MAASS THETA-SERIES AND HECKE OPERATORS

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To the Memory of Hans Maass (1911–1992)

Abstract. The survey part contains definitions and basic properties of harmonic theta-series, Siegel modular forms, and Hecke operators, reminder of the transformation formalism of general harmonic theta-series under modular transformations and Hecke operators with specialization to the case of Maass theta-seies (*). The main new results: infinite sequences of eigenfunctions for all regular Hecke operators on spaces of Maass theta-series are constructed; in the case of Maass theta-series of genus 2 all the eigenfunctions are determined and the relevant zeta functions are expressed as products of two L-functions of corresponding imaginary quadratic rings.

Introduction

The evident importance of the celebrated Shimura–Taniyama conjecture on explicit relation of Hasse zeta functions of elliptic curves and Hecke zeta functions of elliptic modular forms makes it quite natural to look for generalizations.

In 1980, H.Yoshida ([11], p. 243) had stated a two-dimentional analog of the Shimura–Taniyama conjecture. It asserts that the zeta-function of a two-dimentional Abelian variety over the field of rational numbers is (up to a finite number of Euler factors) the (Andrianov) zeta function of a cusp form of weight 2 for a congruence subgroup of the Siegel modular group of genus 2. Yoshida had also given an example supporting the conjecture. In 1993, R. Salvati-Manni and J. Top [10] had considered others interesting examples confirming the Yosida conjecture.

In view of possible relations of zeta functions of algebraic varieties and zeta functions of modular forms, it would be interesting to comput more zeta functions of concrete modular forms. The zeta functions of modular forms are defined as certain Euler products associated to common eigenfunctions of sufficiently complete

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families of Hecke operators. The most common sources for the search of eigenfunctions give spaces of theta-series of integral positive definite quadratic forms weighted by harmonic polynomials. General formulae for modular transformation of harmonic theta series and for their transformation under Hecke operators were obtained in our works [5] and [6], respectively, and in §1–§3 we propose a survey of the transformation formulae.

In 1977, H. Maass [9] had introduced important class of harmonic theta-series of the form

$$(*)\qquad \Theta_\nu(Z,\,Q,\,\chi)=\sum_{N\in\mathbb{Z}_m^m}\chi(\det N)(\det N)^\nu\exp(\pi i\mathrm{trace}({}^tNQNZ)),$$

where Z belongs to the Siegel upper half-plane of genus m, Q is the matrix of an integral positive definite quadratic form in m variables, χ a Dirichlet character, and $\nu=0,\,1$ (if $\nu=0$ it is understood that $\chi(\det N)(\det N)^\nu=\chi(\det N)$ for all N). In the same paper Maass had established automorphic properties of the series and indicated a number of cases, where the series are not identically zero. He had also suggested that "die Thetareihen (*) in der von Andrianov[1] begründeten Theorie der Dirichletreihen zu Modulformen zweiten Grades und ihrer Verallgemeinerung eine wichtige Rolle spilen werden". At that time I was rather young and did not pay much attention on the remark of the old man, besides, the action of Hecke operators on theta-series have not been yet investigated. Now, when I am approaching the age of Maass, I can see that he was right and can provide some positive evidences. In §3 (Proposition 7) we prove that if the proper class number of Q is 1, then, for every χ, the Maass series is an eigenfunction for all regular Hecke operators. Theorem 12 of §4 establishes that for $m=2$ linear combinations of the Maass series with characters of the group of proper classes of binary quadratic forms (with respect to the Gauss composition) are eigenfunctions for all regular Hecke operators with explicitely given eigenvalues. The proof is based on combination of explicit formulae for the action of Hecke opertors on harmonical theta-series and the technique relating Hecke operators on modular forms of genus 2 with Gauss composition of binary quadratic forms introduced in [1]. This result allows us to show that the (Andrianov) zeta function of each such eigenfunction is the product of two L–functions of the corresponding imaginary quadratic ring (§5, Theorem 15).

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Notation. We reserve the letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the set of positive rational integers, the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. $\mathbb{A}_n^m$ is the set of all $m \times n$-matrices with entries in a set $\mathbb{A}$, $\mathbb{A}^n = \mathbb{A}_1^n$, and $\mathbb{A}_n = \mathbb{A}_n^1$.

If M is a matrix, ${}^tM$ always denotes the transpose of M, if entries of M belong $\mathbb{C}$, $\overline{M}$ is the matrix with complex conjugate entries. For a symmetric matrix Q we write $Q[M] = {}^tMQM$ if the product on the right is defined.

We denote by

$$\mathbb{E}^m = \left\{ Q = (q_{\alpha\beta}) \in \mathbb{Z}_m^m \;\middle|\; {}^tQ = Q, \quad q_{11}, q_{22}, \dots, q_{mm} \in 2\mathbb{Z} \right\}$$

the set of all *even matrices* of order m, i.e. the set of matrices of integral quadratic forms in m variables

$$\mathbf{q}(X) = \frac{1}{2}\, {}^tXQX \qquad ({}^tX = (x_1, \dots, x_m)).$$

We recall that the *level* q of an invertible matrix $Q \in \mathbb{E}^m$ (and of the corresponding form) is the least positive integer satisfying $qQ^{-1} \in \mathbb{E}^m$, and the *divisor* d of a nonzero $Q \in \mathbb{E}^m$ (and of the corresponding form) is the greatest positive integer such that $d^{-1}Q \in \mathbb{E}^m$; the matrices and forms of divisor 1 are called *primitive.*

§1. Harmonic theta-series and Maass theta-series

Here we shall remind basic definitions and properties of harmonic theta-series of integral positive definite quadratic forms. For details and proofs see [5], [7], or [8].

Let $Q \in \mathbb{E}^m$ be the matrix of an integral positive definite quadratic form $\mathbf{q}$ in m variables. For $n \in \mathbb{N}$, let P be a polynomial on $\mathbb{C}_n^m$. Suppose that the polynomial is *homogeneous of exponent* ν *and harmonic with respect to the quadratic form* $\mathbf{q}$. The conditions on P mean that it satisfies

$$P(VA) = (\det A)^\nu P(V) \qquad (\forall\, V \in \mathbb{C}_n^m \text{ and } A \in GL_n(\mathbb{C}))$$

and is of the form $P(V) = P_0(SV)$, where S is a real $n \times n$-matrix such that $Q = {}^tSS$, and polynomial P_0 satisfies the Laplace equation

$$\Delta P_0 = \sum_{\alpha,\beta} \frac{\partial^2 P_0(V)}{(\partial v_{\alpha\beta})^2} = 0.$$

The *theta-series of genus* n *of the matrix* Q *with the coefficient polynomial* P *and characteristic*

$$L \in L_n(Q) = \left\{ L \in \mathbb{Z}_n^m / q\mathbb{Z}_n^m \;\middle|\; QL \equiv 0 \pmod{q} \right\},$$

where q is the *level* of Q, is defined by

$$\Theta_P(Z,Q|L) = \sum_{N\in\mathbb{Z}_n^m} P(N+q^{-1}L)e\{Q[N+q^{-1}L]Z\}, \tag{1.1}$$

where Z belongs to the *upper half-plane of genus* n,

$$Z \in \mathbb{H}^n = \left\{ Z = X + \sqrt{-1}Y \in \mathbb{C}_n^n \;\middle|\; {}^tZ = Z, \quad Y > 0 \right\}, \tag{1.2}$$

and, for a square matrix A, we write

$$e\{A\} = e^{\pi\sqrt{-1}\operatorname{trace} A}. \tag{1.3}$$

The theta-series converge absolutely and uniformly on compact subsets of $\mathbb{H}^n$. If U is an integral unimodular matrix of order m, then, since $U\mathbb{Z}_n^m = \mathbb{Z}_n^m$, by replacing N by UN in (1.1), we get the identities

$$\begin{aligned}\Theta_P(Z,\, Q|L) &= \sum_{N\in\mathbb{Z}_n^m} P(U(N+q^{-1}U^{-1}L))e\{Q[U(N+q^{-1}U^{-1}L)]Z\} \\ &= \Theta_{P|U}(Z,\, Q[U]|U^{-1}L) \qquad \forall U \in GL_m(\mathbb{Z}),\end{aligned} \tag{1.4}$$

where $(P|U)(V) = P(UV)$.

Examples. 1. The constant polynomial $P(V) = 1$ is clearly a harmonic polynomial of exponent $\nu = 0$ with respect to every quadratic form $\mathbf{q}$, and the theta-series (1.1) turns into the ordinary *theta-series*

$$\Theta_0(Z,\, Q|L) = \sum_{N\in\mathbb{Z}_n^m} e\{Q[N+q^{-1}L]Z\} \qquad (Z\in\mathbb{H}^n,\, L\in L_n(Q)) \tag{1.5}$$

of genus n *of the form* $\mathbf{q}$ (*or matrix* Q) *with characteristic* L.

2. If $n = m$, the polynomial

$$P(V) = \det V \qquad (V = (v_{\alpha\beta}) \in \mathbb{C}_m^m)$$

is clearly homogeneous of exponent 1 and is linear in each of m^2 variables $v_{\alpha\beta}$. It follows that the polynomial $P_0(T) = P(S^{-1}T)$ is also linear in each of the variables for every nonsingular $m\times m$–matrix S and so it satisfies the Laplace equation. Thus, the polynomial is homogeneous of exponent 1 and harmonic with respect to every positive definite quadratic form $\mathbf{q}(X)$ in m variables. The corresponding theta-series of genus m is

$$\begin{aligned}&\Theta_1(Z,\, Q|L) \\ &\quad= \sum_{N\in\mathbb{Z}_m^m} \det(N+q^{-1}L)e\{Q[N+q^{-1}L]Z\} \qquad (Z\in\mathbb{H}^m,\, L\in L_m(Q)).\end{aligned} \tag{1.6}$$

3. Maass theta-series (*) are linear combinations of series (1.5) with $n = m$ or series (1.6), depending on whether $\nu = 0$ or 1: if χ is a character modulo g, then we have

$$(1.7)\quad \Theta_\nu(Z, Q, \chi)$$
$$= \sum_{R \in \mathbb{Z}^m_m / g\mathbb{Z}^m_m} \chi(\det R) g^{\nu m} \sum_{N \in \mathbb{Z}^m_m} \det{}^\nu \left(N + \frac{qg}{qg^2} R\right) e\{g^2 Q \left[N + \frac{qg}{qg^2} R\right] Z\}$$
$$= g^{\nu m} \sum_{R \in \mathbb{Z}^m_m / g\mathbb{Z}^m_m} \chi(\det R) \Theta_\nu(Z, g^2 Q | qgR),$$

where q is the level of Q (note that the level of $g^2 Q$ is qg^2, and, for every $R \in \mathbb{Z}^m_m$, $g^2 Q(qgR) \equiv 0 \pmod{qg^2}$).

The following theorem is a particular case of well-known transformation formulas for harmonic theta-functions (see, for example, [5], Theorems 4.1-4.3). In order to formulate the theorem it is convenient to consider the theta-series $\Theta_P(Z, Q|L)$ with various characteristics $L \in L_n(Q)$ as entries of the *theta-column*

$$\boldsymbol{\Theta}_P(Z, Q) = \begin{pmatrix} \vdots \\ \Theta_P(Z, Q|L_\alpha) \\ \vdots \end{pmatrix}$$

of order

$$l_n(Q) = \#(L_n(Q)) \overset{L \to qQ^{-1}L'}{=} \#(\mathbb{Z}^m_n / Q\mathbb{Z}^m_n) = (\det Q)^n,$$

where we assume that all different residue classes of $L_n(Q)$ are ordered in a fixed way.

