QUANTUM COHOMOLOGY AND MORSE THEORY ON THE LOOP SPACE OF TORIC VARIETIES

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ABSTRACT. On a symplectic manifold X, the quantum product defines a complex, one parameter family of flat connections called the A-model or Dubrovin connections. Let $\hbar$ denote the parameter. Associated to them, is the quantum $\mathcal{D}$-module $\mathcal{D}/I$ over the Heisenberg algebra of first order differential operators on a complex torus. An element of I, gives a relation in the quantum cohomology of M, by taking the limit as $\hbar \to 0$. Givental [7], discovered that there should be a structure of a $\mathcal{D}$-module on the S^1-equivariant Floer cohomology of the universal covering $\widetilde{\mathcal{L}X}$ of the loop space $\mathcal{L}X$ of X and conjectured that the two modules should be equal. A Duistermaat-Heckman type integral over semi-infinite cycles in $\widetilde{\mathcal{L}X}$, plays formally the role of Fourier transform between S^1-equivariant cohomology of $\widetilde{\mathcal{L}X}$ and differential operators. We attempt to compute this integral by localization. We conjecture that localization holds, as long as we do the following two things: first, fixed components are found by considering maps from curves with at least one marked point, since a marked point should be thought of as a trivial circle, shifted by the deck transformation corresponding to the degree of the curve. Secondly, the a priori infinite dimensional, normal bundle to fixed components, is replaced by an index bundle of the derivative of the deck transformation on $\widetilde{\mathcal{L}X}$. The conjecture is proven, for toric manifolds with $\int_d c_1 > 0$, for all nonzero classes d of rational curves in X, where the linear σ- model compactification is used. In that case, the function generating the quantum $\mathcal{D}$-module, is identified as the generating function of S^1-equivariant Eüler classes of index bundles, of the deck transformations on $\widetilde{\mathcal{L}X}$, corresponding to classes of rational curves in X. A relation with the Morse theory of the symplectic action functional, is also revealed. For the general case, we use the space of stable maps and the computation of the derivative of the deck transformation, is based on the description of vector bundles on Riemann surfaces with a marked point, in terms of the semi-infinite Grassmannian. An explicit conjecture is made, about how to handle this case.

1. INTRODUCTION

In this paper we will study the quantum cohomology and more generally the quantum $\mathcal{D}$-module structure, of symplectic manifolds by relating it to a certain Duistermaat-Heckmann type integral, over a semi-infinite cycle in the loop space of the manifold. A relation to Morse theory of the unperturbed symplectic action functional on the loop space, is also revealed.

This program was initiated by Givental in [7] and provided the inspiration for the methodology applied later in [8] in the context of Kontsevich's space of stable maps.

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In order to describe the main theorem and conjecture, let us briefly recall a few things about quantum cohomology. We follow mainly Givental [8] in this introductory exposition.

Let (X,ω) be a symplectic manifold and choose a compatible almost complex structure J. Let also $\mathcal{C}$ denote a genus g Riemann surface. A pseudo-holomorphic curve, is a map $f:\mathcal{C}\to X$ whose derivative is complex linear. Unless otherwise specified, from now on by a pseudo-holomorphic curve we will always mean one that has genus 0. Let $d\in H_2(X,\mathbb{Z})$ be a homology class. Kontsevich [10], invented the correct space parameterizing pseudo-holomorphic curves in X with k marked (smooth) points. It is called the space of stable maps and it will be denoted it by $M_k(X,d)$. To obtain a compactification $\overline{M}_k(X,d)$, we must also allow reducible curves with at worst double point singularities. In that case, by a map, we mean of course a collection of pseudo-holomorphic maps, each defined on one of the irreducible components of the domain, agreeing on the double points.

Let us consider $\overline{M}_3(X,d)$. Its elements are equivalence classes of 4-tuples, $(f,x_1.x_2,x_3)$ where f is the map and the x_i 's are the marked points. The 4-tuple must satisfy the stability condition that it has at most a discrete group of automorphisms. Two 4-tuples are equivalent, if there is an biholomorphism that takes one to the other.

The space $\overline{M}_3(X,d)$ is at worst an orbifold (Kontsevich [10]), if X is convex (i.e., $H^1(\mathcal{C},f^*TX)=0$ for all stable f). Main examples of convex spaces are homogeneous spaces. Moreover $\overline{M}_3(X,d)$ comes equipped with three evaluation maps $ev_i:\overline{M}_3(X,d)\mapsto X$ for $i=1,2,3$ given by $ev_i(f,x_1.x_2,x_3)=f(x_i)$. Let a,b be classes in $H^{2*}(X,\mathbb{C})$. Let $(\ ,\ )$ denote the intersection pairing. Let also $p_1,\dots p_r$ be classes in the Kähler cone $\mathcal{K}$ of X, which form a basis of $H^2(X,\mathbb{Z})$. The Kähler cone is the cone in $H^2(X,\mathbb{R})$ which consists of all classes whose integral over any (pseudo-) holomorphic curve is non-negative. Finally let $d_i=\int_d p_i$ and $q_i=e^{t_i}$ be complex variables (which can be thought of as coordinates on a complex torus). The quantum product $a*b$ is defined by the property that

$$(a*b,c)=\sum_d q^d\int_{\overline{M}_3(X,d)} {ev_1}^*(a)\wedge {ev_2}^*(b)\wedge {ev_3}^*(c),$$

where $q^d=\prod_{i=1}^r {q_i}^{d_i}$ and the sum is over all homology classes d of pseudo-holomorphic curves. The number $(a*b,c)_d=\int_{\overline{M}_3(X,d)} {ev_1}^*(a)\wedge {ev_2}^*(b)\wedge {ev_3}^*(c)$, is called a Gromov-Witten invariant (of the symplectic structure). It should be thought of geometrically, as counting the number of curves in homology class d, meeting classes dual to a, b and c, when the number of such curves is finite. Otherwise it is 0. The (small) quantum cohomology ring of X, is the ring $SQH^*(X)=H^{2*}(X,\mathbb{Z})\bigotimes\mathbb{C}[[q_1,\dots,q_r]]$ equipped with the quantum product.

The quantum product is commutative and associative. This last property is highly nontrivial and makes for many interesting consequences by itself. It turns out that the associativity can be reformulated as the flatness of the following complex one parameter family of connections

$$\nabla_\hbar=\hbar d-\sum_{i=1}^r\frac{dq_i}{q_i}\wedge p_i*$$

acting on elements of $SQH^*(X)$, where $\hbar$ denotes the complex parameter. This is called the Dubrovin or A-model connection and as we'll see shortly, it is a more fundamental object than the quantum product. Givental in his remarkable paper [8] found a formula for flat sections of $\nabla_\hbar$. It's clear that for flat sections of $\nabla_\hbar$, quantum multiplication by p_i is translated to differentiation and therefore we may expect that relations in the quantum ring may be translated to differential equations. This was formulated explicitly by Givental in the following fashion: We may associate to $\nabla_\hbar$ a certain $\mathcal{D}$-module $\mathcal{D}/I$ over the algebra of Heisenberg differential operators. This has the property that if the operator $D(\hbar q_i \frac{\partial}{\partial q_i}, q_i, \hbar)$ is in the ideal I, then the relation $D(p_i *, q_i, 0) = 0$ holds in the quantum cohomology ring $SQH^*(X)$. Therefore the $\mathcal{D}$-module appears to be the real quantum object while the quantum ring arises as its "semi-classical approximation" when $\hbar \to 0$!

To describe the $\mathcal{D}$-module $\mathcal{D}/I$ we need to introduce a new ingredient. This is the line bundle L_1 over $M_2(X,d)$ which is the universal tangent line at the first marked point, i.e., the line bundle whose fiber over $[\mathcal{C},(x_1,x_2),f]$ is the tangent line to $\mathcal{C}$ at the first marked point. Denote by ${L_1}^*$ the dual to L_1, i.e. the universal cotangent line at the first marked point.

Let ψ_1 denote the first Chern class of ${L_1}^*$. Givental's result [8] is the following: Let G be the $H^{2*}(X,\mathbb{C})$ valued function defined as:

$$G(t_1,\dots,t_n,\hbar) = e^{(t_1p_1+\cdots+t_rp_r)/\hbar}(1+\sum_d q^d ev_{1*}(\frac{1}{\hbar-\psi_1})), \tag{1}$$

where d ranges over all non-zero homology classes of pseudo-holomorphic curves and $ev_1 : \overline{M}_2(X,d) \to X$ is evaluation at the first marked point. Then the ideal I is generated by all polynomial differential operators that annihilate the components of G.

The object of this paper is to *compute the quantum $\mathcal{D}$-module and specifically the function G, by integrating over semi-infinite cycles in the loop space.* As will be explained shortly, each component of G, is given by such an integral, which is a sort of *Fourier transform*, from the S^1-equivariant cohomology of the universal covering of the loop space, to the algebra of differential operators in the variables $t_1,\dots,t_r$. By fundamental semi-infinite cycle in the loop space, we mean the set of all loops which are boundaries of holomorphic discs in X. Constraining the center of the disc to be in a representative of a cycle in X, produces a semi-infinite cycle for any homology cycle in X.

The connection between pseudo-holomorphic curves and the loop space is well known and is of course due to the fact that the gradient flow of the action functional is by pseudo-holomorphic cylinders. This implies, that the semi-infinite cycles are Morse theoretic cycles for the action functional. The relation between the quantum $\mathcal{D}$-module and the loop space via the flow of the action functional, was first explained by Givental in [7].

Such semi-infinite cycles are of course infinite dimensional, even in the case where we bound the energy of the maps from discs and this is the main problem in working with them. On the other hand, one tool we do have, is the obvious S^1 symmetry that may allow localization to fixed components.

We attempt then, to compute the "Fourier transform" integrals by S^1 localization. The two main elements of our approach are: first, the computation of the fixed locus, involves maps from curves to X, with at least one marked point. The idea being, that a marked point is a trivial circle, shifted by the deck transformation corresponding to the degree of the curve. Secondly, the infinite dimensional normal *bundle to a fixed component, has to be replaced by a finite dimensional (possibly virtual) index bundle, of the derivative of the deck transformations.*

We show that this approach gives the right answer for the case of toric manifolds. In particular, we will explicitly construct $\widetilde{\mathcal{L}X}$ and compute the deck transformations, which turn out to be Fredholm maps. A relation with the Morse theory of the symplectic action functional is also revealed. Moreover, we will construct a sequence of finite dimensional spaces, approximating $\widetilde{\mathcal{L}X}$ and prove a version of the previous result in that case.

We also study the case of a general symplectic manifold X using the spaces of stable maps. We make an explicit conjecture about the computation in that case.

To explain in more detail, let us consider the space $\mathcal{L}X$ of free contractible loops in X. We can define the action functional H by

$$H(\gamma) = \int_{D_\gamma} \omega,$$

where γ is a contractible loop and D_γ a disc contracting it. It is multi-valued if there are homologically non-trivial spheres. To resolve the ambiguity we lift it to the covering space $\widetilde{\mathcal{L}X}$ of $\mathcal{L}X$, with covering group the group of spherical classes in X. Assume for simplicity that X is simply connected, then $H_2(X,\mathbb{Z})$ is generated by spherical classes. Now as was mentioned before, H has the remarkable property that its flow lines are pseudo-holomorphic cylinders. Moreover H is a Hamiltonian function with respect to the obvious circle action on $\widetilde{\mathcal{L}X}$ and the symplectic form induced from the symplectic form on X. The critical manifolds correspond to trivial loops and are copies of X, one for every degree $d \in H_2(X,\mathbb{Z})$, i.e., for every floor of the cover. Denote by X_0 the copy on which H has value 0 and by X_d its translation by d.

A formal application of the S^1 equivariant localization theorem suggests that the (Floer) S^1 equivariant cohomology of $\widetilde{\mathcal{L}X}$ should be simply $FH^*{}_{S^1}(\widetilde{\mathcal{L}X}) = H^*(X,\mathbb{C}[[\tilde{q},\tilde{q}^{-1}]])(\hbar)$ where $\mathbb{C}[[\tilde{q},\tilde{q}^{-1}]]$ is the group ring of the covering group.

Givental's observation is that $FH^*{}_{S^1}(\widetilde{\mathcal{L}X})$ bears the structure of a $\mathcal{D}$-module over the Heisenberg algebra of differential operators. This is shown by extending the classes $\{p_1,\ldots,p_r\}$ to equivariant classes $\{P_1,\ldots,P_r\}$ (see (29)). Then if we think of the P_k acting by multiplication and the $\tilde{q_k}^d$ by pullback it is easy to show (32) that

$$[P_j, \tilde{q_k}] = \delta_{j,k}\hbar\tilde{q_k}$$

Givental conjectures that this $\mathcal{D}$-module is the quantum $\mathcal{D}$-module.

Let now Δ denote the fundamental semi-infinite cycle, i.e. the set of elements in $\widetilde{\mathcal{L}X}$ which are restrictions to the boundary, of (pseudo-)holomorphic maps of the standard two-disc, to X. Let also $\check{\Delta}$ denote formally, the Poincare dual to Δ and tP denote $\sum_{i=1}^r t_iP_i$

Then consider formally, the integral:

$$\int_{\widetilde{\mathcal{L}X}} e^{\frac{tP}{\hbar}}\check{\Delta} = \int_{\Delta} e^{\frac{tP}{\hbar}}. \tag{2}$$

This integral is a sort of Fourier transform taking relations in the S^1-equivariant cohomology of $\widetilde{\mathcal{L}X}$ to differential operators in the t_i.

This should then be the component of G, corresponding to the fundamental class of X.

Our main task, is to try and make sense of this integral. We try to use localization in S^1-equivariant cohomology in $\widetilde{\mathcal{L}X}$ or in Δ and as already mentioned, one of the main ideas is to use a "renormalized" normal bundle to the fixed components. This renormalized bundle, turns out to be the index bundle of the derivative of the deck transformations in directions defined by (pseudo-) holomorphic maps of curves, to X. The maps of curves, with at least one marked point, are involved since they correspond to the fixed components of the circle action, based on the idea, that a marked point is a trivial circle.

The main results we prove are, first: if X is a positive toric manifold and $D\tilde{q}^d$ denotes the derivative of $\tilde{q}^d$, then there is the exact sequence of bundles, over X_0:

$$0 \to \mathcal{N}_{M_d(X)} X_0 \to {\mathcal{N}_0}^+ \to {\mathcal{N}_0}^+ \to 0, \tag{3}$$

where ${\mathcal{N}_0}^+$ denotes the (infinite dimensional) positive normal bundle to X_0 in $\widetilde{\mathcal{L}X}$, with respect to H. The second map is $pr_+ oD\tilde{q}^d$ where pr_+ is the orthogonal projection onto ${\mathcal{N}_0}^+$. Finally $M_d(X) \subset \widetilde{\mathcal{L}X}$, is the linear σ-model compactification of holomorphic maps of degree d to X and $\mathcal{N}_{M_d(X)} X_0$ is the normal bundle of X_0 in $M_d(X)$.

Secondly we have that:

$$G = e^{(t_1 p_1 + \cdots + t_r p_r)/\hbar} \sum_{d \in \check{\mathcal{K}}} q^d \frac{1}{e_{S^1}(Ind(pr_+ oD\tilde{q}^d|_{X_0}))}, \tag{4}$$

where $Ind(pr_+ oD\tilde{q}^d|_{X_0})$ denotes the index bundle defined on X_0, by the Fredholm bundle map $pr_+ oD\tilde{q}^d$, where $q_k = e^{t_k}$, $d_k = \int_d \omega_k$ and $q^d = {q_1}^{d_1} \dots {q_1}^{d_1}$.

This is theorem (4) in section (5).

The infinite dimensional space $\widetilde{\mathcal{L}X}$ is explicitly constructed in that section and $\tilde{q}^d$ is computed. Moreover, a version of this theorem where finite dimensional approximations to $\widetilde{\mathcal{L}X}$ are used instead, is also proven.

