geq 2$.

Finally if $b(F') \geq 2$, then $F' = F'_1 \sqcup F'_2$ with $F'_1,\, F'_2 \in \mathfrak{F}$. Using Lemma 2.4 repeatedly, we have

$$(F \diamond F') \diamond F'' = (F \diamond (F'_1 \sqcup F'_2)) \diamond F'' = ((F \diamond F'_1) \sqcup F'_2) \diamond F'' = (F \diamond F'_1) \sqcup (F'_2 \diamond F'').$$

In the same way, we have

$$F \diamond (F' \diamond F'') = (F \diamond F'_1) \sqcup (F'_2 \diamond F'').$$

This again proves the associativity. □

To summarize, our proof of the associativity (11) has been reduced to the special case when the forests $F, F', F'' \in \mathfrak{F}$ are chosen such that

(a) $n := \mathrm{d}(F) + \mathrm{d}(F') + \mathrm{d}(F'') = k+1 \geq 1$ with the assumption that the associativity holds when $n \leq k$, and
(b) the forests are of breadth one, that is, they are trees.

If either one of the trees is $\bullet$ which is the identity under the product $\diamond$, then the associativity is clear.

So it remains to consider the case when F, F', F'' are all in $\lfloor \mathfrak{F} \rfloor$. Then $F = \lfloor \overline{F} \rfloor, F' = \lfloor \overline{F}' \rfloor, F'' = \lfloor \overline{F}'' \rfloor$ with $\overline{F}, \overline{F}', \overline{F}'' \in \mathfrak{F}$. To deal with this case and similar situations later, we prove the following general fact on Rota–Baxter operators on not necessarily associative algebras.

Lemma 2.10. *Let R be a $\mathbf{k}$-module with a multiplication $\cdot$ that is not necessarily associative. Let $\lfloor\ \rfloor_R : R \to R$ be a $\mathbf{k}$-linear map such that the Rota–Baxter identity holds:*

$$\lfloor x \rfloor_R \cdot \lfloor x' \rfloor_R = \big\lfloor x \cdot \lfloor x' \rfloor_R \big\rfloor_R + \big\lfloor \lfloor x \rfloor_R \cdot x' \big\rfloor_R + \lambda \lfloor x \cdot x' \rfloor_R,\ \forall\, x,\, x' \in R. \tag{12}$$

Let $x, x'x''$ be in R. If

$$(x \cdot x') \cdot x'' = x \cdot (x' \cdot x''),$$

then we say that (x, x', x'') is an **associative triple** *for the product $\cdot$. Now for any $y, y', y'' \in R$, if all the triples*

$$(y, y', y''),\ (\lfloor y \rfloor_R, y', y''),\ (y, \lfloor y' \rfloor_R, y''),\ (y, y', \lfloor y'' \rfloor_R),\ (\lfloor y \rfloor_R, y', \lfloor y'' \rfloor_R), \tag{13}$$

$$(\lfloor y \rfloor_R, \lfloor y' \rfloor_R, y''),\ (y, \lfloor y' \rfloor_R, \lfloor y'' \rfloor_R) \tag{14}$$

are associative triples for $\cdot$, then $(\lfloor y \rfloor_R, \lfloor y' \rfloor_R, \lfloor y'' \rfloor_R)$ is an associative triple for $\cdot$.

Proof. Using Eq. (12) and bilinearity of the product $\cdot$, we have

$$\begin{aligned}
(\lfloor y \rfloor_R \cdot \lfloor y' \rfloor_R) \cdot \lfloor y'' \rfloor_R &= \big(\lfloor \lfloor y \rfloor_R \cdot y' \rfloor_R + \lfloor y \cdot \lfloor y' \rfloor_R \rfloor_R + \lambda \lfloor y \cdot y' \rfloor \big) \cdot \lfloor y'' \rfloor_R \\
&= \lfloor \lfloor y \rfloor_R \cdot y' \rfloor_R \cdot \lfloor y'' \rfloor_R + \lfloor y \cdot \lfloor y' \rfloor_R \rfloor_R \cdot \lfloor y'' \rfloor_R + \lambda \lfloor y \cdot y' \rfloor_R \cdot \lfloor y'' \rfloor_R \\
&= \lfloor \lfloor \lfloor y \rfloor_R \cdot y' \rfloor_R \cdot y'' \rfloor_R + \lfloor \big(\lfloor y \rfloor_R \cdot y' \big) \cdot \lfloor y'' \rfloor_R \rfloor_R + \lambda \lfloor \big(\lfloor y \rfloor_R \cdot y' \big) \cdot y'' \rfloor_R \\
&\quad + \lfloor \lfloor y \cdot \lfloor y' \rfloor_R \rfloor_R \cdot y'' \rfloor_R + \lfloor \big(y \cdot \lfloor y' \rfloor_R \big) \cdot \lfloor y'' \rfloor_R \rfloor_R + \lambda \lfloor \big(y \cdot \lfloor y' \rfloor_R \big) \cdot y'' \rfloor_R \\
&\quad + \lambda \lfloor \lfloor y \cdot y' \rfloor_R \cdot y'' \rfloor_R + \lambda \lfloor \big(y \cdot y' \big) \cdot \lfloor y'' \rfloor_R \rfloor_R + \lambda^2 \lfloor \big(y \cdot y' \big) \cdot y'' \rfloor_R.
\end{aligned}$$

Applying the associativity of the second triple in Eq. (14) to $\big(y\cdot\lfloor y'\rfloor_R\big)\cdot\lfloor y''\rfloor_R$ in the fifth term above and then using Eq. (12) again, we have

$$\begin{aligned}
&(\lfloor y\rfloor_R\cdot\lfloor y'\rfloor_R)\cdot\lfloor y''\rfloor_R\\
=\;&\lfloor\lfloor\lfloor y\rfloor_R\cdot y'\rfloor_R\cdot y''\rfloor_R+\lfloor\big(\lfloor y\rfloor_R\cdot y'\big)\cdot\lfloor y''\rfloor_R\rfloor_R+\lambda\lfloor\big(\lfloor y\rfloor_R\cdot y'\big)\cdot y''\rfloor_R\\
&+\lfloor\lfloor y\cdot\lfloor y'\rfloor_R\rfloor_R\cdot y''\rfloor_R+\lfloor y\cdot\lfloor\lfloor y'\rfloor_R\cdot y''\rfloor_R\rfloor_R+\lfloor y\cdot\lfloor y'\cdot\lfloor y''\rfloor_R\rfloor_R\rfloor_R\\
&+\lambda\lfloor y\cdot\lfloor y'\cdot y''\rfloor_R\rfloor_R+\lambda\lfloor\big(y\cdot\lfloor y'\rfloor_R\big)\cdot y''\rfloor_R\\
&+\lambda\lfloor\lfloor y\cdot y'\rfloor_R\cdot y''\rfloor_R+\lambda\lfloor\big(y\cdot y'\big)\cdot\lfloor y''\rfloor_R\rfloor_R+\lambda^2\lfloor\big(y\cdot y'\big)\cdot y''\rfloor_R.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\lfloor y\rfloor_R\cdot\big(\lfloor y'\rfloor_R\cdot\lfloor y''\rfloor_R\big)=\lfloor y\rfloor_R\cdot\Big(\lfloor\lfloor y'\rfloor_R\cdot y''\rfloor_R+\lfloor y'\cdot\lfloor y''\rfloor_R\rfloor_R+\lambda\lfloor y'\cdot y''\rfloor_R\Big)\\
=\;&\lfloor\lfloor y\rfloor_R\cdot\big(\lfloor y'\rfloor_R\cdot y''\big)\rfloor_R+\lfloor y\cdot\lfloor\lfloor y'\rfloor_R\cdot y''\rfloor_R\rfloor_R+\lambda\lfloor y\cdot\big(\lfloor y'\rfloor_R\cdot y''\big)\rfloor_R\\
&+\lfloor\lfloor y\rfloor_R\cdot\big(y'\cdot\lfloor y''\rfloor_R\big)\rfloor_R+\lfloor y\cdot\lfloor y'\cdot\lfloor y''\rfloor_R\rfloor_R\rfloor_R+\lambda\lfloor y\cdot\big(y'\cdot\lfloor y''\rfloor_R\big)\rfloor_R\\
&+\lambda\lfloor\lfloor y\rfloor_R\cdot\big(y'\cdot y''\big)\rfloor_R+\lambda\lfloor y\cdot\lfloor y'\cdot y''\rfloor_R\rfloor_R+\lambda^2\lfloor y\cdot\big(y'\cdot y''\big)\rfloor_R.
\end{aligned}$$

Applying the associativity of the first triple in Eq. (14) to $\lfloor y\rfloor_R\cdot(\lfloor y'\rfloor_R\cdot y'')$ in the first term above and then using Eq. (12) again, we have

$$\begin{aligned}
&\lfloor y\rfloor_R\cdot\big(\lfloor y'\rfloor_R\cdot\lfloor y''\rfloor_R\big)=\lfloor\lfloor\lfloor y\rfloor_R\cdot y'\rfloor_R\cdot y''\rfloor_R+\lfloor\lfloor y\cdot\lfloor y'\rfloor_R\rfloor_R\cdot y''\rfloor_R\\
&+\lambda\lfloor\lfloor y\cdot y'\rfloor_R\cdot y''\rfloor_R+\lfloor y\cdot\lfloor\lfloor y'\rfloor_R\cdot y''\rfloor_R\rfloor_R+\lambda\lfloor y\cdot\big(\lfloor y'\rfloor_R\cdot y''\big)\rfloor_R\\
&+\lfloor\lfloor y\rfloor_R\cdot\big(y'\cdot\lfloor y''\rfloor_R\big)\rfloor_R+\lfloor y\cdot\lfloor y'\cdot\lfloor y''\rfloor_R\rfloor_R\rfloor_R+\lambda\lfloor y\cdot\big(y'\cdot\lfloor y''\rfloor_R\big)\rfloor_R\\
&+\lambda\lfloor\lfloor y\rfloor_R\cdot\big(y'\cdot y''\big)\rfloor_R+\lambda\lfloor y\cdot\lfloor y'\cdot y''\rfloor_R\rfloor_R+\lambda^2\lfloor y\cdot\big(y'\cdot y''\big)\rfloor_R.
\end{aligned}$$

Now by the associativity of the triples in Eq. (13), the i-th term in the expansion of $(\lfloor y\rfloor_R\cdot\lfloor y'\rfloor_R)\cdot\lfloor y''\rfloor_R$ matches with the $\sigma(i)$-th term in the expansion of $\lfloor y\rfloor_R\cdot\big(\lfloor y'\rfloor_R\cdot\lfloor y''\rfloor_R\big)$. Here the permutation $\sigma\in\Sigma_{11}$ is

$$\left(\begin{array}{c} i\\ \sigma(i)\end{array}\right)=\left(\begin{array}{ccccccccccc}1&2&3&4&5&6&7&8&9&10&11\\1&6&9&2&4&7&10&5&3&8&11\end{array}\right). \tag{15}$$

This proves the lemma. □

To continue the proof of Theorem 2.5, we apply Lemma 2.10 to the situation where R is $\mathbf{k}\,\mathcal{F}$ with the multiplication $\cdot=\diamond$, the Rota–Baxter operator $\lfloor\ \rfloor_R=\lfloor\ \rfloor$ and the triple $(y,y',y'')=(\overline{F},\overline{F}',\overline{F}'')$. By the induction hypothesis on n, all the triples in Eq. (13) and (14) are associative for $\diamond$. So by Lemma 2.10, the triple (F,F',F'') is associative for $\diamond$. This completes the induction and therefore the proof of the first part of Theorem 2.5.

(b). We just need to prove that $P_{\mathcal{F}}(F)=\lfloor F\rfloor$ is a Rota–Baxter operator of weight λ. This is immediate from Eq. (7). □

3. Free Rota–Baxter algebras over a module

We will construct the free unitary Rota–Baxter algebra over a $\mathbf{k}$-module by expressing elements in the Rota–Baxter algebra in terms of forests from Section 2, in addtion with angles decorated by elements from the $\mathbf{k}$-module. These decorated forests will be introduced in Section 3.1. The free unitary Rota–Baxter algebra will be constructed in Section 3.2. In Section 3.3, we also give a similar construction of nonunitary Rota–Baxter algebra over a $\mathbf{k}$-module in terms of the ladder-free forests introduced in Corollary 2.6. When the $\mathbf{k}$-module is taken to be free over a set, we obtain the free unitary Rota–Baxter algebra over the set. This will be considered in Section 4.

3.1. **Rooted forests with angular decoration by a module.** Let M be a non-zero $\mathbf{k}$-module. Let F be in $\mathcal{F}$ with ℓ leafs. We let $(F;M)$ denote the tensor power $M^{\otimes(\ell-1)}$ labeled by F. In other words,

$$(F;M)=\{(F;\mathfrak{m}) \mid \mathfrak{m}\in M^{\otimes(\ell-1)}\} \tag{16}$$

with the $\mathbf{k}$-module structure coming from the second component and with the convention that $M^{\otimes 0}=\mathbf{k}$. We can think of $(F;M)$ as the tensor power of M with exponent F with the usual power $M^{\otimes n}, n\geq 0$, corresponding to $(F;M)$ when F is the forest $\bullet^{\sqcup(n+1)}$, such as,

$$(\bullet;M)=M^{\otimes 0}=\mathbf{k},\quad (\bullet\sqcup\bullet;M)=M,\quad (\bullet\sqcup\bullet\sqcup\bullet;M)=M^{\otimes 2},\cdots.$$

So we will also use the notation $M^{\otimes F}=(F;M)$.

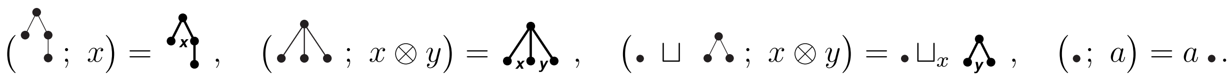
Definition 3.1. *We call* $(F;M)$ *the* **module of the forest** F **with angular decoration by** M*, and call* $(F;\mathfrak{m})$*, for an* $\mathfrak{m}\in M^{\otimes(\ell(F)-1)}$*, an angularly decorated forest* F *with the decoration tensor* $\mathfrak{m}$*.*

Also define the depth and breadth of $(F;\mathfrak{m})$ by

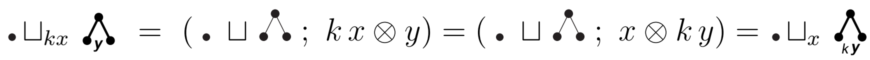
$$\mathrm{d}(F;\mathfrak{m})=\mathrm{d}(F),\quad b(F;\mathfrak{m})=b(F).$$

We now give a more intuitive tree interpretation of $(F;M)$. Let $(F;\mathfrak{m})$ be an angularly decorated forest with a pure tensor $\mathfrak{m}=a_1\otimes\cdots\otimes a_{\ell-1}\in M^{\otimes(\ell-1)}$, $\ell\geq 2$. We picture $(F;\mathfrak{m})$ as the forest F with its angles between adjacent leafs (either from the same tree or from adjacent trees) decorated by $a_1,\cdots,a_{\ell-1}$ from the left most angle to the right most angle. If $\ell(F)=1$, so F is a ladder tree with only one leaf, then $(F;a)$, $a\in\mathbf{k}$, is interpreted as the multiple aF of the ladder tree F.

For example, we have



By the multilinear property of the tensor product, a scalar in $\mathbf{k}$ can be moved across the tensor product sign. In terms of angularly decorated forests, this means that a scalar factor can be moved from one angle of a forest to another angle. For example,



Let F be a forest in $\mathcal{F}$ with ℓ leafs. If $\mathfrak{m}=\sum_i \mathfrak{m}_i$ is not a pure tensor, but a sum of pure tensors $\mathfrak{m}_i$ in $M^{\otimes(\ell-1)}$, we can picture $(F;\mathfrak{m})$ as a sum $\sum_i (F;\mathfrak{m}_i)$ of the forest F with decorations from the pure tensors. Likewise, if F is a linear combination $\sum_i c_iF_i$ of forests F_i with the same number of leaves ℓ and if $\mathfrak{m}=a_1\otimes\cdots\otimes a_{\ell-1}\in M^{\otimes(\ell-1)}$, we also use $(F;\mathfrak{m})$ to denote the linear combination $\sum_i c_i(F_i;\mathfrak{m})$.

Let $D=(F;\mathfrak{m})$ and $D'=(F';\mathfrak{m}')$ be two angularly decorated forests, consisting of forests F and F' decorated by pure tensors $\mathfrak{m}=a_1\otimes\cdots\otimes a_n$ and $\mathfrak{m}'=a'_1\otimes\cdots\otimes a'_{n'}$. Let $a\in M$.

We let $D \otimes_a D'$ denote the new decorated rooted forest with a decorating in between the decorated forests F' and F''. More precisely we define

$$D \otimes_a D' = (F \sqcup F'; a_1 \otimes \cdots \otimes a_n \otimes a \otimes a'_1 \otimes \cdots \otimes a'_{n'}). \tag{17}$$

The decorated forest on the right hand side is defined since the number of angles of $F \sqcup F'$ is

$$\ell(F \sqcup F') - 1 = \ell(F) + \ell(F') - 1 = n + 1 + n' + 1 - 1 = n + n' + 1$$

agreeing with the length of the decorating tensor. As an example, $\big(\bullet\,; \mathbf{1}\big) \otimes_a \big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}; b\big) = \big(\bullet \sqcup \begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix};\ a \otimes b\big)$. This is also denoted by $\big(\bullet\,; \mathbf{1}\big) \sqcup_a \big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}; b\big)$ in our earlier notation. In general the notations $D \otimes_a D'$ and $D \sqcup_a D'$ can be used without contradiction. We use the tensor product to emphasize the module structure. We note that only elements in M, not in $\mathbf{k}$, can be used to decorate between two decorated forests.

Let $(F; \mathfrak{m})$ be an angular decoration of the forest F by a pure tensor $\mathfrak{m}$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ be the decomposition of F into trees. We consider the corresponding decomposition of decorated forests. If $b = 1$, then F is a tree and $(F; \mathfrak{m})$ has no further decompositions. If $b > 1$, then there is the relation

$$\ell(F) = \ell(T_1) + \cdots + \ell(T_b).$$

Denote $\ell_i = \ell(T_i), 1 \leq i \leq b$. Then

$$(T_1; a_1 \otimes \cdots \otimes a_{\ell_1 - 1}),\ (T_2; a_{\ell_1+1} \otimes \cdots \otimes a_{\ell_1+\ell_2-1}), \cdots, (T_b; a_{\ell_1+\cdots+\ell_{b-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_b})$$

are well-defined angularly decorated trees for the trees T_i with $\ell(T_i) > 1$. If $\ell(T_i) = 1$, then $a_{\ell_{i-1}+\ell_i-1} = a_{\ell_{i-1}}$ and we use the convention $(T_i; a_{\ell_{i-1}+\ell_i-1}) = (T_i; \mathbf{1})$. With this convention, we have,

$$\begin{aligned}(F; a_1 \otimes \cdots \otimes a_{\ell-1}) \;&=\; (T_1; a_1 \otimes \cdots \otimes a_{\ell_1-1}) \otimes_{a_{\ell_1}} (T_2; a_{\ell_1+1} \otimes \cdots \otimes a_{\ell_1+\ell_2-1}) \otimes_{a_{\ell_1+\ell_2}} \\ &\qquad \cdots \otimes_{a_{\ell_1+\cdots+\ell_{b-1}}} (T_b; a_{\ell_1+\cdots+\ell_{b-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_b}).\end{aligned}$$

We call this the **standard decomposition** of $(F; \mathfrak{m})$ and abbreviate it as

$$(F; \mathfrak{m}) = (T_1; \mathfrak{m}_1) \otimes_{u_1} (T_2; \mathfrak{m}_2) \otimes_{u_2} \cdots \otimes_{u_{b-1}} (T_b; \mathfrak{m}_b). \tag{18}$$

In other words,

$$(T_i; \mathfrak{m}_i) = \begin{cases} (T_i; a_{\ell_1+\cdots+\ell_{i-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_i-1}), & \ell_i > 1, i < b, \\ (T_i; a_{\ell_1+\cdots+\ell_{i-1}+1} \otimes \cdots \otimes a_{\ell_1+\cdots+\ell_i}), & \ell_i > 1, i = b, \\ (T_i; \mathbf{1}), & \ell_i = 1 \end{cases} \tag{19}$$

and $u_i = a_{\ell_1+\cdots+\ell_i}$. For example,

$$\big(\bullet \sqcup \begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\\ &&\bullet\end{smallmatrix} \sqcup \begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}; v \otimes x \otimes w \otimes y\big) = \big(\bullet; \mathbf{1}\big) \otimes_v \big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\\ &&\bullet\end{smallmatrix}; x\big) \otimes_w \big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}; y\big) = \bullet \otimes_v \begin{smallmatrix}&\bullet&\\ \bullet&x&\bullet\\ &&\bullet\end{smallmatrix} \otimes_w \begin{smallmatrix}&\bullet&\\ \bullet&y&\bullet\end{smallmatrix}$$

We display the following simple property for later applications.

Lemma 3.2. *Let $F \neq \bullet$. In the standard decomposition (18) of $(F; \mathfrak{m})$, if $T_i = \bullet$ for some $1 \leq i \leq b$, then $b > 1$ and the corresponding factor $(T_i; \mathfrak{m}_i)$ is $(T_i; \mathbf{1})$.*

Proof. Let $F \neq \bullet$ and let $F = T_1 \sqcup \cdots \sqcup T_b$ be its standard decomposition. Suppose $T_i = \bullet$ for some $1 \leq i \leq b$ and $b = 1$. Then $F = T_i = \bullet$, a contradiction. So $b > 1$, and by our convention, $(T_i; \mathfrak{m}_i) = (T_i; \bullet)$. $\square$

3.2. **Free Rota–Baxter algebra over a module as decorated forests.** We define the $\mathbf{k}$-module
$$Ш^{\mathrm{NC}}(M)=\bigoplus_{F\in\mathcal{F}}(F;M)=\bigoplus_{F\in\mathcal{F}}M^{\otimes F}.$$
and define a product $\overline{\diamond}$ on $Ш^{NC}(M)$ by using the product $\diamond$ on $\mathcal{F}$ in Section 2.2. We start by defining $D\overline{\diamond}D'$ for any two angularly decorated forests $D=(F;\mathfrak{m})\in(F;M)$ and $D'=(F';\mathfrak{m}')\in(F';M)$ with $\mathfrak{m}$ and $\mathfrak{m}'$ being pure tensors, and then extending by biadditivity.

Let $T(M)=\oplus_{n\geq 0}M^{\otimes n}$ be the tensor algebra and let $\overline{\otimes}$ be its product, so for $\mathfrak{m}\in M^{\otimes n}$ and $\mathfrak{m}'\in M^{\otimes n'}$, we have

$$\mathfrak{m}\overline{\otimes}\mathfrak{m}'=\begin{cases}\mathfrak{m}\otimes\mathfrak{m}'\in M^{\otimes n+n'}, & \text{if } n>0, n'>0,\\ \mathfrak{m}\mathfrak{m}'\in M^{\otimes n'}, & \text{if } n=0, n'>0,\\ \mathfrak{m}'\mathfrak{m}\in M^{\otimes n}, & \text{if } n>0, n'=0,\\ \mathfrak{m}'\mathfrak{m}\in\mathbf{k}, & \text{if } n=n'=0.\end{cases}\tag{20}$$

Here the products in the second and third case are scalar product and in the fourth case is the product in $\mathbf{k}$. In other words, $\overline{\otimes}$ identifies $\mathbf{k}\otimes M$ with M by the structure map $\mathbf{k}\otimes M\to M$ of the $\mathbf{k}$-module.