Theorem 1. *Let Q be even positive definite matrix of order m, and P a homogeneous polynomial on $\mathbb{C}^m_n$ of an integral nonnegative exponent ν and harmonic with respect to the quadratic form with matrix Q. Then the following assertions hold.*

(1). *For every integral symplectic matrix*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n = Sp_n(\mathbb{Z}) = \left\{ M \in \mathbb{Z}^{2n}_{2n} \;\middle|\; {}^tM J_n M = J_n \right\},$$

where

$$(1.8)\qquad J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix},$$

there is a square matrix $\Lambda_Q(M)$ of order l, independent of Z, such that the theta-column satisfies the functional equation

$$\mathbf{\Theta}_P\left((AZ+B)(CZ+D)^{-1},\, Q\right) = \det(CZ+D)^{\nu+m/2}\Lambda_Q(M)\mathbf{\Theta}_P(Z,\, Q).$$

If the order m is even, then the matrices $\Lambda_Q(M)$ satisfy the relations

$$\Lambda_Q(MM') = \Lambda_Q(M)\Lambda_Q(M') \qquad (M,\, M' \in \Gamma^n). \tag{1.9}$$

(2). *Suppose that*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^n(q) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \;\middle|\; C \equiv 0 \pmod{q} \right\}, \tag{1.10}$$

where q is the level of Q. Then the entries $\Lambda_Q(M)_{\alpha\beta}$ of the matrix $\Lambda_Q(M)$ can be defined by the formulae

$$\Lambda_Q(M)_{\alpha\beta} = \begin{cases} \mu_Q(M)e\{q^{-2}Q[L_\beta]\,{}^tDB\} & \text{if } L_\beta \equiv L_\alpha A \pmod{q} \\ 0 & \text{otherwise}, \end{cases}$$

where $\mu_Q(M)$ is an eight root of the unit.

(3). *If the level q of Q is equal to 1, then the order m of Q satisfies $m \equiv 0 \pmod 8$, and*

$$\mu_Q(M) = 1 \qquad (\forall\, M \in \Gamma^n),$$

but if $q > 1$ and m is even, then for every $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^n(q)$ the scalar $\mu_Q(M)$ can be computed by formula

$$\mu_Q(M) = \chi_Q(\det D),$$

where χ_Q is the real Dirichlet character modulo q, satisfying

$$\chi_Q(-1) = (-1)^{m/2},$$
$$\chi_Q(p) = \left(\frac{(-1)^{m/2}\det Q}{p}\right) \quad (\textit{the Legendre symbol})$$

if p is an odd prime number which does not divide q, and

$$\chi_Q(2) = 2^{-m/2} \sum_{r \in \mathbb{Z}^m/2\mathbb{Z}^m} e^{\frac{\pi i}{2}Q[r]}$$

if q is odd.

In principle, when m is even, the relations (1.9) enable one to evaluate multiplier matrices $\Lambda_Q(M)$ for all $M \in \Gamma^n$ if they are known for a system of generators of Γ^n. It is well known that the matrices

$$\begin{cases} U(D) = \begin{pmatrix} {}^tD^{-1} & 0 \\ 0 & D \end{pmatrix} & \text{with} \quad D \in GL_n(\mathbb{Z}), \\ T(B) = \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix} & \text{with} \quad B = {}^tB \in \mathbb{Z}_n^n \end{cases}$$

together with the matrix J_n generate the group Γ^n (see, for example, [2], Theorem 1.3.6). According to Theorem 1(2) or by direct computation, we obtain

$$\Theta_P({}^tD^{-1}ZD^{-1},\, Q|L) = \det(D)^{\nu+m/2}\chi_Q(\det D)\Theta_P(Z,\, Q|L\,{}^tD^{-1})$$

and

$$\Theta_P(Z+B,\, Q|L) = e\{q^{-2}Q[L]B\}\Theta_P(Z,\, Q|L),$$

where $L \in L_n(Q)$, $D \in GL_n(\mathbb{Z})$, and $B = {}^tB \in \mathbb{Z}_n^n$. As to the generator J_n, the inversion formulae for harmonic theta-functions (see [5], Lemma 5.1) give the formulae

$$\Theta_P(-Z^{-1},\, Q|L) = \det(-Z)^{\nu+m/2}\frac{i^{mn/2}}{\sqrt{(\det Q)^n}}\sum_{L' \in L_n(Q)} e\{2q^{-2}\,{}^tLQL'\}\Theta_P(Z,\, Q|L')$$

for $L \in L_n(Q)$.

§2. Modular forms

Here we shall remind basic definitions and properties of Siegel modular forms of integral weights. The *general real positive symplectic group of genus n* consisting of all real symplectic matrices of order $2n$ with positive multipliers,

$$\mathbb{G}^n = GSp_n^+(\mathbb{R}) = \left\{ M \in \mathbb{R}_{2n}^{2n} \;\middle|\; {}^tMJ_nM = \mu(M)J_n, \quad \mu(M) > 0 \right\},$$

is a real Lie group acting as a group of analytic automorphisms on the $n(n+1)/2$-dimensional open complex variety (1.2) by the rule

$$\mathbb{G}^n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto M\langle Z\rangle = (AZ+B)(CZ+D)^{-1} \quad (Z \in \mathbb{H}^n).$$

Acting on the upper half-plane $\mathbb{H}^n$, the general symplectic group operates also on complex-valued functions F on $\mathbb{H}^n$ by *Petersson operators of integral weights* k,

$$\mathbb{G}^n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : F \mapsto F|_k M = \det(CZ+D)^{-k} F(M\langle Z \rangle). \tag{2.1}$$

The Petersson operators satisfy the rules

$$F|_k MM' = (F|_k M)|_k M' \qquad (M, M' \in \mathbb{G}^n). \tag{2.2}$$

Let S be a subgroup of $\mathbb{G}^n$ commensurable with the modular group Γ^n, χ a *character* of S, that is a multiplicative homomorphism of S into nonzero complex numbers with the kernel of finite index in S, and k an integral number. A complex-valued function F on $\mathbb{H}$ is called a (*Siegel*) *modular form of weight* k *and character* χ *for the group* S, if the following conditions are fulfilled:

(i) F is a holomorphic function in $n(n+1)/2$ complex variables on $\mathbb{H}^n$;

(ii) For every matrix $M \in S$, the function F satisfies the functional equation

$$F|_k M = \chi(M) F, \tag{2.3}$$

where $|_k$ is the Petersson operator of weight k;

(iii) If $n = 1$, then every function $F|_k M$ with $M \in \Gamma^1$ is bounded on each subset of $\mathbb{H}^1$ of the form $\mathbb{H}^1_\varepsilon = \{x + iy \in \mathbb{H}^1 | y \geq \varepsilon\}$ with $\varepsilon > 0$.

The set $\mathfrak{M}_k(S, \chi)$ of all modular forms of weight k and character χ for the group S is clearly a linear space over the field $\mathbb{C}$. Each space $\mathfrak{M}_k(S, \chi)$ has finite dimension over $\mathbb{C}$.

For a positive integer q, we shall denote by

$$\Gamma^n(q) = \left\{ M \in \Gamma^n \;\middle|\; M \equiv 1_{2n} \pmod{q} \right\} \tag{2.4}$$

the *principal congruence subgroup of level* q of the modular group. $\Gamma^n(q)$ is clearly a normal subgroup of finite index of the modular group.

A subgroup S of the (*real*) *symplectic group*

$$\mathbb{S}^n = \left\{ M \in \mathbb{G}^n \;\middle|\; \mu(M) = 1 \right\}$$

is called a *congruence subgroup* if it contains a principal congruence subgroup as a subgroup of finite index. A character of such S is said to be a *congruence character* if it is trivial on a principal congruence subgroup contained in S. If S is a congruence subgroup and χ is a congruence character of S, then for each modular form F

of $\mathfrak{M}_k(S,\,\chi)$ and each matrix $M \in \mathbb{G}^n \cap \mathbb{Q}^{2n}_{2n}$ the function $F|_kM$ has a Fourier expansion of the form

$$(F|_kM)(Z) = \sum_{A\in\mathbb{E}^n,\,A\geq 0} f_M(A)e\{AZ/r\} \tag{2.5}$$

with a positive integer r depending on S, χ, and M, which converges absolutely on $\mathbb{H}^n$ and uniformly on every subset of the form

$$\mathbb{H}^n_\varepsilon = \left\{ Z = X + iY \in \mathbb{H}^n \;\middle|\; Y \geq \varepsilon 1_{2n} \right\} \qquad \text{with } \varepsilon > 0. \tag{2.6}$$

The modular form F is called a *cusp form*, if coefficients $f_M(A)$ of the expansion (2.5) satisfy the conditions

$$f_M(A) = 0 \qquad \text{for all } M \in \Gamma^n \quad \text{and } A \in \mathbb{E}^n \quad \text{with } \det A = 0.$$

The subspace of cusp forms of $\mathfrak{M}_k(S,\,\chi)$ will be denoted by $\mathfrak{N}_k(S,\,\chi)$.

Theorem 2. *Let Q be an even positive definite matrix of an even order m, and P a homogeneous polynomial on $\mathbb{C}^m_n$ of an integral nonnegative exponent ν harmonic with respect to the quadratic form with matrix Q. Then each of the theta-series of genus n of the matrix Q with the coefficient form P and characteristic $L \in L_n(Q)$ is a modular form of weight $k = \nu + m/2$ and the unit character for the group $\Gamma^n(q)$, where q is the level of Q:*

$$\Theta_P(Z,\,Q|L) \in \mathfrak{M}_{\nu+m/2}\,(\Gamma^n(q),\,1) = \mathfrak{M}^n_{\nu+m/2}(q). \tag{2.7}$$

If $\nu > 0$, then the theta-series is a cusp form:

$$\Theta_P(Z,\,Q|L) \in \mathfrak{N}_{\nu+m/2}\,(\Gamma^n(q),\,1) = \mathfrak{N}^n_{\nu+m/2}(q) \quad (\nu > 0). \tag{2.8}$$

Proof. Trivial estimates imply that each of the theta-series (1.1) converges absolutely on $\mathbb{H}^n$ and uniformly on its subsets of the form (2.6). It follows that each of the theta-series satisfies the condition (i) and, by Theorem 1(1), the condition (iii) of the definition of modular forms. By Theorem 1, the multiplier matrix $\Lambda_Q(M)$ is the unit matrix of order $l_n(Q)$ if $M \in \Gamma^n(q)$. Thus, each of the function $F(Z) = \Theta_P(Z,\,Q|L)$ satisfies

$$\begin{aligned}&(F|_kM)(Z)\\ &\qquad = \det(CZ+D)^{-k}F(M\langle Z\rangle) = F(Z) \quad \text{for every } M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma^n(q)\end{aligned}$$

and so satisfies the condition (ii) for $S = \Gamma^n(q)$ with $\chi = 1$.

In order to prove (2.8) we note first that the condition $\nu > 0$ implies that $P(T) = 0$ for every matrix $T \in \mathbb{C}_n^m$ with $rank(T) < n$. Really, in this case there is a nonzero complex n-column $\mathbf{b}$ such that $T\mathbf{b} = 0$. Let B be a nonsingular square matrix with the first column $\mathbf{b}$. Then the matrix TB has the zero first column, whence $TBA = TB$ for every matrix A of the form $A = \begin{pmatrix} a & 0 \\ 0 & 1_{n-1} \end{pmatrix}$ with arbitrary $a \in \mathbb{C}$. It follows that

$$P(TBA) = (\det A)^\nu P(TB) = a^\nu P(TB) = P(TB).$$

Thus, $P(TB) = 0$ and so $P(T) = (\det B)^{-\nu} P(TB) = 0$. Now, the Fourier expansion of the series (1.1) can be obviously written in the form

$$\Theta_P(Z, Q|L) = \sum_{A \in \mathbb{E}^n,\, A \geq 0} \left(\sum_{N \in \mathbb{Z}_n^m,\, qQ[N+q^{-1}L]=A} P(N + q^{-1}L) \right) e\{AZ/q\}. \tag{2.9}$$

Since $Q > 0$, the condition $\det A = \det(qQ[N + q^{-1}L]) = 0$ implies the inequality $rank(N + q^{-1}L) < n$, whence, by the above, $P(N + q^{-1}L) = 0$, and so the corresponding Fourier coefficient of the series (2.9) is equal to zero. Then, by Theorem 1(1), we conclude that the function $\Theta_P(Z, Q|L)|_k M$ for every $M \in \Gamma^n$ is a linear combination with constant coefficients of the series $\Theta_P(Z, Q|L_\alpha)$ and so, together with these series, has zero Fourier coefficients corresponding to singular arguments. $\triangle$

In the case of Maass theta-series (*) one can say more:

Proposition 3. *Let Q be an even positive definite matrix of even order m, g a positive integer, and χ a Dirichlet character modulo g. Then the Maass theta-series (*) are modular forms of weight $\nu + m/2$ for the group $\Gamma_0^m(qg^2)$ with the character $\chi\chi_Q$ given by*

$$\Gamma_0^m(qg^2) \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (\chi\chi_Q)(M) = \chi(\det D)\chi_Q(\det D): \tag{2.10}$$

$$\Theta_\nu(Z, Q, \chi) \in \mathfrak{M}_{\nu+m/2}(\Gamma_0^m(qg^2), \chi\chi_Q) \qquad (\nu = 0, 1),$$

where q is the level of Q and χ_Q the character of Q defined in Theorem 1.