In the general case we show first that over $M_2(X, d)$, there is a sequence:

$$0 \to L_1 \oplus {T^v}_{ev_1} M_2(X, d) \to {ev_1}^* {\mathcal{N}_0}^+ \to {ev_1}^* {\mathcal{N}_0}^+ \to 0, \tag{5}$$

which is exact up to homotopy. Here ${T^v}_{ev_1} M_2(X, d)$ denotes the (potentially virtual) vertical tangent bundle, with respect to the map $ev_1 : M_2(X, d) \to X$ and L_1 denotes the line bundle over $M_2(X, d)$ which is the universal tangent line at the first marked point. Moreover, the second map, over a point in $M_2(X, d)$ corresponding to a (pseudo-)holomorphic map $u : \mathcal{C} \to X$, is the Fredholm operator defined as follows: consider the first marked point x_1 and a small disc on $\mathcal{C}$ with local coordinate z, around that point, such that $z(x_1) = \infty$. Identify the space of holomorphic sections of u^*TX over $\mathcal{C}$ minus the disc, with a closed subspace W of $\mathcal{H} = L^2(S^1, \mathbb{C}^n)$. Let $\mathcal{H}^+$ denote the subspace of $L^2(S^1, \mathbb{C}^n)$ corresponding to functions with non negative modes. Then $W = w(\mathcal{H}^+)$ for some Fredholm operator $w : \mathcal{H}^+ \to \mathcal{H}$. Let $pr_+ : \mathcal{H} \to z\mathcal{H}^+$ be the orthogonal projection. Finally, our Fredholm map is the composition: $pr_+ ow|_{z\mathcal{H}^+} : z\mathcal{H}^+ \to z\mathcal{H}^+$. It is independent of all choices, up to homotopy.

Note that if we did not restrict w to $z\mathcal{H}^+$, we would get sections of the full tangent bundle to $M_2(X, d)$ and not of the vertical tangent bundle.

Now, according to our result in the toric case, w should be interpreted as $D\tilde{q}^d$, in the direction defined by u and we will use this notation.

We then formulate the conjecture, that the exact sequence can be extended over all of $\overline{M}_2(X,d)$.

In case the conjecture is true, we would have that:

$$\frac{1}{e_{S^1}(L_1)} = \frac{e_{S^1}(T^v{}_{ev_1}\overline{M}_2(X,d))}{e_{S^1}(Ind(pr_+ oD\tilde{q}^d))}. \tag{6}$$

where e_{S^1} denotes the equivariant Eüler characteristic with respect to an S^1 action. The action here is of course trivial on the base of our bundles but non-trivial on the fiber. In the case of the index bundle it comes from rotating the S^1 that we have marked on our curve $\mathcal{C}$. On L_1 we have $e_{S^1}(L_1) = \hbar - \psi_1$.

This can be used in order to obtain a different and rather more illuminating conjectural expression for the function G, that as we know generates the quantum $\mathcal{D}$-module. We have, conjecturally, that:

$$G = e^{(t_1 p_1 + \cdots + t_r p_r)/\hbar} \sum_{d \in \check{\mathcal{K}}} q^d {ev_1}_* \frac{e_{S^1}(T^v{}_{ev_1}\overline{M}_2(X,d))}{e_{S^1}(Ind(pr_+ oD\tilde{q}^d))}. \tag{7}$$

This formula for G, can be interpreted as similar to the one from the toric case, where the integration has been pulled back from X to $\overline{M}_2(X,d)$ and $e_{S^1}(T^v{}_{ev_1}\overline{M}_2(X,d))$ is the Thom class, that allows us to pull back.

We will show in section (4), that the exact sequence over $M_2(X,d)$ follows immediately, from one over $M_3(X,d)$. This sequence is:

$$0 \to T^v{}_{ev_1} M_3(X,d) \to {ev_1}^* {\mathcal{N}_0}^+ \to {ev_1}^* {\mathcal{N}_0}^+ \to 0, \tag{8}$$

We will show that over $M_3(X,d)$ this sequence is exact up to homotopy.

Here $T^v{}_{ev_1} M_3(X,d)$ denotes the virtual vertical tangent bundle to $M_3(X,d)$ with respect to the map $ev_1 : M_3(X,d) \to X$.

We conjecture that this sequence can be extended over all of $\overline{M}_3(X,d)$.

The argument that allows us to go from (8) to (5) extends without change, in case (8) is extended over $\overline{M}_3(X,d)$, therefore extension of (8) implies extension of (5) over $\overline{M}_2(X,d)$.

The conjecture about extension of (8), will be the subject of a forthcoming paper, so we will not discuss it further here.

These are all explained in section (4).

The structure of the paper is as follows: In the next section we gather the elements of the theory of quantum cohomology that are needed and which are contained mainly in [8]. In the third section we explain the original idea of Givental [7] relating the quantum $\mathcal{D}$ - module and S^1 equivariant Floer homology of the loop space, via a sort of "Fourier" transform of semi-infinite cycles. Then our conjectures are presented on how to compute the "Fourier" transform of cycles arising in Givental's work. In the fifth section the toric case is treated.

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2. The Quantum $\mathcal{D}$-Module

We gather here the basic facts about the quantum $\mathcal{D}$-module that will be needed later on. Let X be a Kähler manifold and let $T_0, T_1, \ldots, T_r, \ldots, T_m$ be a basis of $H^{2*}(X,\mathbb{Z})$ where T_0 is the identity and $T_1, \ldots, T_r$ are classes in the Kähler cone $\mathcal{K}$ that generate $H^2(X,\mathbb{Z})$. Recall that $\mathcal{K}$ is the cone of classes whose integral over holomorphic curves is non-negative. Set $p_i = T_i$ for $i = 1, \ldots r$. The small quantum ring is defined to be the vector space

$$SQH^*(X) = H^{2*}(X,\mathbb{Z}) \bigotimes \mathbb{C}[[q_1, \ldots, q_r]]$$

equipped with the quantum product $*$ already defined in the introduction. That is, if $(\ ,\ )$ denotes the intersection pairing and $a, b, c \in H^{2*}(X,\mathbb{Z})$ we have

$$(a * b, c) = \sum_d q^d \int_{\overline{M}_3(X,d)} ev_1{}^*(a) \wedge ev_2{}^*(b) \wedge ev_3{}^*(c), \tag{9}$$

where $q^d = \prod_{i=1}^r q_i{}^{d_i}$, $q_i = e^{t_i}$, $d_i = \int_d p_i$ and the sum is over all homology classes $d \in H_2(X,\mathbb{Z})$ of holomorphic curves. $\overline{M}_3(X,d)$ is the Kontsevich space of stable maps with three marked points, whose image has genus 0 and degree d in $H_2(X,\mathbb{Z})$. $SQH^*(X)$ is graded if we assign cohomology classes their usual degree and declare $deg(q_1{}^{d_1} \ldots q_r{}^{d_r}) = 2\int_d c_1(TX)$. The reason for this grading is that $\int_{\overline{M}_3(X,d)} ev_1{}^*(a) \wedge ev_2{}^*(b) \wedge ev_3{}^*(c)$ is 0 unless the sum of the degrees of a, b, c is equal to the dimension of $\overline{M}_3(X,d)$ which, as computed in [10], is: $dim_{\mathbb{C}}\overline{M}_3(X,d) = dim_{\mathbb{C}}X + \int_d c_1(T_X)$. Recall that the quantum product is commutative and associative (see [8]).

Introduce now a one parameter family of connections with regular singular points, depending on the complex parameter $\hbar$ and defined by

$$\nabla_\hbar = \hbar d - \sum_{i=1}^r \frac{dq_i}{q_i} \wedge p_i * . \tag{10}$$

$\nabla_\hbar$ are called Dubrovin or A model connections, and act on power series in the q_i with coefficients in $H^{2*}(X,\mathbb{Z})$, in other words on elements of $SQH^*(X)$. As already mentioned, one reason for considering these connections is that flat sections, if they exist, will provide a passage from quantum product to differentiation and from relations in the quantum ring to differential equations. Moreover, motivation for introducing the connections comes from mirror symmetry. In fact in the Calabi-Yau case they are the counterpart of the Gauss-Manin connection corresponding to the mirror family. On the other hand, the reason for having a whole pencil of connections is best understood from the point of view of the loop space and will be explained in the next section.

Now in Givental [8] it is proven that:

Proposition 1. *The connection $\nabla_\hbar$ is flat for any value of $\hbar$.*

The fact that $\nabla_\hbar$ is flat means that we can find flat sections. That is, sections s such that $\nabla_\hbar s = 0$.

One of the remarkable results of [8] is the explicit computation of the flat sections. To describe them we introduce first the line bundle $L_1{}^*$ over $\overline{M}_2(X,d)$ which is the universal cotangent line at the first marked point, i.e., the line bundle whose fiber over $[\mathcal{C}, (x_1, x_2), f]$ is the cotangent line to $\mathcal{C}$ at the first marked point. Let ψ_1

denote the first Chern class of ${L_1}^*$. Choose now basis $T^0, \dots, T^m$ of $H^{2*}(X, \mathbb{Z})$ such that $(T^i, T_j) = \delta_{i,j}$. We still have that $T_0 = 1 \in H^0(X, \mathbb{Z})$ and that $p_i = T_i$ for $i = 1, \dots r$ where $p_i \in \mathcal{K}$ are chosen to be a basis of $H^2(X, \mathbb{Z})$. Givental's result [8](corollary 6.3) is:

Theorem 1. *The sections*

$$s_\beta = e^{plnq/\hbar} T_\beta + \sum_\alpha T^\alpha \sum_{d \in \check{\mathcal{K}}, d \neq 0} q^d < e^{plnq/\hbar} \frac{T_\beta}{\hbar - \psi_1}, T_\alpha >_d$$

for $\beta = 0, \dots m$ are flat and they provide a basis of the space of flat sections.

Here q^d is notation for ${q_1}^{d_1} \dots {q_r}^{d_r}$, $plnq$ is notation for $p_1 lnq_1 + \dots + p_r lnq_r$ and

$$< e^{plnq/\hbar} \frac{T_\beta}{\hbar - \psi_1}, T_\alpha >_d = \int_{\overline{M}_2(X,d)} \frac{{ev_1}^*(e^{plnq/\hbar} T_\beta)}{\hbar - \psi_1} \wedge {ev_2}^*(T_\alpha),$$

where $\overline{M}_2(X, d)$ is the space of genus 0 and degree d stable maps with two marked points. Finally $\check{\mathcal{K}}$ is the cone in $H_2(X, \mathbb{Z})$ consisting of classes of holomorphic, genus 0 curves. It is dual to the Kähler cone $\mathcal{K}$. Note also that the matrix:

$$s_{\alpha,\beta} = (s_\beta, T_\alpha) = (e^{plnq/\hbar} T_\beta, T_\alpha) + \sum_{d \in \check{\mathcal{K}}, d \neq 0} q^d < e^{plnq/\hbar} \frac{T_\beta}{\hbar - \psi_1}, T_\alpha >_d, \tag{11}$$

is the fundamental solution matrix of the flat section equation.

Let us now explain the relation of the A-connection to the small quantum ring. Let G be the following function with values in $H^{2*}(X, \mathbb{C})$:

$$G = e^{plnq/\hbar} (1 + \sum_{d \in \check{\mathcal{K}}, d \neq 0} q^d ev_{1*} (\frac{1}{\hbar - \psi_1})), \tag{12}$$

where $ev_1 : \overline{M}_2(X, d) \to X$ is evaluation at the first marked point. Then G has the property (and is determined by it): $(G, T_\beta) = (s_\beta, 1)$. Indeed,

$$(G, T_\beta) = (e^{plnq/\hbar}, T_\beta) + \sum_{d \in \check{\mathcal{K}}, d \neq 0} q^d \int_X e^{plnq/\hbar} ev_{1*} (\frac{1}{\hbar - \psi_1}) \wedge T_\beta =$$

$$= (e^{plnq/\hbar} T_\beta, 1) + \sum_{d \in \check{\mathcal{K}}, d \neq 0} q^d \int_{\overline{M}_2(X,d)} \frac{1}{\hbar - \psi_1} \wedge {ev_1}^*(e^{plnq/\hbar} T_\beta) = (s_\beta, 1)$$

Therefore we have that

$$G = \sum_\beta (s_\beta, 1) T^\beta. \tag{13}$$

Recall that $T_0 = 1$ and therefore the components of J form the first row of the solution matrix $(s_{\alpha,\beta})$.

The following proposition is due to Givental [8] (corollary 6.4):

Proposition 2. *Let $D(\hbar q_i \frac{\partial}{\partial q_i}, q_i, \hbar)$ be a polynomial differential operator that annihilates the components of G. Then the relation $D(p_i *, q_i, 0) = 0$ holds in $SQH^*(X)$.*

Let $\mathcal{D}$ denote the Heisenberg algebra of differential operators on holomorphic function on a torus with coordinates $q_i = e^{t_i}$. It is by definition generated by the operators $\hbar q_i \frac{\partial}{\partial q_i} = \hbar \frac{\partial}{\partial t_i}$ and multiplication by $q_i = e^{t_i}$. Let I be the ideal of all polynomial differential operators $D(\hbar q_i \frac{\partial}{\partial q_i}, q_i, \hbar)$ that annihilate the components of G.

Definition 1. *The $\mathcal{D}$ - module $\mathcal{D}/I$ is called the quantum cohomology $\mathcal{D}$ - module of X.*

The proposition above shows that the real quantum object is the $\mathcal{D}$ - module or equivalently the A model connection, while the quantum ring should be considered as the semi-classical limit where $\hbar \to 0$. Our objective is to compute the $\mathcal{D}$ - module in terms of the loop space of X. We shall turn to this next.

3. Equivariant Floer theory

Lets start first by considering the S^1-equivariant Floer homology of the unperturbed action functional H in the case of a general symplectic manifold which is not necessarily toric.

Let (X, ω) be a compact symplectic manifold. Let J be a compatible or calibrated almost structure on X. By this we mean that $\omega(v, Jv) \geq 0$ for all nonzero $v \in TX$ and $\omega(Jv, Jw) = \omega(v, w)$. The symplectic form ω along with J define an invariant metric g on TX by $g(v, w) = \omega(v, Jw)$. Let $\mathcal{L}X$ be the space of smooth maps $\gamma : S^1 \to X$ such that $\gamma(S^1)$ is contractible. We call $\mathcal{L}X$ the *loop space* of X. The loop space inherits a symplectic structure Ω and an almost complex structure which we shall denote also by J. To describe them lets first consider the tangent bundle $T\mathcal{L}X$. The tangent space of $\mathcal{L}X$ at a loop γ is $T_\gamma \mathcal{L}X = \Gamma(\gamma^* TX)$, where Γ denotes the space of sections. In other words an element of $T_\gamma \mathcal{L}X$ is a vector field along the loop γ. Consider now the Kähler cone $\mathcal{K} \subset H^2(X, \mathbb{R})$ of X. We have defined $\mathcal{K}$ to be the cone of classes in $H^2(X, \mathbb{R})$ whose integral over any pseudo-holomorphic curve is greater than or equal to zero. Assume that $\mathcal{K}$ is spanned by the classes of symplectic two forms $p_1, \ldots, p_r$. Let v and w be elements of $T_\gamma \mathcal{L}X$ then we define:

$$\mathcal{P}_k|_\gamma(v, w) = \int_{S^1} p_k(v(t), w(t))dt. \tag{14}$$

It is not hard to show that the $\mathcal{P}_k$ are also symplectic. Moreover J induces an almost complex structure by $(Jv)(t) = J(v(t))$. Finally $T_\gamma \mathcal{L}X$ becomes pre-Hilbert with the inner product

$$g_\gamma(v, w) = \Omega_\gamma(v, Jw), \tag{15}$$

where

$$\Omega_\gamma(v, w) = \int_{S^1} \omega(v(t), w(t))dt.$$

Introduce now *action functionals*

$$H_k(\gamma) = \int_{D_\gamma} P_k, \tag{16}$$

for $k = 1 \ldots r$ and

$$H(\gamma) = \int_{D_\gamma} \omega, \tag{17}$$

where D_γ is a disk contracting the loop γ. These are in general not well defined since different disks contracting the same loop will not have the same symplectic areas. The ambiguity in H_k is clearly given by the periods

$$\int_S p_k,$$

where S is a sphere obtained by gluing two different disks contracting γ, along their common boundary. The functions H_k become well defined only on the covering of $\mathcal{L}X$ with group of deck transformations the group of spherical periods of the symplectic forms $p_1, \ldots, p_r$. We shall denote this space by $\widetilde{\mathcal{L}X}$. We can describe $\widetilde{\mathcal{L}X}$ explicitly as equivalence classes of pairs (γ, g) where $\gamma : S^1 \to M$ is a loop and $g : D \to X$ is such that $g|_{\partial D} = \gamma$. Define $(\gamma, g_1) \sim (\gamma, g_1)$ if and only if $g_1\#(-g_2)$ represents a class $A \in H_2(X, \mathbb{Z})$ such that $\int_A P_k = 0$ for all $k = 1 \ldots r$. Observe that by definition $\widetilde{\mathcal{L}X}$ carries an action, denoted by $\cdot$ of the group Γ of spherical classes in $H_2(X, \mathbb{Z})$ such that $(A \cdot (\gamma, g))\#(\gamma, -g) = -A$ for all $A \in \Gamma$. Notice that we could have chosen a positive instead of a negative sign in the definition of the action of Γ. The reason for our choice will become apparent later (see footnote (3), page 14). Note also that since ω is a linear combination of $p_1, \ldots, p_r$ it follows that H is the same linear combination of $H_1, \ldots H_r$ and therefore also becomes a well defined function on $\widetilde{\mathcal{L}X}$. Now it is not hard to compute that

$$dH_k|_\gamma(v) = -\int_{S^1} p_k(\dot{\gamma}, v(t))dt, \tag{18}$$

where $\dot{\gamma}$ denotes the vector field tangent to γ.