Definition 3.3. *For tensors $D=(F;\mathfrak{m})\in(F;M)$ and $D'=(F';\mathfrak{m}')\in(F';M)$, define*
$$D\overline{\diamond}D'=(F\diamond F';\mathfrak{m}\overline{\otimes}\mathfrak{m}').\tag{21}$$

The right hand side is well-defined since $\mathfrak{m}\overline{\otimes}\mathfrak{m}'$ has tensor degree
$$\deg(\mathfrak{m}\overline{\otimes}\mathfrak{m}')=\deg(\mathfrak{m})+\deg(\mathfrak{m}')=\ell(F)-1+\ell(F')-1$$
which equals $\ell(F\diamond F')-1$ by Lemma 2.4.(b). By Eq. (6) — (9), we have a more precise expression.

$$D\overline{\diamond}D'=\begin{cases}(\bullet;cc'), & \text{if } D=(\bullet;c), D'=(\bullet;c'),\\ (F;c'\mathfrak{m}), & \text{if } D'=(\bullet,c'), F\neq\bullet,\\ (F';c\mathfrak{m}'), & \text{if } D=(\bullet,c), F'\neq\bullet,\\ (F\diamond F';\mathfrak{m}\otimes\mathfrak{m}'), & \text{if } F\neq\bullet, F'\neq\bullet.\end{cases}\tag{22}$$

Let the standard decomposition of $D=(F;\mathfrak{m})$ be
$$D=(F;\mathfrak{m})=(T_1;\mathfrak{m}_1)\otimes_{u_1}(T_2;\mathfrak{m}_2)\otimes_{u_2}\cdots\otimes_{u_{b-1}}(T_b;\mathfrak{m}_b)$$
given in Eq. (18) and similarly let
$$D'=(F';\mathfrak{m}')=(T_1';\mathfrak{m}_1')\otimes_{u_1'}(T_2';\mathfrak{m}_2')\otimes_{u_2'}\cdots\otimes_{u_{b'-1}'}(T_{b'}';\mathfrak{m}_{b'}')$$
be the standard decomposition of D'. Then by Eq. (6) – (9) and Eq. (21) – (22), it is easy to see that the product $\overline{\diamond}$ can be defined by induction on the sum of the depths $\mathrm{d}=\mathrm{d}(F)$ and $\mathrm{d}'=\mathrm{d}(F')$ in the following way.

If $\mathrm{d}+\mathrm{d}'=0$, then $F=\bullet^{\sqcup i}$ and $F'=\bullet^{\sqcup j}$ for $i,j\geq 1$. If $i=1$, then $D=(F;\mathfrak{m})=(\bullet;c)=c(\bullet;\mathbf{1})$ and we define $D\overline{\diamond}D'=cD'=(F';c\mathfrak{m}')$. Similarly define $D\overline{\diamond}D'$ if $j=1$. If $i>1$ and $j>1$, then $(F;\mathfrak{m})=(\bullet;\mathbf{1})\otimes_{u_1}\cdots\otimes_{u_{b-1}}(\bullet;\mathbf{1})$ with $u_1,\cdots,u_{b-1}\in M$. Similarly, $(F';\mathfrak{m}')=(\bullet;\mathbf{1})\otimes_{u_1'}\cdots\otimes_{u_{b'-1}'}(\bullet;\mathbf{1})$. Then define
$$(F;\mathfrak{m})\overline{\diamond}(F';\mathfrak{m}')=(\bullet;\mathbf{1})\otimes_{u_1}\cdots\otimes_{u_{b-1}}(\bullet;\mathbf{1})\otimes_{u_1'}\cdots\otimes_{u_{b'-1}'}(\bullet;\mathbf{1}).$$

Suppose $D\overline{\diamond}D'$ has been defined for all $D=(F;\mathfrak{m})$ and $D'=(F';\mathfrak{m}')$ with $\mathrm{d}(F)+\mathrm{d}(F')\leq k$ and consider D and D' with $\mathrm{d}(F)+\mathrm{d}(F')=k+1$. Then we define

$$D\overline{\diamond}D' = (T_1;\mathfrak{m}_1)\otimes_{u_1}\cdots\otimes_{u_{b-1}}\big((T_b;\mathfrak{m}_b)\overline{\diamond}(T'_1;\mathfrak{m}'_1)\big)\otimes_{u'_1}\cdots\otimes_{u'_{b'-1}}(T'_{b'};\mathfrak{m}'_{b'}) \tag{23}$$

where

$$\begin{aligned}&(T_b;\mathfrak{m}_b)\overline{\diamond}(T'_1;\mathfrak{m}'_1)\\ =&\begin{cases}(\bullet;\mathbf{1}), & \text{if } T_b=T'_1=\bullet \text{ (so } \mathfrak{m}_b=\mathfrak{m}'_1=\mathbf{1}),\\ (T_b,\mathfrak{m}_b), & \text{if } T'_1=\bullet, T_b\neq\bullet,\\ (T'_1,\mathfrak{m}'_1), & \text{if } T'_1\neq\bullet, T_b=\bullet,\\ \lfloor(T_b;\mathfrak{m})\overline{\diamond}(\overline{F}'_1;\mathfrak{m}')\rfloor+\lfloor(\overline{F}_b;\mathfrak{m})\overline{\diamond}(T'_1;\mathfrak{m}')\rfloor & \\ +\lambda\lfloor(\overline{F}_b;\mathfrak{m})\overline{\diamond}(\overline{F}'_1;\mathfrak{m}')\rfloor, & \text{if } T'_1=\lfloor\overline{F}'_1\rfloor\neq\bullet, T_b=\lfloor\overline{F}_b\rfloor\neq\bullet.\end{cases}\end{aligned} \tag{24}$$

In the last case, we have used the notations $T'_1=\lfloor\overline{F}'_1\rfloor, T_b=\lfloor\overline{F}_b\rfloor$ and applied the induction hypothesis on $\mathrm{d}(F)+\mathrm{d}(F')$ to define the terms in the brackets on the right hand side. Further, for $(F;\mathfrak{m})\in(F;M)$, define $\lfloor(F;\mathfrak{m})\rfloor=(\lfloor F\rfloor;\mathfrak{m})$. This is well-defined since $\ell(F)=\ell(\lfloor F\rfloor)$.

The product $\overline{\diamond}$ is clearly bilinear. So extending it biadditively, we obtain a binary operation

$$Ш^{\mathrm{NC}}(M)\otimes Ш^{\mathrm{NC}}(M)\to Ш^{\mathrm{NC}}(M).$$

For $(F;\mathfrak{m})\in(F;M)$, define

$$P_M(F;\mathfrak{m})=\lfloor(F;\mathfrak{m})\rfloor=(\lfloor F\rfloor\,;\mathfrak{m})\in(\lfloor F\rfloor;M). \tag{25}$$

As commented above, this is well-defined. Thus P_M defines a linear operator on $Ш^{\mathrm{NC}}(M)$. Note that the right hand side is also $(P_{\mathfrak{F}}(F);\mathfrak{m})$. Here $P_{\mathfrak{F}}$ is the Rota–Baxter operator in Eq. (10). Let

$$j_M: M\to Ш^{\mathrm{NC}}(M) \tag{26}$$

be the $\mathbf{k}$-module map sending $a\in M$ to $(\bullet\sqcup\bullet;a)$.

Theorem 3.4. *Let M be a $\mathbf{k}$-module.*

(a) *The pair $(Ш^{\mathrm{NC}}(M),\overline{\diamond})$ is a unitary associative algebra.*
(b) *The triple $(Ш^{\mathrm{NC}}(M),\overline{\diamond},P_M)$ is a unitary Rota–Baxter algebra of weight λ.*
(c) *The quadruple $(Ш^{\mathrm{NC}}(M),\overline{\diamond},P_M,j_M)$ is the free unitary Rota–Baxter algebra of weight λ on the module M.*

Proof. (a) By definition, $(\bullet,\mathbf{1})$ is the unit of the multiplication $\overline{\diamond}$. For the associativity of $\overline{\diamond}$ on $Ш^{\mathrm{NC}}(M)$ we only need to prove

$$(D\overline{\diamond}D')\overline{\diamond}D''=D\overline{\diamond}(D'\overline{\diamond}D'')$$

for any angularly decorated forests $D\in(F;M), D'\in(F';M)$ and $D''\in(F'';M)$. For this we only need to consider the case when the decorations are by pure tensors. So let $D=(F;\mathfrak{m})$, $D'=(F';\mathfrak{m}')$ and $D''=(F'';\mathfrak{m}'')$ with $\mathfrak{m},\mathfrak{m}'$ and $\mathfrak{m}''$ being pure tensors. Then by Eq. (21), we have

$$(D\overline{\diamond}D')\overline{\diamond}D''=\big((F\diamond F')\diamond F'';(\mathfrak{m}\overline{\otimes}\mathfrak{m}')\overline{\otimes}\mathfrak{m}''\big).$$

Similarly,

$$D\overline{\diamond}(D'\overline{\diamond}D'')=\big(F\diamond(F'\diamond F'');\mathfrak{m}\overline{\otimes}(\mathfrak{m}'\overline{\otimes}\mathfrak{m}'')\big).$$

The product $\overline{\diamond}$ is associative by Theorem 2.5. So the first components of the two right hand sides agree. The product $\overline{\otimes}$ in Eq. (20) is the product in the tensor algebra $T(M) := \bigoplus_{n\geq 0} M^{\otimes n}$, so is also associative. Hence the second component of the two right hand sides agree. This proves the associativity of $\overline{\diamond}$.

(b). The Rota–Baxter relation of $\lfloor\ \rfloor$ on $Ш^{\rm NC}(M)$ follows from the Rota–Baxter relation of $\lfloor\ \rfloor$ on $\mathbf{k}\,\mathcal{F}$ in Theorem 2.5. More specifically, it is the last equation in (24).

(c). Let (R, P) be a unitary Rota–Baxter algebra of weight λ. Let $*$ be the multiplication in R and let $\mathbf{1}_R$ be its unit. Let $f : M \to R$ be a $\mathbf{k}$-module map. We will construct a $\mathbf{k}$-linear map $\bar{f} : Ш^{\rm NC}(M) \to R$ by defining $\bar{f}(D)$ for $D = (F; \mathfrak{m}) \in (\mathcal{F}; M)$. We will achieve this by induction on the depth $\mathrm{d}(F)$ of F.

If $\mathrm{d}(F) = 0$, then $F = \bullet^{\sqcup i}$ for some $i \geq 1$. If $i = 1$, then $D = (\bullet; c)$, $c \in \mathbf{k}$. Define $\bar{f}(D) = c\mathbf{1}_R$. In particular, define $\bar{f}(\bullet; \mathbf{1}) = \mathbf{1}_R$. Then $\bar{f}$ sends the unit to the unit. If $i \geq 2$, then $D = (F; \mathfrak{m})$ with $\mathfrak{m} = a_1 \otimes \cdots \otimes a_n \in M^{\otimes n}$ where $n+1$ is the number of leafs $\ell(F)$. Then we define $\bar{f}(\mathfrak{a}) = f(a_1) * \cdots * f(a_n)$.

Assume that $\bar{f}(D)$ has been defined for all $D = (F; \mathfrak{m})$ with $\mathrm{d}(F) \leq k$ and let $D = (F; \mathfrak{m})$ with $\mathrm{d}(F) = k+1$. So $F \neq \bullet$. Let $D = (T_1; \mathfrak{m}_1) \otimes_{u_1} \cdots \otimes_{u_{b-1}} (T_b; \mathfrak{m}_b)$ be the standard decomposition of D given in Eq. (18). For each $1 \leq i \leq b$, T_i is a tree, so it is either $\bullet$ or is of the form $\lfloor \overline{F}_i \rfloor$ for another forest $\overline{F}_i$. By Lemma 3.2, if $T_i = \bullet$, then $b > 1$ and $\mathfrak{m}_i = \mathbf{1}$. We accordingly define

$$\bar{f}(T_i; \mathfrak{m}_i) = \left\{ \begin{array}{ll} \mathbf{1}_R, & \text{if } T_i = \bullet, \\ P(\bar{f}(\overline{F}_i; \mathfrak{m}_i)), & \text{if } T_i = \lfloor \overline{F}_i \rfloor. \end{array} \right. \tag{27}$$

In the later case, $(\overline{F}_i; \mathfrak{m}_i)$ is a well-defined angularly decorated forest since $\overline{F}_i$ has the same number of leafs as the number of leafs of T_i, and then $\bar{f}(\overline{F}_i; \mathfrak{m}_i)$ is defined by the induction hypothesis since $\mathrm{d}(\overline{F}_i) = \mathrm{d}(T_i) - 1 \leq k$. Therefore we can define

$$\bar{f}(D) = \bar{f}(T_1; \mathfrak{m}_1) * f(u_1) * \cdots * f(u_{b-1}) * \bar{f}(T_b; \mathfrak{m}_b) \tag{28}$$

which is well-defined in R.

For any $D = (F; \mathfrak{m}) \in (F; M)$, we have $P_M(D) = (\lfloor F \rfloor; \mathfrak{m}) \in Ш^{\rm NC}(M)$, and by the definition of $\bar{f}$ in Eq. (27) — (28), we have

$$\bar{f}(\lfloor D \rfloor) = P(\bar{f}(D)). \tag{29}$$

So $\bar{f}$ commutes with the Rota–Baxter operators.

Further, Eq. (27) — (28) is clearly the only way to define $\bar{f}$ in order for $\bar{f}$ to be a Rota–Baxter algebra homomorphism that extends f.

It remains to prove that the map $\bar{f}$ defined in Eq. (28) is indeed an algebra homomorphism. For this we only need to check the multiplicativity

$$\bar{f}(D \overline{\diamond} D) = \bar{f}(D) * \bar{f}(D') \tag{30}$$

for all angularly decorated forests $D = (F; \mathfrak{m}), D' = (F'; \mathfrak{m}')$ with pure tensors $\mathfrak{m}$ and $\mathfrak{m}'$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$ be the decompositions of F and F' into trees. Let

$$(F; \mathfrak{m}) = (T_1; \mathfrak{m}_1) \otimes_{u_1} (T_2; \mathfrak{m}_2) \otimes_{u_2} \cdots \otimes_{u_{b-1}} (T_b; \mathfrak{m}_b).$$

and

$$(F'; \mathfrak{m}') = (T'_1; \mathfrak{m}'_1) \otimes_{u'_1} (T'_2; \mathfrak{m}'_2) \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} (T'_{b'}; \mathfrak{m}'_{b'})$$

be their standard decompositions.

We first note that, since $\bar{f}$ sends the identity $(\bullet;\mathbf{1})$ of $Ш^{\rm NC}(M)$ to the identity $\mathbf{1}_R$ of R, the multiplicativity is clear if either one of D or D' is in $(\bullet;\mathbf{k})$, that is, if either one of F or F' is $\bullet$. So we only need to verify the multiplicativity when $F\neq\bullet$ and $F'\neq\bullet$.

We further make the following simplification. By Eq. (28) and Eq. (23), we have

$$\begin{aligned}\bar{f}(D\bar{\diamond}D') &= \bar{f}(T_1;\mathfrak{m}_1)*f(u_1)*\cdots*f(u_{b-1})\\ &\quad *\bar{f}\big((T_b;\mathfrak{m}_b)\bar{\diamond}(T'_1;\mathfrak{m}'_1)\big)*f(u'_1)*\cdots*f(u'_{b'-1})*\bar{f}(T'_{b'};\mathfrak{m}'_{b'})\end{aligned}$$

and

$$\begin{aligned}\bar{f}(D)*\bar{f}(D') &= \bar{f}(T_1;\mathfrak{m}_1)*f(u_1)*\cdots*f(u_{b-1})\\ &\quad *\bar{f}(T_b;\mathfrak{m}_b)*\bar{f}(T'_1;\mathfrak{m}'_1)*f(u'_1)*\cdots*f(u'_{b'-1})*\bar{f}(T'_{b'};\mathfrak{m}'_{b'}).\end{aligned}$$

We thus have

$$\bar{f}((D;\mathfrak{m})\bar{\diamond}(D';\mathfrak{m}'))=\bar{f}(D;\mathfrak{m})*\bar{f}(D';\mathfrak{m}') \tag{31}$$

if and only if

$$\bar{f}((T_b,;\mathfrak{m}_b)\bar{\diamond}(T'_1;\mathfrak{m}'_1))=\bar{f}(T_b;\mathfrak{m}_b)*\bar{f}(T'_1;\mathfrak{m}'_1). \tag{32}$$

So we only need to prove Eq. (32). For this we use induction on the sum of depths $n:=\mathrm{d}(T_b)+\mathrm{d}(T'_1)$ of T_b and T'_1. Then $n\geq 0$. When $n=0$, we have $T_b=T'_1=\bullet$. So by Lemma 3.2, we have $b>1, b'>1$, and

$$(T_b;\mathfrak{m}_b)=(T'_1;\mathfrak{m}'_1)=(T_b;\mathfrak{m}_b)\bar{\diamond}(T'_1;\mathfrak{m}'_1)=(\bullet;\mathbf{1}).$$

Then

$$\bar{f}(T_b;\mathfrak{m}_b)=\bar{f}(T'_1;\mathfrak{m}'_1)=\bar{f}((T_b;\mathfrak{m}_b)\bar{\diamond}(T'_1;\mathfrak{m}'_1))=\mathbf{1}_R.$$

Thus Eq. (32) and hence Eq. (31) holds.

Assume that the multiplicativity holds for $D=T_1\sqcup\cdots\sqcup T_b$ and $D'=T'_1\sqcup\cdots\sqcup T'_{b'}$ in $(\mathcal{F};M)$ with $n=\mathrm{d}(T_b)+\mathrm{d}(T'_1)\leq k$ and take $D,D'\in(\mathcal{F};M)$ with $n=k+1$. So $n\geq 1$. Then at least one of $\mathrm{d}(T_b)$ and $\mathrm{d}(T'_1)$ is not zero. If exactly one of them is zero, so exactly one of T_b and T'_1 is $\bullet$, then by Eq. (24),

$$(T_b;\mathfrak{m}_b)\bar{\diamond}(T'_1;\mathfrak{m}'_1)=\begin{cases}(T_b;\mathfrak{m}_b), & \text{if } T'_1=\bullet, T_b\neq\bullet,\\ (T'_1;\mathfrak{m}'_1), & \text{if } T'_1\neq\bullet, T_b=\bullet.\end{cases}$$

Then

$$\bar{f}((T_b;\mathfrak{m}_b)\bar{\diamond}(T'_1;\mathfrak{m}'_1))=\begin{cases}\bar{f}(T_b;\mathfrak{m}_b), & \text{if } T'_1=\bullet, T_b\neq\bullet,\\ \bar{f}(T'_1;\mathfrak{m}'_1), & \text{if } T'_1\neq\bullet, T_b=\bullet.\end{cases}$$

Then Eq. (32) and hence (31) holds since one factor in $\bar{f}(T_b;\mathfrak{m}_b)*\bar{f}(T'_1;\mathfrak{m}'_1)$ is $\mathbf{1}_R$.

If neither $\mathrm{d}(T_b)$ nor $\mathrm{d}(T'_1)$ is zero, then $T_b=\lfloor\overline{F}_b\rfloor$ and $T'_1=\lfloor\overline{F}'_1\rfloor$ for some forests $\overline{F}_b$ and $\overline{F}'_1$ in $\mathcal{F}$. Then $(T_b;\mathfrak{m}_b)=\lfloor(\overline{F}_b;\mathfrak{m}_b)\rfloor$ and $(T'_1;\mathfrak{m}'_1)=\lfloor(\overline{F}'_1;\mathfrak{m}'_1)\rfloor$. We will take care of this case by the following lemma which will be used again later.

Lemma 3.5. *Let (R_1,P_1) and (R_2,P_2) be not necessarily associative $\mathbf{k}$-algebras R_1 and R_2 together with $\mathbf{k}$-linear endomorphisms P_1 and P_2 that each satisfies the Rota–Baxter identity in Eq. (1). Let $g:R_1\to R_2$ be a $\mathbf{k}$-linear map such that*

$$g\circ P_1=P_2\circ g. \tag{33}$$

Let $x,y\in R_1$ be such that

$$g(xP_1(y))=g(x)\cdot g(P_1(y)),\ \ g(P_1(x)y)=g(P_1(x))\cdot g(y),\ \ g(xy)=g(x)\cdot g(y). \tag{34}$$

Here we have suppressed the product in R_1 *and denote the product in* R_2 *by* $\cdot$. *Then* $g(P_1(x)P_1(y)) = g(P_1(x)) \cdot g(P_1(y))$.

Proof. By the Rota–Baxter relations of P_1 and P_2, Eq. (33) and Eq. (34), we have

$$\begin{aligned}
g(P_1(x)P_1(y)) &= g\big(P_1(P_1(x)y) + P_1(xP_1(y)) + \lambda P_1(xy)\big) \\
&= g(P_1(P_1(x)y)) + g(P_1(xP_1(y))) + g(\lambda P_1(xy)) \\
&= P_2(g(P_1(x)y)) + P_2(g(xP_1(y))) + \lambda P_2(g(xy)) \\
&= P_2(g(P_1(x)) \cdot g(y)) + P_2(g(x) \cdot g(P_1(y))) + \lambda P_2(g(x) \cdot g(y)) \\
&= P_2(P_2(g(x)) \cdot g(y)) + P_2(g(x) \cdot P_2(g(y))) + \lambda P_2(g(x) \cdot g(y)) \\
&= P_2(g(x)) \cdot P_2(g(y)) \\
&= g(P_1(x)) \cdot g(P_1(y)).
\end{aligned}$$

□

Now we apply Lemma 3.5 to our proof with $(R_1, P_1) = (\text{Ш}^{\mathrm{NC}}(M), \lfloor\ \rfloor)$, $(R_2, P_2) = (R, P)$ and $g = \bar{f}$. By the induction hypothesis, Eq. (34) holds for $x = (\overline{F}_b; \mathfrak{m}_b)$ and $y = (\overline{F}'_1; \mathfrak{m}'_1)$. Therefore by Lemma 3.5, $\bar{f}(T_b \overline{\diamond} T'_1) = \bar{f}(T_b) * \bar{f}(T'_1)$. Thus Eq. (31) holds for $n = k+1$. This completes the induction and the proof of Theorem 3.4. □

3.3. **Free nonunitary Rota–Baxter algebras over a module.** We now modify the construction of free unitary Rota–Baxter algebras in Section 3.2 to obtain free nonunitary Rota–Baxter algebras. Since the constructions are quite similar, we will be brief for most parts except for the differences.

As in Corollary 2.6, we let $\mathcal{F}^0$ be the subset of $\mathcal{F}\backslash\{\bullet\}$ consisting of forests that do not contain any $\lfloor \bullet \rfloor$. For any $\mathbf{k}$-module M, define the $\mathbf{k}$-submodule

$$\text{Ш}^{\mathrm{NC},0}(M) = \bigoplus_{F \in \mathcal{F}^0} (F; M)$$

of $\text{Ш}^{\mathrm{NC}}(M)$. Thus $\text{Ш}^{\mathrm{NC},0}(M)$ is generated as an abelian group by pairs $(F; \mathfrak{m})$ where F is in $\mathcal{F}^0$ and $\mathfrak{m}$ is in $M^{\ell(F)-1}$ where $\ell(F)$ is the number of leafs of F. In fact, since $F \in \mathcal{F}^0$, it must have at least two leafs. So $\mathfrak{m}$ is in $M^{\otimes r}$ with $r \geq 1$.

We define a product $\overline{\diamond}$ on $\text{Ш}^{\mathrm{NC},0}(M)$ to be the restriction of $\overline{\diamond}$ on $\text{Ш}^{\mathrm{NC}}(M)$. This product is well-defined since for $D = (F; \mathfrak{m})$ and $D' = (F'; \mathfrak{m})$ in $(\mathcal{F}^0; M)$, we have by definition in Eq. (21),

$$D \overline{\diamond} D' = (F \diamond F'; \mathfrak{m} \overline{\otimes} \mathfrak{m}')$$

where $\mathfrak{m} \overline{\otimes} \mathfrak{m}'$ is the tensor product in Eq. (20). By Corollary 2.6, $F \diamond F'$ is still in $\mathbf{k}\,\mathcal{F}^0$. Thus the right hand side of the above equation is a linear combination of elements from $(\mathcal{F}^0, M)$ and hence is in $\text{Ш}^{\mathrm{NC},0}(M)$.

Also define $\lfloor\ \rfloor : \text{Ш}^{\mathrm{NC},0}(M) \to \text{Ш}^{\mathrm{NC},0}(M)$ to be the restriction of $\lfloor\ \rfloor$ on $\text{Ш}^{\mathrm{NC}}(M)$. This again is well-defined since by Corollary 2.6, $\lfloor \mathcal{F}^0 \rfloor \subseteq \mathcal{F}^0$. Then adapting the notation and proof of Theorem 3.4, we obtain

Theorem 3.6. *Let* M *be a* $\mathbf{k}$*-module.*

(a) *The pair* $(\text{Ш}^{\mathrm{NC},0}(M), \overline{\diamond})$ *is a nonunitary associative algebra.*
(b) *The triple* $(\text{Ш}^{\mathrm{NC},0}(M), \overline{\diamond}, P_M)$ *is a nonunitary Rota–Baxter algebra of weight* λ.
(c) *The quadruple* $(\text{Ш}^{\mathrm{NC},0}(M), \overline{\diamond}, P_M, j_M)$ *is the free nonunitary Rota–Baxter algebra of weight* λ *on the* $\mathbf{k}$*-module* M.