If $\nu = 1$ then the series is a cusp form:

$$\Theta_1(Z, Q, \chi) \in \mathfrak{N}_{1+m/2}(\Gamma_0^m(qg^2), \chi\chi_Q).$$

Proof. Since the level of g^2Q is qg^2, by applying Theorem 1 for a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\Gamma_0^m(qg^2)$ to a theta-series in the right hand side of the relation (1.7), we obtain

$$\begin{aligned}\Theta_\nu(Z,\, g^2Q|qgR)|_{\nu+m/2}M \\ &= \chi_{g^2Q}(\det D)e\{(qg^2)^{-2}(g^2Q)[qgRA]\,{}^tDB\}\Theta_\nu(Z,\, g^2Q|qgRA) \\ &= \chi_Q(\det D)\Theta_\nu(Z,\, g^2Q|qgRA).\end{aligned}$$

Hence, since $A\,{}^tD \equiv 1_m \pmod{qg^2}$, we get

$$\Theta_\nu(Z,\, Q,\, \chi)|_{\nu+m/2}M = g^{\nu m} \sum_{R \in \mathbb{Z}_m^m/g\mathbb{Z}_m^m} \chi(\det R)\Theta_\nu(Z,\, g^2Q|qgR)|_{\nu+m/2}M$$

$$\begin{aligned}&= \chi_Q(\det D)g^{\nu m} \sum_{R \in \mathbb{Z}_m^m/g\mathbb{Z}_m^m} \chi(\det RA\,{}^tD)\Theta_\nu(Z,\, g^2Q|qgRA) \\ &= \chi_Q(\det D)\chi(\det {}^tD)g^{\nu m} \sum_{R \in \mathbb{Z}_m^m/g\mathbb{Z}_m^m} \chi(\det R)\Theta_\nu(Z,\, g^2Q|qgR) \\ &= (\chi\chi_Q)(M)\Theta_\nu(Z,\, Q,\, \chi).\end{aligned}$$

The rest follows from Theorem 2.△

§3. Hecke operators

Here we shall recall basic definitions concerning Hecke operators on the spaces $\mathfrak{M}_k^n(q)$ and specialize the transformation formulas of general harmonic theta-series under regular Hecke operators obtained in [6] to the cases of series $\Theta_P(Z,\, Q|L)$ and Maass theta-series. For details and proofs see the book [2], Chapters 3 and 4 and [6], §5.

Let Δ be a multiplicative semigroup and S a subgroup of Δ such that every double coset SMS of Δ modulo S is a finite union of left cosets SM'. Let us consider the vector space over a field, say, the field $\mathbb{C}$ of complex numbers consisting of all formal finite linear combinations with coefficients in $\mathbb{C}$ of symbols (SM) with $M \in \Delta$ being in one-to-one correspondence with left cosets SM of the set Δ modulo S. The group S naturally acts on this space by right multiplication defined on the symbols (SM) by

$$(SM)\gamma = (SM\gamma), \qquad (M \in \Delta,\, \gamma \in S).$$

We denote by

$$\mathcal{H}(S,\,\Delta) = HS_{\mathbb{C}}(S,\,\Delta)$$

the subspace of all S–invariant elements. The multiplication of elements of $\mathcal{H}(S,\,\Delta)$ given by the formula

$$(\sum_{\alpha} a_\alpha(SM_\alpha))(\sum_{\beta} b_\beta(SN_\beta)) = \sum_{\alpha,\beta} a_\alpha b_\beta(SM_\alpha N_\beta)$$

does not depend on the choice of representatives $M_\alpha \in SM_\alpha$ and $N_\beta \in SN_\beta$, and turns $\mathcal{H}(S,\,\Delta)$ into an associative algebra over $\mathbb{C}$ with the unity element $(S1_S)$, called *the Hecke–Shimura ring* or *HS–ring of Δ relative to S* (*over* $\mathbb{C}$). Elements

$$(M) = (M)_S = \sum_{M_i \in S\backslash SMS} (SM_i), \qquad (M \in \Delta) \tag{3.1}$$

being in one-to-one correspondence with double cosets of Δ modulo S belong to $\mathcal{H}(S,\,\Delta)$ and form a basis of the ring over $\mathbb{C}$. For brevity, the symbols (SM) and (M) will be refered as *left* and *double classes* (*of* Δ *modulo* S), respectively.

We consider the semigroup

$$\Sigma^n = \mathbb{G}^n \bigcap \mathbb{Z}^{2n}_{2n} = \left\{ M \in \mathbb{Z}^{2n}_{2n} \;\middle|\; {}^tMJ_nM = \mu(M)J_n, \quad \mu(M) > 0 \right\}$$

and its subsemigroups

$$\begin{cases} \Sigma^n_q = \left\{ M \in \Sigma^n \;\middle|\; \gcd(\mu(M), q) = 1 \right\} \\ \Sigma^n_0(q) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Sigma^n_q \;\middle|\; C \equiv 0 \pmod{q} \right\}, \\ \Sigma^n(q) = \left\{ M \in \Sigma^n_0(q) \;\middle|\; M \equiv \begin{pmatrix} \mu(M)1_n & 0 \\ 0 & 1_n \end{pmatrix} \pmod{q} \right\} \end{cases} \tag{3.2}$$

with $q \in \mathbb{N}$. It is easy to see that the semigroups satisfy the relation

$$\Sigma^n_0(q) = \Gamma^n_0(q)\Sigma^n(q) = \Sigma^n(q)\Gamma^n_0(q). \tag{3.3}$$

If S is a subgroup of finite index of the modular group Γ^n, then the pair S, Σ^n satisfies the conditions of the definition of HS–rings, as well as each pair S, Δ with $S \subset \Delta \subset \Sigma^n$, and we can define corresponding Hecke–Shimura rings $\mathcal{H}(S,\,\Delta)$. We shall say that a group S satisfying $\Gamma^n(q) \subset S \subset \Gamma^n$ is *q-symmetric* if $S\Sigma^n(q) = \Sigma^n(q)S$. For such S the HS–ring

$$\mathcal{H}_{qr}(S) = \mathcal{H}(S,\,R(S)) \quad \text{with} \quad R(S) = S\Sigma^n(q) = \Sigma^n(q)S$$

is called the *q-regular HS–ring of S*. According to Theorem 3.3.3 of [2], all q-regular HS–rings for given genus n are isomorphic to each other. In particular, by (3.3), the group $\Gamma_0^n(q)$ is q-symmetric, and the q-regular HS–rings of $\Gamma_0^n(q)$ and $\Gamma^n(q)$,

$$\begin{aligned}\mathcal{H}_0^n(q) &= \mathcal{H}_{qr}(\Gamma_0^n(q)) = \mathcal{H}(\Gamma_0^n(q),\, \Sigma_0^n(q)),\\ \mathcal{H}^n(q) &= \mathcal{H}_{qr}(\Gamma^n(q)) = \mathcal{H}(\Gamma^n(q),\, \Sigma^n(q))\end{aligned} \tag{3.4}$$

are isomorphic. The isomorphism can be defined as follows. Let

$$T' = \sum_\alpha a_\alpha(\Gamma_0^n(q)M_\alpha) \in \mathcal{H}_0^n(q), \tag{3.5}$$

where the left classes $(\Gamma_0^n(q)M_\alpha)$ with $a_\alpha \neq 0$ are pairwise distinct. By (3.3), without loss of generality one may assume that all the representatives M_α of the left cosets $\Gamma_0^n(q)M_\alpha$ belong to $\Sigma^n(q)$. Then, obviously, we have

$$\eta : T' \mapsto T = \sum_\alpha a_\alpha(\Gamma^n(q)M_\alpha) \in \mathcal{H}^n(q), \tag{3.6}$$

and the map η is a homomorphic embedding of the ring $\mathcal{H}_0^n(q)$ into $\mathcal{H}^n(q)$. In fact, η is a ring isomorphism, and the inverse isomorphism $\xi : \mathcal{H}^n(q) \mapsto \mathcal{H}_0^n(q)$ is determined by the condition

$$\xi : \sum_\beta b_\beta(\Gamma^n(q)N_\beta) \mapsto \sum_\beta b_\beta(\Gamma_0^n(q)N_\beta). \tag{3.7}$$

The isomorphism of the rings allows one to transfer various constructions from one of the rings to another. For example, *Zharkovskaya homomorphisms* $\Psi^{n,r} = \Psi^{n,r}_{k,\chi}$ of the ring $\mathcal{H}_0^n(q)$ to $\mathcal{H}_0^r(q)$, where $n > r \geq 1$, k is an integer, and χ a Dirichlet character modulo q satisfying $\chi(-1) = (-1)^k$, can be carried out to the rings $\mathcal{H}^n(q)$, so that we get the homomorphisms

$$\Psi^{n,r} = \Psi^{n,r}_{k,\chi} : \mathcal{H}^n(q) \mapsto \mathcal{H}^r(q) \qquad (n > r \geq 1) \tag{3.8}$$

satisfying the condition of commutativity of the diagramm

$$\begin{array}{ccc} \mathcal{H}_0^n(q) & \xrightarrow{\ \eta\ } & \mathcal{H}^n(q) \\ {\scriptstyle \Psi^{n,r}}\big\downarrow & & {\scriptstyle \Psi^{n,r}}\big\downarrow \\ \mathcal{H}_0^r(q) & \xrightarrow{\ \eta\ } & \mathcal{H}^r(q) \end{array} . \tag{3.9}$$

We remind that the Zharkovskaya map from genus n to genus r,

$$\Psi^{n,r} = \Psi^{n,r}_{k,\chi} : \mathcal{H}_0^n(q) \mapsto \mathcal{H}_0^r(q), \tag{3.10}$$

can be defined in the following way. Let $T' \in \mathcal{H}_0^n(q)$ be an element of the form (3.5). One can assume that each representative $M_\alpha \in \Gamma_0^n(q)\backslash\Sigma_0^n(q)$ is choosen in the form

$$M_\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ 0 & D_\alpha \end{pmatrix} \quad \text{with } D_\alpha = \begin{pmatrix} D'_\alpha & * \\ 0 & D''_\alpha \end{pmatrix} \quad \text{and } D'_\alpha \in \mathbb{Z}_r^r.$$

If $A_\alpha = \begin{pmatrix} A'_\alpha & * \\ * & * \end{pmatrix}$ and $B_\alpha = \begin{pmatrix} B'_\alpha & * \\ * & * \end{pmatrix}$ with $r \times r$-blocks A'_α and B'_α, then

$$M'_\alpha = \begin{pmatrix} A'_\alpha & B'_\alpha \\ 0 & D'_\alpha \end{pmatrix} \in \Sigma_0^r(q),$$

and we put

$$\Psi_{k,\chi}^{n,r}(T') = \sum_\alpha a_\alpha |\det D''_\alpha|^{-k} \chi^{-1}(|\det D''_\alpha|)(\Gamma_0^r(q)M'_\alpha).$$

Then, in view of commutativity of the diagramm (3.9), we can define the Zharkovskaya map (3.8) by

$$\Psi^{n,r}(T) = (\eta\Psi'\xi)(T) \in \mathcal{H}^r(q) \qquad (T \in \mathcal{H}^n(q))$$

with maps η and ξ defined by (3.6) and(3.7), where Ψ' is the map (3.10).

