NON-COMMUTATIVE IWASAWA THEORY

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1. Motivation

A typical problem motivating non-commutative Iwasawa theory is the following from the arithmetic of elliptic curves:

Let E be an elliptic curve over $\mathbb{Q}$, p an odd prime number and F_∞ a Galois extension of $\mathbb{Q}$ whose Galois group is a p-adic Lie group of dimension ≥ 1. How does the rank of the Mordell-Weil group $E(L)$ vary as L ranges over all finite extensions of $\mathbb{Q}$ contained in F_∞?

The classical commutative example of this situation is when F_∞ is obtained by adjoining all p-power roots of unity to $\mathbb{Q}$, i.e. $F_\infty = \mathbb{Q}(\mu_{p^\infty})$. In this case, a recent deep theorem of Kato asserts that the rank of $E(F_\infty)$ is finite. Perhaps the simplest non-commutative example is the field $F_\infty = \bigcup_{n\geq 0} \mathbb{Q}(\mu_{p^\infty}, a^{1/p^n})$, $a \in \mathbb{Q}^\times$, $a \neq \pm 1$. In this case the Galois group $G\big(F_\infty/\mathbb{Q}(\mu_p)\big) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$, a semidirect product of two copies of $\mathbb{Z}_p$.

Theorem 1. *(Hachimori-Venjakob) Let $F_m = \mathbb{Q}(\mu_{p^{m+1}}, a^{1/p^m})$, $m \geq 0$. Then*

$$\mathrm{rk}_{\mathbb{Z}}\big(E(F_m)\big) \leq c\, p^m\,,$$

where c is a constant independent of m.

2. Iwasawa theory

Let G be a profinite p-adic Lie group. The Iwasawa algebra of G, denoted by $\Lambda(G)$, is the completed group algebra

$$\Lambda(G) = \lim_{U\subset G} \mathbb{Z}_p[G/U]\,,$$

where U varies over the open normal subgroups of G. This Iwasawa algebra appears to be a very interesting example of a non-commutative algebra which is of great importance in number theory.

A torsion $\Lambda(G)$-module M is said to be **pseudonull** if $Ext^1_{\Lambda(G)}\big(M, \Lambda(G)\big) = 0$. Every M has a maximal pseudonull submodule which we denote by M°.

Theorem 2. *(Coates, Schneider, Sujatha) Let G be a compact p-valued Lie group in the sense of Lazard, and let M be any torsion, finitely generated left $\Lambda(G)$-module. Then there exist non-zero reflexive left-ideals $L_1, \ldots, L_m$ of $\Lambda(G)$ and an exact sequence*

$$0 \to \bigoplus_{i=1}^{m} \Lambda(G)/L_i \to M/M^\circ \to D \to 0$$

where D is a pseudonull $\Lambda(G)$-module.

This structure theorem is a generalization of an analogous theorem when $G = \mathbb{Z}_p^d$.

Numerical examples: Let $E/\mathbb{Q}$ be the curve $X_1(11)$ of conductor 11, let $p = 5$ and $F = \mathbb{Q}(E_{5^\infty})$. The Galois group $G = G(F_\infty/\mathbb{Q})$ is of dimension 4. Let $F_n = \mathbb{Q}(E_{5^{n+1}})$, then Coates-Howson show, using techniques from Iwasawa theory, that $\mathrm{rk}_{\mathbb{Z}} E(F_n) \leq 4(5^{3n} - 5^{2n-1})\ \forall n \geq 1$.

3. A conjecture

Definition. *The* **fine Selmer group** $R(E/F_\infty)$ *of* E/F_∞*, where* F_∞ *is a* p*-adic Lie extension, is defined to be*

$$\ker\left(H^1(F_\infty, E_{p^\infty}) \to \prod_\nu H^1(F_{\infty,\nu}, E_{p^\infty})\right)$$

where the map is the natural restriction map.

Let $Y(E/F_\infty)$ denote the compact Pontryagin dual of $R(E/F_\infty)$. This module has a natural structure as a finitely generated $\Lambda(G)$-module. It is conjectured to always be a torsion $\Lambda(G)$-module when $F_\infty \supset \mathbb{Q}(\mu_{p^\infty})$. We make the following stronger conjecture:

Conjecture (Coates–Sujatha). *For any* p*-adic Lie extension* $F_\infty/\mathbb{Q}$ *of dimension* > 1 *and containing* $\mathbb{Q}(\mu_{p^\infty})$*,* $Y(E/F_\infty)$ *is always pseudonull.*

Numerical example: Let $E/\mathbb{Q}$ be the elliptic curve $y^2 + xy = x^3 - 3x - 3$, of conductor $N = 150$. Let $p = 5$ and $F_\infty = \mathbb{Q}(E_{p^\infty})$. Again $G = G(F_\infty/\mathbb{Q})$ is a p-adic Lie group of dimension 4.

Theorem 3. *(Coates, Sujatha) In the above example, either* $Y(E/F_\infty)$ *is pseudonull as a* $\Lambda(G)$*-module or* $E(F_\infty)$ *has no point of infinite order.*

Remark. *We strongly suspect that the latter case in the theorem does not occur.*

References

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