Notice further that $\mathcal{L}X$ and therefore $\widetilde{\mathcal{L}X}$, support an obvious S^1 action $(e^{i\theta}, \gamma(e^{i\phi})) \mapsto \gamma(e^{i(\theta+\phi)})$. If we let Y denote the vector generating the Lie algebra of S^1 and $\underline{\mathbb{Y}}$ the induced vector field on $\widetilde{\mathcal{L}M}$ then we have

$$\underline{\mathbb{Y}}(\gamma) = \dot{\gamma}. \tag{19}$$

Equations (14),(18) and (19) reveal the fact that

$$i_{\underline{\mathbb{Y}}}\mathcal{P}_k = -dH_k, \tag{20}$$

for $k = 1 \ldots l$ and

$$i_{\underline{\mathbb{Y}}}\Omega = -dH. \tag{21}$$

In that case H_k is called a Hamiltonian function for $\mathcal{P}_k$ and H a Hamiltonian for Ω. $\underline{\mathbb{Y}}$ can be thought of as a symplectic gradient of H.

Consider now the flow of H. Let $u(s,t) : \mathbb{R} \times S^1 \to X$ be a flow line. Specifically this means that

$$\frac{\partial u}{\partial s} = \nabla H_{u_s(t)}, \tag{22}$$

where $u_s(t)$ is simply $u(s,t)$. On the other hand since Ω and J are compatible and the metric on $\widetilde{\mathcal{L}X}$ is given by (10) we have that

$$\nabla H = -J\underline{\mathbb{Y}}. \tag{23}$$

Equations (22) and (23) imply then

$$\frac{\partial u}{\partial s} = -J\underline{\mathbb{Y}}(u_s(t)) = -J\frac{\partial u}{\partial t}. \tag{24}$$

Therefore

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0. \tag{25}$$

In other words $u(s,t)$ is a pseudo-holomorphic map.

This is the key reason why quantum cohomology is related to the loop space.

Floer theory is Morse theory for the action functional H on $\widetilde{\mathcal{L}X}$. Notice that the critical manifolds are copies of X, one of them corresponding to trivial loops and the rest translations by the action of the group of deck transformations, i.e., the group of spherical classes in $H_2(X,\mathbb{Z})$. This is easy to see using for example (21) which identifies the critical manifolds as the fixed manifolds of the circle action.

Now the fact that $\widetilde{\mathcal{L}X}$ is infinite dimensional pauses several hard problems one needs to overcome in order to get a well defined theory. For example, for any critical manifold both the negative and positive normal bundles of H are infinite dimensional. Therefore the usual notion of index doesn't make sense. Moreover the standard Morse theoretic method of analyzing the topology of a space simply doesn't work. This is because we cannot describe the change in topology when going through a critical manifold, by a gluing of the negative normal sphere bundle since this is trivial! [1] It was Floer's idea to overcome this problem by constructing a Witten type Morse theory where the index is defined by counting orbits connecting critical manifolds. The key point to doing this in this case, is to use orbits of *bounded energy*. In other words if $u(s,t):\mathbb{R}\times S^1\to X$ is a flow line, i.e. satisfies (25), then define the energy of u by :

$$E(u)=\frac{1}{2}\int_0^1\int_{-\infty}^{\infty}|\frac{\partial u}{\partial s}|^2+|\frac{\partial u}{\partial t}|^2. \tag{26}$$

We say that u has bounded energy if $E(u)$ is finite. In fact it is easy to compute that when u is (pseudo-)holomorphic as is the case for flow lines, then

$$E(u)=\int_0^1\int_{-\infty}^{\infty}u^*\omega.$$

Still to get a well behaved theory we have to perturb the flow equation by an extra term using a periodic Hamiltonian. The critical manifolds then become points and the theory can be used to prove the well known Arnold conjecture for periodic Hamiltonians.

Floer [5] was able to rigorously construct a homology theory, now called Floer homology,[2] using these perturbed holomorphic cylinders connecting periodic orbits. He then showed that Floer homology is isomorphic, with respect to additive structure, to the singular homology of M with coefficients in an appropriate ring of Laurent series. It should be mentioned at this point that in [16], [12] and [14] it is proved (in each work with different methods) that in fact the isomorphism between Floer and singular homology, respects the ring structures if $H^{2*}(X,\mathbb{Z})$ is equipped with the *Quantum product*, and Floer cohomology with the so called, pair of pants product.

The unperturbed Morse-Bott-Floer theory has been worked out to a certain extent by Ruan and Tian in [16].

Following their paper, the space of connecting orbits between two critical levels that differ by a class $d\in H_2(X,\mathbb{Z})$ should be taken to consist of maps

$$u:\mathbb{R}\times S^1\to X$$

[1] The negative normal bundle is trivial since any infinite dimensional bundle is trivial. Moreover the infinite dimensional sphere is homotopicaly trivial.

[2] Floer constructed the theory and proved the Arnold conjecture for so called monotone manifolds (this means that the first Chern class of the manifold is a positive multiple of the symplectic form). For general symplectic manifolds the theory was constructed in [11].

that

1. are J- holomorphic
2.
$$E(u) = \int_{\mathbb{R}\times S^1} u^*\omega < \infty$$
3.
$$\lim_{s\to-\infty} u(s,t) = point$$

and

$$\lim_{s\to\infty} u(s,t) = point,$$

in other words the infinite cylinder closes up at the ends to give a sphere with two point removed.

4.The homology class of the image of u is d.

If we denote the set of such maps u by $\mathcal{M}_d$ then we expect this space to have the same dimension as the space of holomorphic spheres of degree d, the expected dimension of which, is

$$dim\ X + \int_d c_1(TX).$$

Indeed the calculation of Morrison [13](p. 277) shows that if $\phi : \mathbb{P}^1 \to X$ is holomorphic and X is a complex manifold then

$$\chi(\phi^*TX) = h^0(\mathbb{P}^1, \phi^*TX) - h^1(\mathbb{P}^1, \phi^*TX) = dim\ X + \int_d c_1(TX). \tag{27}$$

In any case we wish to consider Floer S^1 *equivariant cohomology* of $\widetilde{\mathcal{L}X}$ so following Givental [7], we bypass all that and try to use localization technics instead. The localization theorem relates the equivariant cohomology of a space with a torus acting on it, to that of the fixed components of the action. One way [1] of proving this theorem rests on an analysis of the $H^*(point) = H^*(\mathbb{P}^\infty) = \mathbb{C}[\hbar]$ - module structure of the equivariant cohomology ring. We refer to this paper or [2] for the more general statement which refers to a torus action. For our purposes we only need the S^1 case. The result [1] then is that:

Theorem 2. *Let M be an S^1 (finite dimensional) compact manifold. Let F denote the (possibly disconnected) fixed manifold of the action and let $i : F \to X$ be the inclusion map. Then*

$$i^* : H_{S_1}{}^*(M) \to H_{S^1}{}^*(F)$$

induces an isomorphism after localization to the field of rational functions $\mathbb{C}(\hbar)$.

Notice that since F is fixed it follows that $H_{S^1}{}^*(F) = H^*(F) \otimes \mathbb{C}[\hbar]$. So the meaning of this theorem is that, if the fixed manifolds are $\{F_\alpha\}$ then there is an isomorphism

$$\Phi : H_{S_1}{}^*(M) \to \bigoplus_\alpha H^*(F_\alpha, \mathbb{C}(\hbar))$$

and so

$$\Phi(a) = \sum_\alpha \lambda_\alpha C_\alpha$$

where $C_\alpha \in H^*(F_\alpha)$ and λ_α is a *rational* function in $\hbar$

We would like now to apply this to the space $\widetilde{\mathcal{L}X}$. Since it is infinite dimensional this is only a *formal* application, not a rigorous one.

With this qualification, since the fixed manifolds are copies of X, we expect, after Givental [7], the S^1 equivariant Floer cohomology of $\widetilde{\mathcal{L}X}$ to be as an additive object,

$$FH^*{}_{S^1}(\widetilde{\mathcal{L}X}) = H^*(X, \mathbb{C}[[\tilde{q}, \tilde{q}^{-1}]](\hbar)), \tag{28}$$

where $\mathbb{C}[[\tilde{q}, \tilde{q}^{-1}]]$ is notation for the group ring Λ of the group Γ of spherical classes in $H_2(X, \mathbb{Z})$. In other words, instead of using a direct sum notation, we have used the group ring to enumerate the fixed components of the action.

To be more specific, elements of the ring are formal series

$$\lambda = \sum_{d \in \Gamma} \lambda_d \tilde{q}^d$$

where $\lambda_d \in \mathbb{C}$, $\tilde{q}^d = e^{2\pi i d}$ and we declare $deg\ \tilde{q}^d = 2 \int_d c_1(TX)$.

Assume X is Kähler and simply connected. X being simply connected implies that $\Gamma = H_2(X, \mathbb{Z})$. We have choosen a basis $\{p_1, \dots, p_r\}$ of the Kähler cone, now let $\{A_1, \dots, A_r\}$ be the dual basis of $\Gamma = H_2(M, \mathbb{Z})$ in the sense that $\int_{A_j} p_k = \delta_{j,k}$. Then if $d = \sum_{i=1}^r d_i A_i$ we let $\tilde{q}_k = e^{2\pi i A_k}$ and $\tilde{q}^d = e^{2\pi i d} = \prod_{k=1}^r \tilde{q}_k{}^{d_k}$. In this fashion the group ring Λ can be identified with the ring of formal Laurent series $\Lambda = \mathbb{C}[[\tilde{q}_1, \tilde{q}_1^{-1}, \dots \tilde{q}_r, \tilde{q}_r^{-1}]]$.

Now recall we have defined associated kähler classes $\mathcal{P}_k$ on $\mathcal{L}X$. Denote by the same name the pullbacks on $\widetilde{\mathcal{L}X}$. Let $d_\hbar = d + \hbar i_{\underline{\mathbb{Y}}}$ be the Cartan differential, where $\underline{\mathbb{Y}}$ is defined by (19). Introduce now the equivariant differential forms

$$P_k = \mathcal{P}_k + \hbar H_k, \tag{29}$$

and

$$P = \Omega + \hbar H. \tag{30}$$

Then equations (20) and (21) imply that

$$d_\hbar P_k = d_\hbar P = 0$$

for $k = 1 \dots r$.

By definition of the A_j we have:

$$\int_{A_j} \omega_k = \delta_{j,k}.$$

Recall we have defined

$$H_k(\gamma, g) = \int_D g^* \omega_k.$$

Recall also that Γ acts on $\widetilde{\mathcal{L}X}$ as the group of covering transformations (so in fact $\widetilde{\mathcal{L}X}/\Gamma = \mathcal{L}X$). Now if we identify $\tilde{q}^d$ with the covering transformation corresponding to $d \in \Gamma$ we have that

$$\tilde{q}_j^* H_k(\gamma, g) = H_k(\tilde{q}_j \cdot (\gamma, g)) = H_k((\gamma, g)) - \int_{A_j} \omega_k = H_k((\gamma, g)) - \delta_{j,k}, \tag{31}$$

where $\tilde{q}_j^*$ denotes the pullback.

Moreover, if we denote by P_k wedge product by the equivariantly closed form P_k and also denote simply by $\tilde{q}_k$ the action of $\tilde{q}_k$ by pullback then we claim that

$$[P_j, \tilde{q}_k] = \delta_{j,k} \hbar \tilde{q}_k. \tag{32}$$

The proof is a simple calculation. First notice that:

$$\tilde{q_k}(P_j) = \Omega_j + \hbar\tilde{q}_k^* H_j = \Omega_j + \hbar H_j - \hbar\delta_{j,k}\hbar = P_j - \delta_{j,k}\hbar.$$

Next let α be an equivariant form then

$$[P_j, \tilde{q_k}]\alpha = P_j\tilde{q_k}\alpha - \tilde{q_k}(P_j)\wedge\tilde{q_k}\alpha = P_j\tilde{q_k}\alpha - (P_j - \delta_{j,k}\hbar)\wedge\tilde{q_k}\alpha = \delta_{j,k}\hbar\tilde{q_k}\alpha.$$

Moreover, the operators P_j and $\tilde{q_k}$ for $j,k = 1\dots r$ satisfy the relations

$$[P_j, P_k] = [\tilde{q}_j, \tilde{q_k}] = 0. \tag{33}$$

Now let $t_1, \dots, t_r$ be coordinates on $\mathbb{C}^r$. Then we have

$$[\hbar\frac{\partial}{\partial t_j}, e^{t_k}] = \delta_{j,k}e^{t_k}, \tag{34}$$

where e^{t_k} is thought of as an operator acting by multiplication on functions of $e^{t_1}, \dots e^{t_r}$ and $\hbar\frac{\partial}{\partial t_j}$ for $j = 1\dots r$ also act on such functions. The algebra $\mathcal{D}$ of operators generated by $e^{t_1}, \dots e^{t_r}$ and $\hbar\frac{\partial}{\partial t_1}, \dots, \hbar\frac{\partial}{\partial t_r}$ is called the Heisenberg algebra of differential operators. Relations (32) and (33) say that the S^1 equivariant Floer cohomology ${FH_{S^1}}^*(\widetilde{\mathcal{L}X})$ carries the structure of a module [3] over the Heisenberg algebra $\mathcal{D}$!

In our discussion of the A model connection, in the previous section, we also encountered a $\mathcal{D}$-module. That one consisted of operators which kill the first row of the solution matrix of the flat section equation for the A connection. Givental's conjecture is that *the two $\mathcal{D}$ - modules are in fact the same*!

Of course there is no chance of proving this unless a rigorous S^1 equivariant Floer theory of the unperturbed action functional is constructed. In case X is a toric variety though, we will construct a model for the space $\widetilde{\mathcal{L}X}$ in section (5). If X is also positive, then we shall be able to prove that the $\mathcal{D}$ - module of our model is indeed the same as the quantum $\mathcal{D}$ - module.

Notice than in the previous section, we used coordinates q_j which are related to the t_j by $q_j = e^{t_j}$. It is also clear, that series in the q_j with values in $H^{2*}(X,\mathbb{C})$ can be thought of as sections of a trivial bundle with fiber $H^{2*}(X,\mathbb{C})$, over the (algebraic) torus obtained by the lattice $H^2(X,\mathbb{Z})$ (via complexification and exponentiation). In particular this torus can be thought of as the (affine) toric variety associated to a fan consisting of a single cone, namely the Kähler cone of X. The q_j are then identified with the toric coordinates.