Proof. Since the product $\overline{\diamond}$ on $Ш^{NC,0}(M)$ is the restriction of the associative product $\overline{\diamond}$ on $Ш^{NC}(M)$, it is still associative. This proves (a). For the same reason, the Rota–Baxter relation (1) holds for the restriction of $\lfloor\ \rfloor$, proving (b). Part (c) is proved in the same way as the unitary case with the following modification. Let $(R, *, P)$ be a nonunitary Rota–Baxter algebra. In the recursive definition of $\bar{f}$ in Eq. (28), when $(T_i; \mathfrak{m}_i) = (\bullet; \mathbf{1})$, simply delete the factor $\bar{f}(T_i; \mathfrak{m}_i)$ instead of letting it be $\mathbf{1}_R$ which is not defined. Alternatively, augment R to a unitary $\mathbf{k}$-algebra $\tilde{R} = \mathbf{k1}_R \oplus R$ with unit $\mathbf{1}_R$. Of course $\tilde{R}$ can not be expected to be a Rota–Baxter algebra. But it does not matter since we only need the algebra structure on $\tilde{R}$ to obtain a Rota–Baxter algebra structure on R. For $D = (F; \mathfrak{m}) \in (F; M)$ with $F \in \mathcal{F}^0$, just define $\bar{f}(D)$ as in Eq. (28). Note that F has at least two leafs, so $\mathfrak{m}$ is in $M^{\otimes r}$ with $r \geq 1$. Then it follows by induction that $\bar{f}(D)$ is always in R. Then the rest of the proof goes through. $\square$

4. Free Rota–Baxter algebra over a set

This section serves two purposes. The first one is to use the construction of free Rota–Baxter algebra over a module as decorated forests to obtain a similar construction of a free Rota–Baxter algebra over a set. This is given in Section 4.1 where we display a canonical basis of the free Rota–Baxter algebra in terms of forests decorated by the set. The second purpose is to give another construction of free Rota–Baxter algebra over a set, as bracketed words. This is given in Section 4.2.

4.1. **Free Rota–Baxter algebra over a set as decorated forests.** Either by the general principle of forgetful functors or by a direct checking, the free Rota–Baxter algebra $Ш^{NC}(X)$ over a set X is easily seen to be the free Rota–Baxter algebra over the $\mathbf{k}$-module M when M is taken to be the free $\mathbf{k}$-module $\mathbf{k}\,X = \oplus_{x\in X}\mathbf{k}\,x$ with basis X. Thus we can easily obtain a construction of $Ш^{NC}(X)$ by decorated forests from the construction of $Ш^{NC}(M)$ in Section 3.

For any $n \geq 1$, the tensor power $M^{\otimes n}$ has a natural basis

$$X^n = \{(x_1, \cdots, x_n) \mid x_i \in X,\ 1 \leq i \leq n\}.$$

Accordingly, for any rooted forest $F \in \mathcal{F}$, with $\ell = \ell(F) \geq 2$, the set

$$X^F := (F; X) := \{(F; (x_1, \cdots, x_{\ell-1})) := (F; x_1 \otimes \cdots \otimes x_{\ell-1}) \mid x_i \in X,\ 1 \leq i \leq \ell - 1\}$$

form a basis of $M^{\otimes F} = (F; M)$ defined in Eq. (16). Note that when $\ell(F) = 1$, $(F; M) = \mathbf{k}\,F$ has a basis $(F; X) := \{(F; \mathbf{1})\}$. In summary, every $(F; M), F \in \mathcal{F}$, has a basis

$$X^F := (F; X) = \{(F; \vec{x}) \mid \vec{x} \in X^{\ell(F)-1}\}, \tag{35}$$

with the convention that $X^0 = \{\mathbf{1}\}$. Thus the disjoint union

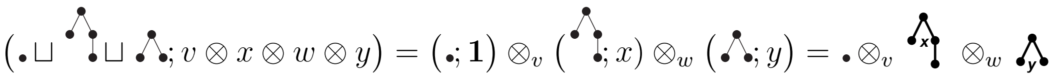
$$(\mathcal{F}; X) := \coprod_{F\in\mathcal{F}} (F; X). \tag{36}$$

forms a basis of $Ш^{NC}(X) := Ш^{NC}(M)$. We call $(\mathcal{F}; X)$ the set of angularly decorated rooted forests with decoration set X. As in Section 3.1, they can be pictured as rooted forests with adjacent leafs decorated by elements from X. In fact this picture is more precise since there is no need to be concerned with the issue of moving a scalar from one angle to another one as in the module case.

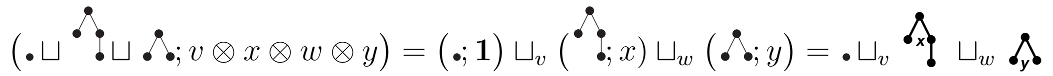
Likewise, for $(F;\vec{x}) \in (\mathcal{F};X)$, the decomposition (18) gives the **standard decomposition**

$$(F;\vec{x}) = (T_1;\vec{x}_1) \sqcup_{u_1} (T_2;\vec{x}_2) \sqcup_{u_2} \cdots \sqcup_{u_{b-1}} (T_b;\vec{x}_b) \tag{37}$$

where $F = T_1 \sqcup \cdots \sqcup T_b$ is the decomposition of F into trees and $\vec{x}$ is the vector concatenation of the elements of $\vec{x}_1, u_1, \vec{x}_2, \cdots, u_{b-1}, \vec{x}_b$ which are not the unit $\mathbf{1}$. Here we have used $\sqcup_u$ instead of $\otimes_u$ to indicate that we are considering concatenations of basis elements, not tensor products. For example, let v, w, x, y be in X, then the decomposition



from Eq. (18) gives the decomposition



As a corollary of Theorem 3.4, we have

Theorem 4.1. *Let* $Ш^{NC}(X)$ *be the free* $\mathbf{k}$*-module with the basis* $(\mathcal{F};X)$*. For* $D = (F;(x_1,\cdots,x_b))$*,* $D' = (F';(x'_1,\cdots,x'_{b'}))$ *in* $(\mathcal{F};X)$*, define*

$$D\overline{\diamond}D' = \begin{cases} (\bullet;\mathbf{1}), & \text{if } F = F' = \bullet, \\ D, & \text{if } F' = \bullet, F \neq \bullet, \\ D', & \text{if } F = \bullet, F' \neq \bullet, \\ (F \diamond F';(x_1,\cdots,x_b,x'_1,\cdots,x'_{b'})), & \text{if } F \neq \bullet, F' \neq \bullet, \end{cases} \tag{38}$$

where $\diamond$ *is defined in Eq. (7) and (9). Define*

$$P_X : Ш^{NC}(X) \to Ш^{NC}(X), \quad P_X(F;(x_1,\cdots,x_b)) = (\lfloor F \rfloor;(x_1,\cdots,x_b)),$$

and

$$j_X : X \to Ш^{NC}(X), \quad j_X(x) = (\bullet \sqcup \bullet;(x)), \quad x \in X.$$

Then the quadruple $(Ш^{NC}(X), \overline{\diamond}, P_X, j_X)$ *is the free Rota–Baxter algebra over* X*.*

Proof. The product $\overline{\diamond}$ in Eq. (38) is defined to be the restriction of the product $\overline{\diamond}$ in Eq. (22) to $(\mathcal{F};X)$. Since $(\mathcal{F};X)$ is a basis of $Ш^{NC}(X)$, the two products coincide. So $Ш^{NC}(X)$ and $Ш^{NC}(M)$ are the same as Rota-Baxter algebras. Then as commented at the beginning of this section, $Ш^{NC}(X)$ is the free Rota–Baxter algebra over X. □

As with Theorem 3.6, the same proof there also gives

Theorem 4.2. *The subalgebra* $Ш^{NC,0}(X)$ *of* $Ш^{NC}(X)$ *generated by the* $\mathbf{k}$*-basis* $(\mathcal{F}^0;X)$*, with the same product* $\overline{\diamond}$*, Rota–Baxter operator* P_X *and set map* j_X*, is the free nonunitary Rota–Baxter algebra over* X*.*

4.2. **Free Rota–Baxter algebra over a set as bracketed words.** For the purposes of the study and applications of free Rota-Baxter algebras, it is desirable to express a basis of the algebra in terms of words. We will give a recursive definition of these words, in analogy to the construction in [23], but will derive it from the decorated rooted forest of the free Rota–Baxter algebra. There is also an explicit (non-recursive) definition of these words which has been omitted for lack of space. Similar results hold for free nonunitary Rota–Baxter algebras in Theorem 4.2.

4.2.1. *Recursive definition of* $(\mathcal{F};X)$ *by words.* We first extend the universal property of rooted forests in § 2 to a broader context. Let X be a set. The category of mapped monoids and free mapped monoid over X are defined in the same way as in the case of mapped semigroups in Definition 2.2. We have the following construction of free mapped monoid over X.

For any set Y, let $S(Y)$ be the free semigroup generated by Y, let $M(Y)$ be the free monoid generated by Y and let $\lfloor Y\rfloor$ be the set $\{\lfloor y\rfloor \big| y\in Y\}$ which is just another copy of Y whose elements are denoted by $\lfloor y\rfloor$ for distinction. We recursively define a direct system $\{\mathfrak{S}_n, i_{n,n+1}:\mathfrak{S}_n\to\mathfrak{S}_{n+1}\}$ of free semigroups and a direct system $\{\mathfrak{M}_n,\tilde{i}_{n,n+1}:\mathfrak{M}_n\to\mathfrak{M}_{n+1}\}$ of free monoids, both with injective transition maps. We do this by first letting $\mathfrak{S}_0=S(X)$ and $\mathfrak{M}_0=M(X)$, and then define

$$\mathfrak{S}_1=S(X\cup\lfloor\mathfrak{M}_0\rfloor)=S(X\cup\lfloor M(X)\rfloor),\quad \mathfrak{M}_1=M(X\cup\lfloor\mathfrak{M}_0\rfloor)$$

with $i_{0,1}$ and $\tilde{i}_{0,1}$ being the natural injection

$$\begin{array}{ll} i_{0,1}: & \mathfrak{S}_0=S(X)\hookrightarrow\mathfrak{S}_1=S(X\cup\lfloor\mathfrak{M}_0\rfloor),\\ \tilde{i}_{0,1}: & \mathfrak{M}_0=M(X)\hookrightarrow\mathfrak{M}_1=M(X\cup\lfloor\mathfrak{M}_0\rfloor).\end{array}$$

We remark that elements in $\lfloor M(X)\rfloor$ are only symbols indexed by elements in $M(X)$. In particular, $\lfloor\mathbf{1}\rfloor$ is not the identity. We identify $\mathfrak{S}_0$ and $\mathfrak{M}_0$ with their images in $\mathfrak{S}_1$ and $\mathfrak{M}_1$. In particular, $\mathbf{1}\in\mathfrak{M}_0$ is sent to $\mathbf{1}\in\mathfrak{M}_1$.

Inductively assume that $\mathfrak{S}_{n-1}$ and $\mathfrak{M}_{n-1}$ have been defined for $n\geq 2$, with the embeddings

$$i_{n-2,n-1}:\mathfrak{S}_{n-2}\hookrightarrow\mathfrak{S}_{n-1} \text{ and } \tilde{i}_{n-2,n-1}:\mathfrak{M}_{n-2}\to\mathfrak{M}_{n-1}.$$

We define

$$\mathfrak{S}_n:=S(X\cup\lfloor\mathfrak{M}_{n-1}\rfloor),\quad \mathfrak{M}_n:=M(X\cup\lfloor\mathfrak{M}_{n-1}\rfloor)=\mathfrak{S}_n\cup\{\mathbf{1}\}. \tag{39}$$

We also have the injections

$$\lfloor\mathfrak{M}_{n-2}\rfloor\hookrightarrow\lfloor\mathfrak{M}_{n-1}\rfloor \text{ and } X\cup\lfloor\mathfrak{M}_{n-2}\rfloor\hookrightarrow X\cup\lfloor\mathfrak{M}_{n-1}\rfloor,$$

yielding injective maps of free semigroups and free monoids

$$\begin{array}{rcl}\mathfrak{S}_{n-1} &=& S(X\cup\lfloor\mathfrak{M}_{n-2}\rfloor)\hookrightarrow S(X\cup\lfloor\mathfrak{M}_{n-1}\rfloor)=\mathfrak{S}_n,\\ \mathfrak{M}_{n-1} &=& M(X\cup\lfloor\mathfrak{M}_{n-2}\rfloor)\hookrightarrow M(X\cup\lfloor\mathfrak{M}_{n-1}\rfloor)=\mathfrak{M}_n.\end{array}$$

We finally define the semigroup

$$\mathfrak{S}(X)=\varinjlim\mathfrak{S}_n$$

and monoid

$$\mathfrak{M}(X)=\varinjlim\mathfrak{M}_n$$

with identity (the image of) $\mathbf{1}$.

The same proof for Proposition 2.3 gives

Proposition 4.3. *Let $j_X:X\to\mathfrak{M}(X)$ be the natural embedding.*

(a) *The triple $(\mathfrak{M}(X),(\lfloor\ \rfloor),j_X)$ is the free mapped monoid on X.*
(b) *The triple $(\mathbf{k}\,\mathfrak{M}(X),(\lfloor\ \rfloor),j_X)$ is the free mapped unitary algebra on X.*

Let Y, Z be two subsets of $\mathfrak{M}(X)$. Define the **alternating products**

$$\begin{aligned}\Lambda_X(Y,Z) &= \Big(\bigcup_{r\geq 1} (Y\lfloor Z\rfloor)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(Y\lfloor Z\rfloor)^r Y\Big)\\ &\bigcup\Big(\bigcup_{r\geq 1}(\lfloor Z\rfloor Y)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(\lfloor Z\rfloor Y)^r\lfloor Z\rfloor\Big).\end{aligned} \tag{40}$$

It is again a subset of $\mathfrak{M}(X)$. Using this, we construct a sub-system $\{\mathfrak{X}_n\}$ of $\{\mathfrak{S}_n\}$ and a sub-system $\{\tilde{\mathfrak{X}}_n\}$ of $\{\mathfrak{M}_n\}$ by the following recursion. Let

$$\mathfrak{X}_0 = S(X),\ \tilde{\mathfrak{X}}_0 = S(X)\cup\{\mathbf{1}\} = M(X).$$

In general, for $n\geq 1$, define

$$\mathfrak{X}_n = \Lambda_X(\mathfrak{X}_0,\tilde{\mathfrak{X}}_{n-1}),\ \tilde{\mathfrak{X}}_n = \mathfrak{X}_n\cup\{\mathbf{1}\}. \tag{41}$$

More precisely,

$$\begin{aligned}\mathfrak{X}_n &= \Big(\bigcup_{r\geq 1}(\mathfrak{X}_0\lfloor\tilde{\mathfrak{X}}_{n-1}\rfloor)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(\mathfrak{X}_0\lfloor\tilde{\mathfrak{X}}_{n-1}\rfloor)^r\mathfrak{X}_0\Big)\\ &\bigcup\Big(\bigcup_{r\geq 1}(\lfloor\tilde{\mathfrak{X}}_{n-1}\rfloor\mathfrak{X}_0)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(\lfloor\tilde{\mathfrak{X}}_{n-1}\rfloor\mathfrak{X}_0)^r\lfloor\tilde{\mathfrak{X}}_{n-1}\rfloor\Big).\end{aligned} \tag{42}$$

So for instance, $\mathfrak{X}_1 = \Lambda_X(S(X),\tilde{\mathfrak{X}}_0)$, $\tilde{\mathfrak{X}}_1 = \mathfrak{X}_1\cup\{\mathbf{1}\}$. Further, define

$$\mathfrak{X}_\infty = \bigcup_{n\geq 0}\mathfrak{X}_n = \varinjlim \mathfrak{X}_n, \tag{43}$$

$$\tilde{\mathfrak{X}}_\infty = \mathfrak{X}_\infty\cup\{\mathbf{1}\} = \bigcup_{n\geq 0}\tilde{\mathfrak{X}}_n = \varinjlim\tilde{\mathfrak{X}}_n. \tag{44}$$

Here the last equations in Eq. (43) and (44) follow since $\mathfrak{X}_1\supseteq\mathfrak{X}_0$, $\tilde{\mathfrak{X}}_1\supseteq\tilde{\mathfrak{X}}_0$ and, assuming $\mathfrak{X}_n\supseteq\mathfrak{X}_{n-1}$ and $\tilde{\mathfrak{X}}_n\supseteq\tilde{\mathfrak{X}}_{n-1}$, we get $\mathfrak{X}_{n+1} = \Lambda_X(\mathfrak{X}_0,\tilde{\mathfrak{X}}_n)\supseteq\Lambda_X(\mathfrak{X}_0,\tilde{\mathfrak{X}}_{n-1}) = \mathfrak{X}_n$ and thus $\tilde{\mathfrak{X}}_{n+1}\supseteq\tilde{\mathfrak{X}}_n$.

Definition 4.4. *Elements in $\tilde{\mathfrak{X}}_\infty$ are called* **unitary Rota–Baxter parenthesized words (RBWs)**.

The following properties of RBWs are easily verified.

Lemma 4.5. (a)

$$\lfloor\tilde{\mathfrak{X}}_\infty\rfloor\subseteq\tilde{\mathfrak{X}}_\infty. \tag{45}$$

(b) *Every RBW $\mathbf{x}\neq\mathbf{1}$ has a unique decomposition*

$$\mathbf{x} = \mathbf{x}_1\cdots\mathbf{x}_b, \tag{46}$$

where $\mathbf{x}_i$, $1\leq i\leq b$, is alternatively in $S(X)$ or in $\lfloor\mathfrak{X}_\infty\rfloor$. This decomposition will be called the **standard decomposition** *of* $\mathbf{x}$.

For example, for $x \in X$, the standard decomposition of

$$\mathbf{x} = x^2\lfloor\lfloor x\rfloor x\rfloor x\lfloor x^2\rfloor x^3\lfloor x\lfloor x\rfloor\rfloor$$

is

$$\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4\mathbf{x}_5\mathbf{x}_6$$

where

$$\mathbf{x}_1 = x^2, \mathbf{x}_2 = \lfloor\lfloor x\rfloor x\rfloor, \mathbf{x}_3 = x, \mathbf{x}_4 = \lfloor x^2\rfloor, \mathbf{x}_5 = x^3, \mathbf{x}_6 = \lfloor x\lfloor x\rfloor\rfloor.$$

Proof. (a) is clear since $\lfloor\mathfrak{X}_{n-1}\rfloor \subseteq \mathfrak{X}_n$ by Eq. (42).

For (b), consider $\mathbf{x} \in \tilde{\mathfrak{X}}_\infty$. Since $\mathbf{x} \neq \mathbf{1}$, it is in $\mathfrak{X}_n \subseteq \mathfrak{S}_n$ for some $n \geq 0$. Since $\mathfrak{S}_n$ is the free semigroup generated by $X \cup \lfloor\mathfrak{M}_{n-1}\rfloor$, $\mathbf{x}$ has a unique decomposition

$$\mathbf{x} = y_1 \cdots y_t$$

with $y_i \in X \cup \lfloor\mathfrak{M}_{n-1}\rfloor$. By the definition of $\mathfrak{X}_n$ in Eq. (42), there cannot be two consecutive y_is that are both in $\lfloor\mathfrak{M}_{n-1}\rfloor$. In other words, any two $y_i \in \lfloor\mathfrak{M}_{n-1}\rfloor$ are separated by an element of $S(X)$. The factors in $\lfloor\mathfrak{M}_{n-1}\rfloor$ and the elements of $S(X)$ that either separate these factors in $\lfloor\mathfrak{M}_{n-1}\rfloor$ or before the first of these factors or after the last of these factors (if there is any) in $\mathbf{x}$ give the decomposition. $\square$

4.2.2. *Bijection of the two sets.* We next define an injective map η from $(\mathcal{F}; X)$ to $\mathfrak{M}(X)$, identifying $(\mathcal{F}; X)$ with $\tilde{\mathfrak{X}}_\infty$.

Theorem 4.6. *There is a unique bijection*

$$\eta : (\mathcal{F}; X) \to \tilde{\mathfrak{X}}_\infty \tag{47}$$

with the properties that

(a) $\eta(\bullet; \mathbf{1}) = \mathbf{1}$;

(b) *if* $(F; \vec{x}) = (F_1; \vec{x}_1) \sqcup_u (F_2; \vec{x}_2)$ *with* $(F_i; \vec{x}_i) \in (\mathcal{F}; X), i = 1, 2,$ *and* $u \in X$*, then*

$$\eta(F; \vec{x}) = \eta(F_1; \vec{x}_1)u\eta(F_2; \vec{x}_2). \tag{48}$$

(c) *if* $(F; \vec{x})$ *with* $F = \lfloor\overline{F}\rfloor$ *for* $\overline{F} \in \mathcal{F}$*, then*

$$\eta(\lfloor\overline{F}\rfloor; \vec{x}) = \lfloor\eta(\overline{F}; \vec{x})\rfloor. \tag{49}$$

We recall that $\mathbf{1}$ is the identity of $M(X)$, so

$$x\mathbf{1} = \mathbf{1}x = x, \ \forall \ x \in X \cup \{\mathbf{1}\}. \tag{50}$$

Proof. We define $\eta(F; \vec{x})$ by induction on the depth $\mathrm{d} := \mathrm{d}(F)$ and show that it has properties (a) – (c). Along the way we will see that such a map is unique.

When $\mathrm{d} = 0$, we have $F = \bullet^{\sqcup i}$ for some $i \geq 1$. When $i = 1$, we have $(F; \vec{x}) = (\bullet; \mathbf{1})$ and we define $\eta(F; \vec{x}) = \mathbf{1}$, as required by the properties. When $i > 1$, define $\eta(F; \vec{x}) = x_1 \cdots x_{i-1}$. Here $\vec{x} = (x_1, \cdots, x_{i-1})$.

Assume that $\eta(F; \vec{x})$ has been defined for $d \leq k$ satisfying the required properties. Let $(F; \vec{x}) \in (\mathcal{F}; X)$ with $\mathrm{d}(F) = k + 1$. Let

$$(F; \vec{x}) = (T_1; \vec{x}_1) \sqcup_{u_1} (T_2; \vec{x}_2) \cdots (T_{b-1}; \vec{x}_{b-1}) \sqcup_{u_{b-1}} (T_b; \vec{x}_b)$$

be the standard decomposition of $(F;\vec{x})$. For each $i=1,\cdots,b$, the tree T_i is either $\bullet$ or $\lfloor \overline{F}_i\rfloor$ for a forest $\overline{F}_i\in\mathcal{F}$ with $\mathrm{d}(\overline{F}_i)\le k$. We accordingly define

$$\eta(F_i;\vec{x}_i)=\left\{\begin{array}{ll} \mathbf{1}, & \text{if } F_i=\bullet,\\ \lfloor \eta(\overline{F}_i;\vec{x}_i)\rfloor, & \text{if } F_i=\lfloor \overline{F}_i\rfloor. \end{array}\right.$$

Here in the second case we have used the induction hypothesis and Eq. (45). We then define

$$\eta(F;\vec{x})=\eta(F_1;\vec{x}_1)u_1\eta(F_2;\vec{x}_2)u_2\cdots u_{b-1}\eta(F;\vec{x}_b). \tag{51}$$

This is clearly the one and only way to define $\eta(F;\vec{x})$ with the required properties. This completes the inductive definition of η.