Hecke–Shimura rings act on modular forms by means of linear representation given by *Hecke operators.* Let, say, F belongs to the space $\mathfrak{M}_k(S) = \mathfrak{M}_k(S,\, 1)$ of modular forms of integral weight k and the trivial character $\chi = 1$ for a subgroup S of finite index in the modular group Γ^n, and let

$$T = \sum_i a_\alpha(SM_\alpha) \in \mathcal{H}(S,\, \Sigma^n).$$

Then, as it easily follows from definitions of modular forms and HS–rings, and the properties (2.2), (2.3) of the Petersson operators (2.1), the function

$$F\|T = F\|_k T = \sum_\alpha a_\alpha F|_k M_\alpha$$

does not depend on the choice of representatives $M_\alpha \in SM_\alpha$ and again belongs to the space $\mathfrak{M}_k(S)$. These operators are called *Hecke operators* (*of weight k for the group S*). The Hecke operators corresponding to elements of regular HS–rings are called *regular.* It follows from the definition of multiplication in HS–rings and (2.2) that the map $T \mapsto \|T$ is a linear representation of the ring $\mathcal{H}(S)$ on the space $\mathfrak{M}_k(S)$. The subspace of cusp forms $\mathfrak{N}_k(S)$ is invariant with respect to all of the Hecke operators.

The following theorem expresses the images of harmonic theta-series under regular Hecke operators in the form of linear combinations of similar theta-series with the coefficients explicitely given in terms of certain trigonometric sums.

Theorem 4. *Let Q be an even positive definite matrix of even order m and level q, and let P be a homogeneous polynomial on $\mathbb{C}_n^m$ of exponent ν and harmonic with respect to the quadratic form with matrix Q. Suppose that*

$$T = \sum_{\alpha} a_{\alpha}(\Gamma^n(q)M_{\alpha})$$

is an element of the Hecke–Shimura ring $\mathcal{H}^n(q)$ with $n \geq 1$ such that all representatives $M_{\alpha} \in \Sigma^n(q)$ have the same multiplier $\mu(M_{\alpha}) = \mu > 0$. Let us assume also that in the case $n < m$ the element T belongs to the image of the ring $\mathcal{H}^m(q)$ under the Zharkovskaya map (3.8) from genus m to genus n with $k = m/2$ and the character $\chi = \chi_Q$ defined in Theorem 1,

$$T = \Psi^{m,n}_{m/2,\chi_Q}(\widetilde{T}) \qquad \text{with} \quad \widetilde{T} \in \mathcal{H}^m(q) \quad (n < m).$$

Then the image of a theta-series (1.1) under the Hecke operator $\|_{\nu+m/2}T$ can be written in the form

$$\begin{aligned}(3.11)\quad &\Theta_P(Z,\, Q|L)\|_{\nu+m/2}T \\ &= \sum_{D \in \Delta(Q,\mu)/\Lambda} I(D, Q, \Psi^{n,m}(T))\Theta_{P|\mu^{-1}D}(Z,\, \mu^{-1}Q[D]|\mu D^{-1}L),\end{aligned}$$

where

$$\begin{aligned}(3.12)\quad &\Delta(Q,\mu) \\ &= \left\{ D \in \mathbb{Z}_m^m \;\middle|\; \det D = \mu^{m/2},\ \mu^{-1}Q[D] \in \mathbb{E}^m \right\}, \quad \Lambda = \Lambda^m = SL_m(\mathbb{Z}),\end{aligned}$$

$$\Psi^{n,m}(T) = \Psi^{n,m}_{m/2,\chi_Q}(T) = \begin{cases} \Psi^{n,m}_{m/2,\chi_Q}(T) & \text{if } n > m \\ T & \text{if } n = m \\ \widetilde{T} \in (\Psi^{m,n}_{m/2,\chi_Q})^{-1}(T) & \text{if } n < m, \end{cases}$$

$I(D, Q, \widetilde{T})$ with $\widetilde{T} \in \mathcal{H}^m(q)$ are the trigonometric sums defined by the following conditions: if

$$T' = \xi(\widetilde{T}) = \sum_{\beta} b_{\beta}\left(\Gamma_0^m(q)\begin{pmatrix} A_{\beta} & B_{\beta} \\ 0 & D_{\beta}\end{pmatrix}\right) \in \mathcal{H}_0^m(q)$$

with ${}^tA_{\beta}D_{\beta} = \mu 1_m$, then

$$\begin{aligned}(3.13)\quad &I(D, Q, \widetilde{T}) = I(D, Q, T') \\ &= \sum_{\beta;\, D\,{}^tD_{\beta} \equiv 0 \pmod{\mu}} b_{\beta}|\det D_{\beta}|^{-m/2}\chi_Q^{-1}(|\det D_{\beta}|)e\{\mu^{-2}Q[D]\,{}^tD_{\beta}B_{\beta}\},\end{aligned}$$

$e\{\cdots\}$ being the exponent (1.3), and where

$$(P|\mu^{-1}D)(V) = P(\mu^{-1}DV).$$

In particular, the image of the theta-series (1.5) with $n = m$ and (1.6) under the Hecke operator $\|_{\nu+m/2}T$ with $T \in \mathcal{H}^m(q)$ can be written in the form

$$\begin{aligned}(3.14)\quad \Theta_\nu(Z,\, Q|L)\|_{\nu+m/2}T \\ = \mu^{-\nu m/2} \sum_{D\in\Delta(Q,\mu)/\Lambda} I(D,Q,T)\Theta_\nu(Z,\, \mu^{-1}Q[D]|\mu D^{-1}L),\end{aligned}$$

where $\nu = 0$ in the first case, and $\nu = 1$ in the second.

Proof. The formulae (3.11) were proved in [6], Theorem 5.3 with the sums taken over representatives sets

$$\left\{D \in \mathbb{Z}_m^m \;\middle|\; |\det D| = \mu^{m/2},\, \mu^{-1}Q[D] \in \mathbb{E}^m\right\}/GL_m(\mathbb{Z}).$$

Since, clearly, each such set can be replaced by the set $\{D \in \Delta(Q,\mu)/\Lambda\}$, the formulas follow.

As to the formulas (3.14), it is sufficient to note that $\det(\mu^{-1}D) = \mu^{-m/2}$ for all $D \in \Delta(Q,\mu)$. $\triangle$

In connection with the formulae (3.11) we note that

$$(3.15)\qquad \mu D^{-1} \in \Delta\left(\mu^{-1}Q[D],\, \mu\right) \quad \text{if} \quad D \in \Delta(Q,\, \mu) \quad \text{and} \quad \gcd(\mu, q) = 1.$$

Indeed, let us set $D' = \mu D^{-1}$ and $Q' = \mu^{-1}Q[D]$. Then the matrix D' can be also written as $\{Q'\}^{-1}\,{}^tDQ$. It follows that the products of D' by $\det D = \mu^{m/2}$ or $\det Q' = \det Q$ are both integral. Since the numbers are coprime, D' is integral. The inclusion follows.

Proposition 5. *Let Q be an even positive definite matrix of even order m, g a positive integer, and χ a Dirichlet character modulo g. Suppose that*

$$T = \sum_\alpha a_\alpha(\Gamma^n(qg^2)M_\alpha)$$

is an element of the regular Hecke–Shimura ring $\mathcal{H}^m(qg^2)$ such that all representatives $M_\alpha \in \Sigma^n(qg^2)$ (see (3.2)) have the same multiplier $\mu(M_\alpha) = \mu$. Then the *image of the Maass theta-series (*) under the Hecke operator* $\|_{\nu+m/2}T$ *is again a linear combination of Maass theta-series of the form*

$$\begin{aligned}(3.16)\quad & \Theta_\nu(Z,\,Q,\,\chi)\|_{\nu+m/2}T \\ &= \chi(\mu)^{-m/2}\mu^{-\nu m/2}\sum_{D\in\Delta(Q,\mu)/\Lambda} I(D,g^2Q,T)\Theta_\nu(Z,\,\mu^{-1}Q[D],\,\chi),\end{aligned}$$

where $I(\cdots)$ *are the trigonometric sums* (3.13).

Proof. By (1.7) and (3.14), we have

$$\Theta_\nu(Z,\,Q,\,\chi)\|_{\nu+m/2}T = g^{\nu m}\sum_{R\in\mathbb{Z}_m^m/g\mathbb{Z}_m^m}\chi(\det R)\Theta_\nu(Z,\,g^2Q|qgR)\|_{\nu+m/2}T$$

$$\begin{aligned}&= \mu^{-\nu m/2}g^{\nu m}\sum_{R\in\mathbb{Z}_m^m/g\mathbb{Z}_m^m}\chi(\det R)\\ &\qquad\times\sum_{D\in\Delta(g^2Q,\mu)/\Lambda} I(D,g^2Q,T)\Theta_\nu(Z,\,\mu^{-1}g^2Q[D]|qg\mu D^{-1}R)\end{aligned}$$

$$\begin{aligned}&= \mu^{-\nu m/2}g^{\nu m}\sum_{D\in\Delta(g^2Q,\mu)/\Lambda} I(D,g^2Q,T)\\ &\qquad\times\sum_{R\in\mathbb{Z}_m^m/g\mathbb{Z}_m^m}\chi(\det\mu D^{-1}R)\chi(\det\mu D^{-1})^{-1}\Theta_\nu(Z,\,\mu^{-1}g^2Q[D]|qg\mu D^{-1}R)\end{aligned}$$

$$\begin{aligned}&= \chi(\mu)^{-m/2}\mu^{-\nu m/2}\sum_{D\in\Delta(Q,\mu)/\Lambda} I(D,g^2Q,T)\\ &\qquad\times g^{\nu m}\sum_{R'\in\mathbb{Z}_m^m/g\mathbb{Z}_m^m}\chi(\det R')\Theta_\nu(Z,\,\mu^{-1}g^2Q[D]|qgR')\end{aligned}$$

$$= \chi(\mu)^{-m/2}\mu^{-\nu m/2}\sum_{D\in\Delta(Q,\mu)/\Lambda} I(D,g^2Q,T)\Theta_\nu(Z,\,\mu^{-1}Q[D],\,\chi),$$

since clearly $\Delta(g^2Q,\mu) = \Delta(Q,\mu)$ if μ is prime to qg^2, the matrix $R' = \mu D^{-1}R$ runs together with R the set $\mathbb{Z}_m^m/g\mathbb{Z}_m^m$, and the level of $\mu^{-1}Q[D]$ with $D\in\Delta(Q,\mu)$ is again q. $\triangle$

It is sometimes convenient to rewrite the above transformation formulae in somewhat different form. For this we have to do several preliminary remarks. We shall say that an even matrix Q' is *properly similar* to a nonsingular matrix $Q \in \mathbb{E}^m$, $Q' \sim_p Q$, if it can be written in the form $Q' = \mu^{-1}Q[D]$ with $D \in \Delta(Q, \mu)$, where μ is coprime with the level q of Q. It follows from (3.15) that the relation of proper similarity is symmetric. Besides, it is clearly reflexive and transitive. Thus, the set of all nonsingular matrices of $\mathbb{E}^m$ is disjoint union of *ps-classes*

$$ps\{Q\} = \left\{Q' \in \mathbb{E}^m \;\middle|\; Q' \sim_p Q\right\} \qquad (Q \in \mathbb{E}^m,\, \det Q \neq 0).$$

It is easy to see that all matrices of the ps-class of a matrix Q have the same signature, determinant, level, and divisor as Q (see Notation). Further, we recall that two matrices Q, Q' of $\mathbb{E}^m$ are said to be *properly equivalent*, $Q \simeq_p Q'$, if $Q' = Q[U]$ with $U \in \Lambda = SL_m(\mathbb{Z})$. All matrices properly equivalent to a matrix Q form the *pe-class of* Q,

$$pe\{Q\} = \left\{Q' = Q[U] \;\middle|\; U \in \Lambda\right\}. \tag{3.17}$$

According to the reduction theory of integral quadratic forms, the ps-class of every nonsingular matrix $Q \in \mathbb{E}^m$ is a finite union of the pe-classes:

$$ps\{Q\} = \bigcup_{j=1}^{h_p(Q)} pe\{Q_j\}.$$

The quantity $h_p(Q)$ we shall call the *proper class number of* Q.