Now having a $\mathcal{D}$-module how can we associate a flat connection ?

Recall that from (28) we have:

$$FH^*{}_{S^1}(\widetilde{\mathcal{L}X}) = H^*(M, \mathbb{C}[\tilde{q}_1, \tilde{q}_1^{-1} \dots, \tilde{q}_r\tilde{q}_r^{-1}](\hbar)). \tag{35}$$

Therefore $H^{2*}(X,\mathbb{Z})$ is embedded in $FH^*{}_{S^1}(\widetilde{\mathcal{L}X})$. Consider again the basis $T_0, \dots, T_m$ of $H^{2*}(X,\mathbb{Z})$. We then have

$$P_k \wedge (T_0, \dots, T_m) = \tilde{A}^k(T_0, \dots, T_m), \tag{36}$$

where $\tilde{A}^k$ is a matrix with coefficients functions of $\tilde{q_k}$ and $\hbar$. The coefficient functions are expected to be holomorphic so they will not contain any of the ${\tilde{q_k}}^{-1}$.

[3]The reason for the choice of sign in the action of Γ on $\widetilde{\mathcal{L}M}$ is precisely so that we end up with e^{t_j} instead of e^{-t_j}.

Define now a pencil of connections $\tilde{\nabla}_\hbar$ acting on series in the $\tilde{q_k}$ with values in $H^{2*}(X,\mathbb{C})$ by:

$$\tilde{\nabla}_\hbar = \hbar d - \sum_{k=1}^{r} \frac{d\tilde{q_k}}{\tilde{q_k}} P_k \wedge. \tag{37}$$

The connection $\tilde{\nabla}_\hbar$ is expected to be equal to the A model connection $\nabla_\hbar$ considered in our discussion of quantum cohomology based on stable maps. This means for example that $\tilde{\nabla}_\hbar$ should be flat, i.e., that flat sections should exist. If $\sigma = \sum_{j=0}^{m} f_j T_j$ is a section then

$$\tilde{\nabla}_\hbar \sigma = 0$$

is equivalent to the system

$$\hbar \tilde{q_k} \frac{\partial}{\partial \tilde{q_k}} (f_0, \dots, f_m)^t = \tilde{A}^k (f_0, \dots, f_m)^t \quad for\ k = 1 \dots r. \tag{38}$$

Flatness of course means that (38) is integrable. To shed some more light we note that (38) can equally be written as

$$P_k \wedge (T_0, \dots, T_m)(f_0, \dots, f_m)^t = (T_0, \dots, T_m) \hbar \tilde{q_k} \frac{\partial}{\partial \tilde{q_k}} (f_0, \dots, f_m)^t. \tag{39}$$

In other words the $(m+1)$-tuple $(f_0, \dots, f_m)^t$ defines a $\mathcal{D}$ - module homomorphism between $FH^*{}_{S^1}(\widetilde{\mathcal{L}X})$ and the sheaf $\mathcal{O}$ of holomorphic functions on the torus. Therefore we can reformulate our discussion in an invariant fashion by saying that an element in $Hom_{\mathcal{D}}(FH^*{}_{S^1}(\widetilde{\mathcal{L}X}), \mathcal{O})$ defines a locally constant sheaf $\mathcal{V}$ over the torus. This sheaf defines in turn by the standard procedure a flat connection on the sheaf $\mathcal{U} = \mathcal{V} \otimes \mathcal{O}$. As an aside we note that this may remind the reader of the construction of the Gauss-Manin connection associated to a family of varieties. The locally constant sheaf is there, the one associated to the integral cohomology of the fiber. This is no accident since mirror symmetry identifies, in the case X is Calabi-Yau, the A-model connection with the Gauss-Manin connection of a certain family of Calabi-Yau manifolds.

Instead of concentrating on the connection lets look now at the $\mathcal{D}$-module itself and try to find a presentation or at least some relations. We have seen up to now that $FH^*{}_{S^1}(\widetilde{\mathcal{L}X})$ is generated by $H^{2*}(X,\mathbb{Z})$ over the ring Λ. Geometrically an equivariant Floer cycle associated to an element $T \in H^{2*}(X,\mathbb{Z})$ can be constructed as the boundary loops of all holomorphic discs whose center lies in a cycle representing the Poincaré dual of T. Now notice that the standard way to go between the Heisenberg algebra and its presentation in terms of the P_k and $\tilde{q_k}$ is via the *Fourier transform.* This way relations that involve the later can be transformed to differential equations that involve the former. Indeed if Γ is in $FH^*{}_{S^1}(\widetilde{\mathcal{L}X})$ and we denote $t_1 P_1 + \cdots + t_r P_r$ by tP then consider the pairing

$$(e^{tP/\hbar}, \Gamma) = \int_{\widetilde{\mathcal{L}X}} e^{tP/\hbar} \Gamma. \tag{40}$$

We claim that

$$(e^{tP/\hbar}, R(P, \tilde{q}, \hbar)\Gamma) = R(\hbar \frac{\partial}{\partial t}, e^t, \hbar)(e^{tP/\hbar}, \Gamma). \tag{41}$$

In other words that the map

$$\mathcal{F} : FH^*{}_{S^1}(\widetilde{\mathcal{L}X}) \to \mathcal{O}$$

given by

$$\mathcal{F}(\Gamma) = (e^{tP/\hbar}, \Gamma)$$

is an element of $Hom_{\mathcal{D}}(FH^*{}_{S^1}(\widetilde{\mathcal{L}X}), \mathcal{O})$.

Indeed we can do a bit better than that. If $C_T \in FH^*{}_{S^1}(\widetilde{\mathcal{L}X})$ is such that it has the same localization $T \in H^{2*}(X, \mathbb{Z})$ on every critical manifold, then

$$(e^{tP/\hbar}C_T, R(P, \tilde{q}, \hbar)\Gamma) = R(\hbar\frac{\partial}{\partial t}, e^t, \hbar)(e^{tP/\hbar}C_T, \Gamma), \tag{42}$$

and therefore the map

$$\mathcal{F}_T : FH^*{}_{S^1}(\widetilde{\mathcal{L}X}) \to \mathcal{O}$$

given by

$$\mathcal{F}_T(\Gamma) = (e^{tP/\hbar}C_T, \Gamma), \tag{43}$$

is an element of $Hom_{\mathcal{D}}(FH^*{}_{S^1}(\widetilde{\mathcal{L}X}), \mathcal{O})$.

The reason is that

$$(e^{tP/\hbar}C_T, \tilde{q_k}\Gamma) = (\tilde{q_k}^{-1}e^{tP/\hbar}C_T, \Gamma) = e^{t_k}(e^{tP/\hbar}C_T, \Gamma)$$

and

$$(e^{tP/\hbar}C_T, P_k\Gamma) = (P_k e^{tP/\hbar}C_T, \Gamma) = (\hbar\frac{\partial}{\partial t_k}e^{tP/\hbar}C, \Gamma) = \hbar\frac{\partial}{\partial t_k}(e^{tP/\hbar}C_T, \Gamma).$$

Thus we can find the differential operators and compute solution by computing $(e^{tP/\hbar}, \Gamma)$ if we can write down Γ and compute the integral.

Now let Δ denote the fundamental Floer cycle corresponding to the fundamental cycle of X, i.e., the set of all boundary loops of holomorphic discs in X. Let $\check{\Delta}$ denote formally the Poincaré dual to Δ and take $\Gamma = \check{\Delta}$ in the above. We note that if the cohomology of X is generated by classes in $H^2(X, \mathbb{Z})$, then polynomials $R(P, \tilde{q}, \hbar)$ such that $R(P, \tilde{q}, \hbar)\Delta = 0$ generate all relations. So if I_0 is the ideal generated by such polynomials then $FH^*{}_{S^1}(\widetilde{\mathcal{L}X}) = \mathbb{C}[P, \tilde{q}, \hbar]/I_0$. The reason for this is that if $f(p_1, \ldots, p_r)$ is a polynomial in the generators $\{p_1, \ldots, p_r\}$ of $H^2(X, \mathbb{Z})$ then the corresponding Floer cycle is $\Delta_f = f(P_1, \ldots, P_r)\Delta$ since out of all loops (boundaries of holomorphic discs) that have their center in X, this pics the ones that are in the cycle Poincaré dual to $f(p_1, \ldots, p_r)$. It is clear now that any polynomial $R_1(P, \tilde{q}, \hbar)$ such that $R_1(P, \tilde{q}, \hbar)\Delta_f = 0$ induces a relation $R_1(P, \tilde{q}, \hbar)f(P_1, \ldots, P_r)\Delta = 0$. Therefore relations stemming form Δ generate all relations.

Up to this point our discussion of S^1 equivariant Floer theory of the unperturbed action functional has followed Givental's paper [7]. We would like now to propose a conjecture about how to regularize the integral in (43). In the last section we shall prove a version of it for toric manifolds.

4. A conjecture on the regularization of the Fourier transform of the Floer fundamental cycle

Recall first that we have chosen a basis $\{T_0, \ldots, T_m\}$ of $H^{2*}(X, \mathbb{Z})$. We arrange that $T_0 = 1$. Choose also a dual basis $\{T^0, \ldots, T^m\}$ of $H^{2*}(X, \mathbb{Z})$ such that $(T_i, T^j) = \delta_{i,j}$, where the pairing is the Poincaré pairing. Now to compute the integral in (43) for $T = T_\beta$ and since the integrand is an equivariantly closed form, we could attempt to formally use a localization theorem in equivariant cohomology.

The theorem we need is a stronger version of theorem (2) mentioned before and it is due independently to Berline-Vergne [3] and Atiyah-Bott [1].

Theorem 3. *Let $\mathbb{T}$ be a torus acting on a (finite dimensional) compact manifold X and let α be an equivariantly closed form in the Cartan model. Then*

$$\int_M \alpha = \sum_F \int_F \frac{\alpha_{|F}}{e_{\mathbb{T}}(\mathcal{N}_F)}$$

Where the sum is over all the fixed components F of the action and $e_{\mathbb{T}}(\mathcal{N}_F)$ indicates the $\mathbb{T}$ equivariant Eüler class of the normal bundle, $\mathcal{N}_F$ to the fixed component F. By $\alpha_{|F}$ we denote the pullback of α to F by the inclusion of F into M.

Since $\widetilde{\mathcal{L}X}$ is infinite dimensional an application of this theorem in our case can only be done in a formal fashion. This formal application gives:

$$\mathcal{F}_{T_\beta}(\check{\Delta}) = \int_{\widetilde{\mathcal{L}M}} e^{tP/\hbar} C_{T_\beta}\check{\Delta} = \sum_{d\in H_2(X,\mathbb{Z})} \int_{X_d} T_\beta e^{\sum_{k=1}^r t_k(p_k/\hbar + \int_d p_k)} \frac{\check{\Delta}_{|X_d}}{e_{S^1}(\mathcal{N}_d)}, \tag{44}$$

where we have used the following notation. First recall that the action functional H is a function on $\widetilde{\mathcal{L}X}$ whose critical manifolds are the fixed components of the S^1 action and therefore are just copies of X. Denote the copy of X such that $H_{|X} = 0$ by X_0. The action of $\tilde{q}^d$ maps X_0 to another copy of X which we denote by X_d. $\mathcal{N}_d$ denotes the normal bundle to X_d. This is of course an infinite dimensional bundle. Notice now that $\mathcal{N}_d$ carries a representation of S^1 (as a sub-bundle of $T\widetilde{\mathcal{L}X}_{|X_d}$) and splits to the direct sum of vector bundles according to the weights of this representation. The Eüler class $e_{S^1}(\mathcal{N}_d)$ is therefore some infinite product which in general, will be divergent. Moreover recall that $P_k = \mathcal{P}_k + \hbar H_k$ and $\mathcal{P}_{k|X_d} = p_k$ and finally $H_{k|X_d} = \int_d p_k$.

We see then, that our first attempt to apply localization fails due to the infinite dimensionality of the normal bundle to X_d in $\widetilde{\mathcal{L}X}$. To try to get around this we will start by thinking in a dual fashion: Let A denote the set of pseudo-holomorphic maps $u : D \to X$ where D denotes the unit disc with complex coordinate $z = re^{i\theta}$. Recall that $\check{\Delta}$ denotes formally, the Poincaré dual to Δ, which is the set of all loops which are boundaries of holomorphic discs. Let $p^+ : A \to \widetilde{\mathcal{L}X}$ be defined by $p^+(u) = u|_{r=1}(e^{i\theta})$. Then $p^+(A) = \Delta$. We will now attempt to compute $\mathcal{F}_{T_\beta}(\check{\Delta})$ as an integral over Δ and to do this we have to identify the fixed components of the circle action and normal bundles to them. It's clear that for a loop in Δ to be fixed by the circle action, it has to actually be a point. *This means that fixed components come from pseudo-holomorphic maps $u : \mathbb{P}^1 \to X$ which are not initially in Δ but should be added in as limit points.* These trivial loops are of course interpreted as points in X_d and in the simplest case, we will in fact get all of X_d.

We see from our discussion up to here, that the answer really depends on our compactification of the space of pseudo-holomorphic maps from genus 0 curves. Moreover, it becomes clear, that contributions only arise for degrees d, which may be represented by pseudo-holomorphic maps from genus 0 curves i.e. for $d \in \check{\mathcal{K}}$.

Finally for now, let us mention that the simple situation described before, of maps only from $\mathbb{P}^1$ where a whole copy of X will appear for any degree $d \in \check{\mathcal{K}}$ does actually arise in the case of toric manifolds and the so-called gauged, linear σ-model compactification of genus 0 holomorphic maps. This will be explained in the next section.

To continue our attempt to use localization we denote by F_d the submanifold of X_d which arises as trivial loops coming from degree d maps. We also denote by $\mathcal{N}_\Delta F_d$ the normal bundle to F_d in Δ

We then have:

$$\mathcal{F}_{T_\beta}(\check{\Delta}) = \int_\Delta e^{tP/\hbar} C_{T_\beta} = \sum_{d\in H_2(X,\mathbb{Z})} \int_{F_d} T_\beta e^{\sum_{k=1}^r t_k(p_k/\hbar + \int_d p_k)} \frac{1}{e_{S^1}(\mathcal{N}_\Delta F_d)}, \tag{45}$$

Now though, we see that we still have a similar problem as before, since $\mathcal{N}_\Delta F_d$ is infinite dimensional. *However, here the number of incoming directions to F_d is finite, since they correspond to degree d, genus 0 curves.* On the other hand, outgoing directions are given by the restriction of ${\mathcal{N}_d}^+$ to F_d where $\mathcal{N}_d$ denotes the normal bundle to X_d in $\widetilde{\mathcal{L}X}$ and $\mathcal{N}_d = {\mathcal{N}_d}^+ \bigoplus {\mathcal{N}_d}^-$ is the decomposition of $\mathcal{N}_d$ to positive and negative normal bundles. Now since the circle action on $\widetilde{\mathcal{L}X}$ is lifted from that on $\mathcal{L}X$ and the action functional is lifted from a multivalued function on $\mathcal{L}X$, all the ${\mathcal{N}_d}^+$ can be identified. *Of course they are infinite dimensional and the same at every level d and as a first attempt to guess the correct approach we propose to simply ignore them. This crude argument, will be replaced by the use of a Fredholm map later on.*

We are nevertheless, still left with a rather complicated state of affairs. The way forward, depends on the compactification of the space of degree d maps. In case we use the space of stable maps in order to model the space of connecting flowlines, we may think that we need to have two marked points, namely we should use $\overline{M}_2(X,d)$. The two marked points could be interpreted as the initial and final points of the flow and $F_d = ev_2(\overline{M}_2(X,d))$. It is a subtle point, that this in fact, *is not* the way that we will proceed. *Instead of attempting to define a Fredholm map using the flow of the action functional in the loop space, we will use the deck transformation maps $\tilde{q}^d$.* This will be clear in what follows in this section and in the next when we look at toric manifolds. Nevertheless, there is still a relation with Morse theory of the action functional. This just reflects the fact that as already pointed out, the deck transformation will map the positive normal bundle at X_0 to the positive normal bundle at X_d.