To prove that η is a bijection, we explicitly display the inverse $\rho:\tilde{\mathfrak{X}}_\infty\to(\mathcal{F};X)$ of η. Let $\mathbf{x}\in\tilde{\mathfrak{X}}_\infty$. If $\mathbf{x}=\mathbf{1}$, we define $\rho(\mathbf{x})=(\bullet;\mathbf{1})$. So we just need to define $\rho(\mathbf{x})$ for $\mathbf{x}\ne\mathbf{1}$. Then $\mathbf{x}\in\mathfrak{X}_\infty=\cup_{n\ge 0}\mathfrak{X}_n$. We use induction on n to define

$$\rho:\mathfrak{X}_n\to(\mathcal{F}_n;X)$$

such that ρ is the inverse of $\eta:(\mathcal{F}_n;X)\backslash\{(\bullet;\mathbf{1})\}\to\mathfrak{X}_n$. When $n=0$, $\mathfrak{X}_n=S(X)$. Then $\mathbf{x}=x_1\cdots x_t$ with $x_i\in X$, $1\le i\le t$, $t\ge 1$. We define

$$\rho(\mathbf{x})=(\bullet^{\sqcup(t+1)};(x_1,\cdots,x_t)).$$

Then evidently, ρ is the inverse of $\eta:(\mathcal{F}_0;X)\backslash\{(\bullet;\mathbf{1})\}\to\mathfrak{X}_0$.

Assume that the inverse ρ of

$$\eta:(\mathcal{F}_k;X)\backslash\{(\bullet;\mathbf{1})\}\to\mathfrak{X}_k$$

has been defined for $k\ge 0$. For $\mathbf{x}\in\mathfrak{X}_{k+1}$, it has a unique factorization

$$\mathbf{x}=\mathbf{x}_1\cdots\mathbf{x}_t$$

with $\mathbf{x}_i$ alternatively in the two union components of $S(X)\cup\lfloor\tilde{\mathfrak{X}}_k\rfloor$. We accordingly define

$$\rho(\mathbf{x})=\tilde{\rho}(\mathbf{x}_1)\cdots\tilde{\rho}(\mathbf{x}_t),$$

Here when $\mathbf{x}_i=\lfloor\overline{\mathbf{x}}_i\rfloor\in\lfloor\tilde{\mathfrak{X}}_k\rfloor$ with $\overline{\mathbf{x}}_i\in\tilde{\mathfrak{X}}_k$, define

$$\tilde{\rho}(\mathbf{x}_i)=\lfloor\rho(\overline{\mathbf{x}}_i)\rfloor$$

using the induction hypothesis; while when $\mathbf{x}_i\in S(X)$ with $\mathbf{x}_i=x_1\cdots x_r\in X^r$, define $\tilde{\rho}(\mathbf{x}_i)$ to be the $(\bullet;\mathbf{1})-x_j$ alternating product

$$\tilde{\rho}(\mathbf{x}_i)=\left\{\begin{array}{ll} (\bullet;\mathbf{1})\sqcup_{x_1}(\bullet;\mathbf{1})\cdots(\bullet;\mathbf{1})\sqcup_{x_r}, & \text{if } i=1,\\ \sqcup_{x_1}(\bullet;\mathbf{1})\cdots\sqcup_{x_r}(\bullet;\mathbf{1}), & \text{if } i=t,\\ \sqcup_{x_1}(\bullet;\mathbf{1})\cdots(\bullet;\mathbf{1})\sqcup_{x_r}, & \text{if } i\ne 1,t. \end{array}\right.$$

This completes the inductive definition of ρ. It is easy to check that it is the inverse of η. As an illustrating example, for $\mathbf{x}=x_1x_2\lfloor x_3\lfloor\mathbf{1}\rfloor\rfloor x_4x_5\lfloor x_6\rfloor$ with $x_i\in X, 1\le i\le 6$, we have

$$\begin{array}{rcl} \rho(\mathbf{x}) &=& \tilde{\rho}(x_1x_2)\tilde{\rho}(\lfloor x_3\lfloor\mathbf{1}\rfloor\rfloor)\tilde{\rho}(x_4x_5)\tilde{\rho}(\lfloor x_6\rfloor)\\ &=& (\bullet;\mathbf{1})\sqcup_{x_1}(\bullet;\mathbf{1})\sqcup_{x_2}\lfloor\rho(x_3\lfloor\mathbf{1}\rfloor)\rfloor\sqcup_{x_4}(\bullet;\mathbf{1})\sqcup_{x_5}\lfloor\rho(x_6)\rfloor\\ &=& (\bullet;\mathbf{1})\sqcup_{x_1}(\bullet;\mathbf{1})\sqcup_{x_2}\lfloor(\bullet;\mathbf{1})\sqcup_{x_3}\lfloor(\bullet;\mathbf{1})\rfloor\rfloor\sqcup_{x_4}(\bullet;\mathbf{1})\sqcup_{x_5}\lfloor(\bullet;\mathbf{1})\sqcup_{x_6}(\bullet;\mathbf{1})\rfloor. \end{array}$$

We then have, by Eq. (51)

$$\begin{array}{rcl} \eta(\rho(\mathbf{x})) &=& \eta(\bullet;\mathbf{1})x_1\eta(\bullet;\mathbf{1})x_2\eta(\lfloor(\bullet;\mathbf{1})\sqcup_{x_3}\lfloor(\bullet;\mathbf{1})\rfloor\rfloor)x_4\eta(\bullet;\mathbf{1})x_5\eta(\lfloor(\bullet;\mathbf{1})\sqcup_{x_6}(\bullet;\mathbf{1})\rfloor)\\ &=& \mathbf{1}x_1\mathbf{1}x_2\lfloor\eta((\bullet;\mathbf{1})\sqcup_{x_3}\lfloor(\bullet;\mathbf{1}))\rfloor\rfloor)x_4\mathbf{1}x_5\lfloor\eta((\bullet;\mathbf{1})\sqcup_{x_6}(\bullet;\mathbf{1}))\rfloor \end{array}$$

$$\begin{aligned} &= x_1x_2\lfloor \mathbf{1}x_3\lfloor \eta(\bullet;\mathbf{1})\rfloor\rfloor x_4x_5\lfloor \mathbf{1}x_6\mathbf{1}\rfloor \\ &= x_1x_2\lfloor x_3\lfloor \mathbf{1}\rfloor\rfloor x_4x_5\lfloor x_6\rfloor = \mathbf{x}. \end{aligned}$$

□

4.2.3. *The product on free Rota–Baxter algebra in terms of* RBWs. By identifying $\tilde{\mathfrak{X}}_\infty$ with $(\mathcal{F};X)$ through the bijection η in Theorem 4.6, we can use $\tilde{\mathfrak{X}}_\infty$ as an alternative basis of $Ш^{\mathrm{NC}}(X)$.

We now express the product $\bar{\diamond}$ on $Ш^{\mathrm{NC}}(X)$ in terms of the new basis by defining $\mathbf{x}\bar{\diamond}\mathbf{x}'$ for any two words $\mathbf{x}$ and $\mathbf{x}'$ in $\tilde{\mathfrak{X}}_\infty$, and then extending by bilinearity. This is for the convenience of the reader who would like to see the product defined directly in terms of Rota–Baxter words. Other readers can safely skip this part.

For $\mathbf{x}\in\tilde{\mathfrak{X}}_\infty$, define its **depth** $\mathrm{d}(\mathbf{x})$ to be the smallest integer d such that $\mathbf{x}\in\mathfrak{X}_{\mathrm{d}}$. Since η defines a bijection between $(\mathcal{F}_n;X)$ and $\mathfrak{X}_n$ for $n\geq 0$, the map preserves the depths.

We define the product $\mathbf{x}\bar{\diamond}\mathbf{x}'$ inductively on the sum $n:=\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{x}')$ of the depths of $\mathbf{x}$ and $\mathbf{x}'$. So we have $n\geq 0$. If $n=0$, then $\mathbf{x},\mathbf{x}'$ are in $\tilde{\mathfrak{X}}_0=M(X)$ and we define $\mathbf{x}\bar{\diamond}\mathbf{x}'=\mathbf{x}\mathbf{x}'$, the concatenation in $M(X)$.

Suppose $\mathbf{x}\bar{\diamond}\mathbf{x}'$ have been defined for all RBWs $\mathbf{x},\mathbf{x}'\in\tilde{\mathfrak{X}}_\infty$ with $0\leq n\leq k$ and let $\mathbf{x},\mathbf{x}'\in\tilde{\mathfrak{X}}_\infty$ with $n=k+1$. First assume that the breadths $b(\mathbf{x})=b(\mathbf{x}')=1$. Then $\mathbf{x}$ and $\mathbf{x}'$ are in $\mathfrak{X}_0=S(X)$ or $\lfloor\tilde{\mathfrak{X}}_\infty\rfloor$. We accordingly define

$$\mathbf{x}\bar{\diamond}\mathbf{x}' = \begin{cases} \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x},\mathbf{x}'\in S(X),\\ \mathbf{x}\,\mathbf{x}', & \text{if } \mathbf{x}\in S(X), \mathbf{x}\in\lfloor\tilde{\mathfrak{X}}_\infty\rfloor,\\ \mathbf{x}\,\mathbf{x}', & \text{if } \mathbf{x}\in\lfloor\tilde{\mathfrak{X}}_\infty\rfloor, \mathbf{x}'\in S(X),\\ \lfloor\lfloor\overline{\mathbf{x}}\rfloor\bar{\diamond}\,\overline{\mathbf{x}}'\rfloor+\lfloor\overline{\mathbf{x}}\,\bar{\diamond}\lfloor\overline{\mathbf{x}}'\rfloor\rfloor+\lambda\lfloor\overline{\mathbf{x}}\,\bar{\diamond}\,\overline{\mathbf{x}}'\rfloor, & \text{if } \mathbf{x}=\lfloor\overline{\mathbf{x}}\rfloor, \mathbf{x}'=\lfloor\overline{\mathbf{x}}'\rfloor\in\lfloor\tilde{\mathfrak{X}}_\infty\rfloor. \end{cases} \tag{52}$$

Here the product in the first three cases is defined by concatenation, and in the fourth case by the induction hypothesis since for the three products on the right hand side we have

$$\begin{aligned} \mathrm{d}(\lfloor\overline{\mathbf{x}}\rfloor)+\mathrm{d}(\overline{\mathbf{x}}') &= \mathrm{d}(\lfloor\overline{\mathbf{x}}\rfloor)+\mathrm{d}(\lfloor\overline{\mathbf{x}}'\rfloor)-1=\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{x}')-1,\\ \mathrm{d}(\overline{\mathbf{x}})+\mathrm{d}(\lfloor\overline{\mathbf{x}}'\rfloor) &= \mathrm{d}(\lfloor\overline{\mathbf{x}}\rfloor)+\mathrm{d}(\lfloor\overline{\mathbf{x}}'\rfloor)-1=\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{x}')-1,\\ \mathrm{d}(\overline{\mathbf{x}})+\mathrm{d}(\overline{\mathbf{x}}') &= \mathrm{d}(\lfloor\overline{\mathbf{x}}\rfloor)-1+\mathrm{d}(\lfloor\overline{\mathbf{x}}'\rfloor)-1=\mathrm{d}(\mathbf{x})+\mathrm{d}(\mathbf{x}')-2 \end{aligned}$$

which are all less than or equal to k.

Now assume $b(\mathbf{x})>1$ or $b(\mathbf{x}')>1$. Let $\mathbf{x}=\mathbf{x}_1\cdots\mathbf{x}_b$ and $\mathbf{x}'=\mathbf{x}'_1\cdots\mathbf{x}'_{b'}$ be the standard decompositions from Lemma 4.5. We then define

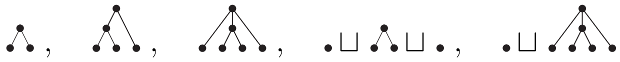
$$\mathbf{x}\bar{\diamond}\mathbf{x}'=\mathbf{x}_1\cdots\mathbf{x}_{b-1}\,(\mathbf{x}_b\bar{\diamond}\mathbf{x}'_1)\;\mathbf{x}'_2\cdots\mathbf{x}_{b'} \tag{53}$$

where $\mathbf{x}_b\bar{\diamond}\mathbf{x}'_1$ is defined by Eq. (52) and the rest is given by the concatenation product.

5. The Rota–Baxter algebra on controlled forests

We now give a variation of the constructions of Rota-Baxter operators on rooted forests in Section 2 by considering a restricted class of forests. This construction will be applied in Section 6 to obtain free Rota-Baxter algebras over another algebra.

5.1. **Controlled rooted forests.** To describe the subset of forests considered here, recall that $\mathcal{T}$ is the set of planar rooted trees and $\mathcal{F}$ is the set of planar forests. We let $\mathcal{F}^r$ be the subset of $\mathcal{F}$ consisting of **controlled forests**, defined to be the forests in which the empty tree $\bullet$ occurs only as the left most or right most branch of a vertex, or the left most or right most tree of the forest. In other words, a forest in $\mathcal{F}^r$ does not contain a subforest $T_1 \sqcup \cdots \sqcup T_n$ in which some $T_i = \bullet$ with $i \neq 1, n$. For example,



are in $\mathcal{F}^r$, while

are not.

We also define $\mathcal{T}^r = \mathcal{T} \cap \mathcal{F}^r$, called the set of **controlled trees**. Thus $\mathcal{T}^r$ consists of planar rooted trees with no subforests $T_1 \sqcup \cdots \sqcup T_n$ in which some $T_i = \bullet$ with $i \neq 1, n$. Any controlled forest is made of controlled trees. The converse is not true. For example,

is made of controlled trees, but it is not a controlled forest.

5.2. **Rota–Baxter algebra on controlled forests.** We next define a product on the free $\mathbf{k}$-module $\mathbf{k}\,\mathcal{F}^r$ generated by the controlled forests $\mathcal{F}^r$.

Recall that $\mathcal{F}$ has an increasing filtration $\mathcal{F}_n, n \geq 0$, of forests with depth less or equal to n. By restriction, we obtain a filtration $\mathcal{F}^r_n := \mathcal{F}^r \cap \mathcal{F}_n$ on $\mathcal{F}^r$, and a filtration $\mathcal{T}^r_n := \mathcal{T}^r \cap \mathcal{F}_n$ on $\mathcal{T}^r$. We will use this filtration on $\mathcal{F}^r$ to define a set map

$$\diamond^r : \mathcal{F}^r \times \mathcal{F}^r \to \mathbf{k}\,\mathcal{F}^r.$$

and then to extend it bilinearly. We first define

$$\bullet \diamond^r F = F \diamond^r \bullet = F.$$

Next, for $F, F' \in \mathcal{F}^r$ that are not $\bullet$, we define $F \diamond^r F'$ by induction on $n := \mathrm{d}(F) + \mathrm{d}(F')$

When $n = 0$, we have $F = \bullet^{\sqcup b}$ and $F' = \bullet^{\sqcup b'}$. Since F and F' are in $\mathcal{F}^r$, we have $b, b' \leq 2$. Since $F, F' \neq \bullet$, we have $b = b' = 2$. Then define

$$F \diamond^r F' = \bullet^{\sqcup 2}.$$

For any $k \geq 0$, assume that $F \diamond^r F'$ has been defined for $n \leq k$. Consider controlled forests F and F' with $n = k + 1$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$ be the decompositions into trees. If $\mathrm{d}(T_b) = \mathrm{d}(T'_1) = 0$, then $T_b = T'_1 = \bullet$. Since $F, F' \neq \bullet$, we have $b > 1, b' > 1$. Then we define

$$F \diamond^r F' = T_1 \sqcup \cdots \sqcup T_{b-1} \sqcup T'_2 \sqcup \cdots \sqcup T'_{b'} \tag{54}$$

If at least one of T_b or T'_1 are not $\bullet$. Then note that a tree is either $\bullet$ or is of the form $\lfloor \overline{F} \rfloor$ for a forest $\overline{F}$ of smaller depth. Thus we can define

$$F \diamond^r F' = T_1 \sqcup \cdots \sqcup (T_b \diamond^r T'_1) \sqcup \cdots \sqcup T'_{b'}, \tag{55}$$

where

$$T_b \diamond^r T'_1 = \begin{cases} T_b, & \text{if } T'_1 = \bullet, T_b \neq \bullet, \\ T'_1, & \text{if } T_b = \bullet, T'_1 \neq \bullet, \\ \lfloor \lfloor \overline{F}_b \rfloor \diamond^r \overline{F}'_1 \rfloor + \lfloor \overline{F}_b \diamond^r \lfloor \overline{F}'_1 \rfloor \rfloor + \lambda \lfloor \overline{F}_b \diamond^r \overline{F}'_1 \rfloor, & \text{if } T_b = \lfloor \overline{F}_b \rfloor, T'_1 = \lfloor \overline{F}'_1 \rfloor. \end{cases} \tag{56}$$

The three products on the right hand of the last equation are well-defined by the induction hypothesis since we have

$$\begin{aligned} \mathrm{d}(\lfloor \overline{F}_b \rfloor) + \mathrm{d}(\overline{F}'_1) &= \mathrm{d}(\lfloor \overline{F}_b \rfloor) + \mathrm{d}(\lfloor \overline{F}'_1 \rfloor) - 1 = \mathrm{d}(T_b) + \mathrm{d}(T'_1) - 1, \\ \mathrm{d}(\overline{F}_b) + \mathrm{d}(\lfloor \overline{F}'_1 \rfloor) &= \mathrm{d}(\lfloor \overline{F}_b \rfloor) + \mathrm{d}(\lfloor \overline{F}'_1 \rfloor) - 1 = \mathrm{d}(T_b) + \mathrm{d}(T'_1) - 1, \\ \mathrm{d}(\overline{F}_b) + \mathrm{d}(\overline{F}'_1) &= \mathrm{d}(\lfloor \overline{F}_b \rfloor) - 1 + \mathrm{d}(\lfloor \overline{F}'_1 \rfloor) - 1 = \mathrm{d}(T_b) + \mathrm{d}(T'_1) - 2 \end{aligned}$$

which are all less than or equal to k. Note that in either case, $T_1 \diamond^r T_2$ is a tree or a sum of trees. This completes the definition of the binary operation $\diamond^r$ on $\mathbf{k}\,\mathcal{F}^r$.

Extending $\diamond^r$ bilinearly, we obtain a binary operation

$$\diamond^r : \mathbf{k}\,\mathcal{F}^r \otimes \mathbf{k}\,\mathcal{F}^r \to \mathbf{k}\,\mathcal{F}^r.$$

For $T \in \mathcal{F}^r$, define

$$P_{\mathcal{F}^r}(T) = \lfloor T \rfloor. \tag{57}$$

Then $P_{\mathcal{F}^r}$ defines a linear operator on $\mathbf{k}\,\mathcal{F}^r$.

We have the following variation of Theorem 5.1 which will be proved in the next subsection.

Theorem 5.1. (a) *The pair $(\mathbf{k}\,\mathcal{F}^r, \diamond^r)$ is a unitary associative algebra.*

(b) *The triple $(\mathbf{k}\,\mathcal{F}^r, \diamond^r, P_{\mathcal{F}})$ is a unitary Rota–Baxter algebra of weight λ.*

(c) *$(\mathbf{k}\,\mathcal{F}^r, \diamond^r, P_{\mathcal{F}})$ is a quotient Rota–Baxter algebra of the Rota–Baxter algebra $(\mathbf{k}\,\mathcal{F}, \diamond, P_{\mathcal{F}})$ in Theorem 2.5.*

Proof. We only need to prove part (c) of Theorem 5.1.

By Proposition 5.4 whose proof we postpone to the next section, there is a surjective $\mathbf{k}$-linear map

$$\delta : \mathbf{k}\,\mathcal{F} \to \mathbf{k}\,\mathcal{F}^r \tag{58}$$

that is multiplicative with respect to the multiplications of $\mathbf{k}\,\mathcal{F}$ and $\mathbf{k}\,\mathcal{F}^r$. Then $\mathbf{k}\,\mathcal{F}^r$ is a quotient associative algebra of $\mathbf{k}\,\mathcal{F}$ by the following elementary fact on algebras.

Let A be a $\mathbf{k}$-algebra and let A' be a $\mathbf{k}$-module equipped with a $\mathbf{k}$-bilinear (not necessarily associative) multiplication. If there is a surjective $\mathbf{k}$-linear map $f : A \to A'$ that preserves the multiplications, then A' is a $\mathbf{k}$-algebra and f is an algebra homomorphism.

By Eq. (56), the operator $\lfloor\ \rfloor$ on $\mathbf{k}\,\mathcal{F}^r$ is a Rota–Baxter operator. By Eq. (63), δ also preserves the Rota–Baxter operators. Thus $\mathbf{k}\,\mathcal{F}^r$ is a quotient Rota–Baxter algebra of $\mathbf{k}\,\mathcal{F}$. □

We will also construct a nonunitary sub-Rota–Baxter algebra in $\mathbf{k}\,\mathcal{F}^r$ by excluding forests containing $\lfloor \bullet \rfloor$. Compare with Corollary 2.6.

Corollary 5.2. *Let $\mathcal{F}^{r,0}$ be the subset of $\mathcal{F}^r$ consisting of forests that do not contain $\lfloor \bullet \rfloor$. The submodule $\mathbf{k}\,\mathcal{F}^{r,0}$ of $\mathbf{k}\,\mathcal{F}^r$ is a nonunitary Rota–Baxter subalgebra of $\mathbf{k}\,\mathcal{F}^r$ under the product $\diamond^r$.*

Proof. We only need to check that $\mathbf{k}\mathcal{F}^{r,0}$ is closed under $\diamond^r$ and $P_{\mathcal{F}} = \lfloor\ \rfloor$.

We remark that a forest F is in $\mathcal{F}^{r,0}$ if and only if $\lfloor F \rfloor$ is in $\mathcal{F}^{r,0}$, as can be easily seen by the definition of $\mathcal{F}^{r,0}$. So $\mathbf{k}\,\mathcal{F}^{r,0}$ is closed under the Rota–Baxter operator $P_{\mathcal{F}}$.

Note also that a forest is in $\mathcal{F}^{r,0}$ if and only if each of its constituent trees is in $\mathcal{F}^{r,0}$ and $\bullet$ is not a middle tree.

Next let $F = T_1 \sqcup \cdots \sqcup T_b$, $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$ be in $\mathcal{F}^r$. We use induction on $n := \mathrm{d}(T_b) + \mathrm{d}(T'_1)$ to prove that $F \diamond^r F'$ is in $\mathcal{F}^r$. If $n = 0$, then $T_b = T'_1 = \bullet$. By Eq. (54), we see that $F \diamond^r F'$ is composed of trees from F or F', so are in $\mathcal{T}^r$. Further, the middle trees of $F \diamond^r F'$ are also middle trees of F or F' and so are not $\bullet$. Therefore, $F \diamond^r F'$ is in $\mathcal{F}^r$.

Assume that $F \diamond^r F'$ is in $\mathcal{F}^r$ for $F, F' \in \mathcal{F}^r$ with $n \leq k$ and consider $F, F' \in \mathcal{F}^r$ with $n = k+1$. Then at least one of T_b and T'_1 is not $\bullet$. So by Eq. (55), we have

$$F \diamond^r F' = T_1 \sqcup \cdots \sqcup (T_b \diamond^r T'_1) \sqcup \cdots \sqcup T'_{b'}, \tag{59}$$

where

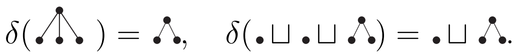
$$T_b \diamond^r T'_1 = \begin{cases} T_b, & \text{if } T'_1 = \bullet, T_b \neq \bullet, \\ T'_1, & \text{if } T_b = \bullet, T'_1 \neq \bullet, \\ \lfloor \lfloor \overline{T}_b \rfloor \diamond^r \overline{T}'_1 \rfloor + \lfloor \overline{T}_b \diamond^r \lfloor \overline{T}'_1 \rfloor \rfloor + \lambda \lfloor \overline{T}_b \diamond^r \overline{T}'_1 \rfloor, & \text{if } T_b = \lfloor \overline{T}_b \rfloor, T'_1 = \lfloor \overline{T}'_1 \rfloor. \end{cases} \tag{60}$$

In the first two cases, $T_b \diamond^r T'_1$ is not $\bullet$. So $F \diamond^r F'$ is in $\mathcal{F}^r$. In the third case, the products in the brackets are defined by the induction hypothesis. So the products are in $\mathcal{F}^r$. Therefore each of the products is in $\mathcal{F}^r$, and are clearly not $\bullet$. Thus $F \diamond^r F'$ is in $\mathcal{F}^r$. $\square$

5.3. **Construction of the quotient map δ.** We first give an intuitive description of the map δ, followed by a rigorous definition which we will use later. Let $T_1 \sqcup \cdots \sqcup T_n$ be a sub-forest of F. If $T_1 = \bullet$ or $T_n = \bullet$, then they are called edge leafs. If $T_i = \bullet$ for a $1 < i < n$, then T_i is called a non-edge leaf. Thus a forest is controlled if and only if it does not contain any non-edge leafs. We define a map

$$\delta : \mathcal{F} \to \mathcal{F}^r \tag{61}$$

by defining $\delta(F)$ to be the controlled forest after deleting all non-edge leafs of all sub-forests of F. For example,



To make this description more precise, we give an inductive definition of $\delta : \mathcal{F} \to \mathcal{F}^r$ on the depth $\mathrm{d}(F)$ of a forest F. If $\mathrm{d}(F) = 0$, then $F = \bullet^{\sqcup i}$ for some $i \geq 1$. We define $\delta(F) = F$ if $i = 1$ and define $\delta(F) = \bullet^{\sqcup 2}$ if $i \geq 2$.