Proposition 6. *With the notation and assumptions of Theorem* 4 *formula* (3.11) *can be rewritten in the form*

$$\begin{aligned}\Theta_P(Z,\, Q|L)\|_{\nu+m/2}T \\ = \sum_{j=1}^{h_p(Q)} \sum_{D \in R_p(Q, \mu Q_j)/E_p(Q_j)} I(D, Q, \Psi^{n,m}(T))\Theta_{P|\mu^{-1}D}(Z,\, Q_j|\mu D^{-1}L),\end{aligned} \tag{3.18}$$

where $Q_1, \dots, Q_{h_p(Q)}$ is a system of representatives of all different pe-classes contained in the ps-class of Q,

$$R_p(Q,\, Q') = \left\{M \in \mathbb{Z}_m^m \;\middle|\; Q[M] = Q',\, \det M > 0\right\} \qquad (Q,\, Q' \in \mathbb{E}^m)$$

is the set of all proper integral representations of Q' by Q, and

$$E_p(Q') = R_p(Q',\, Q') \qquad (Q' \in \mathbb{E}^m,\, \det Q' \neq 0)$$

denotes the group of all proper units of Q'.

In particular, the formula (3.14) *can be rewritten in the form*

$$(3.19)\quad \Theta_\nu(Z,\, Q|L)\|_{\nu+m/2}T = \mu^{-\nu m/2}\sum_{j=1}^{h_p(Q)}\sum_{D\in R_p(Q,\mu Q_j)/E_p(Q_j)} I(D,Q,T)\Theta_\nu(Z,\, Q_j|\mu D^{-1}L).$$

Proof. It is an easy consequense of definitions that the sums (3.13) are independent of the choice of representatives in the decomposition of T' and satisfy

$$(3.20)\qquad I(UDU',\, Q,\, T') = I(D,\, Q[U],\, T') \quad \text{for all } U,\, U' \in GL_m(\mathbb{Z})$$

(see [6], Lemma 3.1). From (1.4) and (3.20) we conclude that the term of the sum on the right in (3.11) corresponding to a matrix $D \in \Delta(Q,\, \mu)$ depends only on the coset $D\Lambda$. On the other hand, each of the matrices $\mu^{-1}Q[D]$ with $D \in \Delta(Q,\, \mu)$ is properly similar to Q and so is properly equivalent to one of the matrices $Q_1, \dots, Q_{h_p(Q)}$, say, $\mu^{-1}Q[D] \simeq_p Q_j$. This means that $\mu^{-1}Q[D][U] = \mu^{-1}Q[DU] = Q_j$ with $U \in \Lambda$. On replacing D by DU, we can assume that $D \in R_p(Q,\, Q_j)$. If $D' = DU'$ is another such matrix, then, clearly, $Q_j[U'] = Q_j$, whence $U' \in E_p(Q_j)$. This proves the formulae. $\triangle$

Similarly, in the same notations we have the following formulae for the action of regular Hecke operators on Maass theta-series:

Proposition 7. *In the notation and assumption of Proposition* 5 *the formulae* (3.16) *can be rewritten in the form*

$$(3\text{-}21)\quad \Theta_\nu(Z,\, Q,\, \chi)\|_{\nu+m/2}T = \chi(\mu)^{-m/2}\mu^{-\nu m/2}\sum_{j=1}^{h_p(Q)}\sum_{D\in R_p(Q,\mu Q_j)/E_p(Q_j)} I(D,g^2Q,T)\Theta_\nu(Z,\, Q_j,\, \chi),$$

where $Q_1, \dots, Q_{h_p(Q)}$ *is a system of representatives of all different pe-classes contained in the ps-class of* Q, *and where sets* $R_p(Q,Q')$, $E_p(Q')$ *were defined in Proposition* 6.

In particular, if $h_p(Q) = 1$, *then, for each* $\nu = 0,\, 1$ *and every Dirichlet character* χ, *the Maass series* $\Theta_\nu(Z,\, Q,\, \chi)$ *is an eigenfunction of the operator* $\|_{\nu+m/2}T$ *with the eigenvalue*

$$\chi(\mu)^{-m/2}\mu^{-\nu m/2}\sum_{D\in R_p(Q,\mu Q)/E_p(Q)} I(D,g^2Q,T).$$

We conclude this section with summary of known at results of computation of the sums (3.13). We remind that the HS–ring $\mathcal{H}_0^n(q)$ is generated over $\mathbb{C}$ by the following double classes of the form (3.1):

$$T^n(p) = \big(diag(\underbrace{1,\dots,1}_{n},\underbrace{p,\dots,p}_{n})\big)_{\Gamma_0^n(q)}, \tag{3.22}$$

and

$$T_j^n(p^2) = \big(diag(\underbrace{1,\dots,1}_{n-j},\underbrace{p,\dots,p}_{j},\underbrace{p^2,\dots,p^2}_{n-j},\underbrace{p,\dots,p}_{j})\big)_{\Gamma_0^n(q)} \quad (j=1,\dots,n), \tag{3.23}$$

where p runs over all prime numbers not dividing q (see [2], Theorem 3.3.23). It follows that the ring $\mathcal{H}^n(q)$ is generated over $\mathbb{C}$ by the classes

$$\widetilde{T}^n(p) = \eta(T^n(p)) \quad \text{and } \widetilde{T}_j^n(p^2) = \eta(T_j^n(p^2)) \quad (j=1,\dots,n), \tag{3.24}$$

where η is the map (3.6) and p again runs over all prime numbers not dividing q.

Proposition 8. *Let Q be an even positive definite matrix of even order m, q the level of Q, and χ_Q the Dirichlet character corresponding to Q. Then the sums (3.13) on elements (3.22)–(3-24) for each prime number p not dividing q can be computed by the following formulae:*

$$I(D,\,Q,\,\widetilde{T}^m(p)) = I(D,\,Q,\,T^m(p)) = p^{m/2}\prod_{j=1}^{m/2}(1+\chi_Q(p)p^{-j})$$

for all $D \in \Delta(Q,\,p)$;

$$I(D,\,Q,\,\widetilde{T}_{m-1}^m(p)) = I(D,\,Q,\,T_{m-1}^m(p))$$

$$= \begin{cases} \chi_Q(p)p^{(2+m-m^2)/2}\dfrac{(p^m-1)}{p-1} + p^{-m^2/2}(\chi_Q(p)p^{m/2}-1) & \textit{if } D \in \Lambda(p1_m), \\ \chi_Q(p)p^{(2+m-m^2)/2} \quad \textit{if } D \in \Lambda diag(1,\underbrace{p,\dots,p}_{m-2},p^2)\Lambda, \\ 0 \quad \textit{otherwise}; \end{cases}$$

$$I(D,\,Q,\,\widetilde{T}_m^m(p^2)) = I(D,\,Q,\,T_m^m(p^2)) = \begin{cases} p^{-m^2/2} & \textit{if } D \in \Lambda(p1_m), \\ 0 & \textit{otherwise}\ . \end{cases}$$

Proof. In [3] (formula (2.19) and Lemma 5.1) the sums $\gamma(Q, D, T)$ similar to the sums (3.11) were defined and computed for $T = T^m(p)$. In §2 of [4] the sums $\gamma(Q, D, T)$ were, in fact, computed for $T = T^m_m(p^2) = (p1_{2m})_{\Gamma_0^m(q)}$ and $T = T^m_{m-1}(p^2)$ (see also Lemma 3.3.32 of [2] for the presentation of $T^m_{m-1}(p^2)$ used in [4]). It directly follows from definitions of the sums that $I(D, Q, T) = \mu^{m/2}\gamma(Q, \mu D^{-1}, T)$. The rest is clear. △

Note that the formulae of Proposition 7 determine sums $I(D, Q, T)$ for all generators of the rings $\mathcal{H}_0^2(q)$ and $\mathcal{H}^2(q)$. We do not know any other results on computation of the sums $I(D, Q, T)$.

§4 Theta-series of binary forms with characters of class groups

In this section we shall construct eigenfunctions of regular Hecke operators in the form of linear combinations of Maass theta-series of binary quadratic forms with characters of class groups of modules in the corresponding quadratic fields.

At first we shall recall basic definitions and facts concerning to modules in algebraic number fields and relations quadratic modules with binary quadratic forms. For more details and proofs see, for example, [2], Appendix 3.

Let $\mathbf{K}$ be a (finite) algebraic number field, i.e. a finite extension of the field $\mathbb{Q}$. Any finitely generated $\mathbb{Z}$–submodule $\mathbf{M}$ of $\mathbf{K}$ is called a *module* in $\mathbf{K}$. Two modules $\mathbf{M}$ and $\mathbf{M}'$ are said to be *similar* if $\mathbf{M}' = \alpha\mathbf{M}$ for some $\alpha \neq 0$ of $\mathbf{K}$. All modules in $\mathbf{K}$ are divided into *classes of similar modules.* A module $\mathbf{M}$ in $\mathbf{K}$ is said to be *full* if $\mathbb{Q}\mathbf{M} = \mathbf{K}$. Each full module in $\mathbf{K}$ has a $\mathbb{Z}$–basis of $n = [\mathbf{K} : \mathbb{Q}]$ elements and vice versa.

A full module in $\mathbf{K}$ that contains 1 and is itself a ring is called an *order* in $\mathbf{K}$. Every order in $\mathbf{K}$ is contained in the maximal order $\mathcal{O}$ of all integeral numbers of $\mathbf{K}$ over $\mathbb{Q}$. If $\mathbf{M}$ is a module in $\mathbf{K}$, then the ring

$$\mathcal{O}(\mathbf{M}) = \left\{ \alpha \in \mathbf{K} \;\middle|\; \alpha\mathbf{M} \subset \mathbf{M} \right\}$$

is called the *ring of multipliers of* $\mathbf{M}$. Similar modules have equal rings of multipliers. The ring of multipliers of a full module is an order in $\mathbf{K}$. For each full module, there is a similar module contained in its ring of multipliers.

Let $\mathcal{O}'$ be an order in the field $\mathbf{K}$ and $\omega_1, \ldots, \omega_n$ a basis of $\mathcal{O}'$. Then the number $d(\mathcal{O}') = \det(tr(\omega_i\omega_j))$, where tr means the trace from $\mathbf{K}$ to $\mathbb{Q}$, is independent of the choice of the basis and is called the *discriminant of* $\mathcal{O}'$. The discriminant of the maximal order $\mathcal{O}$,

$$d = d_K = d(\mathcal{O}),$$

is called the (*absolute*) *discriminant* of $\mathbf{K}$.

For a full module $\mathbf{M}$ with the ring of multipliers $\mathcal{O}' = \mathcal{O}(\mathbf{M})$, let $\alpha_1, \ldots, \alpha_n$ and $\omega_1, \ldots, \omega_n$ be bases of $\mathbf{M}$ and $\mathcal{O}'$, respectively. Then the absolute value of the determinant of the transition matrix from the first basis to the second,

$$N(\mathbf{M}) = |\det(a_{ij})|, \quad \text{where} \quad \alpha_i = \sum a_{ij}\omega_j,$$

is independent of the choices of bases, and is called the *norm* of $\mathbf{M}$. If $\mathbf{M} \subset \mathcal{O}'$, then $N(\mathbf{M}) = [\mathcal{O}' : \mathbf{M}]$ (the index of $\mathbf{M}$ in $\mathcal{O}'$).

Let $\mathbf{M}$ and $\mathbf{M}'$ be two full modules in $\mathbf{K}$. Then the set $\mathbf{M}\mathbf{M}'$ is again a full module in $\mathbf{K}$, which is called the *product of* $\mathbf{M}$ *and* $\mathbf{M}'$. The norm of the product is equal to the product of norms:

$$N(\mathbf{M}\mathbf{M}') = N(\mathbf{M})N(\mathbf{M}'). \tag{4.1}$$

All full modules in $\mathbf{K}$ with a fixed ring of multipliers $\mathcal{O}'$ form a commutative group under multiplication of modules. The quotient group of this group by the subgroup of modules similar to $\mathcal{O}'$ is a finite group $H(\mathcal{O}')$ called the *class group* of $\mathcal{O}'$.

Let $\mathcal{O}'$ be an order in an algebraic number field. Full modules in $\mathbf{K}$ with the ring of multipliers $\mathcal{O}'$ contained in $\mathcal{O}'$ are called *regular ideals of* $\mathcal{O}'$. Each regular ideal of an order in a finite extension $\mathbf{K}$ of $\mathbb{Q}$ is uniquely (up to order) decomposable into product of regular prime ideals.