Now, the gauged, linear σ-model compactification $M_d(X)$, that will be used in the next section, is quite different than stable maps, in two respects. First, degree is not preserved at the limit while it is of course preserved for stable maps. Secondly, the domain of the maps is always a $\mathbb{P}^1$ (no reducible curves). The degree though, can also become 0 and in fact we get several copies of X in the space of maps, one for each degree d_1 such that $\int_{d_1}\omega \leq \int_d \omega$. In that case, $F_d = X_d$ and taking our previous discussion into account, we will use $\mathcal{N}_{M_d}X_d$ for $\mathcal{N}_{\check{\Delta}}F_d$. Note also, that since the domain is always a $\mathbb{P}^1$ there is circle action on the space of maps, by rotation of the source.

Therefore in that case we expect:

$$\mathcal{F}_{T_\beta}(\check{\Delta}) = \int_\Delta e^{tP/\hbar} C_{T_\beta} = \sum_{d\in H_2(X,\mathbb{Z})} \int_{X_d} T_\beta e^{\sum_{k=1}^r t_k(p_k/\hbar + \int_d p_k)} \frac{1}{e_{S^1}(\mathcal{N}_{M_d(X)}X_d)} \tag{46}$$

Let us mention here, that it will also be convenient to consider all the functions at once, by considering the $H^{2*}(X,\mathbb{C})$ valued function:

$$F = \sum_{d\in\check{\mathcal{K}}} e^{\sum_{k=1}^{r} t_k(p_k/\hbar+\int_d p_k)} \frac{1}{e_{S^1}(\mathcal{N}_\Delta X_d)} \tag{47}$$

Then

$$F = \sum_{\beta=1}^{m} \mathcal{F}_{T_\beta}(\check{\Delta})T^\beta.$$

and in our case:

$$F = \sum_{d\in\check{\mathcal{K}}} e^{\sum_{k=1}^{r} t_k(p_k/\hbar+\int_d p_k)} \frac{1}{e_{S^1}(\mathcal{N}_{M_d(X)} X_d)}. \tag{48}$$

We will see in the next section that this is in fact the correct answer in the case of positive toric manifolds. In other words $F = G$.

Now in order to approach the problem in the general case, the first think to observe, is that since $F_d = ev_1(\overline{M}_2(X,d))$, it seems more convenient to try and pull the calculation back to $\overline{M}_2(X,d)$.

Recall that we are also interested in understanding the relation to the loop space and the Morse theory of the action functional.

To do that, we will take our cue from a well known description of vector bundles on Riemann surfaces with a marked point, via Fredholm maps on a Hilbert space. This is explained in [15] (section 8.11), which we largely follow. Consider a genus g Riemann surface $\mathcal{C}$ with a marked point x. Let E denote a rank n holomorphic vector bundle on $\mathcal{C}$. Consider a local holomorphic coordinate z around x such that $z(x) = \infty$. Identify the standard circle S^1 with the circle $|z| = 1$ on $\mathcal{C}$. Let $\mathcal{H} := L^2(S^1,\mathbb{C}^n)$ then $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^+$ and $\mathcal{H}^-$ denote the spaces of elements in the Hilbert space $\mathcal{H}$ with non-negative modes and negative modes respectively. Let us also fix a trivialization of E over $|z| \geq 1$. We may now consider, relative to this trivialization, the closed subspace W of $\mathcal{H}$, consisting of functions, which are the boundary values of holomorphic sections of E over the complement of $|z| \geq 1$. Then (proposition 8.11.10 [15]) the orthogonal projection of W onto $z\mathcal{H}^+$ is Fredholm and $H^0(\mathcal{C},\mathcal{E})$ and $H^1(\mathcal{C},\mathcal{E})$ are respectively its kernel and cokernel. Notice that equivalently we may say that $W = w(\mathcal{H}^+)$ where $w : \mathcal{H}^+ \to \mathcal{H}$ is an operator such that if $\tilde{pr}_+ : \mathcal{H} \to \mathcal{H}^+$ denotes the orthogonal projection, then $\tilde{pr}_+ow$ is Fredholm. Let us denote by $pr_+ : \mathcal{H} \to z\mathcal{H}^+$ the orthogonal projection to $z\mathcal{H}^+$. Then we have for $pr_+ow : \mathcal{H}^+ \to z\mathcal{H}^+$ that $H^0(\mathcal{C},\mathcal{E}) = Ker(pr_+ow)$ and $H^1(\mathcal{C},\mathcal{E}) = coker(pr_+ow)$. Notice now, that we have chosen a trivialization of E, over a disc around the marked point, and therefore an element of $\mathcal{L}^-GL(n,\mathbb{C})$ can act on W without changing the bundle. Its action permutes all the trivializations of E over the disc around the point x. However, it can be proven that W determines a unique, up to homotopy, topological $\mathbb{C}^n$ bundle over $\mathcal{C}$ (with the fiber at x identified with $\mathbb{C}^n$). This is proposition (8.11.6) in [15].

Now in our case, lets consider a (pseudo-)holomorphic map $u : \mathcal{C} \to X$ and a marked point x_1 on $\mathcal{C}$ and take $E = u^*TX$. It is natural to take $x = x_1$ and to consider the corresponding map $pr_+ow : \mathcal{H}^+ \to z\mathcal{H}^+$ as a map from $T|_{ev_1(x_1)}X \bigoplus {\mathcal{N}_0}^+|_{ev_1(x_1)}$ to ${\mathcal{N}_0}^+|_{ev_1(x_1)}$.

We claim then, that the previous discussion can be rephrased as the fact that, *over the open part*, $M_3(X,d)$ the following sequence:

$$0 \to T^v{}_{ev_1} M_3(X,d) \to ev_1{}^* \mathcal{N}_0{}^+ \to ev_1{}^* \mathcal{N}_0{}^+ \to 0, \tag{49}$$

is exact up to homotopy. Here, $T^v{}_{ev_1} M_3(X,d)$ denotes the virtual vertical tangent bundle to $M_3(X,d)$ with respect to the map $ev_1 : M_3(X,d) \to X$. The second map, over the point in $M_2(X,d)$ corresponding to the map u, is just the Fredholm operator, $pr_+ ow|_{z\mathcal{H}^+} : z\mathcal{H}^+ \to z\mathcal{H}^+$.

The reason that $M_3(X,d)$ appears here, is that a smooth genus 0 Riemann surface $\mathcal{C}$, has a three dimensional group of automorphisms. This induces a nontrivial continuous group of automorphisms of a pseudo-holomorphic map defined on $\mathcal{C}$, by which we need to mod out in the construction of the moduli space of stable maps. Marking three points on $\mathcal{C}$ though, leaves us with no continuous group of automorphisms and the virtual tangent space to $M_3(X,d)$ at such a map is given by $H^0(\mathcal{C},\mathcal{E}) - H^1(\mathcal{C},\mathcal{E}) = Ker(pr_+ ow) - Coker(pr_+ ow)$.

We now claim that exactness of (49) implies that *over the open part*, $M_2(X,d)$ the sequence :

$$0 \to L_1 \oplus T^v{}_{ev_1} M_2(X,d) \to ev_1{}^* \mathcal{N}_0{}^+ \to ev_1{}^* \mathcal{N}_0{}^+ \to 0, \tag{50}$$

is exact up to homotopy. Here $T^v{}_{ev_1} M_2(X,d)$ denotes the virtual vertical tangent bundle with respect to the map $ev_1 : M_2(X,d) \to X$ and L_1 denotes the line bundle over $M_2(X,d)$ which is the universal tangent line at the first marked point.

To show how (50) arises, let $ft_3 : M_3(X,d) \to M_2(X,d)$ denote the forgetful map that forgets the third marked point. Along with the evaluation at the third marked point $ev_3 : M_3(X,d) \to X$, it is the universal family over $M_2(X,d)$.

The map ft_3 has a section $s_1 : M_2(X,d) \to M_3(X,d)$ which is choosing the first marked point. Then the exactness of (49) implies the exactness over $s_1(M_2(X,d))$ of :

$$0 \to T^v{}_{ev_1 oft_3} s_1(M_2(X,d)) \to ev_1{}^* \mathcal{N}_0{}^+ \to ev_1{}^* \mathcal{N}_0{}^+ \to 0. \tag{51}$$

But $s_1{}^*(T^v{}_{ev_1 oft_3} s_1(M_2(X,d))) = L_1 \oplus T^v{}_{ev_1} M_2(X,d)$ and exactness of (50) follows.

Notice that nothing in our argument depends on the fact that we are working over $M_2(X,d)$. Therefore, if we manage to extend (49) over $\overline{M}_3(X,d)$, then we immediately have an extension of (50) over $\overline{M}_2(X,d)$.

Notice also, that the family of Fredholm maps defines an index bundle over $M_2(X,d)$ and we may think of every map as giving a shifting, in the splitting of the normal bundle to positive and negative.

We propose now the following:

Conjecture 1. *The sequences (49) and (50) can be extended over all of $\overline{M}_3(X,d)$ and $\overline{M}_2(X,d)$ respectively and remain exact.*

From the above we see that, conjecturally:

$$\frac{1}{e_{S^1}(L_1)} = \frac{e_{S^1}(T^v{}_{ev_1} \overline{M}_2(X,d))}{e_{S^1}(Ind(pr_+ ow))}. \tag{52}$$

where e_{S^1} denotes the equivariant Eüler characteristic with respect to an S^1 action. The action here is of course trivial on the base of our bundles but non-trivial on the fiber. In the case of the index bundle it comes from rotating the S^1 that we have marked on our curve $\mathcal{C}$. On L_1 we have $e_{S^1}(L_1) = \hbar - \psi_1$.

This can be used in order to obtain a different and rather more illuminating, conjectural expression for the function G, that as we know generates the quantum $\mathcal{D}$-module. We have conjecturally that:

$$G = e^{tp/\hbar}(1 + \sum_{d \in \check{\mathcal{K}}} q^d ev_{1*} \frac{e_{S^1}(T^v{}_{ev_1}\overline{M}_2(X,d))}{e_{S^1}(Ind(pr_+ ow))}). \tag{53}$$

Now we may interpret $Ind(pr_+ ow) = L_1 \oplus T^v{}_{ev_1}\overline{M}_2(X,d)$ as the pull back to $\overline{M}_2(X,d)$ of the "renormalized" normal bundle $\mathcal{N}_\Delta F_d$ and $T^v{}_{ev_1}\overline{M}_2(X,d)$ as the Thom class that allows us to pull back the integral to $\overline{M}_2(X,d)$. From the Morse theoretic point of view, our formula is also very suggestive since we may think of L_1 as corresponding to the direction tangent to the flow line itself, while $T^v{}_{ev_1}\overline{M}_2(X,d)$ corresponds to directions tangent to the manifold parameterizing flow lines connecting X_0 and X_d. It may seem strange that $\mathcal{N}_d$ doesn't appear explicitly here but this is due to the fact, that we have implicitly identified all the copies of X, by using the spaces of stable maps to model spaces of flow lines.

We will not go more into the general case here as this is the subject of a forthcoming paper.

Instead we will now go back to the case of toric manifolds. In that case as mentioned earlier, we have an alternate compactification called the (gauged) linear σ-model. For this model the fixed locus F_d is in fact X_d it is thus convenient to use expression (48) and work over X_d instead of over the space of maps. Recall also that according to our conventions, from the previous paragraph, the deck transformation $\tilde{q}^{-d}$ maps X_0 to X_d. Due to the linearity of the model we will see that we may identify the deck transformation and its derivative (linearization). There is a subtle point here that will be clarified further in the next section: We may compute the deck transformation itself that indeed maps X_0 to X_d and thus also its derivative, which maps $\mathcal{N}_0 \to \mathcal{N}_d$. To remain in contact though with the approach we have described here, we will identify all the X_d with X_0, using the deck transformations and consider the derivative of the deck transformation as a map from $\mathcal{N}_0 \to \mathcal{N}_0$.

As support to our conjecture then, we will show in the next section, that there is an exact sequence over X_0 given by:

$$0 \to \mathcal{N}_{M_d} X_0 \to \mathcal{N}_0{}^+ \to \mathcal{N}_0{}^+ \to 0, \tag{54}$$

where the second map is the composition of the derivative of the deck transformation $\tilde{q}^d$ composed with orthogonal projection.

We may now make an interesting observation: from this point of view we see that, the *the Fredholm map w, that we used in the general situation, should be interpreted as the linearization i.e. the derivative, of the deck transformation, in the positive normal directions to* X_0. In the toric case, our model is essentially linear, so instead of just the derivative we are able to compute the deck transformation itself.

We will then show in the next section, that if we use the exact sequence above to compute $D\tilde{q}^d$, then we have that:

$$F = \sum_{d \in \check{\mathcal{K}}} e^{\sum_{k=1}^r t_k(p_k/\hbar + \int_d p_k)} \frac{1}{e_{S^1}(Ind(pr_+ oD\tilde{q}^d))} \tag{55}$$

or equivalently

$$F = e^{(t_1 p_1 + \cdots + t_r p_r)/\hbar} \sum_{d \in \check{\mathcal{K}}} q^d \frac{1}{e_{S^1}(Ind(pr_+ o D\tilde{q}^d))}, \tag{56}$$

where $Ind(pr_+ o D\tilde{q}^d)$ denotes the index bundle defined on X_0 by the Fredholm bundle map $pr_+ o D\tilde{q}^d$. where $q_k = e^{t_k}$, $d_k = \int_d \omega_k$ and $q^d = {q_1}^{d_1} \dots {q_r}^{d_r}$. Finally we will show that $F = G$

We will follow two different strategies in the toric case. The first, will be to consider the actual space $\widetilde{\mathcal{L}X}$ which is infinite dimensional and to work with Fredholm maps.

The second, will be to reformulate everything in terms of a sequence of finite dimensional approximations of $\widetilde{\mathcal{L}X}$, by spaces parameterizing loops of arbitrarily large but finitely many modes (in their Fourier expansion). We can then consider a certain sequence of ratios of Eüler classes that stabilizes for large modes and we use this "*stable ratio*" in place of the equivariant Eüler class of the index bundle . We can then calculate F explicitly. Finally, we invoke the calculation of G by Givental [8] to show that $F = G$.

We will see though that the infinite dimensional version of the story is in fact more straightforward and transparent.

One more observation that needs to be made, is that we may use the negative normal bundle instead of the positive one. This is equivalent to changing the sign of the complex structure on X and from the point of view of Morse theory on the loop space it corresponds to changing the sign of the action functional. Therefore we just replace the up going manifold starting from X_0 with the down going one. Yet another way of thinking about it, is that instead of using the map $p^+ : A \to \widetilde{\mathcal{L}X}$ defined by $p^+(u) = u|_{r=1}(e^{i\theta})$, we consider p^- where $p^-(u) = u|_{r=1}(e^{-i\theta})$ and compute the integral over $p^-(A)$. This has the effect of changing $\hbar$ with $-\hbar$. In the toric case this will appear as an exact sequence:

$$0 \to \mathcal{N}_{M_{-d}(X)} X_0 \to {\mathcal{N}_0}^- \to {\mathcal{N}_0}^- \to 0, \tag{57}$$

where again the second map is the derivative of the deck transformation composed with orthogonal projection. This makes sense, since if a homology class d is represented by a holomorphic map, then $-d$ is represented by an antiholomorphic one.