Suppose $\delta(F)$ has been defined for all F with depth less or equal to $k \geq 0$ and assume that $F \in \mathcal{F}$ has depth $k+1$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ be the decomposition of F into trees. Then each T_i is either $\bullet$ or $\lfloor \overline{F}_i \rfloor$ for a forest $\overline{F}_i$. Let $T_{i_1}, \cdots, T_{i_t}$ be the subsequence (called the **controlled subsequence**) of $T_1, \cdots, T_b$ consisting of T_1, T_b and those T_j with $1 < j < b$ such that T_j is not $\bullet$. Then we define

$$\delta(F) = \delta(T_{i_1}) \sqcup \cdots \sqcup \delta(T_{i_t}), \tag{62}$$

where

$$\delta(T_{i_j}) = \begin{cases} \bullet, & \text{if } T_{i_j} = \bullet, \\ \lfloor \delta(\overline{T}_{i_j}) \rfloor, & \text{if } T_{i_j} \neq \bullet \text{ and so } T_{i_j} = \lfloor \overline{T}_{i_j} \rfloor \end{cases} \tag{63}$$

Here the term $\lfloor \delta(\overline{T}_{i_j}) \rfloor$ is a well-defined element in $\mathcal{F}^r$ since $\delta(\overline{T}_{i_j})$ is a well-defined element in $\mathcal{F}^r$ by the induction hypothesis, and then $\lfloor \delta(\overline{T}_{i_j}) \rfloor$ is a well-defined element in $\mathcal{F}^r$ since $\mathcal{F}^r$ is closed under the operator $\lfloor\ \rfloor$. Further since $T_{i_j} = \bullet$ is possible only if $j = 1$ or t,

$\delta(T_{i_j}) = \bullet$ only if $j = 1$ or t. So $\delta(F)$ is in $\mathcal{F}^r$. This completes the recursion, yielding a map

$$\delta : \mathcal{F} \to \mathcal{F}^r$$

and finally a $\mathbf{k}$-linear map

$$\delta : \mathbf{k}\,\mathcal{F} \to \mathbf{k}\,\mathcal{F}^r \tag{64}$$

after extending by linearity.

We first check an easy fact.

Lemma 5.3. *The map δ preserves the depth of a tree.*

Proof. We use induction on the depth $\mathrm{d}(T)$ of a planar rooted tree T. If $\mathrm{d}(T) = 0$, then $T = \bullet$. Since $\delta(\bullet) = \bullet$, we are done. Suppose $\mathrm{d}(\delta(T)) = \mathrm{d}(T)$ for all trees T with $0 \leq \mathrm{d}(T) \leq k$. Consider a tree T with $\mathrm{d}(T) = k+1$. Then $k+1 \geq 1$. So T is of the form $\lfloor \overline{F} \rfloor$ where $\overline{F}$ is a forest in $\mathcal{F}$ with $\mathrm{d}(\overline{F}) = k$. Let $\overline{F} = \overline{T}_1 \sqcup \cdots \sqcup \overline{T}_r$ be the decomposition of $\overline{F}$ into trees. Then there is a $\overline{T}_{i_0}$, $1 \leq i_0 \leq r$ such that $\mathrm{d}(\overline{T}_{i_0}) = k$. If $k = 0$, then all $\overline{T}_i$ are $\bullet$. Then $\delta(\overline{F}) = \bullet$ or $\bullet \sqcup \bullet$. So $\delta(T) = \lfloor \delta(\overline{F}) \rfloor$ has depth 1. This is also the depth of T. If $k > 0$, then $\delta(\overline{T}_{i_0})$ occurs in the decomposition of $\delta(\overline{F})$. By the induction hypothesis, $\mathrm{d}(\delta(\overline{T}_{i_0})) = \mathrm{d}(\overline{T}_{i_0}) = k$. So $\mathrm{d}(\delta(\overline{F})) = k$ and thus $\mathrm{d}(T) = k+1$. This completes the induction. □

Proposition 5.4. *The map $\delta : \mathbf{k}\,\mathcal{F} \to \mathbf{k}\,\mathcal{F}^r$ is surjective and is multiplicative with respect to the multiplications $\diamond$ in $\mathbf{k}\,\mathcal{F}$ and $\diamond^r$ in $\mathbf{k}\,\mathcal{F}^r$.*

Proof. The map δ is clearly surjective since it maps a controlled forest to itself. To prove the multiplicity, we only need to prove

$$\delta(F \diamond F') = \delta(F) \diamond^r \delta(F') \tag{65}$$

for $F, F' \in \mathcal{F}$. If one of F or F' is $\bullet$, the identity of $\mathbf{k}\,\mathcal{F}$, then since $\delta(\bullet) = \bullet$ is also the identity of $\mathbf{k}\,\mathcal{F}^r$, Eq. (65) holds.

We next assume $F, F' \neq \bullet$ and use induction on the sum $n := \mathrm{d}(F) + \mathrm{d}(F')$.

When $n = 0$, then $F = \bullet^{\sqcup i}$, $F' = \bullet^{\sqcup i'}$ and $i, i' \geq 1$. Since $F, F' \neq \bullet$, we have $i, i' \geq 2$. Then $\delta(F) = \delta(F') = \bullet^{\sqcup 2}$. So $\delta(F) \diamond^r \delta(F') = \bullet^{\sqcup 2}$ by Eq. (54). We also have

$$F \diamond F' = \bullet^{\sqcup (i+i'-1)}$$

with $i + i' - 1 \geq 2$. Thus $\delta(F \diamond F') = \bullet^{\sqcup 2}$, again verifying Eq. (65).

Assume Eq. (65) holds for $F, F' \in \mathcal{F}$ with $\mathrm{d}(F) + \mathrm{d}(F') \geq k \geq 0$ and consider $F, F' \in \mathcal{F}$ with $\mathrm{d}(F) + \mathrm{d}(F') = k+1$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$ be the decomposition of F and F' into trees.

If both T_b and T'_1 are $\bullet$, then since $F, F' \neq \bullet$, we have $b > 1$ and $b' > 1$. Then in the definition (62) of δ, the controlled subsequence of $T_1, \cdots, T_b$ is $T_1, T_{i_2}, \cdots, T_{i_{t-1}}, T_b$ where $i_2, \cdots, i_{t-1}$ are the indices of $1 < j < b$ such that $T_j \neq \bullet$, and the controlled subsequence of $T'_1, \cdots, T'_{b'}$ is $T'_1, T'_{i'_2}, \cdots, T'_{i'_{t'-1}}, T'_{b'}$ where $i'_2, \cdots, i'_{t'-1}$ are the indices of $1 < j' < b'$ such that $T_{j'} \neq \bullet$. Then

$$\delta(F) = \delta(T_1) \sqcup \delta(T_{i_2}) \sqcup \cdots \sqcup \delta(T_{i_{t-1}}) \sqcup \delta(T_b)$$

and

$$\delta(F') = \delta(T'_1) \sqcup \delta(T'_{i'_2}) \sqcup \cdots \sqcup \delta(T'_{i'_{t'-1}}) \sqcup \delta(T'_{b'}).$$

Since $T_b=\delta(T_b)=\bullet$ and $T'_1=\delta(T'_1)=\bullet$, by Eq. (54) we have
$$\delta(F)\diamond^r\delta(F')=\delta(T_1)\sqcup\delta(T_{i_2})\sqcup\cdots\sqcup\delta(T_{i_{t-1}})\sqcup\delta(T'_{i'_2})\cdots\sqcup\delta(T'_{i'_{t'-1}})\sqcup\delta(T'_{b'}).$$
On the other hand, by Eq. (7) and (9), we have
$$F\diamond F'=T_1\sqcup\cdots\sqcup T_b\sqcup T'_2\cdots\sqcup T'_{b'}.$$
Then we have
$$\delta(F\diamond F')=\delta(T_1)\sqcup\delta(T''_2)\sqcup\cdots\sqcup\delta(T''_{t''-1})\sqcup\delta(T'_{b'})$$
where $T''_2,\cdots,T''_{t''-1}$ are the subsequence of the trees $T_2,\cdots,T_b,T'_2,\cdots,T'_{b'-1}$ that are not $\bullet$. Since $T_b=\bullet$, this subsequence is also obtained by starting with the subsequence of $T_2,\cdots,T_{b-1}$ that are not $\bullet$ and then followed by the subsequence of $T'_2,\cdots,T'_{b'-1}$ that are not $\bullet$. This subsequence is exactly $T_{i_2},\cdots T_{i_{t-1}},T'_{i'_2},\cdots T'_{i'_{t'-1}}$. So Eq. (65) holds in this case.

If $T_b=\bullet$ and $T'_1\neq\bullet$, then $b>1$. Then the controlled subsequence of $T_1,\cdots,T_b$ is obtained by starting with T_1, then followed by the subsequence $T_{i_2},\cdots,T_{i_{t-1}}$ of trees $T_2,\cdots,T_{b-1}$ that are not $\bullet$, then followed by T_b. Since $T'_1\neq\bullet$, the controlled subsequence of $T'_1,\cdots,T'_{b'}$ is the subsequence $T'_1,T'_{i'_2},\cdots,T_{i'_{t'-1}}$ of trees $T'_1,\cdots,T'_{b'-1}$ that are not $\bullet$, followed by $T'_{b'}$. Since $T'_1\neq\bullet$, we have $\mathrm{d}(T'_1)>0$. So by Lemma 5.3, $\delta(T'_1)$ has depth greater than 0 and so is not $\bullet$. Then by Eq. (59),
$$\begin{aligned}&\delta(F)\diamond^r\delta(F')\\ =\;&\big(\delta(T_1)\sqcup\delta(T_{i_2})\sqcup\cdots\sqcup\delta(T_{i_{t-1}})\sqcup\delta(T_b)\big)\diamond^r\big(\delta(T'_1)\sqcup\delta(T'_{i_2})\sqcup\cdots\sqcup\delta(T'_{i'_{t'-1}})\sqcup\delta(T'_{b'})\big)\\ =\;&\delta(T_1)\sqcup\delta(T_{i_2})\sqcup\cdots\sqcup\delta(T_{i_{t-1}})\sqcup\delta(T'_1)\sqcup\delta(T'_{i_2})\sqcup\cdots\sqcup\delta(T'_{i'_{t'-1}})\sqcup\delta(T'_{b'}).\end{aligned}$$
On the other hand, by Eq. (7),
$$F\diamond F'=T_1\sqcup\cdots\sqcup T_{b-1}\sqcup T'_1\sqcup\cdots\sqcup T'_{b'}.$$
So $\delta(F\diamond F')$ is the concatenation of the subsequence that starts with $\delta(T_1)$, followed by trees $\delta(T_i)$ from $T_2,\cdots T_{b-1},T'_1,\cdots,T'_{b'-1}$ that are not $\bullet$, followed by $\delta(T'_{b'})$. Since T'_1 is not $\bullet$, this agrees with $\delta(F)\diamond^r\delta(F')$.

The same arguments work if $T_b\neq\bullet$ and $T'_1=\bullet$.

If $T_b\neq\bullet$ and $T'_1\neq\bullet$, then $T_b=\lfloor\overline{F}_b\rfloor$ and $T'_1=\lfloor\overline{F}'_1\rfloor$. Since $\delta(T_b)=\lfloor\delta(\overline{F}_b)\rfloor$ and $\delta(T'_1)=\lfloor\delta(\overline{F}'_1)\rfloor$, we have
$$\delta(F)=\delta(T_1)\sqcup\delta(T_{i_2})\sqcup\cdots\sqcup T_{i_{t-1}}\sqcup\lfloor\delta(\overline{F}_b)\rfloor$$
and
$$\delta(F')=\lfloor\delta(\overline{F}'_1)\rfloor\sqcup\delta(T'_{i'_2})\sqcup\cdots\sqcup\delta(T'_{i'_{t'-1}})\sqcup\delta(T'_{b'}).$$
So we have
$$\delta(F)\diamond^r\delta(F')=\delta(T_1)\sqcup\delta(T_{i_2})\sqcup\cdots\sqcup\delta(T_{i_{t-1}})\sqcup\big(\lfloor\delta(\overline{F}_b)\rfloor\diamond^r\lfloor\delta(\overline{F}'_1)\rfloor\big)\sqcup\delta(T'_{i'_2})\sqcup\cdots\sqcup\delta(T'_{i'_{t'-1}})\sqcup\delta(T'_{b'}).$$
By Eq. (7),
$$F\diamond F'=T_1\sqcup\cdots\sqcup\big(T_b\diamond T'_1\big)\sqcup\cdots\sqcup T'_{b'},$$
where
$$T_b\diamond T'_1=\big\lfloor\lfloor\overline{F}_b\rfloor\diamond\overline{F}'_1\big\rfloor+\lfloor\overline{F}_b\diamond\lfloor\overline{F}'_1\rfloor\rfloor+\lambda\lfloor\overline{T}_b\diamond\overline{T}_1\rfloor.$$
So $T_b\diamond T'_1$ is a tree not equal $\bullet$. Thus we have
$$\delta(F\diamond F')=\delta(T_1)\sqcup\delta(T_{i_2})\sqcup\cdots\sqcup\delta(T_{i_{t-1}})\sqcup\delta\big(T_b\diamond T'_1\big)\sqcup\delta(T'_{i'_2})\sqcup\cdots\sqcup\delta(T'_{i'_{t'-1}})\sqcup\delta(T'_{b'}).$$

By Eq. (63) and the induction hypothesis, we have

$$\begin{aligned}
\delta(T_b \diamond T_1') &= \delta\big(\lfloor \overline{F}_b \diamond T_1' + T_b \diamond \overline{F}_1' + \lambda \overline{F}_b \diamond \overline{F}_1' \rfloor\big)\\
&= \lfloor \delta\big(\overline{F}_b \diamond T_1' + T_b \diamond \overline{F}_1' + \lambda \overline{F}_b \diamond \overline{F}_1'\big) \rfloor\\
&= \big\lfloor \delta(\overline{F}_b) \diamond^r \delta(T_1') + \delta(T_b) \diamond^r \delta(\overline{F}_1') + \lambda \delta(\overline{F}_b) \diamond^r \delta(\overline{F}_1') \big\rfloor\\
&= \big\lfloor \delta(\overline{F}_b) \diamond^r \lfloor \delta(\overline{F}_1') \rfloor + \lfloor \delta(\overline{F}_b) \rfloor \diamond^r \delta(\overline{F}_1') + \lambda \delta(\overline{F}_b) \diamond^r \delta(\overline{F}_1') \big\rfloor\\
&= \lfloor \delta(\overline{F}_b) \rfloor \diamond^r \lfloor \delta(\overline{F}_1') \rfloor.
\end{aligned}$$

So we again have $\delta(F) \diamond^r \delta(F') = \delta(F \diamond F')$. This completes the induction proof of the multiplicativity and hence the proof of Proposition 5.4. ☐

6. Free Rota–Baxter algebra over an algebra

In this section, we use the Rota–Baxter algebra structure on the controlled forests obtained in Section 5 to construct the free Rota–Baxter algebra over another algebra. This construction is more or less parallel to the construction of the free Rota–Baxter algebra over a module obtained in Section 3 but there are two major distinctions. The first one is that the connection with the underlying tree structure is less direct. As a result, it is less straightforward to transport the Rota–Baxter algebra property on controlled forests to obtain the Rota–Baxter algebra property on the free Rota–Baxter algebra. The second distinction is that a restriction on the generating algebra has to be imposed in order to obtain free Rota–Baxter algebra with properties similar to the previous cases. This is not to say that free Rota–Baxter algebras no longer exist without the restriction, just to say that their structures become more subtle to describe.

6.1. **Angularly decorated controlled forests.**

We will impose the following condition on the $\mathbf{k}$-algebra A in this section.

Assumption 6.1. *There is a $\mathbf{k}$-submodule $\check{A}$ of A such that $A = \mathbf{k} \oplus \check{A}$ as a $\mathbf{k}$-module.*

This assumption is satisfied if

(a) either A is the unitarization of a nonunitary $\mathbf{k}$-algebra $\check{A}$,
(b) or A has a $\mathbf{k}$-basis containing $\mathbf{1}$.

The second condition holds if $\mathbf{k}$ is a division ring. So the restriction is quite mild.

We impose this condition to be specific on the ring of scalars for which the Rota-Baxter operator is linear. In other words we would like to exclude the situation where an element a in A is not in $\mathbf{k}$, but a non-zero $\mathbf{k}$-scalar multiple ka of a is in $\mathbf{k}$. This makes a to behave like a scalar. For example, let $A = \mathbb{Z}[x]/(2x-1)$ regarded as a $\mathbb{Z}$-algebra. Suppose $Ш^{NC}(A)$ is the free Rota-Baxter $\mathbf{k}$-algebra over A with Rota-Baxter operator P_A. Then from $2x = 1$ in A, we have $2P_A(x)x = P_A(2x)x = P_A(1)x$ and $2P_A(x)x = P_A(x)2x = P_A(x)$. Thus $P_A(x) = xP_A(1)$. Since $2^n x^n = 1$, we similarly obtain $P_A(x^n) = x^n P(1)$ and thus $P_A(a) = aP_A(1)$ for all $a \in A$. So P_A is not only $\mathbf{k}$-linear, but also A-linear. As a consequence, $Ш^{NC}(A) \cong A \otimes_{\mathbf{k}} Ш^{NC}(\mathbf{k})$ where $Ш^{NC}(\mathbf{k})$ is the free Rota–Baxter algebra over the base ring $\mathbf{k}$. It has the structure of a twisted divided power algebra which was discussed in [7, 34]. In the context of this paper, $Ш^{NC}(\mathbf{k})$ is the subalgebra of $Ш^{NC}(X)$ spanned by the decorated ladder trees. Thus it appears that $Ш^{NC}(A)$ for A without Assumption 6.1 should have a much more degenerated forest structure than what we will considering in this section. To limit the size of this paper, we will not pursue the degenerated case further elsewhere.

Let $A = \mathbf{k}\oplus \check{A}$ be a unitary $\mathbf{k}$-algebra satisfying Assumption 6.1. To obtain the $\mathbf{k}$-module structure of the free Rota–Baxter algebra over A, we use the controlled forests introduced in Section 5 with angular decorations from $\mathbf{k} \subseteq A$ or $\check{A}$, depending on the number of leafs of the forests.

Let F be in $\mathcal{F}^r$ with ℓ leafs. We continue to use the notation $(F; \check{A})$ in Eq. (16). More precisely,

$$(F; \check{A}) = \{(F; \mathfrak{a}) \mid \mathfrak{a} \in \check{A}^{\otimes(\ell-1)}\}$$

with the convention that $\check{A}^{\otimes 0} = \mathbf{k}$. As in the case of module decorated forests in Section 3, we call $(F; \check{A})$ the **module of the forest F with angular decoration by $\check{A}$**, and call $(F; \mathfrak{a})$ an angularly decorated forest F with the decoration tensor $\mathfrak{a}$. Further, let

$$(\mathcal{F}^r; \check{A}) = \bigcup_{F\in\mathcal{F}^r} (F; \check{A}).$$

We also continue to use the other concepts, such as the tensor product of $D' = (F'; \mathfrak{a}')$ and $D'' = (F''; \mathfrak{a}'')$:

$$D' \otimes_a D'' = (F' \sqcup F''; a_1' \otimes \cdots \otimes a_{n'}' \otimes a \otimes a_1'' \otimes \cdots \otimes a_{n''}'') \tag{66}$$

first defined in Eq. (17), and the standard decomposition in Eq. (18) of $D = (F; a_1 \otimes \cdots \otimes a_{\ell-1})$

$$(F; \mathfrak{a}) = (T_1; \mathfrak{a}_1) \otimes_{u_1} (T_2; \mathfrak{a}_2) \otimes_{u_2} \cdots \otimes_{u_{b-1}} (T_b; \mathfrak{a}_b). \tag{67}$$

6.2. **Free Rota–Baxter algebra over A as angularly decorated controlled forests.** We define $Ш^{\mathrm{NC}}(A)$ to be the $\mathbf{k}$-module

$$Ш^{\mathrm{NC}}(A) = \bigoplus_{F\in\mathcal{F}^r} (F; \check{A}).$$

First note that, for $(F; \mathfrak{a}) \in (F; \check{A})$, the element

$$P_A(F; \mathfrak{a}) := (\lfloor F \rfloor ; \mathfrak{a}) \tag{68}$$

is a well-defined element in $(\lfloor F\rfloor; \check{A})$ since the number of leaves of $\lfloor F\rfloor$ is the same as the number of leafs of F. Thus P_A defines a linear operator on $Ш^{\mathrm{NC}}(A)$. Note that the right hand side is also $(P_{\mathcal{F}^r}(F); \mathfrak{a})$. Here $P_{\mathcal{F}^r}$ is the Rota–Baxter operator in Eq. (57).

We now define a product $\overline{\diamond}^r$ on $Ш^{\mathrm{NC}}(A)$ by defining $D\overline{\diamond}^r D'$ for any two angularly decorated forests $D = (F; \mathfrak{a}) \in (F; A)$ and $D' = (F'; \mathfrak{a}') \in (F'; A)$ with $\mathfrak{a}$ and $\mathfrak{a}'$ being pure tensors, and then extending by biadditivity.

As above, we have $D = (F; a_1\otimes\cdots\otimes a_n)$ and $D' = (F'; a_1'\otimes\cdots\otimes a_{n'}')$ with the convention that, if F or F' have one leaf, then the corresponding tensor is in $\mathbf{k}$. We first claim $(\bullet, \mathbf{1}) \in (\bullet; \mathbf{k})$ to be the identity of the multiplication $\overline{\diamond}^r$. So for any $(F; \mathfrak{a}) \in (F; A)$, define

$$(\bullet; c)\overline{\diamond}^r(F; \mathfrak{a}) = (F; \mathfrak{a})\overline{\diamond}^r(\bullet; c) = (F; c\mathfrak{a})$$

for any $c \in \mathbf{k}$.

In the following we define $(F;\mathfrak{a})\overline{\diamond}^r(F';\mathfrak{a}')$ for F and F' different from $\bullet$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$ be the decomposition of F and F' into trees. Let the standard decomposition of $D = (F;\mathfrak{a})$ be

$$D = (F;\mathfrak{a}) = (T_1;\mathfrak{a}_1) \otimes_{u_1} (T_2;\mathfrak{a}_2) \otimes_{u_2} \cdots \otimes_{u_{b-1}} (T_b;\mathfrak{a}_b)$$

given in Eq. (67) and similarly let

$$D' = (F';\mathfrak{a}') = (T'_1;\mathfrak{a}'_1) \otimes_{u'_1} (T'_2;\mathfrak{a}'_2) \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} (T'_{b'};\mathfrak{a}'_{b'})$$

be the standard decomposition of D'. Under these assumptions we define $D\overline{\diamond}^r D'$ by induction on the sum of depth $n := \mathrm{d}(F) + \mathrm{d}(F)$.