Now let $\mathbf{K}$ be a quadratic extension of $\mathbb{Q}$. Every such field $\mathbf{K}$ is of the form $\mathbf{K} = \mathbb{Q}(\sqrt{d_0})$, where $d_0 \neq 0$, 1 is a square-free rational integer. As a basis of the maximal order $\mathcal{O}$ of $\mathbf{K}$ one can take the numbers 1 and ω, where $\omega = (1+\sqrt{d_0})/2$ if $d_0 \equiv 1 \pmod 4$, and $\omega = \sqrt{d_0}$ if $d_0 \equiv 2$ or $3 \pmod 4$. The discriminant $d = d_K = d(\mathbf{O})$ of the field $\mathbf{K}$ and the ring $\mathbf{O}$ is equal to d_0 in the first case, and to $4d_0$ in the second, and in both cases one can write $\mathbf{K} = \mathbb{Q}(\sqrt{d})$. Any order $\mathcal{O}'$ of $\mathbf{K}$ has the form

$$\mathcal{O}_l = \mathbb{Z} + l\omega\mathbb{Z},$$

where l is the index $[\mathcal{O} : \mathcal{O}']$. The discrininant of $\mathcal{O}_l$ is equal to dl^2.

For a full module $\mathbf{M}$ in $\mathbf{K}$ we denote by $\overline{\mathbf{M}}$ the module consisting of the conjugates $\bar{\alpha}$ over $\mathbb{Q}$ of the elements $\alpha \in \mathbf{M}$. The *conjugate module* $\overline{\mathbf{M}}$ is again a full module with the same ring of multipliers as that of $\mathbf{M}$, and the following formula holds

$$\mathbf{M}\overline{\mathbf{M}} = N(\mathbf{M})\mathcal{O}(\mathbf{M}). \tag{4.2}$$

Let $\mathbf{K}$ be a quadratic extension of $\mathbb{Q}$. For $\alpha, \beta, \ldots \in \mathbf{K}$ we shall denote by $\{\alpha, \beta, \ldots\}$ the full module in $\mathbf{K}$ generated over $\mathbb{Z}$ by $\alpha, \beta, \ldots$:

$$\{\alpha, \beta, \ldots\} = \mathbb{Z}\alpha + \mathbb{Z}\beta + \cdots .$$

The following lemma, which is proved, for example, in [2], Lemma A.3.2, is often useful in consideration of quadratic modules.

Lemma 9. *Let* $\mathbf{K}$ *be a quadratic field,* $\gamma \in \mathbf{K}$*,* $\gamma \notin \mathbb{Q}$*, and let* $a\gamma^2 + b\gamma + c = 0$*, where* a*,* b*, and* c *are the rational integers satisfying* $\gcd(a, b, c) = 1$ *and* $a > 0$*. Then the module* $\mathbf{M} = \{1, \gamma\}$ *satisfies*

$$N(\mathbf{M}) = 1/a, \quad \mathcal{O}(\mathbf{M}) = \{1, a\gamma\}, \quad d(\mathcal{O}(\mathbf{M})) = b^2 - 4ac.$$

The relation between quadratic modules and prime numbers is described by the following proposition (the proof can be found, for example, in [1], Proposition 2.3.1 or [2], Proposition A.3.5).

Proposition 10. *Let* $\mathbf{K} = \mathbb{Q}(\sqrt{d})$ *be the quadratic field with discriminant* d*, and* $\mathcal{O}_l$ *the order of* $\mathbf{K}$ *with discriminant* dl^2*. Suppose that* p *is a rational prime number not dividing* l*. Then the existence of a full module* $\mathbf{M}$ *in* $\mathbf{K}$ *satisfying the conditions*

$$\mathcal{O}(\mathbf{M}) = \mathcal{O}_l, \quad \mathbf{M} \subset \mathcal{O}_l, \quad N(\mathbf{M}) = p, \tag{4.3}$$

is equivalent to the solvability of the congruence $x^2 \equiv d \pmod{4p}$*. If this congruence is solvable, and* p *does not divide* d*, then there are precisely two modules* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *with the properties* (4.3)*, and* $\mathfrak{p}' = \overline{\mathfrak{p}}$*. But if the congruence has a solution and* p *divides* d*, then there is exactly one module* $\mathfrak{p}$ *satisfying* (4.3)*, and* $\overline{\mathfrak{p}} = \mathfrak{p}$*. Finally, if the congruence has no solutions, then there is the unique complete module* $\mathfrak{p}$ *satisfying*

$$\mathcal{O}(\mathfrak{p}) = \mathcal{O}_l, \quad \mathfrak{p} \subset \mathcal{O}_l, \quad N(\mathfrak{p}) = p^2,$$

and $\mathfrak{p} = p\mathcal{O}_l$*. All of the listed modules* $\mathbf{M} = \mathfrak{p}$ *are regular prime ideals of the ring* $\mathcal{O}_l$*.*

We conclude the survey part of this section with a description of the correspondence between modules in quadratic fields and integral binary quadratic forms. We restrict ourselves to imaginary quadratic field and positive definite forms, since for applications we shall not need any others.

Let

$$\mathbf{q}(X) = \frac{1}{2}Q[X] = ax^2 + bxy + cy^2 \qquad \left(X = \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

be a binary quadratic form with matrix $Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$. The number $d(\mathbf{q}) = d(Q) = b^2 - 4ac$ is called the *discriminant* of $\mathbf{q}$ and Q. The form is called *integral* if $Q \in \mathbb{E}^2$, i.e. if a, b, and c are rational integers. In this case $\gcd(a, b, c)$ is the divisor of $\mathbf{q}$, and the form is primitive if the divisor equals 1 (see Notation). We remind that two forms $\mathbf{q}$ and $\mathbf{q}'$ with matrices Q and Q' are properly equivalent if

$$\mathbf{q}(UX) = \mathbf{q}'(X) \quad \text{or} \quad {}^tUQU = Q' \quad \text{with } U \in \Lambda^2 = SL_2(\mathbb{Z}).$$

According to the reduction theory of integral quadratic forms (see, for example, [2], Theorem 2.1.12), the set of all integral binary forms with a fixed nonzero discriminant d is a finite union of classes (3.17) of properly equivalent forms. All forms from a fixed class have the same signatur, level, discriminant, and divisor.

Let $\mathbf{K} = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with discriminant $d < 0$, and let $\mathbf{M}$ be a full module in $\mathbf{K}$ with the ring of multipliers $\mathcal{O}(\mathbf{M}) = \mathcal{O}_l$. With every $\mathbb{Z}$ basis α, β of $\mathbf{M}$ ordered by the condition $\Im(\beta/\alpha) > 0$ we associate the binary quadratic form

$$\mathbf{q} = q(\mathbf{M})(X) = \frac{1}{N(M)}(\alpha x + \beta y)(\overline{\alpha} x + \overline{\beta} y) = ax^2 + bxy + cy^2.$$

The form $\mathbf{q}$ is, clearly, positive definite. It follows easily from Lemma 9 that $\mathbf{q}$ is integral and primitive with the discriminant $d(\mathbf{q}) = b^2 - 4ac = dl^2$ equal to the discriminant of the ring of multipliers $\mathcal{O}(\mathbf{M})$. Conversely, if $\mathbf{q}(X) = ax^2+bxy+cy^2$ is a positive definite integral primitive quadratic form of discriminant $b^2-4ac = dl^2$, then the module

$$\mathbf{M} = M(\mathbf{q}) = \left\{a,\ \frac{-b + l\sqrt{d}}{2}\right\} \tag{4.4}$$

is a full module in $\mathbf{K}$ with the ring of multipliers $\mathcal{O}_l$ contained in the ring. It is not difficult to check that the indicated correspondences define a bijection between the set of all classes of equivalent full modules in $\mathbf{K}$ with the ring of multipliers $\mathcal{O}_l$ and the set of all classes of properly equivalent positive definite integral primitive binary quadratic forms of discriminant dl^2.

Let $\mathbf{q}$ and $\mathbf{q}'$ be two integral primitive positive definite binary quadratic forms of the same negative discriminant $-\delta$, and let $Q = Q(\mathbf{q})$ and $Q' = Q(\mathbf{q}')$ be the matrices of the forms. Since the modules $\mathbf{M} = M(\mathbf{q})$ and $\mathbf{M}' = M(\mathbf{q}')$ are full modules in the field $\mathbf{K} = \mathbb{Q}(\sqrt{-\delta})$ with the same ring of multipliers $\mathcal{O}_l$ of discriminant $-\delta = dl^2$, where d is the discriminant of the field $\mathbf{K}$, the same is true for their product $\mathbf{M} \cdot \mathbf{M}'$. We define the product $\mathbf{q} \times \mathbf{q}'$ of the two forms $\mathbf{q}$ and $\mathbf{q}'$ by

$$\mathbf{q} \times \mathbf{q}' = q(\mathbf{M} \cdot \mathbf{M}') \tag{4.5}$$

and denote by $Q \times Q'$ the matrix of $\mathbf{q} \times \mathbf{q}'$:

$$Q \times Q' = Q(\mathbf{q} \times \mathbf{q}'). \tag{4.6}$$

Further, for any full module $\mathbf{N}$ in $\mathbf{K}$ with the ring of multipliers $\mathcal{O}_l$, we set

$$\mathbf{q} \times \mathbf{N} = \mathbf{N} \times \mathbf{q} = q(M(\mathbf{q}) \cdot \mathbf{N}), \tag{4.7}$$

and denote by $Q \times \mathbf{N}$ the matrix of $\mathbf{q} \times \mathbf{N}$:

$$Q \times \mathbf{N} = \mathbf{N} \times Q = Q(\mathbf{q} \times \mathbf{N}). \tag{4.8}$$

The multiplication (4.5) is called the *Gauss composition of quadratic forms.* The proper equivalence class of the product $\mathbf{q} \times \mathbf{q}'$ depends only on proper classes of $\mathbf{q}$ and $\mathbf{q}'$, and the proper equivalence class of $\mathbf{q} \times \mathbf{N}$ depends only on proper class of $\mathbf{q}$ and class of $\mathbf{N}$. Thus, the formulae (4.5)–(4.8) define the products of corresponding classes.

Since the set of equivalence classes of full modules with the ring of multipliers of discriminant $-\delta$ form an Abelian group under multiplication of the classes, the set of proper equivalence classes of integral primitive positive definite binary quadratic forms of the discriminant $-\delta$, can also be endowed with a group structure. We shall denote by $H(-\delta)$ both of the class groups.

The following useful lemma is an easy consequence of definitions.

Lemma 11. *The level q and the discriminant $-\delta$ of an integral primitive positive definite binary quadratic form with matrix Q satisfy the relation $q = \det Q = \delta$.*

We turn now to the action of regular Hecke operators on Maass theta-series (*) of genus 2.

Theorem 12. *Let q be a positive integer, and let $Q_1, \dots, Q_{h(-q)}$ be a system of representatives of all different proper equivalence classes (3.17) of even primitive positive definite matrices of order 2 and level q. Finally, let ψ be a character of the group $H(-q)$ of the proper equivalence classes of matrices of integral primitive positive definite binary quadratic forms of discriminant $-q$ under the composition (4.6), and χ a Dirichlet character modulo a rational integer g. Then the linear combination of Maass theta-series of the form*

$$F = \Theta_\nu(Z, q, \psi, \chi) = \sum_{j=1}^{h(-q)} \psi(Q_j)\Theta_\nu(Z, Q_j, \chi) \qquad (\nu = 0, 1)$$

has the following properties:

(1). *The function F is independent of the choice of representatives Q_j in the proper equivalence classes; it is a modular form of weight $\nu+1$ for the group $\Gamma_0^2(qg^2)$ with the character $\chi\chi_Q$, where χ_Q is the character of an even primitive positive definite matrix Q of order 2 and level q defined in Theorem 1 ; if $\nu = 1$, then F is a cusp form.*

(2). *The modular form F is an eigenfunction for all Hecke operators of the regular Hecke–Shimura ring $\mathcal{H}^2(qg^2)$:*

$$F|_{\nu+1}T = \lambda_F(T)F \qquad (\forall T \in \mathcal{H}^2(qg^2)).$$

(3). *For a prime number p not dividing qg, the eigenvalues of the operators corresponding to the generators* (3.24) *of the ring $\mathcal{H}^2(qg^2)$ let T be one of the images can be given by the formulae:*

$$\lambda_F(\widetilde{T}^2(p)) = \begin{cases} \frac{(p+1)(\psi(\mathfrak{p})+\psi(\overline{\mathfrak{p}}))}{\chi(p)p^\nu} \quad \text{if} \quad \chi_Q(p) = 1, \\ 0 \quad \text{if} \quad \chi_Q(p) = -1; \end{cases} \tag{4.9}$$

$$\lambda_F(\widetilde{T}_1^2(p^2)) = \begin{cases} \frac{\left(p^3+p^2+p-1+(p\psi(\mathfrak{p}))^2+(p\psi(\overline{\mathfrak{p}}))^2\right)}{\chi(p)^2p^{2(\nu+1)}} \quad \text{if} \quad \chi_Q(p) = 1, \\ -\frac{(p^3+p^2+p+1)}{\chi(p)^2p^{2(\nu+1)}} \quad \text{if} \quad \chi_Q(p) = -1, \end{cases} \tag{4.10}$$

where, for the case $\chi_Q(p) = 1$, in both formulae $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ are the conjugate regular prime ideals of the order $\mathcal{O}'$ with discriminant $-q$ in the field $\mathbf{K} = \mathbb{Q}(\sqrt{-q})$ satisfying $\mathfrak{p}\overline{\mathfrak{p}} = \mathcal{O}'$;

$$\lambda_F(\widetilde{T}_2^2(p^2)) = \frac{1}{\chi(p)^2p^{2(\nu+1)}}. \tag{4.11}$$

At first we shall establish some preliminary results.