As a final observation we note that $\mathcal{F}_{T_\beta}$ for all $\beta = 0, \dots, m$ are elements of $Hom_{\mathcal{D}}(FH^*{}_{S^1}(\widetilde{\mathcal{L}M}), \mathcal{O})$ and therefore define flat section of the A-connection. Denote by Δ_α the Floer-Witten cycle corresponding to T_α. Then $\Delta_0 = \Delta$. For a fixed T_β, the functions $\mathcal{F}_{T_\beta}(\check{\Delta}_\alpha)$ simply give $\mathcal{F}_{T_\beta}$ in a basis. They should be

$$\mathcal{F}_{T_\beta}(\check{\Delta}_\alpha) = s_{\alpha,\beta},$$

where $(s_{\alpha,\beta})$ is the fundamental solution matrix (6) of the A model flat section equation. In this fashion we may identify $\mathcal{F}_{T_\beta}$ with s_β namely, the flat section found in theorem (1).

5. The Toric Case

Our goal in this section is to formulate and prove rigorously a version of conjecture (1) formulated in the previous section, in the form of sequences (54) and (57).

First we need to describe the set up. Our main references for toric varieties are Fulton [6] for the algebraic geometric point of view and Audin [2] for the symplectic side. Let X be a compact, smooth, Kähler toric variety. We choose to think of it as a symplectic quotient. To that end, in order to define X we start with an exact sequence of lattices as in:

$$0 \to \mathbb{Z}^l \to \mathbb{Z}^n \to \mathbb{Z}^d \to 0, \tag{58}$$

where the first map is called m and the second π. Tensoring the sequence with $\mathbb{C}$ and exponentiating gives a sequence of algebraic tori :

$$1 \to {\mathbb{C}^*}^l \to {\mathbb{C}^*}^n \to {\mathbb{C}^*}^d \to 1. \tag{59}$$

Now tensoring (58) with $i\mathbb{R}$ and exponentiating gives a sequence of real tori

$$1 \to \mathbb{T}^l \to \mathbb{T}^n \to \mathbb{T}^d \to 1. \tag{60}$$

These sequences define an embedding of ${\mathbb{C}^*}^l$ into ${\mathbb{C}^*}^n$ and of $\mathbb{T}^l$ into $\mathbb{T}^n$. Composing this with the diagonal action of ${\mathbb{C}^*}^n$ on itself defines the action

$$(x_1, \dots, x_n) \mapsto (\prod_{j=1}^{l} {\lambda_j}^{m_{j,1}} x_1, \dots, \prod_{j=1}^{l} {\lambda_j}^{m_{j,n}} x_n). \tag{61}$$

Associated to this, there is the moment map

$$\mu : \mathbb{C}^n \to \mathbb{R}^l$$

given by

$$\mu = \mu_l = \check{m} \circ \mu_n = \frac{1}{2}(\sum_{k=1}^{n} m_{1,k}|x_k|^2, \dots, \sum_{k=1}^{n} m_{l,k}|x_k|^2). \tag{62}$$

If $\lambda \in \mathbb{R}^l$ is a regular value of μ then X is constructed by symplectic reduction as

$$X = X_\lambda = \mu^{-1}(\lambda)/\mathbb{T}^l. \tag{63}$$

X comes equipped with the reduced symplectic form ω_λ. For simplicity we shall just denote it by ω.

X is a Kähler (and at worst) orbifold. The Kähler form is the reduction of the standard Kähler form on $\mathbb{C}^n$. Notice further that there is a cone in $\mathbb{R}^l$ defined by the conditions that it contains λ and that the differential of μ drops rank along its walls. Reducing at any point in the cone gives a space topologically equivalent but with a different Kähler form. In fact we may identify this cone with the Kähler cone $\mathcal{K}$ of X.

Now recall that if $(x_1, \dots, x_n)$ are coordinates on $\mathbb{C}^n$ then they can be thought of as sections of corresponding line bundles L_k over X for $k = 1, \dots, n$. The divisor (x_k) is denoted by D_k. We call these divisors the toric divisors. Let $v_k = \pi(w_k)$ for $k = 1, \dots n$ where $\{w_1, \dots, w_n\}$ is the standard basis of $\mathbb{Z}^n$, then rational equivalences among the D_k are given by the relations

$$\sum_{k=1}^{n} < {e_\nu}^*, v_k > D_k = 0 \quad for \quad \nu = 1, \dots, (n-l), \tag{64}$$

where $\{{e_1}^*, \dots, {e_{n-l}}^*\}$ is the dual of the standard basis of $\mathbb{Z}^{n-l}$. Let $\alpha_k = c_1(L_k)$. The α_k are Poincaré dual to the D_k and (64) gives the additive relations among them.

It is known (see e.g. Fulton [6])that $H^2(X,\mathbb{R})$ is spanned by the α_k and that $H^*(X,\mathbb{R})$ is generated by classes in $H^2(X,\mathbb{R})$.

Now we would like to model somehow the space $\widetilde{\mathcal{L}X}$ defined in section (3) as a covering of the space of free contractible loops in X. We will construct two different kinds of spaces. One is infinite dimensional, it will be denoted $\widetilde{\mathcal{L}_\infty X}$ and it is really the space $\widetilde{\mathcal{L}X}$, using smooth loops in X. The other will be a sequence of finite dimensional spaces $\widetilde{\mathcal{L}_N X}$ that approximate $\widetilde{\mathcal{L}_\infty X}$ as $N \to \infty$. To this end, lets consider first, loops in $\mathbb{C}^n$ with a *finite but large* number of modes $2N$. To be specific we shall consider loops which are in general of the form:

$$\gamma : S^1 \to \mathbb{C}^n \; with \; \gamma(e^{i\theta}) = (\gamma_0(e^{i\theta}), \dots, \gamma_n(e^{i\theta})), \tag{65}$$

where, if we let $z = e^{i\theta}$, then

$$\gamma_k : S^1 \to \mathbb{C}, \tag{66}$$

has Fourier expansion :

$$\gamma_k(z) = \sum_{\nu=-N}^{N} {a_\nu}^k z^\nu. \tag{67}$$

Our model $\widetilde{\mathcal{L}_N X}$ for $\widetilde{\mathcal{L}X}$ will be defined as follows : The space $\widetilde{\mathcal{L}_N \mathbb{C}^n}$ of loops of finite modes in $\mathbb{C}^n$ is parametrized by the Fourier coefficients ${a_\nu}^k$ and therefore is just $\mathbb{C}^{(n+1)(2N+1)}$. Consider the ${\mathbb{C}^*}^l$ (or T^l) action on $\widetilde{\mathcal{L}_N \mathbb{C}^n}$ induced by the action (61) on $\mathbb{C}^n$ defining X. By this we mean that the action on all the coefficients of γ_k is the same as the action on x_k. The moment map attached to this action is:

$$\mu_N = \frac{1}{2}\Big(\sum_{\nu=-N}^{N}\sum_{k=1}^{n} m_{1,k}|{a_\nu}^k|^2, \dots, \sum_{\nu=-N}^{N}\sum_{k=1}^{n} m_{l,k}|{a_\nu}^k|^2\Big). \tag{68}$$

Define $\widetilde{\mathcal{L}_N X}$ as

$$\widetilde{\mathcal{L}_N X} = {\mu_N}^{-1}(\lambda)/T^l, \tag{69}$$

where we have identified the Kähler cones of X and $\widetilde{\mathcal{L}X}$. We can do this since the subsets of $\mathbb{R}^l$ where μ and μ_N drop rank are clearly the same. This is just because μ drops rank at some value if and only if some homogeneous coordinates are forced to be zero. At the same value μ_N drops rank since the corresponding sums of squares are forced to be zero which in turn forces each of the squares to be zero.

We have described a sequence of finite dimensional spaces but it is equally easy to construct the actual universal covering of the space parameterizing smooth loops in X. We will develop the two versions of the theory in parallel and we will see that there are advantages to working with the infinite dimensional space.

The main point up to now is that we would replace loops with finite number of modes by loops that may have infinitely many modes i.e. we would replace N with ∞. The moment map then changes in the same way and will be denoted by μ_∞. Then define $\widetilde{\mathcal{L}_\infty X} = \mu_\infty^{-1}(\lambda)/T^l$. The space $\widetilde{\mathcal{L}_\infty X}$ is an infinite dimensional toric manifold.

Now in general the Kähler cone $\mathcal{K}$ will not necessarily be simplicial, but it can of course be subdivided, to simplicial cones. Pick such a subdivision and consider the simplicial cone containing the value λ.

Let $\{p_1, \ldots p_l\}$ be the basis of that cone such that

$$\omega = \sum_{j=1}^{l} \lambda_j p_j. \tag{70}$$

The fact that the cone is simplicial means that $\{p_1, \ldots p_l\}$ is a basis of $H^2(X, \mathbb{Z})$. Moreover we have that

$$\alpha_k = \sum_{j=1}^{l} m_{j,k} p_j. \tag{71}$$

If d is an element of $H_2(X, \mathbb{Z})$ then we let

$$d_j = \int_d p_j. \tag{72}$$

In that case we may identify d with the vector $(d_1, \ldots, d_l)$.

Next we need to consider the action functional H_N associated to $\widetilde{\mathcal{L}_N X}$. Recall that the action functional assigns to a pair (loop, contracting disc) the symplectic area of the contracting disc. Recall also that we have fixed the standard Kähler form on $\mathbb{C}^n$ which is

$$\omega_0 = \sum_{k=1}^{n} ds_k \wedge dt_k = \frac{i}{2} \sum_{k=1}^{n} dx_k \wedge d\overline{x_k}, \tag{73}$$

where $x_k = s_k + it_k$. Define first

$$H_N(\gamma) = \frac{1}{2\pi} \int_{\gamma(S^1)} \sum_{k=1}^{n} s_k dt_k = \frac{1}{2\pi} \sum_{k=1}^{n} \int_{S^1} \gamma^*(s_k dt_k) = \frac{1}{2\pi} \sum_{k=1}^{n} \int_D u^*(\frac{i}{2} dx_k \wedge d\overline{x_k}), \tag{74}$$

where γ is given by (65). In other words H_N is the (normalized) action functional for loops in $\mathbb{C}^n$ or rather $\widetilde{\mathcal{L}_N \mathbb{C}^n}$ to be exact.

An elementary calculation shows that

$$H_N(\gamma) = \frac{1}{2} \sum_{\nu=-N}^{N} \nu(|a_\nu{}^1|^2 + \cdots + |a_\nu{}^n|^2). \tag{75}$$

To see this, it's enough to notice that if a loop $\gamma : S^1 \to \mathbb{C}$ is given by $\gamma(e^{i\theta}) = e^{ik\theta}$ then $u : D \to \mathbb{C}$ such that $u(re^{i\theta}) = r^k e^{ik\theta}$ contracts that loop. Moreover if $x = re^{i\theta}$ is a coordinate on $\mathbb{C}$ then

$$\int_D u^*(dx \wedge d\overline{x}) = \int_D (du \wedge d\overline{u}) = \int_0^{2\pi} \int_0^1 -2ik^2 r^{2k-1} dr \wedge d\theta = -2\pi i k$$

Now H_N is thus far defined on $\mathbb{C}^{n2N}$, but since it is invariant under the T^l action, it actually drops to a function on $\widetilde{\mathcal{L}_N X}$. We will still call that function by the same name H_N, and it is our action functional.

Consider next the S^1 action on $\widetilde{\mathcal{L}_N X}$. It is induced by rotation on the source circle, namely by the action $e^{i\theta} \mapsto e^{i(\theta+\phi)}$. This action induces an action on the Fourier coefficients of a loop γ by

$$a_\nu{}^k \mapsto e^{ik\phi} a_\nu{}^k. \tag{76}$$

It's clear that H_N is the Hamiltonian function corresponding to this action on $\widetilde{\mathcal{L}_N\mathbb{C}^n}$ and consequently on $\widetilde{\mathcal{L}_N X}$. This is in accordance with the general theory. As we saw in equation (21), the action functional is indeed the Hamiltonian of the circle action.

As a consequence the fixed components of the circle action on $\widetilde{\mathcal{L}_N X}$ coincide with the critical manifolds of H_N. We expect those to be copies of X and to correspond to homology classes $d \in H_2(X, \mathbb{Z})$. Recall that we may identify the class d with its period vector $(d_1, \ldots, d_l)$ as in (72). Now the fixed components of the circle action can be identified as follows: The action of T^l on $\mu^{-1}(\lambda) \in \widetilde{\mathcal{L}_N\mathbb{C}^n} = \mathbb{C}^{n2N}$ that defines $\widetilde{\mathcal{L}_N X}$ is induced by the action in (61). When we take $\lambda_1 = z^{d_1}, \ldots, \lambda_l = z^{d_l}$, this becomes an S^1 action. Components in $\mu^{-1}(\lambda)$ where the S^1 action from (76) coincides with the one appearing as a one parameter subgroup of the T^l action as above, will lead to fixed components in $\widetilde{\mathcal{L}_N X} = \mu^{-1}(\lambda)/T^l$. Thus making the substitution

$$\lambda_1 = z^{d_1}, \ldots, \lambda_l = z^{d_l}$$

in (61) shows immediately that loops of the form

$$\gamma(z) = (a^1{}_{\sum_{j=1}^{l} m_{j,1} d_j} z^{\sum_{j=1}^{l} m_{j,1} d_j}, \ldots, a^n{}_{\sum_{j=1}^{l} m_{j,n} d_j} z^{\sum_{j=1}^{l} m_{j,n} d_j}), \tag{77}$$

form a fixed component of the circle action on $\widetilde{\mathcal{L}_N X}$. The T^l action on $\mathbb{C}^{n(2N+1)}$ restricted to loops in $\mathbb{C}^n$ of the form (77), restricts to the action (61) defining X. Therefore the reduction of the space of loops of the form (77) will indeed be exactly a copy of X. We shall name this component X_d. Notice that there is a more illuminating way to write (77). According to (71) and (72) we have

$$\sum_{j=1}^{l} m_{j,k} d_j = \int_d \alpha_k, \tag{78}$$

where α_k, as we said earlier, is Poincaré dual to the toric divisor D_k. Therefore we see that X_d consists of loops of the form :

$$\gamma(z) = (a^1{}_{\int_d \alpha_1} z^{\int_d \alpha_1}, \ldots, a^n{}_{\int_d \alpha_n} z^{\int_d \alpha_n}). \tag{79}$$

This immediately tells us that in order to be able to study X_d we must take $N \geq max\{\int_d \alpha_1, \ldots, \int_d \alpha_n\}$ which we will assume from now on whenever discussing X_d. To recapitulate the set up, up to now we have defined the spaces $\widetilde{\mathcal{L}_N X}$ which as $N \to \infty$ approximate $\widetilde{\mathcal{L}X}$ and action functionals

$$H_N : \widetilde{\mathcal{L}_N X} \to \mathbb{R}. \tag{80}$$

We have also described the critical manifolds X_d of H_N, which are copies of X and of course coincide with the fixed components of the circle action on $\widetilde{\mathcal{L}_N X}$.

Moreover notice that since H_N is the Hamiltonian of an S^1 action it follows from general theory that it is a perfect Morse-Bott function. This is explained for example in Audin [2]. In our case it also obvious from (75) which shows that indices of H_N are even numbers. The so called lacunary principle (see e.g. Bott [4]) then guarantees that H_N is perfect. It's also clear that H_N is non-degenerate in the normal directions.

Let us see here, how this goes over in the case of the space $\widetilde{\mathcal{L}_\infty X}$. It is clear, that all we have to do is replace N with infinity to define H_∞ and everything else remains the same, with one important exception of course. H_∞ doesn't have a finite index. Both, positive and negative normal bundles to a fixed component X_d are infinite dimensional.