When $n = 0$, then $F = \bullet^{\sqcup b}$ and $F' = \bullet^{\sqcup b'}$. Since $F, F' \neq \bullet$, by Lemma 3.2, we have $b \geq 2$ and $b' \geq 2$. Since F and F' are in $\mathfrak{F}^r$, we have $b = b' = 2$. Then $D = (F;a)$, $D = (F';a')$ with $a, a' \in \check{A}$. By Assumption 6.1, $aa' = c + v$ for unique $c \in \mathbf{k}$ and $v \in \check{A}$. We define

$$D\overline{\diamond}^r D' = (\bullet; c) + (\bullet^{\sqcup 2}; v). \tag{69}$$

Assume that $\overline{\diamond}^r$ has been define for $D = (F;\mathfrak{a})$ and $D' = (F';\mathfrak{a}')$ with $0 \leq k \leq n := d + d'$ and consider D and D' with $n = k + 1$. We then use induction on the sum of breadth $m := b + b$ of F and F'. Then $m \geq 2$. If $m = 2$, then $b = b' = 1$ and F and F' are both trees. Since we are considering F and F' different from $\bullet$, we have $F = \lfloor \overline{F} \rfloor$ and $F' = \lfloor \overline{F}' \rfloor$. Further $(\overline{F};\mathfrak{a})$ and $(\overline{F}';\mathfrak{a}')$ are in $(\mathfrak{F}^r;\check{A})$ with $\mathrm{d}(\overline{F}) < d$ and $\mathrm{d}(\overline{F}') < d'$. We then define

$$(F;\mathfrak{a})\overline{\diamond}^r(F';\mathfrak{a}') = \lfloor (\overline{F};\mathfrak{a})\overline{\diamond}^r(F';\mathfrak{a}') \rfloor + \lfloor (F;\mathfrak{a})\overline{\diamond}^r(\overline{F}';\mathfrak{a}') \rfloor + \lambda \lfloor (\overline{F};\mathfrak{a})\overline{\diamond}^r(\overline{F}';\mathfrak{a}') \rfloor. \tag{70}$$

Here the sums $\mathrm{d}(\overline{F}) + \mathrm{d}(F'_1), \mathrm{d}(F) + \mathrm{d}(\overline{F}')$ and $\mathrm{d}(\overline{F}) + \mathrm{d}(\overline{F}')$ are all less than or equal to k. So by the induction hypothesis the products

$$(\overline{F};\mathfrak{a})\overline{\diamond}^r(F';\mathfrak{a}'), \quad (F;\mathfrak{a})\overline{\diamond}^r(\overline{F}';\mathfrak{a}'), \quad (\overline{F};\mathfrak{a})\overline{\diamond}^r(\overline{F}';\mathfrak{a}')$$

are all well-defined elements in $\text{Ш}^{\mathrm{N}C}(A)$. Then each term on the right hand side of Eq. (70) is defined since $\text{Ш}^{\mathrm{N}C}(A)$ is shown to be closed under the bracket operation $\lfloor\ \rfloor$.

Now assume that $(F;\mathfrak{a})\overline{\diamond}^r(F';\mathfrak{a}')$ has been defined either when $n := \mathrm{d}(F) + \mathrm{d}(F') \leq k$ or when $n = k + 1$ and $\ell \geq m := b(F) + b(F') \geq 2$. Consider $D = (F;\mathfrak{a})$ and $D = (F;\mathfrak{a}')$ in $(\mathfrak{F}^r;\check{A})$ with $n = k + 1$ and $m = \ell + 1$. We distinguish several cases.

Case 1. If $T_b = T'_1 = \bullet$ in the standard decomposition of D and D', then by Lemma 3.2, $(T_b;\mathfrak{a}_b) = (\bullet;\mathbf{1})$, $b > 1$, $(T'_1;\mathfrak{a}'_1) = (\bullet;\mathbf{1})$ and $b' > 1$. By Assumption 6.1, $u_{b-1}u'_1 = c + v$ for unique $c \in \mathbf{k}$ and $v \in \check{A}$. Define

$$\begin{aligned} D\overline{\diamond}^r D' :\ =&\ c\big((T_1;\mathfrak{a}_1) \otimes_{u_1} \cdots \otimes_{u_{b-2}} (T_{b-1};\mathfrak{a}_{b-1})\big)\overline{\diamond}^r\big((T'_2;\mathfrak{a}'_2) \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} (T'_{b'};\mathfrak{a}'_{b'})\big) \\ &+(T_1;\mathfrak{a}_1) \otimes_{u_1} \cdots \otimes_{u_{b-2}} (T_{b-1};\mathfrak{a}_{b-1}) \otimes_v (T'_2;\mathfrak{a}'_2) \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} (T'_{b'};\mathfrak{a}'_{b'}). \end{aligned} \tag{71}$$

The product in the first term is defined by the induction hypothesis since $(T_1;\mathfrak{a}_1) \otimes_{u_1} \cdots \otimes_{u_{b-2}} (T_{b-1};\mathfrak{a}_{b-1})$ has depth less or equal to that of D and breadth less than that of D, and similarly for the other product factor.

Case 2. If exactly one of T_b and T'_1 is $\bullet$, then the corresponding decorated tree is $(\bullet;\mathbf{1})$ by Lemma 3.2. We define

$$D\overline{\diamond}^r D' = (T_1;\mathfrak{a}_1) \otimes_{u_1} \cdots \otimes_{u_{b-1}} \big((T_b;\mathfrak{a}_b)\overline{\diamond}^r(T'_1;\mathfrak{a}'_1)\big) \otimes_{u'_1} \cdots \otimes_{u'_{b'-1}} (T'_{b'};\mathfrak{a}'_{b'}) \tag{72}$$

where

$$(T_b;\mathfrak{a}_b)\overline{\diamond}^r(T_1';\mathfrak{a}_1') = \begin{cases} (T_b,\mathfrak{a}_b), & \text{if } T_1' = \bullet, T_b \neq \bullet, \\ (T_1',\mathfrak{a}_1'), & \text{if } T_1' \neq \bullet, T_b = \bullet. \end{cases} \tag{73}$$

Case 3. If none of T_b or T_1' is $\bullet$, then $T_b = \lfloor \overline{F}_b \rfloor, T_1' = \lfloor \overline{F}_1' \rfloor$ for $\overline{F}_b, \overline{F}_1' \in \mathfrak{F}^r$. Then define $D\overline{\diamond}^r D'$ by Eq. (72) with

$$(T_b;\mathfrak{a}_b)\overline{\diamond}^r(T_1';\mathfrak{a}_1') = \lfloor (T_b;\mathfrak{a}_b)\overline{\diamond}^r(\overline{T}_1';\mathfrak{a}_1')\rfloor + \lfloor (\overline{T}_b;\mathfrak{a}_b)\overline{\diamond}^r(T_1';\mathfrak{a}_1')\rfloor + \lambda\lfloor (\overline{T}_b;\mathfrak{a}_b)\overline{\diamond}^r(\overline{T}_1';\mathfrak{a}_1')\rfloor \tag{74}$$

Note that, as in Eq. (70), each product inside the brackets of the right hand side is well-defined by the induction hypothesis and then the right hand side is defined by the closeness of $Ш^{\mathrm{N}C}(A)$ under the $\lfloor\ \rfloor$ operation. Then define $D\overline{\diamond}^r D'$ by Eq. (72).

This completes the definition of $\overline{\diamond}^r$ when $m = d + d' = \ell + 1$ and thus completes the definition of $\overline{\diamond}^r$.

We record the following simple properties of $\overline{\diamond}^r$ for later applications.

Lemma 6.2. *Given angularly decorated controlled forests D and D' such that either D does not end with a $\bullet$ or D' does not start with a $\bullet$. If $D = D_1 \otimes_u D_2$, then*

$$(D_1 \otimes_u D_2)\overline{\diamond}^r D' = D_1 \otimes_u (D_2 \overline{\diamond}^r D').$$

If $D' = D_1' \otimes_{u'} D_2'$, then

$$D\overline{\diamond}^r(D_1' \otimes_{u'} D_2') = (D\overline{\diamond}^r D_1') \otimes_{u'} D_2'.$$

Proof. The proof is the same as for Lemma 2.4, using Eq. (72). □

Extending $\overline{\diamond}^r$ biadditively, we obtain a binary operation

$$Ш^{\mathrm{N}C}(A) \otimes Ш^{\mathrm{N}C}(A) \to Ш^{\mathrm{N}C}(A).$$

Let

$$j_A : A \to Ш^{\mathrm{N}C}(A) \tag{75}$$

be the map sending $a \in \check{A}$ to $(\bullet \sqcup \bullet; a)$ and sending $c \in \mathbf{k}$ to $(\bullet; c)$.

Theorem 6.3. *Let A be an $\mathbf{k}$-algebra satisfying Assumption 6.1.*

(a) *The pair $(Ш^{\mathrm{N}C}(A), \overline{\diamond}^r)$ is a unitary $\mathbf{k}$-algebra.*
(b) *The quadruple $(Ш^{\mathrm{N}C}(A), \overline{\diamond}^r, P_A, j_A)$ is the free unitary Rota–Baxter algebra of weight λ on the algebra A.*

6.3. **Proof of Theorem 6.3.** (a). We first prove that $Ш^{\mathrm{N}C}(A)$ is a unitary associative algebra. Since we have defined $(\bullet; \mathbf{1})$ to be the identity of $\overline{\diamond}^r$, we only need to prove the associativity of $\overline{\diamond}^r$. Under additional assumptions on A, the associativity can be achieved by applying a similar argument for Theorem 5.1 and proving that $Ш^{\mathrm{N}C}(A)$ is a quotient of a free Rota–Baxter algebra $Ш^{\mathrm{N}C}(M)$ for a suitably chosen $\mathbf{k}$-module M. But it turns out that the construction of the quotient map in general requires associativity. So we will prove the associativity directly.

To prove the associativity, we just need to verify

$$(D\overline{\diamond}^r D')\overline{\diamond}^r D'' = D\overline{\diamond}^r(D'\overline{\diamond}^r D'') \tag{76}$$

for angularly decorated controlled forests $D = (F; \mathfrak{a}), D' = (F'; \mathfrak{a}'), D'' = (F''; \mathfrak{a}'') \in (\mathcal{F}^r, \check{A})$. If either one of F, F', F'' is $\bullet$, then the corresponding decorated forest is $(\bullet; c)$, $c \in \mathbf{k}$. Then both sides equal c times the product of the other two decorated forests. Thus we will consider the case when none of F, F' and F'' is $\bullet$. Under this assumption, we will prove the associativity by induction on the sum of the depths $n := \mathrm{d}(F) + \mathrm{d}(F') + \mathrm{d}(F'')$.

If $n = 0$, then all of F, F', F'' have depth zero and so $F = \bullet^{\sqcup i}$, $F' = \bullet^{\sqcup i'}$ and $F'' = \bullet^{\sqcup i''}$, for $i, i', i'' \geq 1$. Since they are in $\mathcal{F}^r$, we also have $i, i', i'' \leq 2$. Since none of them is $\bullet$ we have $i = i' = i'' = 2$. Thus $D^{(j)} = (\bullet \sqcup \bullet; a^{(j)})$ with $a^{(j)} \in \check{A}$, $0 \leq j \leq 2$ (taking $a^{(0)} = a$). By Assumption 6.1, for any $x, y \in \check{A}$, we have

$$xy = c_{x,y} + v_{x,y} \tag{77}$$

for unique $c_{x,y} \in \mathbf{k}$ and $v_{x,y} \in \check{A}$. Using this notation, we have $aa' = c_{a,a'} + v_{a,a'}$ for unique $c_{a,a'} \in \mathbf{k}$ and $v_{a,a'} \in \check{A}$. Further we have

$$(aa')a'' = (c_{a,a'} + v_{a,a'})a'' = c_{a,a'}a'' + v_{a,a'}a'' = c_{a,a'}a'' + c_{v_{a,a'},a''} + v_{v_{a,a'},a''},$$

for unique $c_{v_{a,a'},a''} \in \mathbf{k}$ and $c_{a,a'}a'' + v_{v_{a,a'},a''} \in \check{A}$. Similarly,

$$a(a'a'') = a(c_{a',a''} + v_{a',a''}) = c_{a',a''}a + av_{a',a''} = c_{a',a''}a + c_{a,v_{a',a''}} + v_{a,v_{a',a''}}$$

for unique $c_{a,v_{a',a''}} \in \mathbf{k}$ and $c_{a',a''}a + v_{a,v_{a',a''}} \in \check{A}$. Since $(aa')a'' = a(a'a'')$, by the uniqueness of the decomposition in Eq. (77), we have

$$c_{v_{a,a'},a''} = c_{a,v_{a',a''}}, \quad c_{a,a'}a'' + v_{v_{a,a'},a''} = c_{a',a''}a + v_{a,v_{a',a''}}. \tag{78}$$

According to Eq. (69) and the above decompositions of $(aa')a''$ and $a(a'a'')$, we have

$$\begin{aligned}
(D \overline{\diamond}^r D') \overline{\diamond}^r D'' &= \big(c_{a,a'}(\bullet; \mathbf{1}) + (\bullet \sqcup \bullet; v_{a,a'})\big) \overline{\diamond}^r (\bullet \sqcup \bullet; a'') \\
&= c_{a,a'}(\bullet \sqcup \bullet; a'') + c_{v_{a,a'},a''}(\bullet; \mathbf{1}) + (\bullet \sqcup \bullet; v_{v_{a,a'},a''}) \\
&= c_{v_{a,a'},a''}(\bullet; \mathbf{1}) + \big(\bullet \sqcup \bullet; (c_{a,a'}a'' + v_{v_{a,a'},a''})\big).
\end{aligned}$$

Similarly,

$$D \overline{\diamond}^r (D' \overline{\diamond}^r D'') = c_{a,v_{a',a''}}(\bullet; \mathbf{1}) + (\bullet \sqcup \bullet; (c_{a',a''}a + v_{a,v_{a',a''}})).$$

By Eq. (78), the two equations agrees. So we have proved Eq. (76) when $n = 0$.

Assume that the associativity (76) holds for $n \leq k$ and assume that $D, D', D'' \in (\mathcal{F}^r; \check{A})$ satisfy $n = \mathrm{d}(D) + \mathrm{d}(D') + \mathrm{d}(D'') = k + 1$. We next reduce the breadths of the decorated forests D, D' and D''.

Lemma 6.4. *If the associativity*

$$(D \overline{\diamond}^r D') \overline{\diamond}^r D'' = D \overline{\diamond}^r (D' \overline{\diamond}^r D'')$$

holds when D, D' and D'' in $(\mathcal{F}^r; \check{A})$ are decorated trees, then it holds when they are decorated forests.

Proof. We use induction on the sum of breadths $m := b(D) + b(D') + b(D'')$. Then $m \geq 3$. The case when $m = 3$ is the assumption of the lemma. Assume the associativity holds for $3 \leq m \leq j$ and take $D, D', D'' \in (\mathcal{F}^r, \check{A})$ with $m = j + 1$. Then $j + 1 \geq 4$. So at least one of $b(D), b(D')$ and $b(D'')$ is greater than or equal to 2. Further, as shown earlier in the proof, if one of F, F' or F'' is $\bullet$, then the associativity is clear. So we can assume none of F, F' or F'' is $\bullet$. Then by Lemma 3.2, if the standard decomposition of D ends (resp. starts) with a $(\bullet; c)$ with $c \in \mathbf{k}$, then $b(D) > 1$ and D ends (resp. starts) with a $(\bullet; \mathbf{1})$. The same is true for D' and D''. We consider the following two conditions.

(i) Either D does not end with a $(\bullet; \mathbf{1})$ or D' does not start with a $(\bullet; \mathbf{1})$;

(ii) Either D' does not end with a $(\bullet; \mathbf{1})$ or D'' does not start with a $(\bullet; \mathbf{1})$.

Then the proof of the lemma is divided into the following four cases.

Case 1. Both Conditions (i) and (ii) are satisfied. Then Lemma 6.2 applies. Assume $b(D) \geq 2$. Then $D = D_1 \otimes_u D_2$ with $D_1, D_2 \in \mathfrak{F}^r$ and $u \in \check{A}$. Then by Lemma 6.2, we have

$$\begin{aligned}(D \overline{\diamond}^r D')\overline{\diamond}^r D'' &= ((D_1 \otimes_u D_2)\overline{\diamond}^r D')\overline{\diamond}^r D'' \\ &= (D_1 \otimes_u (D_2 \overline{\diamond}^r D'))\overline{\diamond}^r D'' \\ &= D_1 \otimes_u ((D_2 \overline{\diamond}^r D')\overline{\diamond}^r D'').\end{aligned}$$

The last equation follows since if D' does not end with a $(\bullet; \mathbf{1})$, then nor does $D_2 \overline{\diamond}^r D'$. Similarly,

$$\begin{aligned}D\overline{\diamond}^r (D' \overline{\diamond}^r D'') &= (D_1 \otimes_u D_2)\overline{\diamond}^r (D' \overline{\diamond}^r D'') \\ &= D_1 \otimes_u (D_2 \overline{\diamond}^r (D' \overline{\diamond}^r D'')).\end{aligned}$$

Thus

$$(D \overline{\diamond}^r D')\overline{\diamond}^r D'' = D \overline{\diamond}^r (D' \overline{\diamond}^r D'')$$

whenever

$$(D_2 \overline{\diamond}^r D')\overline{\diamond}^r D'' = D_2 \overline{\diamond}^r (D' \overline{\diamond}^r D'')$$

which follows from the induction hypothesis.

A similar proof works if $b(D'') \geq 2$.

Finally if $b(D') \geq 2$, then $D' = D'_1 \otimes_u D'_2$ with $D'_1, D'_2 \in (\mathfrak{F}^r; \check{A})$ and $u \in \check{A}$. Using Lemma 6.2 repeatedly, we have

$$\begin{aligned}(D \overline{\diamond}^r D')\overline{\diamond}^r D'' &= (D \overline{\diamond}^r (D'_1 \otimes_u D'_2))\overline{\diamond}^r D'' \\ &= ((D \overline{\diamond}^r D'_1) \otimes_u D'_2)\overline{\diamond}^r D'' \\ &= (D \overline{\diamond}^r D'_1) \otimes_u (D'_2 \overline{\diamond}^r D'').\end{aligned}$$

In the same way, we have

$$D \overline{\diamond}^r (D' \overline{\diamond}^r D'') = (D \overline{\diamond}^r D'_1) \otimes_u (D'_2 \overline{\diamond}^r D'').$$

This again proves the associativity.

Case 2. Now assume that only Condition (ii) is satisfied. Then by Lemma 3.2, we have $b := b(D) > 1$, $b' := b(D') > 1$ and in the standard decompositions of D and D', we have $D_b = (\bullet; \mathbf{1})$ and $D'_1 = (\bullet; \mathbf{1})$. So we have

$$D = (T_1; \mathfrak{a}_1) \sqcup_{u_1} \cdots \sqcup_{u_{b-2}} (T_{b-1}; \mathfrak{a}_{b-1}) \sqcup_{u_{b-1}} (\bullet; \mathbf{1})$$

and

$$D' = (\bullet; \mathbf{1}) \sqcup_{u'_1} (T'_2; \mathfrak{a}'_2) \sqcup_{u'_2} \cdots \sqcup_{u'_{b'-1}} (T'_{b'}; \mathfrak{a}'_{b'}).$$

For notational simplicity, we denote $x = u_{b-1}$ and $y = u'_1$. Then by Eq. (71),

$$\begin{aligned}(D \overline{\diamond}^r D')\overline{\diamond}^r D'' = \;& c_{x,y}\big((D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1})\overline{\diamond}^r (D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} D'_{b'})\big)\overline{\diamond}^r D'' \\ & + (D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,y}} D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} D'_{b'})\overline{\diamond}^r D''\end{aligned}$$

$$\begin{aligned} &= c_{x,y}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r \big((D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-}} D'_{b'}) \overline{\diamond}^r D''\big) \\ &\quad + D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,y}} D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} (D'_{b'} \overline{\diamond}^r D''), \end{aligned}$$

where the second equation follows from the induction hypothesis on the sum of breadth m and Lemma 6.2. For the same reason, we have

$$\begin{aligned} D \overline{\diamond}^r (D' \overline{\diamond}^r D'') &= D \overline{\diamond}^r \big(D'_1 \otimes_{u'_1} \cdots \otimes_{u'_{b'-1}} (D'_b \overline{\diamond}^r D'')\big) \\ &= c_{x,y}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r \big(D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} (D'_{b'} \overline{\diamond}^r D'')\big) \\ &\quad + D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,y}} D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} (D'_{b'} \overline{\diamond}^r D''). \end{aligned}$$

This verifies the associativity.

Case 3. Assume that only Condition (i) is satisfied. Then the same argument as the previous case works.

Case 4. Assume none of Conditions (i) or (ii) is satisfied. Then by by Lemma 3.2, we have $b := b(D) > 1$, $b' := b(D') > 1$, $b'' := b(D'') > 1$ and in the standard decompositions of D, D' and D'', we have $D_b = D'_1 = D'_{b'} = D''_1 = (\bullet; \mathbf{1})$.

Subcase 4.1. We first consider the subcase when $b' > 2$. Then $D' = (\bullet; \mathbf{1}) \otimes_{u'_1} \cdots \otimes_{u'_{b'-1}} (\bullet; \mathbf{1})$ with $b' - 1 \neq 1$. Denote $x = u_{b-1}, y = u'_1, z = u'_{b'-1}, w = u''_1$. By Eq. (71), we have

$$\begin{aligned} (D \overline{\diamond}^r D') \overline{\diamond}^r D'' &= \big(c_{x,y}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r (D'_2 \otimes_{u'_2} \cdots \otimes_z D'_{b'})\big) \overline{\diamond}^r D'' \\ &\quad + (D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,y}} D'_2 \otimes_{u'_2} \cdots \otimes_z D'_{b'}) \overline{\diamond}^r D'' \end{aligned}$$

Applying the induction hypothesis on m to the first term and applying Eq. (71) to the second term, it gives

$$\begin{aligned} & c_{x,y}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r \big((D'_2 \otimes_{u'_2} \cdots \otimes_z D'_{b'}) \overline{\diamond}^r D''\big) \\ & + c_{z,w}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,y}} D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-2}} D'_{b'-1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}) \\ & + D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,y}} D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-2}} D'_{b'-1} \otimes_{v_{z,w}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}. \end{aligned}$$

Applying Eq. (71) to the first term, we obtain

$$\begin{aligned} & (D \overline{\diamond}^r D') \overline{\diamond}^r D'') \\ = \; & c_{x,y} c_{z,w}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r \big((D'_2 \otimes_{u'_2} \cdots \otimes_{b'-2} D'_{b'-1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''})\big) \\ & + c_{x,y}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r \big(D'_2 \otimes_{u'_2} \cdots \otimes_{b'-2} D'_{b'-1} \otimes_{v_{z,w}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}\big) \\ & + c_{z,w}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,y}} D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-2}} D'_{b'-1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}) \\ & + D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,y}} D'_2 \otimes_{u'_2} \cdots \otimes_{u'_{b'-2}} D'_{b'-1} \otimes_{v_{z,w}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}. \end{aligned}$$

Using the same arguments for $D \overline{\diamond}^r (D' \overline{\diamond}^r D'')$, we see that it matches with $(D \overline{\diamond}^r D') \overline{\diamond}^r D''$.

Subcase 4.2. We finally consider the subcase when D' has breadth $b' = 2$. Then $u'_{b'-1} = u'_1$ and $D' = (\bullet; \mathbf{1}) \otimes_{u'_1} (\bullet; \mathbf{1})$. Denote $x = u_{b-1}, y = u'_1 (= u'_{b'-1}), w = u''_1$. Using the same arguments, we have

$$\begin{aligned} (D \overline{\diamond}^r D') \overline{\diamond}^r D'' &= \big(c_{x,y}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r (\bullet; \mathbf{1})\big) \overline{\diamond}^r D'' \\ &\quad + (D_1 \otimes_{u_1} \cdots \otimes_{u_{b-1}} D_{b-1} \otimes_{v_{x,y}} (\bullet; \mathbf{1})) \overline{\diamond}^r D'' \\ &= c_{x,y}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r D'' \\ &\quad + (D_1 \otimes_{u_1} \cdots \otimes_{u_{b-1}} D_{b-1} \otimes_{v_{x,y}} (\bullet; \mathbf{1})) \overline{\diamond}^r D''. \end{aligned}$$

Note that D_{b-1} is not in $(\bullet;\mathbf{k})$ by the restriction on controlled forests. Applying Eq. (71) – (74) to the right hand side then gives

$$\begin{aligned}
& c_{x,y}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_w D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D_{u''_{b''}} \\
& + c_{v_{x,y},w}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}) \\
& + D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{v_{x,y},w}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''} \\
= \;& D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{c_{x,y}w + v_{v_{x,y},w}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D_{u''_{b''}} \\
& + c_{v_{x,y},w}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}),
\end{aligned}$$

where the right hand side is obtained by combining the first term and the third term on the left hand side, using the linearity of tensor products. Using the same arguments, we also verify

$$\begin{aligned}
D \overline{\diamond}^r (D' \overline{\diamond}^r D'') &= D \overline{\diamond}^r \big(c_{y,w}((\bullet;\mathbf{1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''})\big) \\
&\quad + D \overline{\diamond}^r ((\bullet;\mathbf{1}) \otimes_{v_{y,w}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}) \\
&= c_{y,w} D \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}) \\
&\quad + c_{x,v_{y,w}}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}) \\
&\quad + D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,v_{y,w}}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''} \\
&= c_{y,w}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_x D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}) \\
&\quad + c_{x,v_{y,w}}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}) \\
&\quad + D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1} \otimes_{v_{x,v_{y,w}}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''} \\
&= D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} \otimes_{c_{y,w}x + v_{x,v_{y,w}}} D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''} \\
&\quad + c_{x,v_{y,w}}(D_1 \otimes_{u_1} \cdots \otimes_{u_{b-2}} D_{b-1}) \overline{\diamond}^r (D''_2 \otimes_{u''_2} \cdots \otimes_{u''_{b''-1}} D''_{b''}).
\end{aligned}$$

Then the associativity follows from Eq. (78).