Lemma 13. *Each of the Maass series* (*) *satisfies the relations*

$$\Theta_\nu(Z, Q[U], \chi) = \chi(\det U)^{-1}(\det U)^{-\nu}\Theta_\nu(Z, Q, \chi) \qquad (\forall U \in GL_m(\mathbb{Z})).$$

In particular, it depends only on the proper equivalence class (3.17) *of Q.*

Proof. The relation follows directly from the definition by the change $N \mapsto U^{-1}N$.△

Lemma 14. *Let Q be an even primitive positive definite matrix of order 2, q the level of Q, and χ_Q the corresponding Diriclet character modulo q. Let d be the discriminant of the imaginary quadratic field $\mathbf{K} = \mathbb{Q}(\sqrt{-q})$ so that $-q = dl^2$, and $\mathcal{O}_l$ is the order in $\mathbf{K}$ with discriminant $-q$. For a prime number p not dividing q and a positive integer β, we denote by $\Pi(p^\beta)$ the linear operator on functions Φ of proper equivalence classes of matrices Q' properly similar to Q given by*

$$\begin{aligned} \left(\Phi \circ \Pi(p^\beta)\right)(Q) &= \sum_{D \in \left\{\Lambda\begin{pmatrix}1 & 0\\ 0 & p^\beta\end{pmatrix}\Lambda \cap \Delta(Q,\, p^\beta)\right\}/\Lambda} \Phi(p^{-\beta}Q[D]), \\ \left(\Phi \circ \Pi(p^\beta)\right)(Q) &= 0 \quad \text{if} \quad \Lambda\begin{pmatrix}1 & 0\\ 0 & p^\beta\end{pmatrix}\Lambda \cap \Delta(Q,\, p^\beta) = \emptyset, \end{aligned} \tag{4.12}$$

where $\Delta(Q,\,\mu)$ *are the sets* (3.12)*, and* $\Lambda=\Lambda^2=SL_2(\mathbb{Z})$*. Then the following formulae hold:*

(1). *If* $\chi_Q(p)=1$*, then there are precisely two regular prime ideals* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *of norm* p *in the ring* $\mathcal{O}_l$ *conjugate with each other,* $\mathfrak{p}'=\overline{\mathfrak{p}}$*, and the formula is true*

$$\left(\Phi\circ\Pi(p^\beta)\right)(Q)=\Phi(Q\times\mathfrak{p}^\beta)+\Phi(Q\times\overline{\mathfrak{p}}^\beta),$$

where $\times$ *means the multiplication* (4.8)*.*

(2). *If* $\chi_Q(p)=-1$ *, then the ring* $\mathcal{O}_l$ *contains no regular prime ideals of norm* p*, and the formula is true*

$$\left(\Phi\circ\Pi(p^\beta)\right)(Q)=0.$$

Proof. First, we introduce on the set of all pairs $(u,\,v)$ of coprime integers the equivalence relation $\sim \pmod{p^\beta}$ as on the "projective line modulo p^β":

$$(u',\,v')\sim(u,\,v)\pmod{p^\beta}\Leftrightarrow u'\equiv tu,\quad v'\equiv tv\pmod{p^\beta}$$

and denote by $R(p^\beta)$ a set of integral matrices U of order 2, where the first column runs a system of reptesentatives for the classes $\sim \pmod{p^\beta}$ of all coprime pairs and the second row is fixed by the condition $U\in\Lambda$. Then it is easy to see that the following decomposition into different right cosets modulo Λ holds

$$\Lambda\begin{pmatrix}1&0\\0&p^\beta\end{pmatrix}\Lambda=\bigcup_{U\in R(p^\beta)}U\begin{pmatrix}1&0\\0&p^\beta\end{pmatrix}\Lambda. \tag{4.13}$$

Now, let $\mathbf{q}(x,y)=ax^2+bxy+cy^2$ be the quadratic form with matrix Q. We consider the congruence

$$\mathbf{q}(x,y)=ax^2+bxy+cy^2\equiv 0\pmod{p^\beta}. \tag{4.14}$$

Let us call a solution $(u,\,v)$ of the congruence *primitive* if the integers u, v are coprime. An easy consideration of the corresponding quadratic space over the field of p elements shows that if $\chi_Q(p)=1$, then the congruence $x^2\equiv d \pmod{4p}$ is solvable, and the congruence (4.14) has precisely two nonequivalent modulo p^β primitive solutions, and these solutions already are not equivalent modulo p, but if $\chi_Q(p)=-1$ the congruence $x^2\equiv d\pmod{4p}$ is unsolvable, and the congruence (4.14) has no primitive solutions (see, for example, [2], Proof of Proposition 4.3.14).

On the other hand, note that a matrix of the form

$$U\begin{pmatrix}1&0\\0&p^\beta\end{pmatrix}\qquad\text{with}\quad U=\begin{pmatrix}u&u_1\\v&v_1\end{pmatrix}\in\Lambda$$

belongs to the set $\Delta(Q, p^\beta)$ if and only if the pair u, v is a primitive solution of (4.14). Hence, the part (2) follows from Proposition 10.

Let $\chi_Q(p) = 1$, and (u, v), (u', v') be two nonequivalent primitive solutions of (4.14). Since p does not divide $q = \det Q$, we may assume that $\mathbf{q}(u, v) = p^\beta a_1$ and $\mathbf{q}(u', v') = p^\beta a_2$, where a_1 and a_2 are not divisible by p. We can choose the set $R(p^\beta)$ so that

$$U_1 = \begin{pmatrix} u & u_1 \\ v & v_1 \end{pmatrix} \in R(p^\beta) \quad \text{and} \quad U_2 = \begin{pmatrix} u' & u_1' \\ v' & v_1' \end{pmatrix} \in R(p^\beta).$$

Let us set for $i = 1, 2$

$$Q_i = Q[U_i] = \begin{pmatrix} 2p^\beta a_i & b_i \\ b_i & 2c_i \end{pmatrix}, \quad Q_i' = p^{-\beta} Q\left[U_i \begin{pmatrix} 1 & 0 \\ 0 & p^\beta \end{pmatrix}\right] = \begin{pmatrix} 2a_i & b_i \\ b_i & 2p^\beta c_i \end{pmatrix}.$$

Then according to (4.22), (4.25), and the above considerations, we can write

$$\Phi(Q) \circ \Pi(p^\beta) = \Phi(Q_1') + \Phi(Q_2').$$

Now, let us consider the modules in $\mathbf{K}$ given by

$$\mathbf{M}_1 = \left\{p^\beta, \frac{-b_1 + l\sqrt{d}}{2}\right\}, \qquad \mathbf{M}_2 = \left\{p^\beta, \frac{-b_2 + l\sqrt{d}}{2}\right\}$$

and show at first that the modules are regular ideals of $\mathcal{O}_l$ of norm p^β. Let us consider, for example, the module $\mathbf{M}_1$. The number $\gamma_1 = (-b_1 + l\sqrt{d})/2p^\beta$ is a root of the polynomial $p^\beta x^2 + b_1 x + a_1 c_1$. Since $q = 4ac - b^2 = 4p^\beta a_1 c_1 - b_1^2$ is not divisible by p, it follows that b_1 is not divisible by p, and so the coefficients of the polynomial are coprime. Then, by Lemma 9, the ring of multipliers of the module $\mathbf{M}_1 = p^\beta\{1, \gamma_1\}$ is $\mathcal{O}_l$, and its norm $N(\mathbf{M}_1) = N(p^\beta)/p^\beta = p^\beta$. It is also clear that $\mathbf{M}_1 \subset \mathcal{O}_l$. Next, we shall show that the ideals $\mathbf{M}_1$ and $\mathbf{M}_2$ are coprime, i.e. that $\mathbf{M}_1 + \mathbf{M}_2 = \mathcal{O}_l$. The numbers $(b_2 - b_1)/2$ and p^β are obviously both contained in $\mathbf{M}_1 + \mathbf{M}_2$. Hence, every integral linear combination of the numbers is contained in the sum; in particular, $s = \gcd((b_2 - b_1)/2, p^\beta)$ is contained in $\mathbf{M}_1 + \mathbf{M}_2$. Thus, we only have to prove that $s = 1$. Let us set

$$U_1^{-1} U_2 = \begin{pmatrix} v_1 & -u_1 \\ -v & u \end{pmatrix} \begin{pmatrix} u' & u_1' \\ v' & v_1' \end{pmatrix} = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} = T \in \Lambda.$$

Since the pairs (u, v) and (u', v') are not equivalent modulo p, it follows that $t_3 = uv' - vu'$ is not divisible by p. By comparing the corresponding entries of matrices in the obvious relation $Q_1[T] = Q_1[U_1^{-1} U_2] = Q_2$, we obtain the relations

$$p^\beta a_1 t_1^2 + b_1 t_1 t_3 + c_1 t_3^2 = p^\beta a_2 \text{ and } 2p^\beta a_1 t_1 t_2 + b_1(t_1 t_4 + t_2 t_3) + 2c_1 t_3 t_4 = b_2.$$

Since t_3 is not divisible by p, the first relation implies that $b_1t_1+c_1t_3 \equiv 0 \pmod p$. Since $t_1t_4 - t_2t_3 = 1$, the second relation implies that $(b_2-b_1)/2 \equiv t_3(b_1t_2+c_1t_4) \pmod{p^\beta}$. Thus, if $(b_2-b_1)/2$ is divisible by p, then we have congruences

$$b_1t_1 + c_1t_3 \equiv b_1t_2 + c_1t_4 \equiv 0 \pmod p,$$

which would imply the congruences $b_1 \equiv c_1 \equiv 0 \pmod p$, contrary to the assumption that $q = 4p^\beta a_1c_1 - b_1^2$ is not divisible by p. From what we have proved, by the theorem on prime ideals factorization of regular ideals of the ring $\mathcal{O}_l$ and Proposition 10, it follows that $\mathbf{M}_1 = \mathfrak{p}^\beta$ and $\mathbf{M}_2 = \overline{\mathfrak{p}}^\beta$, where $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ are the only regular prime ideals of norm p in $\mathcal{O}_l$. Let $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}'_1$, and $\mathbf{q}'_2$ be the quadratic forms with the matrices Q_1, Q_2, Q'_1, and Q'_2, respectively, and let $M(\mathbf{q}_1)$, $M(\mathbf{q}_2)$, $M(\mathbf{q}'_1)$, and $M(\mathbf{q}'_2)$ be the modules (4.4) corresponding to the forms. We set $\delta_1 = (-b_1+l\sqrt{d})/2$. Since clearly $\delta_1^2 = -b_1\delta_1 - p^\beta a_1c_1$ and, as it was noted above, b_1 is prime to p, it follows that

$$\begin{aligned} M(\mathbf{q}'_1)\mathfrak{p}^\beta &= \{a_1,\, \delta_1\}\{p^\beta,\, \delta_1\} = \{a_1p^\beta,\, a_1\delta_1,\, p^\beta\delta_1,\, -b_1\delta_1 - p^\beta a_1c_1\} \\ &= \{a_1p^\beta,\, \delta_1\} = M(\mathbf{q}_1). \end{aligned}$$

Hence, by (4.2), we obtain

$$M(\mathbf{q}'_1) = \frac{1}{p^\beta}M(\mathbf{q}'_1)\mathfrak{p}^\beta\overline{\mathfrak{p}}^\beta = \frac{1}{p^\beta}M(\mathbf{q}_1)\overline{\mathfrak{p}}^\beta.$$

Since the form $\mathbf{q}_1$ is properly equivalent to $\mathbf{q}$, it follows that the last module is equivalent to $M(\mathbf{q})\overline{\mathfrak{p}}^\beta$. Thus, the matrix Q'_1 of $\mathbf{q}'_1$ is properly equivalent to the matrix $Q \times \overline{\mathfrak{p}}^\beta$. Similarly, Q'_2 is properly equivalent to $Q \times \mathfrak{p}^\beta$. $\triangle$

Proof of the theorem. The part (1) follows from the definition of the function F and Proposition 3.