We note now the first advantage of looking at $\widetilde{\mathcal{L}_\infty X}$ and that is, that we may easily identify the deck transformation $\tilde{q}^{-d} : \widetilde{\mathcal{L}_\infty X} \to \widetilde{\mathcal{L}_\infty X}$. It is just the action induced on $\widetilde{\mathcal{L}_\infty X}$ by the mapping

$$\gamma(z) = (\gamma_1(z), \dots, \gamma_n(z)) \to (z^{\int_d \alpha_1}\gamma_1(z), \dots, z^{\int_d \alpha_n}\gamma_n(z)) \tag{81}$$

where, $\gamma_k : S^1 \to \mathbb{C}$, has Fourier expansion : $\gamma_k(z) = \sum_{\nu=-\infty}^{\infty} a_\nu{}^k z^\nu$. Clearly $\gamma(z)$ and $\tilde{q}^{-d}\gamma(z)$ correspond to the same loop in X but they are different as elements in $\widetilde{\mathcal{L}_\infty X}$. Moreover considered as a map on homogenous coordinates, it is the very familiar from functional analysis, shifting map on a Hilbert space. In other words: a_ν^k to $a^k_{\nu+\int_d \alpha_1}$, for $k = 1 \dots n$. Or to be even more explicit, let $\tilde{L}_{k,\nu}$ denote the bundle, over $\widetilde{\mathcal{L}_\infty X}$ (or $\widetilde{\mathcal{L}_N X}$), associated to $a_\nu{}^k$ and let $T := \bigoplus_\nu \tilde{L}_{1,\nu} \oplus \dots \oplus \bigoplus_\nu \tilde{L}_{n,\nu}$, then $\tilde{q}^{-d}$ defines a Fredholm bundle map from T to itself. Not to overburden the notation, we will denote by $\tilde{q}^{-d}$, both the map on T and on $\widetilde{\mathcal{L}_\infty X}$.

Let us note, that we have an Eüler sequence:

$$0 \to \mathbb{C}^l \to T \to T\widetilde{\mathcal{L}_\infty X} \to 0, \tag{82}$$

where $\mathbb{C}^l$ above denotes the trivial rank l complex bundle over $\widetilde{\mathcal{L}_\infty X}$.

Note also, that this maps cannot be defined on $\widetilde{\mathcal{L}_N X}$ even if we take $N \geq max(\int_d \alpha_1, \dots, \int_d \alpha_n)$.

To go further, let us define E_d to be a sub-bundle of T given by:

$$E_d = \bigoplus_{\nu \neq \int_d \alpha_1} \tilde{L}_{1,\nu} \oplus \dots \oplus \bigoplus_{\nu \neq \int_d \alpha_n} \tilde{L}_{n,\nu}, \tag{83}$$

then the normal bundle $\mathcal{N}_d$ to X_d in $\widetilde{\mathcal{L}_\infty X}$ is:

$$\mathcal{N}_d = E_d|_{X_d}. \tag{84}$$

This follows from the description of X_d as

$$X_d = (a^1{}_{\int_d \alpha_1}, \dots, a^n{}_{\int_d \alpha_1})/T^l, \tag{85}$$

where it is implied that all the other coordinates are zero. Therefore X_d is given as the complete intersection of the toric divisors, whose sum is E_d.

Now, $\tilde{q}^{-d}$ induces a map from sections of E_0 to sections of E_d. Moreover, $\tilde{q}^d$ induces a corresponding map shifting down, in a similar fashion.

Notice though, that considered as a Fredholm bundle map, either on T or between E_0 and E_d, our shifting map has neither kernel nor cokernel.

Let us define now $E_d{}^+ = \bigoplus_{\nu > \int_d \alpha_1} \tilde{L}_{1,\nu} \oplus \dots \oplus \bigoplus_{\nu > \int_d \alpha_n} \tilde{L}_{n,\nu}$. Let also $E_d{}^- = \bigoplus_{\nu < \int_d \alpha_1} \tilde{L}_{1,\nu} \oplus \dots \oplus \bigoplus_{\nu < \int_d \alpha_n} \tilde{L}_{n,\nu}$. Then $E_d = E_d{}^+ \oplus E_d{}^-$.

While $\tilde{q}^d$ does not induce a map on E_0 (because it doesn't preserve it) we may nevertheless consider the map induced on $E_0{}^+$, by first restricting $\tilde{q}^d$ from T to $E_0{}^+$ and then projecting back to $E_0{}^+$.

This Fredholm bundle map from ${E_0}^+$ to itself, over $\widetilde{\mathcal{L}_\infty X}$, is surjective but clearly has a non-trivial kernel. It is the bundle

$$T_d = \bigoplus_{\nu=1}^{\int_d \alpha_1} \tilde{L}_{1,\nu} \oplus \cdots \oplus \bigoplus_{\nu=1}^{\int_d \alpha_n} \tilde{L}_{n,\nu}.$$

Therefore we have an exact sequence of bundles over $\widetilde{\mathcal{L}_\infty X}$:

$$0 \to T_d \to {E_0}^+ \to {E_0}^+ \to 0 \tag{86}$$

At this point we can hardly fail to introduce the *gauged, linear σ-model* for holomorphic maps from $\mathbb{P}^1$ to X. We shall denote it by $M_d(X)$. To define it, simply consider loops $\gamma(z) = (\gamma_1(z), \ldots, \gamma_n(z))$ where now, $\gamma_k(z) = \sum_{\nu=0}^{\int_d \alpha_k} {a_\nu}^k z^\nu$. The coefficients in such an expansion are in $\mathbb{C}^{\int_d c_1(TX)+dimX+l}$ and there is as usual a $\mathbb{T}^l$ action with an associated moment map $\mu_d : \mathbb{C}^{\int_d c_1(TX)+dimX+l} \to \mathbb{R}^l$ given by $\mu_d = \frac{1}{2}(\sum_{\nu=0}^{\int_d \alpha_1} \sum_{k=1}^n m_{1,k}|{a_\nu}^k|^2, \ldots, \sum_{\nu=0}^{\int_d \alpha_n} \sum_{k=1}^n m_{l,k}|{a_\nu}^k|^2)$. We then define $M_d(X) = {\mu_d}^{-1}(\lambda)/\mathbb{T}^l$. It parameterizes holomorphic maps of degree $\kappa \preceq d$ from $\mathbb{P}^1 \to X$, where we use the notation $\kappa \preceq d$ to signify that $\int_\kappa \alpha_i \leq \int_d \alpha_i$ for $i = 1, \ldots, n$. In contrast with the space of stable maps, degree can drop at the limit of a family of maps. In particular $M_d(X)$ contains all X_κ for $\kappa \preceq d$ and thus we find ourselves in the situation leading to equations (46) and (48). Let us then compute $e_{S^1}(N_{M_d(X)}X_d)$. It is clear that[4]:

$$e_{S^1}(N_{M_d(X)}X_d) = \prod_{\nu=1}^{\int_d \alpha_1} (\alpha_1 - \nu\hbar) \cdots \prod_{\nu=1}^{\int_d \alpha_n} (\alpha_n - \nu\hbar) \tag{87}$$

and that:

$$e_{S^1}(N_{M_d(X)}X_0) = \prod_{\nu=1}^{\int_d \alpha_1} (\alpha_1 + \nu\hbar) \cdots \prod_{\nu=1}^{\int_d \alpha_n} (\alpha_n + \nu\hbar) \tag{88}$$

And from equation (48) it follows that:

$$F = \sum_{d \in \check{\mathcal{K}}} e^{\sum_{k=1}^r t_k(p_k/\hbar + \int_d p_k)} \frac{1}{e_{S^1}(\mathcal{N}_{M_d(X)}X_0)} = \tag{89}$$

$$e^{(t_1p_1+\cdots+t_lp_l)/\hbar} \sum_{d \in \check{\mathcal{K}}} q^d \, \frac{1}{\prod_{\nu=1}^{\int_d \alpha_1} (\alpha_1 + \nu\hbar) \ldots \prod_{\nu=1}^{\int_d \alpha_n} (\alpha_n + \nu\hbar)},$$

where q^d stands for ${q_1}^{d_1} \ldots {q_l}^{d_l} = e^{t_1d_1+\cdots+t_ld_l}$ as usual.

Finally, according to Givental's computation of the function

$$G = e^{plnq/\hbar}(1 + \sum_{d \in \check{\mathcal{K}}, d \neq 0} q^d ev_{1*}(\frac{1}{\hbar - \psi_1}))$$

in [9] (Theorem (0.1), page (3) and its corollary: Example (a) page (4)) we have that, if $\int_d c_1(TX) > 0$ for all $d \in \check{\mathcal{K}}$ and $d \neq 0$, then the function F as computed above is indeed equal to the function G and therefore it generates the quantum $\mathcal{D}$-module.

[4]For more explanation on this, see the proof of proposition (4) below, where a similar calculation is done.

We should make a comment here about how it is possible that the same answer, for the quantum $\mathcal{D}$-module, arises from the rather naive linear σ-model and the much more complicated space of stable maps. The two spaces have the same open part and differ with respect to the compactification. According to Givental's calculation ([8], page 28), the contributions to the calculation of G for $X = \mathbb{P}^n$ are of two kinds. The first is exactly of the type corresponding to the linear σ-model compactification. The second kind happens to vanish ([8], lemma 9.7). In a way then, the problem is linearized, since instead of having a summation over trees, we end up with a summation over chains ([8], page 28). This is why, in the toric case the linear σ-model gives the correct answer.

We should also explain here, that we have taken $\mathcal{N}_{M_d(X)}X_0$ above, instead of $\mathcal{N}_{M_d(X)}X_d$ since we have identified X_d and X_0 by the deck transformation. In any case, using $\mathcal{N}_{M_d(X)}X_d$ also gives a function that generates the quantum $\mathcal{D}$-module, since we have just changed $\hbar$ to $-\hbar$.

We would like now to relate this discussion, to the one we had in the previous section, using Fredholm maps, about the general situation.

In that case using spaces of stable maps and the evaluation map, we implicitly identified all the X_d. Here we do this by using the deck transformation $\tilde{q}^d$: we have that $\tilde{q}^d(X_d) = X_0$.

Let $\mathcal{N}_d^+ = E_d{}^+|_{X_d}$ and $\mathcal{N}_d^- = E_d{}^-|_{X_d}$. Then $\mathcal{N}_d = \mathcal{N}_d^+ \oplus \mathcal{N}_d^-$. We will prove the following:

Proposition 3. *$\mathcal{N}_d^+$ and $\mathcal{N}_d^-$ are respectively, the positive and negative normal bundles to X_d in $\widetilde{\mathcal{L}_\infty X}$, with respect to the action functional H_∞. An analogous statement is true for $\widetilde{\mathcal{L}_N X}$.*

We postpone the proof for a bit later.

First we note that by restriction of (86) to X_0, we get an exact sequence:

$$0 \to T_d|_{X_0} \to \mathcal{N}^+{}_0 \to \mathcal{N}^+{}_0 \to 0, \tag{90}$$

where the second map is $pr_+ o\tilde{q}^d$.

To be explicit $\tilde{q}^d$ on $\mathcal{N}_0{}^+ = E_0|_{X_0}{}^+ = \bigoplus_{\nu>0} L_{1,\nu} \oplus \cdots \oplus \bigoplus_{\nu>0} L_{n,\nu} \to \bigoplus_\nu L_{1,\nu} \oplus \cdots \oplus \bigoplus_\nu L_{n,\nu}$ sends the section corresponding to $\gamma(z) = (\gamma_1(z), \dots, \gamma_n(z))$ where $\gamma_k(z) = \sum_{\nu>0} a_\nu{}^k z^\nu$ to the section corresponding to $(z^{-\int_d \alpha_1}\gamma_1(z), \dots, z^{-\int_d \alpha_n}\gamma_n(z))$.

Then pr_+ is the projection.

Moreover it is clear that $T_d|_{X_0} = \mathcal{N}_{M_d(X)}X_0$. This statement makes sense, as $M_d(X)$ is a toric submanifold of $\widetilde{\mathcal{L}_\infty X}$, given by the vanishing of (infinitely many) homogeneous coordinates. To conclude, we just have to notice that X_0 is a complete intersection in $M_d(X)$, of the divisors whose sum is T_d. Therefore we have the exact sequence:

$$0 \to \mathcal{N}_{M_d(X)}X_0 \to \mathcal{N}^+{}_0 \to \mathcal{N}^+{}_0 \to 0 \tag{91}$$

Notice finally, that in fact $\tilde{q}^d$ restricted to $\mathcal{N}_0^+$, is exactly the derivative $D\tilde{q}^d$ of $\tilde{q}^d$, due to linearity of this map.

As promised, we have thus shown that:

Theorem 4.

$$G = e^{(t_1p_1+\cdots+t_lp_l)/\hbar} \sum_{d\in\check{\mathcal{K}}} q^d \frac{1}{e_{S^1}(Ind(pr_+ oD\tilde{q}^d|_{X_0}))}, \tag{92}$$

where $Ind(pr_+ oD\tilde{q}^d|_{X_0})$ denotes the index bundle defined on X_0 by the Fredholm bundle map $pr_+ oD\tilde{q}^d$.

We see now, that *the map w from the previous section should be interpreted as the derivative of the deck transformation, in the direction defined by the map u.* Moreover, for a map $u : \mathbb{P}^1 \to X$, $\tilde{q}^d(\mathcal{N}_0^+|_{u(0)})$ is the space W, in the semi-infinite Grassmannian, from the previous section. Finally, in accordance to what we did then, we project back to $\mathcal{N}_0^+|_{u(0)}$.

One way of thinking about it, is that we identify X_0 and X_d via the deck transformation but they have different normal bundles in $\widetilde{\mathcal{L}_\infty X}$.

A fundamental observation that comes out of the previous discussion, is that the Fredholm bundle map between positive normal bundles, that we constructed using the deck transformation, doesn't map between fibers over points which are upper and lower endpoints of the flow of the action functional in loop space. We saw that the fiber over $(a_0^1, \dots, a_0^n)$ in X_0 is mapped to the fiber over the point in X_d corresponding to the loop $(a_0^1 z^{\int_d \alpha_1}, \dots, a_0^n z^{\int_d \alpha_n})$. *In other words there is an identification of all the X_d and in particular all the F_d as submanifolds of X_0, via the deck transformations.* This matches perfectly with our approach in the previous section where the Fredholm map was a self map on a Hilbert space attached to only one marked point on the curve. To be even more explicit, it is not right to think of the Fredholm map mapping a Hilbert space attached to one marked point, to one attached on another marked point. That would correspond in the toric model to considering a map $\gamma(z) = (\gamma_1(z), \dots, \gamma_n(z))$ where $\gamma_k(z) = \sum_{\nu=0}^{\int_d \alpha_k} {a_\nu}^k z^\nu$ and associating to it a Fredholm map from a Hilbert space at $(a_0^1, \dots, a_0^n)$ in X_0 to one at $(a^1_{\int_d \alpha_1}, \dots, a^n_{\int_d \alpha_n})$.

Let us also note that we are assuming $\int_d \alpha_k > 0$ for all k and d in the Kähler cone in order to only have kernel in our exact sequence . Otherwise we have also cokernel and this just means that we have obstructions in the deformation theory. From our point of view, it is not really a complication since we just work with the index bundle which is well defined in K-theory. This is an example, of how as described also in the general situation, in the previous section, we may have virtual bundles appearing.