Now we have verified all the cases to complete the inductive proof of Lemma 6.4 □

To summarize, our proof of the associativity has been reduced to the special case when the decorated forests $D, D', D'' \in (\mathcal{F}^r, \check{A})$ are chosen such that

(i) $n := \mathrm{d}(D) + \mathrm{d}(D') + \mathrm{d}(D'') = k+1 \geq 1$ with the assumption that the associativity holds when $n \leq k$, and
(ii) the elements are of breadth one and are not $(\bullet;\mathbf{1})$.

By the second condition, all the three elements are decorated trees and are all in $\lfloor(\mathcal{F}^r;\check{A})\rfloor = (\lfloor\mathcal{F}^r\rfloor;\check{A})$. Then $D = \lfloor\overline{D}\rfloor, D' = \lfloor\overline{D}'\rfloor, D'' = \lfloor\overline{D}''\rfloor$ with $\overline{D}, \overline{D}', \overline{D}'' \in (\mathcal{F}^r;\check{A})$. Apply Lemma 2.10 to our situation where $R = Ш^{\mathrm{NC}}(A)$ with the multiplication $\cdot = \overline{\diamond}^r$, the Rota–Baxter operator $\lfloor\ \rfloor_R = \lfloor\ \rfloor$ and the triple $(y, y', y'') = (\overline{D}, \overline{D}', \overline{D}'')$. By the induction hypothesis on n, $\overline{\diamond}^r$ is associative for all the triples in Eq. (13) and (14). So by Lemma 2.10, $\overline{\diamond}^r$ is associative for the triple (D, D', D''). This completes the induction and therefore the proof of the first part of Theorem 6.3.

(b). The Rota–Baxter operator property of $\lfloor\ \rfloor$ on $Ш^{NC}(A)$ follows from Eq. (70). We next prove the universal property of a free Rota–Baxter algebra.

Let (R,P) be a unitary Rota–Baxter algebra of weight λ. Let $f: A \to R$ be a $\mathbf{k}$-algebra homomorphism. We will construct a $\mathbf{k}$-linear map $\bar{f}: Ш^{NC}(A) \to R$ by defining $\bar{f}(D)$ for $D = (F;\mathfrak{a}) \in (\mathcal{F}^r; A)$. We will achieve this by induction on the depth $\mathrm{d}(F)$ of $F \in \mathcal{F}^r$.

If $\mathrm{d}(F) = 0$, then $F = \bullet^{\sqcup i}$ with $i = 1, 2$ by the restriction on $\mathcal{F}^r$. If $i = 1$, then by convention, $D = (\bullet; c)$, $c \in \mathbf{k}$. Define

$$\bar{f}(D) = f(c) = c \in R. \tag{79}$$

In particular,

$$\bar{f}(\bullet; \mathbf{1}) = \mathbf{1}_R, \tag{80}$$

the unit of R. So $\bar{f}$ is unitary. If $i = 2$, then $D = (\bullet \sqcup \bullet; a)$ with $a \in \check{A}$. Define

$$\bar{f}(D) = f(a) \in R. \tag{81}$$

Assume that $\bar{f}(D)$ has been defined for all $D = (F;\mathfrak{a})$ with $\mathrm{d}(F) \leq k$ and let $D = (F;\mathfrak{a})$ with $\mathrm{d}(F) = k+1$. Then $\mathrm{d}(F) > 1$, so $F \neq \bullet$. Let

$$D = (T_1; \mathfrak{a}_1) \otimes_{u_1} \cdots \otimes_{u_{b-1}} (T_b; \mathfrak{a}_b)$$

be the standard decomposition of D given in Eq. (67). For each $1 \leq i \leq b$, T_i is a tree, so it is either $\bullet$ or is of the form $\lfloor \overline{F}_i \rfloor$ for another forest $\overline{F}_i$. Suppose $T_i = \bullet$ for some i, then since F is not $\bullet$, we have $b > 1$ and $(T_i; \mathfrak{a}_i) = (\bullet; \mathbf{1})$ by Lemma 3.2. We define

$$\bar{f}(T_i; \mathfrak{a}_i) = \begin{cases} \mathbf{1}_R, & \text{if } T_i = \bullet, \\ P(\bar{f}(\overline{F}_i; \mathfrak{a}_i)), & \text{if } T_i = \lfloor \overline{F}_i \rfloor. \end{cases} \tag{82}$$

Note that $(\overline{F}_i; \mathfrak{a}_i)$ is a well-defined angularly decorated forest since $\overline{F}_i$ has the same number of leafs as the number of leafs as T_i, and then $\bar{f}(\overline{F}_i; \mathfrak{a}_i)$ is defined by the induction hypothesis since $\mathrm{d}(\overline{F}_i) = \mathrm{d}(T_i) - 1 \leq k$. Therefore we can define

$$\bar{f}(D) = \bar{f}(T_1; \mathfrak{a}_1) * f(u_1) * \cdots * f(u_{b-1}) * \bar{f}(T_b; \mathfrak{a}_b)) \tag{83}$$

which is well-defined in R.

For any $D = (F;\mathfrak{a}) \in (F; A)$, we have $P_A(D) = (\lfloor F \rfloor; \mathfrak{a}) \in (\lfloor F \rfloor; A)$, and by the definition of $\bar{f}$ (Eq. (83)), we have

$$\bar{f}(\lfloor D \rfloor) = P(\bar{f}(D)). \tag{84}$$

So $\bar{f}$ commutes with the Rota–Baxter operators.

Thus to prove that the map $\bar{f}$ defined in Eq. (79) – (83) is a unitary Rota–Baxter algebra homomorphism, we only need to check the multiplicativity

$$\bar{f}(D \overline{\diamond} D) = \bar{f}(D) * \bar{f}(D') \tag{85}$$

for all angularly decorated controlled forests $D, D' \in (\mathcal{F}^r, \check{A})$. We can further assume that $D = (F;\mathfrak{m}), D' = (F';\mathfrak{m}')$ with pure tensors $\mathfrak{m}$ and $\mathfrak{m}'$. Let $F = T_1 \sqcup \cdots \sqcup T_b$ and $F' = T'_1 \sqcup \cdots \sqcup T'_{b'}$ be the decomposition of F and F' into trees. Let

$$(F;\mathfrak{m}) = (T_1;\mathfrak{m}_1) \otimes_{u_1} (T_2;\mathfrak{m}_2) \otimes_{u_2} \cdots \otimes_{u_{b-1}} (T_b;\mathfrak{m}_b)$$

and

$$(F';\mathfrak{m}') = (T'_1;\mathfrak{m}'_1) \otimes_{u'_1} (T'_2;\mathfrak{m}'_2) \otimes_{u'_2} \cdots \otimes_{u'_{b'-1}} (T'_{b'};\mathfrak{m}'_{b'})$$

be their standard decompositions.

We first note that, since $\bar{f}$ sends the identity $(\bullet;\mathbf{1})$ of $Ш^{NC}(A)$ to the identity $\mathbf{1}_R$ of R, the multiplicativity is clear if either one of D or D' is in $(\bullet;\mathbf{k})$, that is, if either one of F or F' is $\bullet$. So we only need to prove

Lemma 6.5. *The multiplicity (85) holds for all $D=(F;\mathfrak{a}), D'=(F';\mathfrak{a}')\in(\mathcal{F}^r;\breve{A})$ with $F\neq\bullet$ and $F'\neq\bullet$.*

Proof. We use induction on the sum of depth $n:=\mathrm{d}(F)+\mathrm{d}(F')$.

If $n=0$, then since F and F' are not $\bullet$ and are controlled forests, we have $F=F'=\bullet\sqcup\bullet$. Thus $D=(\bullet\sqcup\bullet,a)$ and $D'=(\bullet\sqcup\bullet;a')$ with $a,a'\in\breve{A}$. Then using the notation of Eq. (69) and Eq. (79)-(81), we have

$$\bar{f}(D\overline{\diamond}^r D')=\bar{f}((\bullet;c)+(\bullet\sqcup\bullet;v))=c\mathbf{1}_R+f(v).$$

This agrees with

$$\bar{f}(D)*\bar{f}(D')=f(a)*f(a')=f(aa')=f(c+v)=c\mathbf{1}_R+f(v).$$

Assume that the multiplicativity has been proved for $D=(F;\mathfrak{a})$ and $D'=(F';\mathfrak{a}')$ with $0\leq k\leq n$ and consider D and D' with $n=k+1$. We then use induction on the sum of the breadths $m:=b+b'$ of F and F'. If $m=2$, then $b=b'=1$ and F and F' are both trees. Since $F,F'\neq\bullet$ by assumption, we have $F=\lfloor\overline{F}\rfloor$ and $F'=\lfloor\overline{F}'\rfloor$. Further $(\overline{F};\mathfrak{a})$ and $(\overline{F}';\mathfrak{a}')$ are in $(\mathcal{F}^r;A)$ with $\mathrm{d}(\overline{F})<d$ and $\mathrm{d}(\overline{F}')<d'$. So by the induction hypothesis on n, we have

$$\begin{aligned}
\bar{f}((\overline{F};\mathfrak{a})\overline{\diamond}^r(F';\mathfrak{a}'))&=\bar{f}(\overline{F};\mathfrak{a})*\bar{f}(F';\mathfrak{a}'),\\
\bar{f}((F;\mathfrak{a})\overline{\diamond}^r(\overline{F}';\mathfrak{a}'))&=\bar{f}(F;\mathfrak{a})*\bar{f}(\overline{F}';\mathfrak{a}'),\\
\bar{f}((\overline{F};\mathfrak{a})\diamond(\overline{F}';\mathfrak{a}'))&=\bar{f}(\overline{F};\mathfrak{a})*\bar{f}(\overline{F}';\mathfrak{a}').
\end{aligned}$$

Therefore by Lemma 3.5, we have

$$\bar{f}((F;\mathfrak{a})\overline{\diamond}^r(F';\mathfrak{a}'))=\bar{f}(F;\mathfrak{a})*\bar{f}(F';\mathfrak{a}').$$

Now assume that the multiplicativity has been verified when $n:=\mathrm{d}(F)+\mathrm{d}(F')\leq k$, and when $n=k+1$ and $2\leq m:=b(F)+b(F')\leq\ell$. Consider $D=(F;\mathfrak{a})$ and $D'=(F';\mathfrak{a}')$ in $(\mathcal{F}^r;\breve{A})$ with $n=k+1$ and $m=\ell+1$. Let the standard decomposition of $D=(F;\mathfrak{a})$ and $D'=(F';\mathfrak{a}')$ be

$$D=(F;\mathfrak{a})=(T_1;\mathfrak{a}_1)\otimes_{u_1}(T_2;\mathfrak{a}_2)\otimes_{u_2}\cdots\otimes_{u_{b-1}}(T_b;\mathfrak{a}_b)$$

and

$$D'=(F';\mathfrak{a}')=(T_1';\mathfrak{a}_1')\otimes_{u_1'}(T_2';\mathfrak{a}_2')\otimes_{u_2'}\cdots\otimes_{u_{b'-1}'}(T_{b'}';\mathfrak{a}_{b'}').$$

If $T_b=T_1'=\bullet$, then by Lemma 3.2, we have $(T_b;\mathfrak{a}_b)=(\bullet;\mathbf{1})$, $b>1$, $(T_1';\mathfrak{a}_1')=(\bullet;\mathbf{1})$ and $b'>1$. By Assumption 6.1, $u_{b-1}u_1'=c+v$ for unique $c\in\mathbf{k}$ and $v\in\breve{A}$. Then using Eq. (71), the induction hypothesis on the breadths, Eq. (83) and the notation $*_w=*w*$, we have

$$\begin{aligned}
&\bar{f}(D\diamond D')\\
=\;&\bar{f}\Big(c\big((T_1;\mathfrak{a}_1)\otimes_{u_1}\cdots\otimes_{u_{b-2}}(T_{b-1};\mathfrak{a}_{b-1})\big)\overline{\diamond}^r\big((T_2';\mathfrak{a}_2')\otimes_{u_2'}\cdots\otimes_{u_{b'-1}'}(T_{b'}';\mathfrak{a}_{b'}')\big)\Big)\\
&+\bar{f}\big((T_1;\mathfrak{a}_1)\otimes_{u_1}\cdots\otimes_{u_{b-2}}(T_{b-1};\mathfrak{a}_{b-1})\otimes_v(T_2';\mathfrak{a}_2')\otimes_{u_2'}\cdots\otimes_{u_{b'-1}'}(T_{b'}';\mathfrak{a}_{b'}')\big)\\
=\;&c\bar{f}\big((T_1;\mathfrak{a}_1)\otimes_{u_1}\cdots\otimes_{u_{b-2}}(T_{b-1};\mathfrak{a}_{b-1})\big)*\bar{f}\big((T_2';\mathfrak{a}_2')\otimes_{u_2'}\cdots\otimes_{u_{b'-1}'}(T_{b'}';\mathfrak{a}_{b'}')\big)
\end{aligned}$$

$$
\begin{aligned}
&+\bar{f}\big((T_1;\mathfrak{a}_1)\otimes_{u_1}\cdots\otimes_{u_{b-2}}(T_{b-1};\mathfrak{a}_{b-1})\otimes_v(T'_2;\mathfrak{a}'_2)\otimes_{u'_2}\cdots\otimes_{u'_{b'-1}}(T'_{b'};\mathfrak{a}'_{b'})\big)\\
=\;& c\bar{f}(T_1;\mathfrak{a}_1)*_{f(u_1)}\cdots*_{f(u_{b-2})}\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*\bar{f}(T'_2;\mathfrak{a}'_2)*_{f(u'_2)}\cdots*_{f(u'_{b'-1})}\bar{f}(T'_{b'};\mathfrak{a}'_{b'})\\
&+\bar{f}(T_1;\mathfrak{a}_1)*_{f(u_1)}\cdots*_{f(u_{b-2})}\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*_{f(v)}\bar{f}(T'_2;\mathfrak{a}'_2)*_{f(u'_2)}\cdots*_{f(u'_{b'-1})}\bar{f}(T'_{b'};\mathfrak{a}'_{b'}).
\end{aligned}
$$

On the other hand, by Eq. (83), we have

$$
\begin{aligned}
\bar{f}(D)*\bar{f}(D') \;=\;& \big(\bar{f}(T_1;\mathfrak{a}_1)*_{f(u_1)}\cdots*_{f(u_{b-2})}\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*_{f(u_{b-1})}\bar{f}(T_b;\mathfrak{a}_b)\big)\\
&*\big(\bar{f}(T'_1;\mathfrak{a}'_1)*_{f(u'_1)}\bar{f}(T'_2;\mathfrak{a}'_2)*_{f(u'_2)}\cdots*_{f(u'_{b'-1})}\bar{f}(T'_{b'};\mathfrak{a}'_{b'})\big)
\end{aligned}
$$

Since $(T_b;\mathfrak{a}_1)=(T'_1;\mathfrak{a}'_1)=(\bullet;\mathbf{1})$, in the above equation, we have

$$
\begin{aligned}
&\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*_{f(u_{b-1})}\bar{f}(T_b;\mathfrak{a}_b)*\bar{f}(T'_1;\mathfrak{a}'_1)*_{f(u'_1)}\bar{f}(T'_2;\mathfrak{a}'_2)\\
=\;&\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*f(u_{b-1})*f(u'_1)*\bar{f}(T'_2;\mathfrak{a}'_2)\\
=\;&\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*f(u_{b-1}u'_1)*\bar{f}(T'_2;\mathfrak{a}'_2)\\
=\;&\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*f(c+v)*\bar{f}(T'_2;\mathfrak{a}'_2)\\
=\;&c\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*\bar{f}(T'_2;\mathfrak{a}'_2)+\bar{f}(T_{b-1};\mathfrak{a}_{b-1})*f(v)*\bar{f}(T'_2;\mathfrak{a}'_2).
\end{aligned}
$$

This proves the multiplicativity (30) when $T_b=T'_1=\bullet$.

If one of T_b, T'_1 is not $\bullet$, then $D\overline{\diamond}^r D'$ is defined by Eq. (72). So we have by Eq. (83),

$$
\begin{aligned}
\bar{f}(D\diamond D') \;=\;& \bar{f}\big((T_1;\mathfrak{a}_1)\otimes_{u_1}\cdots\otimes_{u_{b-1}}\big((T_b;\mathfrak{a}_b)\diamond(T'_1;\mathfrak{a}'_1)\big)\otimes_{u'_1}\cdots\otimes_{u'_{b'-1}}(T'_{b'};\mathfrak{a}'_{b'})\big)\\
=\;& \bar{f}\big(T_1;\mathfrak{a}_1)*_{f(u_1)}\cdots*_{f(u_{b-1})}\bar{f}\big((T_b;\mathfrak{a}_b)\diamond(T'_1;\mathfrak{a}'_1)\big)*_{f(u'_1)}\cdots*_{f(u'_{b'-1})}\bar{f}(T'_{b'};\mathfrak{a}'_{b'}).
\end{aligned}
$$

Also by Eq. (83), we have

$$
\begin{aligned}
\bar{f}(D)*\bar{f}(D') \;=\;& \big(\bar{f}\big(T_1;\mathfrak{a}_1)\big)*_{f(u_1)}\cdots*_{f(u_{b-1})}\bar{f}(T_b;\mathfrak{a}_b)\big)\\
&*\big(\bar{f}(T'_1;\mathfrak{a}'_1)*_{f(u'_1)}\cdots*_{f(u'_{b'-1})}\bar{f}(T'_{b'};\mathfrak{a}'_{b'})\big).
\end{aligned}
$$

Thus to prove the multiplicativity, we only need to prove

$$
\bar{f}\big((T_b;\mathfrak{a}_b)\overline{\diamond}^r(T'_1;\mathfrak{a}'_1)\big)=\bar{f}(T_b;\mathfrak{a}_b)*\bar{f}(T'_1;\mathfrak{a}'_1). \tag{86}
$$

For this we distinguish three cases. First assume $T_b=\bullet$. Then $(T_b;\mathfrak{a}_b)=(\bullet;\mathbf{1})$ and by Eq. (72) – (73), we have

$$
\bar{f}((T_b;\mathfrak{a}_b)\overline{\diamond}^r(T'_1;\mathfrak{a}'_1))=\bar{f}(T'_1;\mathfrak{a}'_1)=\bar{f}(T_b;\mathfrak{a}_b)*\bar{f}(T'_1;\mathfrak{a}'_1)
$$

since $\bar{f}(\bullet;\mathbf{1})=\mathbf{1}_R$. Next assume $T'_1=\bullet$. This is treated in the same way. Finally assume that none of T_b or T'_1 is $\bullet$. Then $T_b=\lfloor\overline{F}_b\rfloor, T'_1=\lfloor\overline{F}'_1\rfloor$ for $\overline{F}_b,\overline{F}'_1\in\mathfrak{F}^r$. Then as in Eq. (8), the sums $\mathrm{d}(T_b)+\mathrm{d}(\overline{F}'_1), \mathrm{d}(\overline{F}_b)+\mathrm{d}(T'_1)$ and $\mathrm{d}(\overline{F}_b)+\mathrm{d}(\overline{F}'_1)$ are all less than or equal to k. So by the induction hypothesis we have

$$
\begin{aligned}
\bar{f}\big((T_b;\mathfrak{a}_b)\overline{\diamond}^r(\overline{F}'_1;\mathfrak{a}'_1)\big) &= \bar{f}(T_b;\mathfrak{a}_b)*\bar{f}(\overline{F}'_1;\mathfrak{a}'_1),\\
\bar{f}\big((\overline{F}_b;\mathfrak{a}_b)\overline{\diamond}^r(T'_1;\mathfrak{a}'_1)\big) &= \bar{f}(\overline{F}_b;\mathfrak{a}_b)*\bar{f}(T'_1;\mathfrak{a}'_1),\\
\bar{f}\big((\overline{F}_b;\mathfrak{a}_b)\overline{\diamond}^r(\overline{F}'_1;\mathfrak{a}'_1)\big) &= \bar{f}(\overline{F}_b;\mathfrak{a}_b)*\bar{f}(\overline{F}'_1;\mathfrak{a}'_1).
\end{aligned}
$$

Then by Lemma 3.5, Eq. (86) and thus the multiplicativity in Eq. (85) hold. This verifies the case when $n=k+1$ and $m=\ell+1$ and thus finishes the inductive proof of Lemma 6.5. $\square$

This completes the proof of the existence of $\bar{f}: Ш^{NC}(A) \to R$.

It remains to prove that Eq. (83) is the only way to define $\bar{f}$ in order to be a Rota–Baxter algebra homomorphism extending f. For this we first prove the following lemma which is interesting on its own right.