Since the elements (3.24) generate the ring $\mathcal{H}^2(qg^2)$ over $\mathbb{C}$, in order to prove the parts (2) and (3) it will be sufficient to show that the function F is an eigenfunction for all Hecke operators corresponding to the elements T of the form (3.24) with eigenvalues given by the formulae (4.9)–(4.11).

By Proposition 5, we can write

$$\begin{aligned} &F|_{\nu+1}T \\ &\quad= \frac{1}{\chi(\mu)\mu^\nu}\sum_{j=1}^{h(-q)}\psi(Q_j)\sum_{D\in\Delta(Q_j,\mu)/\Lambda} I(D,\, g^2Q_j,\, T)\Theta_\nu(Z,\, \mu^{-1}Q_j[D],\, \chi), \end{aligned}$$

where $\mu = \mu(T)$ equals p for the first of the elements (3.24) and p^2 for the second and the third of the elements.

Let $T = \eta(T^2(p))$. By the definition (3.13) of the sums $I(D, Q, T)$ and Proposition 8, for all $D \in \Delta(Q_j, p)$ we have

$$I(D, Q_j, \eta(T^2(p))) = I(D, Q_j, T^2(p)) = p(1 + \chi_Q(p)p^{-1}) = p + \chi_Q(p),$$

since all of the matrices Q_j have the same character as the matrix Q of part (1). If the set $\Delta(Q_j, p)$ is not empty, it is clearly contained in the double coset $\Lambda \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Lambda$. Therefore, using the notation (4.12), we can rewrite the above formula for $T = \eta(T^2(p))$ in the form

$$F\|_{\nu+1}T = \frac{p + \chi_Q(p)}{\chi(p)p^\nu} \sum_{j=1}^{h(-q)} \psi(Q_j)\Theta_\nu(Z, Q_j, \chi) \circ \Pi(p),$$

which, by Lemma 14, is equal to

$$\begin{aligned}
&\frac{p + \chi_Q(p)}{\chi(p)p^\nu} \sum_{j=1}^{h(-q)} \psi(Q_j)\big(\Theta_\nu(Z, Q_j \times \mathfrak{p}, \chi) + \Theta_\nu(Z, Q_j \times \overline{\mathfrak{p}}, \chi)\big) \\
&= \frac{p + \chi_Q(p)}{\chi(p)p^\nu} \Bigg(\psi(\overline{\mathfrak{p}}) \sum_{j=1}^{h(-q)} \psi(Q_j \times \mathfrak{p})\Theta_\nu(Z, Q_j \times \mathfrak{p}, \chi) \\
&+ \psi(\mathfrak{p}) \sum_{j=1}^{h(-q)} \psi(Q_j \times \overline{\mathfrak{p}})\Theta_\nu(Z, Q_j \times \overline{\mathfrak{p}}, \chi)\Bigg) \\
&= \frac{p + 1}{\chi(p)p^\nu}(\psi(\mathfrak{p}) + \psi(\overline{\mathfrak{p}})) \sum_{j=1}^{h(-q)} \psi(Q_j)\Theta_\nu(Z, Q_j, \chi)
\end{aligned}$$

if $\chi_Q(p) = 1$, and is 0 if $\chi_Q(p) = -1$. This proves the formulae (4.9).

Now, let $T = \eta(T_1^2(p^2))$. Then, by Proposition 8, with the notation (4.12), we obtain

$$\begin{aligned}
F\|_{\nu+1}T = \frac{1}{\chi(p^2)p^{2\nu}} \Bigg(&\big(\chi_Q(p)(p+1) + p^{-2}(\chi_Q(p)p - 1)\big) \sum_{j=1}^{h(-q)} \psi(Q_j)\Theta_\nu(Z, Q_j, \chi) \\
&+ \chi_Q(p) \sum_{j=1}^{h(-q)} \psi(Q_j)\Theta_\nu(Z, Q_j, \chi) \circ \Pi(p^2)\Bigg).
\end{aligned}$$

If $\chi_Q(p) = 1$, then, by Lemma 14, the last expression is equal to

$$\frac{1}{\chi(p^2)p^{2\nu}}\Bigg((p+1+p^{-1}-p^{-2})\sum_{j=1}^{h(-q)}\psi(Q_j)\Theta_\nu(Z,\,Q_j,\,\chi) + \sum_{j=1}^{h(-q)}\psi(Q_j)\big(\Theta_\nu(Z,\,Q_j\times\mathfrak{p}^2,\,\chi)+\Theta_\nu(Z,\,Q_j\times\overline{\mathfrak{p}}^2,\,\chi)\big)\Bigg)$$

$$= \frac{1}{\chi(p)^2p^{2(\nu+1)}}(p^3+p^2+p-1+(p\psi(\mathfrak{p}))^2+(p\psi(\overline{\mathfrak{p}}))^2)F.$$

But if $\chi_Q(p) = -1$ we have just

$$F\|_{\nu+1}T = \frac{-(p^3+p^2+p+1)}{\chi(p)^2p^{2(\nu+1)}}F.$$

It proves the formula (4.10).

Finally, let $T = \eta(T_2^2(p^2))$. By Proposition 8, or directly, by the definition of Hecke operators, we obtain

$$F\|_{\nu+1}T = \frac{1}{\chi(p^2)p^{2\nu}}\sum_{j=1}^{h(-q)}\psi(Q_j)p^{-2}\Theta_\nu(Z,\,Q_j,\,\chi) = \frac{1}{\chi(p^2)p^{2(\nu+1)}}F,$$

which proves the formula (4.11) and the theorem.$\triangle$

§5 Zeta functions of Maass theta-series of binary forms

Let $F \in \mathfrak{M}_k(\Gamma_0^2(q))$ be a nonzero eigenfunction for all Hecke operators of the regular Hecke–Shimura ring $\mathcal{H}^2(q)$:

$$F\|_kT = \lambda_F(T)F \qquad (\forall T \in \mathcal{H}^2(q)).$$

As it was noted in §3, the ring $\mathcal{H}^2(q)$ is generated over $\mathbb{C}$ by the elements (3.14) for all prime number p not dividing q. For any such p we introduce the *p-polynomial* of the eigenfunction F by

$$(5.1)\quad \mathcal{Q}_{p,F}(t) = 1 - p^{2k-3}\lambda_F(\widetilde{T}(p))t + p^{4k-6}\lambda_F(p\widetilde{T}_1(p^2)+p(p^2+1)\widetilde{T}_2(p^2))t^2 - p^{6k-6}\lambda_F(\widetilde{T}(p)\widetilde{T}_2(p^2))t^3 + p^{8k-6}\lambda_F(\widetilde{T}_2(p^2)^2)t^4$$

and define the *zeta function* of F as the Euler product

$$(5.2)\qquad Z_F(s) = \prod_{p\nmid q}\mathcal{Q}_{p,F}(p^{-s})^{-1}$$

(compare with [1], §1.3 and [2], §4.3.2). One can show that the product converges absolutely and uniformly in a right half-plane of the complex variable s.

Theorem 15. *Let $-q = dl^2$ be the discriminant of an even primitive positive definite matrix Q of order 2, where d is the discriminant of the field $\mathbf{K} = \mathbb{Q}(\sqrt{-q})$, and let $Q_1, \dots, Q_{h(-q)}$ be a system of representatives of all different proper equivalence classes* (3.17) *of even primitive positive definite matrices of order 2 and discriminant $-q$. Finally, let ψ be a character of the group $H(-q)$ of the proper equivalence classes under the composition* (4.6), *g a rational integer, and χ a Dirichlet character modulo g. Suppose that the linear combination of Maass theta-series*

$$F = \Theta_\nu(Z, q, \psi, \chi) = \sum_{j=1}^{h(-q)} \psi(Q_j)\Theta_\nu(Z, Q_j, \chi) \qquad (\nu = 0, 1)$$

is not identically zero. Then the zeta function (5.2) *of the eigenfunction F satisfies the identity*

$$Z_F(s) = L_{\mathcal{O}_l}(s-\nu, \psi, \overline{\chi})L_{\mathcal{O}_l}(s-\nu+1, \psi, \overline{\chi}), \tag{5.3}$$

where $\mathcal{O}_l$ is the subring of index l in the maximal order of $\mathbf{K}$, and

$$L_{\mathcal{O}_l}(s, \psi, \overline{\chi}) = \prod_{\mathfrak{p}} \left(1 - \frac{\psi(\mathfrak{p})\overline{\chi}(N(\mathfrak{p}))}{N(\mathfrak{p})^s}\right)^{-1}, \tag{5.4}$$

with $\mathfrak{p}$ running over all regular prime ideals of the ring $\mathcal{O}_l$ not dividing gq is a L-function of $\mathcal{O}_l$.

Proof. According to Theorem 12, the function F is a modular form of weight $k = \nu + 1$ and character $\chi\chi_Q$ for the group $\Gamma_0^2(q')$ of the level $q' = qg^2$, and it is an eigenfunction for all Hecke operators corresponding to elements (3.24) for all prime numbers $p \nmid q'$ with eigenvalues given by formulae (4.9)–(4.11). By substituting the eigenvalues to formulae for coefficients of polynomials (5.1), we obtain that if $\chi_Q(p) = 1$, then

$$\begin{aligned}\mathcal{Q}_{p,F}(t) &= 1 - p^{\nu-1}(p+1)(\psi(\mathfrak{p}) + \psi(\overline{\mathfrak{p}})\overline{\chi}(p)t \\ &\quad + p^{2\nu-3}(p^3+p^2+p-1+p^2\psi(\mathfrak{p})^2 + p^2\psi(\overline{\mathfrak{p}})^2 + p^2+1)\overline{\chi}(p)^2t^2 \\ &\quad - p^{3\nu-2}(p+1)(\psi(\mathfrak{p}) + \psi(\overline{\mathfrak{p}}))\overline{\chi}(p)^3t^3 + p^{4\nu-2}\overline{\chi}(p)^4t^4 \\ &= (1-p^\nu\psi(\mathfrak{p})\overline{\chi}(p)t)(1-p^\nu\psi(\overline{\mathfrak{p}})\overline{\chi}(p)t)(1-p^{\nu-1}\psi(\mathfrak{p})\overline{\chi}(p)t)(1-p^{\nu-1}\psi(\overline{\mathfrak{p}})\overline{\chi}(p)t),\end{aligned}$$

but if $\chi_Q(p) = -1$, then

$$\begin{aligned}\mathcal{Q}_{p,F}(t) &= 1 + p^{2\nu-3}(-p^3-p^2-p-1+p^2+1)\overline{\chi}(p)^2t^2 + p^{4\nu-2}\overline{\chi}(p)^4t^4 \\ &= (1-p^{2\nu}\overline{\chi}(p^2)t^2)(1-p^{2\nu-2}\overline{\chi}(p^2)t^2).\end{aligned}$$

Since, by Proposition 10, in the first case $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ are the only regular prime ideals of the ring $\mathcal{O}_l$ of the norm $N(\mathfrak{p}) = N(\overline{\mathfrak{p}}) = p$, and in the second case the principal ideal $\mathfrak{p} = p\mathcal{O}_l$ is the only regular prime ideal of the norm $N(\mathfrak{p}) = p^2$, the above calculations and factorization (5.4) show that the infinite products in both sides of relation (5.3) have equal p-factors for every prime number p not dividing q'. $\triangle$

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