There is a corresponding sequence involving negative normal bundles and the map $pr_- o\tilde{q}^{-d}$ that gives a formula where $\hbar$ changes to $-\hbar$. This is:

$$0 \to \mathcal{N}_{M_{-d}(X)} X_0 \to {\mathcal{N}_0}^- \to {\mathcal{N}_0}^- \to 0. \tag{93}$$

It is now time to give the proof of the proposition (3). We give the proof for $\widetilde{\mathcal{L}_N X}$ and the proof for $\widetilde{\mathcal{L}_\infty X}$ follows from taking $N = \infty$

Recall that $\widetilde{\mathcal{L}_N X} = {\mu_N}^{-1}(\lambda)/T^l$, where $\lambda \in \mathbb{R}^l$ has components $\lambda = (\lambda_1, \dots, \lambda_l)$. Therefore according to (68) we have

$$\frac{1}{2} \sum_{\nu=-N}^{N} \sum_{k=1}^{n} m_{1,k} |{a_\nu}^k|^2 = \lambda_1, \dots, \frac{1}{2} \sum_{\nu=-N}^{N} \sum_{k=1}^{n} m_{l,k} |{a_\nu}^k|^2 = \lambda_l. \tag{94}$$

Now the action functional H_N can be restricted to ${\mu_N}^{-1}(\lambda)$ by using the relations (94). Once we do this, then we have the function on $\widetilde{\mathcal{L}_N X}$ since H_N is invariant under the T_l action. In order to look in the normal directions of X_d and using the description of X_d found in (79), we work as follows:

$$2H_N(\gamma) = \sum_{\nu=-N}^{N} \nu(|a_\nu{}^1|^2 + \cdots + |a_\nu{}^n|^2) = \tag{95}$$

$$= \sum_{\nu \neq \int_d \alpha_1} \nu |a_\nu{}^1|^2 + \cdots + \sum_{\nu \neq \int_d \alpha_n} \nu |a_\nu{}^n|^2 + (\int_d \alpha_1)|a^1{}_{\int_d \alpha_1}|^2 + \cdots + (\int_d \alpha_n)|a^n{}_{\int_d \alpha_n}|^2.$$

Now substituting (78) in (95) we find :

$$2H_N = \sum_{\nu \neq \int_d \alpha_1} \nu |a_\nu{}^1|^2 + \cdots + \sum_{\nu \neq \int_d \alpha_n} \nu |a_\nu{}^n|^2 + \sum_{j=1}^{l} m_{j,1} d_j |a^1{}_{\int_d \alpha_1}|^2 + \cdots + \sum_{j=1}^{l} m_{j,n} d_j |a^n{}_{\int_d \alpha_n}|^2. \tag{96}$$

Rearranging this sum gives :

$$2H_N = \sum_{\nu \neq \int_d \alpha_1} \nu |a_\nu{}^1|^2 + \cdots + \sum_{\nu \neq \int_d \alpha_n} \nu |a_\nu{}^n|^2 + d_1 \sum_{k=1}^{n} m_{1,k} |a^k{}_{\int_d \alpha_k}|^2 + \cdots + d_l \sum_{k=1}^{n} m_{l,k} |a^k{}_{\int_d \alpha_k}|^2. \tag{97}$$

Now we may use (94) to obtain

$$2H_N = 2\sum_{j=1}^{l} d_j \lambda_j + \sum_{\nu \neq \int_d \alpha_1} \nu |a_\nu{}^1|^2 + \cdots + \sum_{\nu \neq \int_d \alpha_n} \nu |a_\nu{}^n|^2 -$$

$$- \sum_{k=1}^{n} \sum_{\nu \neq \int_d \alpha_k} d_1 m_{1,k} |a_\nu{}^k|^2 - \cdots - \sum_{k=1}^{n} \sum_{\nu \neq \int_d \alpha_k} d_l m_{l,k} |a_\nu{}^k|^2.$$

Rearranging the sum once more we find :

$$H_N = \sum_{j=1}^{l} d_j \lambda_j + \frac{1}{2} \sum_{\nu \neq \int_d \alpha_1} (\nu - \sum_{j=1}^{l} d_j m_{j,1}) |a_\nu{}^1|^2 + \cdots + \frac{1}{2} \sum_{\nu \neq \int_d \alpha_n} (\nu - \sum_{j=1}^{l} d_j m_{j,n}) |a_\nu{}^n|^2$$

and finally

$$H_N = \int_d \omega + \frac{1}{2} \sum_{\nu \neq \int_d \alpha_1} (\nu - \int_d \alpha_1) |a_\nu{}^1|^2 + \cdots + \frac{1}{2} \sum_{\nu \neq \int_d \alpha_n} (\nu - \int_d \alpha_n) |a_\nu{}^n|^2. \tag{98}$$

Notice that the fact that $\sum_{j=1}^{l} d_j \lambda_j = \int_d \omega$, follows from (70) and (72). Moreover since H_N is quadratic, computing the Hessian is immediate and the proof of the proposition is completed.

To be able to consider these Fredholm maps we need to work with infinite dimensional Hilbert spaces. The reason becomes clear if we try to consider the shifting map on a finite dimensional vector space. For example, while the forward shifting has only cokernel on a Hilbert space, on a finite dimensional vector space it also has kernel which in way is "artificial", at least from the geometric point of view.

We will see that we may get around this difficulty and work with finite dimensional spaces, especially in the case of toric manifolds by using the extra information that we have, namely the knowledge not just of the derivative of the deck transformation but the deck transformation itself. To be more precise, instead of using the derivative of the deck transformation and the induced map on normal bundles, we will use the deck transformation at the level of the bundles ${E^+}_{d,N}$ i.e. the bundle of homogeneous coordinates on $\widetilde{\mathcal{L}_N X}$.

We will prove the following finite dimensional version of the theorem::

Proposition 4. *Let X be a toric manifold. For every class $d \in H_2(X,\mathbb{Z})$, there is an integer $N(d)$ such that the ratio of equivariant Eüler classes,*

$$\frac{e_{S^1}({E^+}_{d,N}|_{X_0})}{e_{S^1}({E^+}_{0,N}|_{X_0})} = \frac{e_{S^1}({E^+}_{d,N}|_{X_0})}{e_{S^1}({\mathcal{N}^+}_{0,N})} \tag{99}$$

remains constant for all $N \geq N(d)$.

In other words this ratio of Eüler classes stabilizes. This allows us to define the *stable ratio.*

Definition 2. *Define the stable ratio $\frac{e_{S^1}({E^+}_d|_{X_0})}{e_{S^1}({E^+}_0|_{X_0})} = \frac{e_{S^1}({E^+}_d|_{X_0})}{e_{S^1}({\mathcal{N}^+}_0)}$ to be the common ratio $\frac{e_{S^1}({E^+}_{d,N}|_{X_0})}{e_{S^1}({E^+}_{0,N}|_{X_0})} = \frac{e_{S^1}({E^+}_{d,N}|_{X_0})}{e_{S^1}({\mathcal{N}^+}_{0,N})}$*
for all $N \geq N(d)$

Theorem 5. *Let X be a smooth toric variety of Picard number l, and assume that for every $d \in \check{\mathcal{K}} - \{0\}$ we have $\int_d c_1(T_X) > 0$. Then the stable ratio is equal to $e_{S^1}(\mathcal{N}_{M_d} X_0)$ and therefore if we let*

$$F = e^{(t_1 p_1 + \cdots + t_l p_l)/\hbar} \sum_{d \in \check{\mathcal{K}}} q^d \, \frac{e_{S^1}({E^+}_d|_{X_0})}{e_{S^1}({E^+}_0|_{X_0})}, \tag{100}$$

then

$$F = G, \tag{101}$$

where

$$G = e^{(t_1 p_1 + \cdots + t_l p_l)/\hbar}(1 + \sum_{d \in \check{\mathcal{K}}, d \neq 0} q^d ev_{1*}(\frac{1}{\hbar - \psi_1}))$$

is the function defined in (12) (since $q_i = e^{t_i}$).

We start by the proof of proposition (4): The normal bundle of the subset of $\mathbb{C}^{n2N}$ whose quotient by the T^l action is X_d, is trivial of course and has fiber coordinates given by all the variables in $\mathbb{C}^{n2N}$ except for $(a_{\int_d \alpha_1}, \ldots, a_{\int_d \alpha_1})$. The normal bundle of X_d is the quotient of the normal bundle to the subset $(a_{\int_d \alpha_1}, \ldots, a_{\int_d \alpha_1})$. Recall that $\tilde{L}_{k,\nu}$ is the bundle over $\widetilde{\mathcal{L}_N X}$ associated to ${a_\nu}^k$ and define

Of course $\tilde{L}_{k,\nu_1}$ is isomorphic to $\tilde{L}_{k,\nu_2}$ for all ν_1 and ν_2 if we don't take into account the S^1 action but they are different as S^1 equivariant bundles according to (76). Now let

$${L^d}_{k,\nu} = \tilde{L}_{k,\nu}|_{X_d}, \tag{102}$$

then we have that

$${L^d}_{k,\nu} \text{ isomorphic to } L_k \text{ for all } \nu, \tag{103}$$

where again the isomorphism is taken in the usual sense, not the S^1 equivariant.

It is now straight forward to compute the S^1 equivariant Eüler class of each of the line bundles. We have that, if

$$E^+{}_{d,N} = \bigoplus_{\nu>\int_d \alpha_1}^{N} \tilde{L}_{1,\nu} \oplus \cdots \oplus \bigoplus_{\nu>\int_d \alpha_n}^{N} \tilde{L}_{n,\nu}, \tag{104}$$

then

$$\mathcal{N}^+{}_{d,N} = E^+{}_{d,N}|_{X_d} = \bigoplus_{\nu>\int_d \alpha_1}^{N} L^d{}_{1,\nu} \oplus \cdots \oplus \bigoplus_{\nu>\int_d \alpha_n}^{N} L^d{}_{n,\nu}. \tag{105}$$

Note now, that our calculation from equation (98) gives the following information: since H_N is hamiltonian for the S^1 action, we may read off the weights of the action. We see then that the weights are shifted exactly so that the homogeneous coordinates of X_d have weight 0.

Finally since the S^1 action is given by (76) and using (98) we find that

$$e_{S^1}(L^d{}_{k,\nu}) = c_1(L_{k,\nu}) + (\nu - \int_d \alpha_k)\hbar = \alpha_k + (\nu - \int_d \alpha_k)\hbar, \tag{106}$$

and therefore

$$e_{S^1}(\mathcal{N}^+{}_{d,N}) = \prod_{\nu>0}^{N-\int_d \alpha_1} (\alpha_1 + \nu\hbar)\ldots \prod_{\nu>0}^{N-\int_d \alpha_n} (\alpha_n + \nu\hbar). \tag{107}$$

As a special case it follows that

$$e_{S^1}(\mathcal{N}^+{}_{0,N}) = \prod_{\nu>0}^{N}(\alpha_1 + \nu\hbar)\ldots \prod_{\nu>0}^{N}(\alpha_n + \nu\hbar). \tag{108}$$

We then compute the ratio

$$\frac{e_{S^1}(E^+{}_{d,N}|_{X_0})}{e_{S^1}(E^+{}_{0,N}|_{X_0})} = \frac{e_{S^1}(E^+{}_{d,N}|_{X_0})}{e_{S^1}(\mathcal{N}^+{}_{0,N})} = \tag{109}$$
$$\frac{\prod_{\nu=\int_d \alpha_1+1}^{N}(\alpha_1 + \nu\hbar)\ldots\prod_{\nu=\int_d \alpha_n+1}^{N}(\alpha_n + \nu\hbar)}{\prod_{\nu=1}^{N}(\alpha_1 + \nu\hbar)\ldots\prod_{\nu=1}^{N}(\alpha_n + \nu\hbar)} =$$
$$\frac{1}{\prod_{\nu=1}^{\int_d \alpha_1}(\alpha_1 + \nu\hbar)\ldots\prod_{\nu=1}^{\int_d \alpha_n}(\alpha_n + \nu\hbar)}.$$

It follows that indeed the ratio $\frac{e_{S^1}(E^+{}_{d,N}|_{X_0})}{e_{S^1}(E^+{}_{0,N}|_{X_0})}$ is independent of N as long as N is greater than $N(d) = max\{\int_d \alpha_1, \ldots, \int_d \alpha_n\}$. This concludes the proof of Proposition (4).

The proof of Theorem (4) follows now easily from the proof of Proposition (4). First note that the *stable ratio* is

$$\frac{e_{S^1}(E^+{}_d|_{X_0})}{e_{S^1}(E^+{}_0|_{X_0})} = \frac{1}{\prod_{\nu=1}^{\int_d \alpha_1}(\alpha_1 + \nu\hbar)\ldots\prod_{\nu=1}^{\int_d \alpha_n}(\alpha_n + \nu\hbar)}. \tag{110}$$

We may now compute the function F of (100). We find that

$$F = e^{(t_1 p_1 + \cdots + t_l p_l)/\hbar} \sum_{d \in \check{\mathcal{K}}} q^d \, \frac{e_{S^1}(E^+{}_d|_{X_0})}{e_{S^1}(E^+{}_0|_{X_0})} = \tag{111}$$

$$= e^{(t_1p_1+\dots+t_lp_l)/\hbar}\sum_{d\in\check{\mathcal{K}}} q^d \frac{1}{\prod_{\nu=1}^{\int_d\alpha_1}(\alpha_1+\nu\hbar)\dots\prod_{\nu=1}^{\int_d\alpha_n}(\alpha_n+\nu\hbar)},$$

where q^d stands for ${q_1}^{d_1}\dots {q_l}^{d_l} = e^{t_1d_1+\dots+t_ld_l}$ as usual.

Finally, according to Givental's computation of the function

$$G = e^{plnq/\hbar}(1+\sum_{d\in\check{\mathcal{K}},d\neq 0} q^d ev_{1*}(\frac{1}{\hbar - c}))$$

in [9] (Theorem (0.1), page (3) and its corollary: Example (a) page (4)) we have that, if $\int_d c_1(TX) > 0$ for all $d\in\check{\mathcal{K}}$ and $d\neq 0$, then the function F as computed above is indeed equal to the function G and therefore it generates the quantum $\mathcal{D}$ - module. This concludes the proof of theorem (4). It may finally be useful to consider a simple example in order to clarify things a bit more.

Example 1. *Let us consider the simplest example which is the complex projective space $\mathbb{P}^n$. Let ω be the class dual to a hyperplane. The Kähler cone is a half line and is generated by ω. The toric divisor classes α_i are also all equal to the class dual to a hyperplane. For d in $H_2(\mathbb{P}^n,\mathbb{Z})$ let $d_1 = \int_\omega d = \int_d \alpha_i$. Let $q_1 = e^{t_1}$. Then the function F of (100) becomes*

$$F = e^{t_1\omega/\hbar}\sum_{d_1=0}^{\infty} {q_1}^{d_1} \frac{1}{\prod_{\nu=1}^{d_1}(\omega+\nu\hbar)^{n+1}}.$$

We may now expand F in the basis $\{1,\omega,\omega^2,\dots,\omega^n\}$:

$$F = \sum_{i=0}^{n} f_i(e^{t_1},\hbar)\omega^i.$$

Let $< a,b >= \int_{\mathbb{P}^n} a\wedge b$ where a and b are cohomology classes in $\mathbb{P}^n$. Clearly we have $f_i =< F,\omega^{n-i} >$ Moreover notice that $< a,b >= Res_0 ab\frac{d\omega}{\omega^{n+1}}$. Therefore

$$f_i = Res_0\sum_{d_1=0}^{\infty} {q_1}^{d_1} \frac{\omega^{n-i}e^{t_1\omega/\hbar}}{\prod_{\nu=1}^{d_1}(\omega+\nu\hbar)^{n+1}}\frac{d\omega}{\omega^{n+1}}.$$

The easiest one to compute is f_0 :

$$f_0(e^{t_1},\hbar) = Res_0\sum_{d_1=0}^{\infty} {q_1}^{d_1} \frac{\omega^{n}e^{t_1\omega/\hbar}}{\prod_{\nu=1}^{d_1}(\omega+\nu\hbar)^{n+1}}\frac{d\omega}{\omega^{n+1}}.$$

Therefore

$$f_0(e^{t_1},\hbar) = \sum_{d_1=0}^{\infty}\frac{e^{t_1d_1}}{\hbar^{d_1(n+1)}(d_1!)^{n+1}}.$$

The function f_0 is annihilated by the differential operator $R(\hbar\frac{\partial}{\partial t_1}, e^{t_1},\hbar) = (\hbar\frac{\partial}{\partial t_1})^{n+1} - e^{t_1}$. The quantum $\mathcal{D}$ - module of $\mathbb{P}^n$ is the Heisenberg algebra modulo the ideal generated by R. Finally the corresponding relation in the quantum ring of $\mathbb{P}^n$ is $R(p,q,0) = 0$ i.e., $p^{n+1} = q$. Indeed, the quantum cohomology of $\mathbb{P}^n$ is $\mathbb{C}[p,q]/(p^{n+1} = q)$ where p^{n+1} is computed by the quantum multiplication and p is the class of the hyperplane. For a computation of the quantum cohomology in terms of the space of stable maps see for example [8]*. Notice also that the rest of the f_i are also annihilated by R. In fact we get a complete basis of solutions of the equation $R = 0$.*

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