Lemma 6.6. *Let $(F;\mathfrak{a}) \in (\mathcal{F}^r;\check{A})$ with $\mathfrak{a}$ a pure tensor and let*

$$(F;\mathfrak{a}) = (T_1;\mathfrak{a}_1) \otimes_{u_1} (T_2;\mathfrak{a}_2) \otimes_{u_2} \cdots \otimes_{u_{b-1}} (T_b;\mathfrak{a}_b)$$

be the standard decomposition of $(F;\mathfrak{a})$. Using the notation $\overline{\diamond}^r_{u_i} = \overline{\diamond}^r(\bullet \sqcup \bullet; u_i)\,\overline{\diamond}^r$, we have

$$(F;\mathfrak{a}) = (T_1;\mathfrak{a}_1)\overline{\diamond}^r_{u_1}(T_2;\mathfrak{a}_2)\overline{\diamond}^r_{u_2}\cdots\overline{\diamond}^r_{u_{b-1}}(T_b;\mathfrak{a}_b). \tag{87}$$

Proof. We use induction on b. There is nothing to prove when $b=1$. We next verify for $b=2$ which will be needed later in the induction. We need to prove

$$(T_1;\mathfrak{a}_1) \otimes_{u_1} (T_2;\mathfrak{a}_2) = (T_1;\mathfrak{a}_1)\overline{\diamond}^r_{u_1}(T_2;\mathfrak{a}_2) := (T_1;\mathfrak{a}_1)\overline{\diamond}^r(\bullet \sqcup \bullet; u_1)\overline{\diamond}^r(T_2;\mathfrak{a}_2).$$

First note that

$$(\bullet \sqcup \bullet; u_1) = (\bullet;\mathbf{1}) \otimes_{u_1} (\bullet;\mathbf{1}).$$

If $T_1 \neq \bullet$ and $T_2 \neq \bullet$, then by the associativity of the product $\overline{\diamond}^r$ that we have just proved and Eq. (72) – (74), we have

$$\begin{aligned}
(T_1;\mathfrak{a}_1)\overline{\diamond}^r(\bullet \sqcup \bullet; u_1)\overline{\diamond}^r(T_2;\mathfrak{a}_2) &= (T_1;\mathfrak{a}_1)\overline{\diamond}^r\big((\bullet;\mathbf{1}) \otimes_{u_1} (\bullet;\mathbf{1})\big)\overline{\diamond}^r(T_2;\mathfrak{a}_2) \\
&= \Big(\big((T_1;\mathfrak{a}_1)\overline{\diamond}^r(\bullet;\mathbf{1})\big) \otimes_{u_1} (\bullet;\mathbf{1})\Big)\overline{\diamond}^r(T_2;\mathfrak{a}_2) \\
&= \big((T_1;\mathfrak{a}_1) \otimes_{u_1} (\bullet;\mathbf{1})\big)\overline{\diamond}^r(T_2;\mathfrak{a}_2) \\
&= (T_1;\mathfrak{a}_1) \otimes_{u_1} \big((\bullet;\mathbf{1})\overline{\diamond}^r(T_2;\mathfrak{a}_2)\big) \\
&= (T_1;\mathfrak{a}_1) \otimes_{u_1} (T_2;\mathfrak{a}_2).
\end{aligned}$$

If $T_1 = \bullet$ and $T_2 \neq \bullet$, then $(T_1;\mathfrak{a}_1) = c(\bullet;\mathbf{1})$ with $c \in \mathbf{k}$. Then since $(\bullet;\mathbf{1})$ is the identity, we have

$$\begin{aligned}
(T_1;\mathfrak{a}_1)\overline{\diamond}^r(\bullet \sqcup \bullet; u_1)\overline{\diamond}^r(T_2;\mathfrak{a}_2) &= c(\bullet \sqcup \bullet; u_1)\overline{\diamond}^r(T_2;\mathfrak{a}_2) \\
&= c\big((\bullet;\mathbf{1}) \otimes_{u_1} (\bullet;\mathbf{1})\big)\overline{\diamond}^r(T_2;\mathfrak{a}_2) \\
&= c\big(\bullet;\mathbf{1}) \otimes_{u_1} \big((\bullet;\mathbf{1})\overline{\diamond}^r(T_2;\mathfrak{a}_2)\big) \\
&= c\big(\bullet;\mathbf{1}) \otimes_{u_1} (T_2;\mathfrak{a}_2) \\
&= (T_1;\mathfrak{a}_1) \otimes_{u_1} (T_2;\mathfrak{a}_2).
\end{aligned}$$

The same proof works for $T_1 \neq \bullet$ and $T_2 = \bullet$. Finally, if $T_1 = T_2 = \bullet$, so $(T_i;\mathfrak{a}_i) = (\bullet; c_i)$ with $c_i \in k$, $i = 1,2$. Then since $(\bullet;\mathbf{1})$ is the identity, we have

$$(T_1;\mathfrak{a}_1)\overline{\diamond}^r(\bullet \sqcup \bullet; u_1)\overline{\diamond}^r(T_2;\mathfrak{a}_2) = c_1c_2(\bullet \sqcup \bullet; u_1) = c_1c_2(\bullet;\mathbf{1}) \otimes_{u_1} (\bullet;\mathbf{1}) = (T_1;\mathfrak{a}_1) \otimes_{u_1} (T_2;\mathfrak{a}_2).$$

Now let $n \geq 2$ and assume that the lemma has been proved for $(F;\mathfrak{a}) \in (\mathcal{F}^r;\check{A})$ with $b(F) \leq n$. Consider $(F;\mathfrak{a}) \in (\mathcal{F}^r;\check{A})$ with $b(F) = n+1$. Then we have

$$\begin{aligned}
(F;\mathfrak{a}) &= (T_1;\mathfrak{a}_1) \otimes_{u_1} \cdots \otimes_{u_{n-1}} (T_n;\mathfrak{a}_n) \otimes_{u_n} (T_{n+1};\mathfrak{a}_{n+1}) \\
&= \big((T_1;\mathfrak{a}_1)\overline{\diamond}^r_{u_1}\cdots\overline{\diamond}^r_{u_{n-1}}(T_n;\mathfrak{a}_n)\big) \otimes_{u_n} (T_{n+1};\mathfrak{a}_{n+1}) \\
&= \big((T_1;\mathfrak{a}_1)\overline{\diamond}^r_{u_1}\cdots\overline{\diamond}^r_{u_{n-2}}(T_{n-1};\mathfrak{a}_{n-1})\big)\overline{\diamond}^r_{u_{n-1}}\big((T_n;\mathfrak{a}_n) \otimes_{u_n} (T_{n+1};\mathfrak{a}_{n+1})\big)
\end{aligned}$$

$$\begin{aligned} &= \big((T_1;\mathfrak{a}_1)\overline{\diamond}^r_{u_1}\cdots\overline{\diamond}^r_{u_{n-2}}(T_{n-1};\mathfrak{a}_{n-1})\big)\overline{\diamond}^r_{u_{n-1}}\big((T_n;\mathfrak{a}_n)\overline{\diamond}^r_{u_n}(T_{n+1};\mathfrak{a}_{n+1})\big)\\ &= (T_1;\mathfrak{a}_1)\overline{\diamond}^r_{u_1}\cdots\overline{\diamond}^r_{u_{n-1}}(T_n;\mathfrak{a}_n)\overline{\diamond}^r_{u_n}(T_{n+1};\mathfrak{a}_{n+1}).\end{aligned}$$

Here the second equation follows from the induction hypothesis, the third equation follows from Eq. (72) – (74), the fourth equation follows from the case when $b=2$ and the fifth equation follows from the associativity of $\overline{\diamond}^r$. This completes the induction. □

Now the uniqueness of $\bar{f}$ is clear since if $\bar{f}':ш^{\rm NC}(A)\to R$ were another unitary Rota–Baxter algebra homomorphism extending f, that is, $\bar{f}'\circ j_A=f$. Then since $\bar{f}'$ is unitary, Eq. (80) and hence (79) must be satisfied. Since $\bar{f}'\circ j_A=f$, Eq. (81) must be satisfied. Since $\bar{f}'$ is multiplicative, by Lemma 6.6 we see that Eq. (83) must be satisfied. Since $\bar{f}'$ preserves the Rota–Baxter operators, Eq. (82) must be satisfied. Thus we have $\bar{f}'=\bar{f}$.

We have therefore completed the proof of Theorem 6.3.

6.4. **Free nonunitary Rota–Baxter algebras over an algebra.** We now modify the construction of the free unitary Rota–Baxter algebras over a unitary algebra in Section 6.2 to obtain the free nonunitary Rota–Baxter algebra over a nonunitary algebra. We will just give a sketch of the construction since they are quite similar.

6.4.1. *Angularly decorated ladder-free forests.* Similar to Corollary 5.2, we let $\mathcal{F}^{r,0}$ be the subset of $\mathcal{F}^r\backslash\{\bullet\}$ consisting of forests that do not contain $\lfloor\bullet\rfloor$. Further, for any nonunitary algebra $\check{A}$, define

$$(\mathcal{F}^{r,0};\check{A})=\bigcup_{F\in\mathcal{F}^{r,0}}(F;\check{A})$$

to be the set of angularly decorated forests from $\mathcal{F}^{r,0}$ with decoration set $\check{A}$. Thus $(\mathcal{F}^{r,0};\check{A})$ consists of pairs $(F;\mathfrak{a})$ where F is in $\mathcal{F}^{r,0}$ and $\mathfrak{a}$ is in $A^{\ell(F)-1}$ where $\ell(F)$ is the number of leafs of F.

We define $ш^{\rm NC,\,0}(\check{A})$ to be the $\mathbf{k}$-module $\bigoplus_{F\in\mathcal{F}^{r,0}}(F;\check{A})$. Then it is a submodule of $ш^{\rm NC}(A)$. Here $A=\mathbf{k}\oplus\check{A}$ is defined to be the unitarization of $\check{A}$. We define a product $\overline{\diamond}^r$ on $ш^{\rm NC,\,0}(A)$ to be the restriction of $\overline{\diamond}^r$ on $ш^{\rm NC}(A)$. For $D=(F;\mathfrak{a})$ and $D'=(F';\mathfrak{a}')$ in $(\mathcal{F}^{r,0};\check{A})$, by Corollary 2.6, $F\diamond^r F'$ is still in $\mathbf{k}\,\mathcal{F}^{r,0}$. Further, since now $\check{A}$ is closed under multiplication, in the decomposition $xy=c+v$ of $x,y\in\check{A}$ into $c\in\mathbf{k}$ and $v\in\check{A}$, we have $c=0$. Thus in the recursive definition (69) – (74) of $D\overline{\diamond}^r D'$ is a linear combination of elements from $(\mathcal{F}^{r,0},\check{A})$ and hence is in $ш^{\rm NC,\,0}(A)$.

Also define $P_{\check{A}}:=\lfloor\ \rfloor:ш^{\rm NC,\,0}(\check{A})\to ш^{\rm NC,\,0}(\check{A})$ to be the restriction of $P_A=\lfloor\ \rfloor$ on $ш^{\rm NC}(A)$. This is well-defined since by Corollary 5.2, we have $\lfloor\mathcal{F}^{r,0}\rfloor\subseteq\mathcal{F}^{r,0}$. Let $j_{\check{A}}:\check{A}\to ш^{\rm NC,\,0}(\check{A})$ to be the restriction to $\check{A}$ of $j_A:A\to ш^{\rm NC}(A)$ in Eq. (75). Then adapting the proof of Theorem 6.3, we obtain

Theorem 6.7. *Let $\check{A}$ be a nonunitary $\mathbf{k}$-algebra.*

(a) *The pair $(ш^{\rm NC,\,0}(\check{A}),\overline{\diamond}^r)$ is a nonunitary associative algebra.*
(b) *The quadruple $(ш^{\rm NC,\,0}(\check{A}),\overline{\diamond}^r,P_{\check{A}},j_{\check{A}})$ is the free nonunitary Rota–Baxter algebra of weight λ on the algebra $\check{A}$.*

7. Unitarization of Rota–Baxter algebras

The unitarization process of associative algebras is simple and well-known. For any nonunitary algebra A (even if A does have an identity), define $\tilde{A} := \mathbf{k} \oplus A$ with component wise addition and with product defined by

$$(a, x)(b, y) = (ab, ay + bx + xy).$$

Then $\tilde{A}$ is a unitary algebra with identity $(\mathbf{1}, 0)$ and a natural embedding

$$u_A : A \to \tilde{A},\ x \mapsto (0, x).$$

Further $\tilde{A}$ is the unitarization of A, characterized by the property that, for any unitary algebra B and nonunitary algebra homomorphism $f : A \to B$, there is a unique unitary algebra homomorphism $\tilde{f} : \tilde{A} \to B$ such that $f = \tilde{f} \circ u_A$. To generalize this process to Rota–Baxter algebras turns out to be much more involved since, after formally adding a unit $\mathbf{1}$ to a nonunitary Rota–Baxter algebra (A, P), we also need to add its images under the Rota–Baxter operator P and its iterations, such as $P(\mathbf{1}), P(xP(\mathbf{1}))$, etc. Then it is not clear in general how these new elements should behave to form a Rota–Baxter algebra, except possibly in special cases (see Proposition 7.4 below). We will start with the unitarization of free Rota–Baxter algebras and then take care of the case of a general Rota–Baxter algebra by regarding it as a quotient of a free Rota–Baxter algebra. Let us first give the definition.

Definition 7.1. Fix a weight λ in the base ring $\mathbf{k}$. Let (A, P) be a nonunitary Rota–Baxter $\mathbf{k}$-algebra. A unitarization of A is a unitary Rota–Baxter algebra $(\tilde{A}, \tilde{P})$ with a nonunitary Rota-Baxter algebra homomorphism $u_A : A \to \tilde{A}$ such that for any unitary Rota–Baxter algebra B and a homomorphism $f : A \to B$ of nonunitary Rota–Baxter algebras, there is a unique homomorphism $\tilde{f} : \tilde{A} \to B$ of unitary Rota-Baxter algebras such that $f = \tilde{f} \circ u_A$.

7.1. Unitarization of free Rota–Baxter algebras.

Let X be a set. Let $Ш^{\mathrm{NC}}(X)$ and $Ш^{\mathrm{NC},0}(X)$ be the free unitary and nonunitary Rota–Baxter algebras in Theorem 6.3 and Theorem 6.7. Let $\tilde{j}_X : X \to Ш^{\mathrm{NC}}(X)$ and $j_X : X \to Ш^{\mathrm{NC},0}(X)$ be the canonical embeddings. Regarding $Ш^{\mathrm{NC}}(X)$ as a nonunitary Rota–Baxter algebra, then by the universal property of the free nonunitary Rota–Baxter algebra $Ш^{\mathrm{NC},0}(X)$, there is a unique homomorphism $u_X : Ш^{\mathrm{NC},0}(X) \to Ш^{\mathrm{NC}}(X)$ of nonunitary Rota–Baxter algebras such that $\tilde{j}_X = u_X \circ j_X$.

Theorem 7.2. *For any set X, the free unitary Rota–Baxter algebra $Ш^{\mathrm{NC}}(X)$, with the nonunitary Rota–Baxter algebra homomorphism $u_X : Ш^{\mathrm{NC},0}(X) \to Ш^{\mathrm{NC}}(X)$, is the unitarization of the free nonunitary Rota–Baxter algebra $Ш^{\mathrm{NC},0}(X)$.*

Proof. Let (B, Q) be a unitary Rota–Baxter algebra and let $f : Ш^{\mathrm{NC},0}(X) \to B$ be a homomorphism of nonunitary Rota–Baxter algebras. Let $f' = f \circ j_X : X \to B$, then by the freeness of the unitary Rota–Baxter algebra $Ш^{\mathrm{NC}}(X)$, there is a unique homomorphism $\bar{f}' : Ш^{\mathrm{NC}}(X) \to B$ of unitary Rota–Baxter algebras such that $f' = \bar{f}' \circ \tilde{j}_X$.

$$\begin{tikzcd}
X \arrow[rr, "j_X"] \arrow[dd, "\tilde{j}_X"'] \arrow[ddrr, "f'"] & & \text{Ш}^{\mathrm{N}C,0}(X) \arrow[dd, "f"] \arrow[ddll, "u_X"'] \\
& & \\
\text{Ш}^{\mathrm{N}C}(X) \arrow[rr, "\bar{f}'"] \arrow[rr, bend right, "g"'] & & B
\end{tikzcd}$$

We have

$$\bar{f}' \circ u_X \circ j_X = \bar{f}' \circ \tilde{j}_X = f' = f \circ j_X.$$

By the freeness of $\text{Ш}^{\mathrm{N}C,0}(X)$, we have $\bar{f}' \circ u_X = f$. Suppose there is another unitary Rota–Baxter algebra homomorphism $g : \text{Ш}^{\mathrm{N}C}(X) \to B$ such that $g \circ u_X = f$. Then

$$g \circ \tilde{j}_X = g \circ u_X \circ j_X = f \circ j_X = f' = \tilde{f}' \circ \tilde{j}_X.$$

So $g = \tilde{f}'$ by the universal property of the free unitary Rota–Baxter algebra $\text{Ш}^{\mathrm{N}C}(X)$. $\square$

7.2. **Unitarization of Rota–Baxter algebras.** We now construct the unitarization of any given nonunitary Rota–Baxter algebra A. We use the following diagram to keep track of the maps that we will introduced below.

(88)
$$\begin{tikzcd}
J \arrow[d, "\mathrm{incl}"'] & & X \arrow[dll, "j_X"'] \arrow[drr, "\tilde{j}_X"] \arrow[dddll, "g"] \arrow[dddrr, "\tilde{g}"'] & & \tilde{J} \arrow[d, "\mathrm{incl}"] \\
\text{Ш}^{\mathrm{N}C,0}(X) \arrow[rrrr, "u_X"] \arrow[drr, "h"] \arrow[dd, "\bar{g}"'] & & & & \text{Ш}^{\mathrm{N}C}(X) \arrow[dll, "\tilde{h}"'] \arrow[dd, "\bar{\tilde{g}}"] \\
& & B & & \\
A \cong \text{Ш}^{\mathrm{N}C,0}(X)/J \arrow[rrrr, "u_A"] \arrow[urr, "f"'] & & & & \tilde{A} = \text{Ш}^{\mathrm{N}C}(X)/\tilde{J} \arrow[ull, bend right, "\tilde{f}"'] \arrow[ull, bend left, "\tilde{f}'"]
\end{tikzcd}$$

Let X be a generating set of A as a nonunitary Rota–Baxter algebra with $g : X \hookrightarrow A$ being the inclusion map. Let $\text{Ш}^{\mathrm{N}C,0}(X)$ be the free nonunitary Rota–Baxter algebra over X with the canonical embedding $j_X : X \to \text{Ш}^{\mathrm{N}C,0}(X)$. Then there is a unique nonunitary Rota–Baxter algebra homomorphism $\bar{g} : \text{Ш}^{\mathrm{N}C,0}(X) \to A$ such that $g = \bar{g} \circ j_X$. Since X is a generating set of A, $\bar{g}$ is surjective. So $A \cong \text{Ш}^{\mathrm{N}C,0}(X)/J$ where J is the kernel of $\bar{g}$ and is a Rota–Baxter ideal of $\text{Ш}^{\mathrm{N}C,0}(X)$. Recall from Theorem 7.2 that we have the unitarization $u_X : \text{Ш}^{\mathrm{N}C,0}(X) \to \text{Ш}^{\mathrm{N}C}(X)$. Let $\tilde{J}$ be the Rota–Baxter ideal of $\text{Ш}^{\mathrm{N}C}(X)$ generated by $u_X(J)$, and define

$$\tilde{A} = \text{Ш}^{\mathrm{N}C}(X)/\tilde{J}$$

with $\bar{\tilde{g}} : Ш^{NC}(X) \to \tilde{A}$ being the quotient Rota–Baxter homomorphism. Let $\tilde{g} = \bar{\tilde{g}} \circ \tilde{j}_X$. Then $\bar{\tilde{g}} : Ш^{NC}(X) \to \tilde{A}$ is the unique unitary Rota–Baxter algebra homomorphism induced from the set map $\tilde{g}$. So the notation $\bar{\tilde{g}}$ is justified.

Now since $u_X(J) \subseteq \tilde{J}$, we have $(\bar{\tilde{g}} \circ u_X)(J) = 0$. Thus $\ker(\bar{\tilde{g}} \circ u_X) \supseteq J$. Therefore, there is a unique homomorphism

$$u_A : A \cong Ш^{NC,0}(X)/J \to \tilde{A} \cong Ш^{NC}(X)/\tilde{J}$$

of nonunitary Rota–Baxter algebras such that

$$u_A \circ \bar{g} = \bar{\tilde{g}} \circ u_X.$$

Theorem 7.3. *With the above notation, the nonunitary Rota–Baxter algebra homomorphism*

$$u_A : A \to \tilde{A}$$

gives the unitarization of A.

By the uniqueness of Rota–Baxter unitarization, for a different choices of the generating set X of A, the unitarization we obtain are isomorphic.

Proof. Let B be a unitary Rota–Baxter algebra and let let $f : A \to B$ be a nonunitary Rota–Baxter algebra homomorphism. Let $h = f \circ \bar{g}$. By Theorem 7.2, there is a unique unitary Rota–Baxter algebra homomorphism $\tilde{h} : Ш^{NC}(X) \to B$ such that $\tilde{h} \circ u_X = h$. Then

$$\ker \tilde{h} \supseteq u_X(\ker h) \supseteq u_X(\ker \bar{g}) = J.$$

Since $\tilde{h}$ is a Rota–Baxter ideal of $Ш^{NC}(X)$ and $\tilde{J}$ is the Rota–Baxter ideal of $Ш^{NC}(X)$ generated by J, we must have $\ker \tilde{h} \supseteq \tilde{J}$. Therefore, there is a unique

$$\tilde{f} : \tilde{A} \to B$$

such that $\tilde{h} = \bar{\tilde{g}} \circ \tilde{f}$. Now

$$\tilde{f} \circ u_A \circ \bar{g} = \tilde{f} \circ \bar{\tilde{g}} \circ u_X = \tilde{h} \circ u_X = h = f \circ \bar{g}.$$

Since $\bar{g}$ is surjective, we have $\tilde{f} \circ u_A = f$. So the existence of $\tilde{f}$ in Definition 7.1 is proved.

To prove the uniqueness of $\tilde{f}$, suppose there is also a unitary Rota–Baxter algebra homomorphism $\tilde{f}' : \tilde{A} \to B$ such that $\tilde{f}' \circ u_A = f$. Then we have

$$\begin{aligned} \tilde{f}' \circ \bar{\tilde{g}} \circ u_X &= \tilde{f}' \circ u_A \circ \bar{g} \\ &= f \circ \bar{g} \\ &= \tilde{f} \circ u_A \circ \bar{g} \\ &= \tilde{f} \circ \bar{\tilde{g}} \circ u_X \\ &= \tilde{h} \circ u_X \\ &= h. \end{aligned}$$

So $\tilde{f}' \circ \bar{\tilde{g}} : Ш^{NC,0}(X) \to B$, as well as $\tilde{h}$ is the unitarization of $h : Ш^{NC,0}(X) \to B$. By the uniqueness of this unitarization, proved in Theorem 7.2, we have

$$\tilde{f}' \circ \bar{\tilde{g}} = \tilde{h} = \tilde{f} \circ \bar{\tilde{g}}.$$

Since $\bar{\tilde{g}}$ is surjective, we have $\tilde{f}' = \tilde{f}$, as needed. $\square$

7.3. **Unitarization with idempotent Rota–Baxter operators.** We end our discussion on unitariness of Rota–Baxter algebras with an simple case.

Proposition 7.4. *Let a (R, P) be Rota–Baxter algebra of weight λ such that $P^2 = -\lambda P$. The unitarization $\tilde{R} := \mathbf{k1} \oplus R$ of R together with the extension of P to $\tilde{P} : \tilde{R} \to \tilde{R}$,*

$$\tilde{P}(m, a) := \big(-\lambda m, P(a)\big), \ \ \forall m \in \mathbf{k}, \ a \in R,$$

forms a unitary Rota–Baxter $\mathbf{k}$*-algebra of weight λ such that $\tilde{P}^2 = -\lambda\tilde{P}$.*

Other results on such Rota–Baxter operators can be found in [6] where they are called pseudo-idempotent.

Proof. We first show that $\tilde{P} : \tilde{R} \to \tilde{R}$ satisfies the Rota–Baxter relation of weight λ

$$\tilde{P}(m, a)\tilde{P}(n, b) = \tilde{P}\big((m, a)\tilde{P}(n, b)\big) + \tilde{P}\big(\tilde{P}(m, a)(n, b)\big) + \lambda\tilde{P}\big((m, a)(n, b)\big) \tag{89}$$

for $(m, a), (n, b) \in \tilde{R}$. For the left hand side, we have

$$\begin{aligned}
\tilde{P}(m, a)\tilde{P}(n, b) &= \big(-\lambda m, P(a)\big)\big(-\lambda n, P(b)\big) \\
&= \big(\lambda^2 mn, -\lambda m P(b) - \lambda n P(a) + P(a)P(b)\big) \\
&= \big(\lambda^2 mn, -\lambda m P(b) - \lambda n P(a) + P(aP(b)) + P(P(a)b) + \lambda P(ab)\big).
\end{aligned}$$

For the right hand side, we have

$$\begin{aligned}
\tilde{P}\big((m, a)\tilde{P}(n, b)\big) &= \tilde{P}\big((m, a)(-\lambda n, P(b))\big) \\
&= \tilde{P}(-\lambda mn, mP(b) - \lambda na + aP(b)) \\
&= \big(\lambda^2 mn, mP^2(b) - \lambda n P(a) + P(aP(b))\big) \\
&= \big(\lambda^2 mn, -\lambda m P(b) - \lambda n P(a) + P(aP(b))\big),
\end{aligned}$$

where we used idempotency of P in the third equality. For the other terms we similarly find

$$\begin{aligned}
\tilde{P}\big(\tilde{P}(m, a)(n, b)\big) &= \tilde{P}\big((-\lambda m, P(a))(n, b)\big) \\
&= \big(\lambda^2 mn, -\lambda m P(b) - \lambda n P(a) + P(P(a)b)\big)
\end{aligned}$$

$$\begin{aligned}
\tilde{P}\big((m, a)(n, b)\big) &= \tilde{P}\big(mn, na + mb + ab\big) \\
&= \big(-\lambda mn, mP(b) + nP(a) + P(ab)\big).
\end{aligned}$$

From these equations, Eq. (89) is immediately verified. Note that the addition in $\tilde{R} = \mathbf{k} \oplus R$ is defined componentwise.

Finally,

$$\tilde{P}^2(m, a) = \tilde{P}(-\lambda m; P(a)) = ((-\lambda)^2 m; P^2(a)) = (\lambda^2 m; -\lambda P(a)) = -\lambda\tilde{P}(m, a).$$

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