EXTENSIONS WITH ALMOST MAXIMAL DEPTH OF RAMIFICATION

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Introduction

§1 of [BVZ] was devoted to a general discussion of the extensions L/K which satisfy the condition

$$[L:K] \mid \mathcal{D}_{L/K}. \tag{*}$$

Here L/K is a wildly ramified Galois extension of complete discretely valued fields of characteristic 0 with the residue field of prime characteristic p, $\mathcal{D}_{L/K}$ is the different of L/K. The importance of the condition (*) is clarified in [V, BVZ, BLV]. Roughly speaking, it appeared that (*) is equivalent to the fact that some ideals of $\mathcal{O}_L$ have a non-trivial decomposition into a direct sum of $\mathcal{O}_K[G]$-modules, where $\mathcal{O}_K$ is the ring of integers in K, $G = \mathrm{Gal}(L/K)$.

The condition (*) means that the depth of ramification in L/K has its "almost maximal" value. For example, consider the case $[L:K] = p$. Then we have $v_L(\mathcal{D}_{L/K}) = (p-1)(h+1) = pd + p - 1$, where h is the ramification jump of L/K, and $d = d_K(L/K)$ is the depth of ramification. The condition (*) can be read here as $pe_K \leqslant (p-1)(h+1)$ which is equivalent to $h \geqslant \frac{pe_K}{p-1} - 1$, or to $d \geqslant e_K - \frac{p-1}{p}$. Recall that the maximal possible values for h and d are $\frac{pe_K}{p-1}$ and e_K respectively.

Consider the ramification filtration on $\mathrm{Gal}(L/K)$. It yields a canonical decomposition of L/K into a tower of elementary abelian p-extensions. We can further decompose L/K into a tower of cyclic extensions of degree p: $K = K_0 \subset K_1 \subset \dots \subset K_n = L$. The following observations were reflected in [BVZ].

A. The condition (*) for L/K implies (*) for all intermediate extensions in L/K. (See [BVZ, Prop. 1.5].)

B. The condition (*) for the "upper level" L/K_{n-1} implies (*) for the whole L/K. (See [ibid].)

C. The condition (*) for the "lower level" K_1/K does not imply (*) for the whole L/K. However, if the depth of ramification for K_1/K is maximal, i. e., equals e, then L/K is cyclic, and the depth of ramification is maximal on all levels. (Recall that K_1/K is a subextension of degree p in the subextension of L/K corresponding to the first ramification jump.) In particular, (*) for L/K holds. (This is well known, see, e. g., [Se, Ch. IV, §2, Ex. 3f].)

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D. The condition (*) determines the Galois group of L/K, namely, forces it to be cyclic when $p > 2$. (See [BVZ, Prop. 1.5].)

E. If we know a priori that L/K is cyclic, then the condition (*) for the "lower level" K_1/K implies (*) for the whole L/K. (It follows from [BVZ, Lemma 1.2.1].)

The aim of the present article is to consider the general case without the assumption that L/K is wildly ramified. In other words, we admit here that $\overline{L}/\overline{K}$ can be inseparable and try to find reasonable generalizations of the above A,B,C,D,E.

§§1–3 contain some preliminary results. In §4 we establish a general structure theorem for extensions satisfying (*) under the condition that $p-1|e$. In §5 we give a complete classification of cyclic extensions satisfying (*). §6 and §7 contains the proofs of the following theorems, descrbing the structure of abelian AMR extensions. In the statements we use the notion of independently ramified extensions, explained in the beginning of §3.

Theorem 1. *Suppose that $p-1|e$. Let E/K be an abelian p-extension. Denote by E_1/K the maximal subextension of E/K of exponent p. Then the following conditions are equivalent.*

1. E/K satisfies (*).

2. E_1/K satisfies (*).

3. $E = MN$, where M/K and N/K are independently ramified, M/K is cyclic and satisfies (*), N/K is a maximally ramified, i. e., its depth of ramification equals ne, where $p^n = [N : K]$.

Theorem 2. *Suppose that $p-1 \nmid e$. Let L/K be an abelian p-extension. Write L as $K^1 \dots K^m$, where K^i/K are linearly disjoint cyclic, $i = 1, \dots, m$. Then L/K satisfies (*) if and only if the following 3 conditions are satisfied.*

1. The subextension of degree p in K^i/K satisfies (*) for all $i = 1, \dots, m$.

2. The subextension of degree p in K^i/K are independently ramified, $i = 1, \dots, m$.

3. $\varepsilon \sum_{i=1}^{m} \left(1 - \frac{1}{p^{r_i}}\right) \leqslant 1 - \frac{1}{p^r}$, where $p^{r_i} = [K^i : K]$, $i = 1, \dots, m$; $r = \max(r_1, \dots, r_m)$; $\varepsilon = \frac{pe}{p-1} - \left[\frac{pe-1}{p-1}\right]$.

Finally, in §8 we establish some further properties of composites of cyclic extensions satisfying (*).

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Notation and terminology

p is a prime number; *we assume throughout the article that $p > 2$.*

Throughout the article the letter K denotes a complete discretely valued field of characteristic 0 or p with the residue field of characteristic p. For such a field K, we use the following notation:

- $v = v_K$: the valuation on K, as well as its non-normalized extension onto the algebraic closure of K;
- $\mathcal{O} = \mathcal{O}_K$: the valuation ring in K;
- $\mathfrak{M} = \mathfrak{M}_K = \{\, a \in \mathcal{O}_K : v(a) > 0 \,\}$: the maximal ideal of this ring;
- $\mathfrak{p}(i) = \mathfrak{p}_K(i) = \mathfrak{M}_K^i$ for any $i \in \mathbb{Z}$;

- $U_K = \mathcal{O}_K^*$;
- $U_{i,K} = 1 + \mathfrak{p}_K(i)$ for $i \geqslant 1$;
- $\overline{K}$: the residue field of K;
- $\overline{a}$: the residue class in $\overline{K}$ of $a \in \mathcal{O}_K$;
- $e = e_K = v_K(p)$: the absolute ramification index of K;
- $e(L/K)$: the ramification index of a finite extension L/K;
- $\mathcal{D}_{L/K}$: the different of a finite extension L/K;
- ζ_m: a primitive mth root of unity in the algebraic closure of K.

A finite extension L/K is said to be:

- tame, if $(p, e(L/K)) = 1$;
- wild, if $[L : K] = e(L/K)$ is a power of p;
- ferocious, if $[L : K] = [\overline{L} : \overline{K}]_{insep}$;
- a wf-extension if the ramification subfield (i. e., the maximal tame subextension) in L/K coincides with K.

In other words, a wf-extension is an extension which can be represented by a tower of extensions of degree p, each of them is either wild or ferocious.

Let M/L be a finite separable extension of algebraic extensions of K. Depth of ramification of M/L is defined as

$$d_K(M/L) = \inf\left\{ v_K\left(\frac{\operatorname{Tr}_{M/L} y}{y}\right) \middle| y \in M^* \right\}.$$

Let $\operatorname{char} K = 0$. We say that an extension L/K is *almost maximally ramified* (AMR), if $[L : K] | \mathcal{D}_{L/K}$. An extension L/K is *maximally ramified* (MR), if $[L : K] = p^s$, and $d_K(L/K) = se$.

Let L/K be a finite Galois extension, $\sigma \in \operatorname{Gal}(L/K)$. The ramification number of σ is defined as

$$h(\sigma) = \min_{a \in L^*} v_L(a^{\sigma-1} - 1).$$

If $h(\sigma)$ is the same for all $\sigma \neq 1$, it is called the ramification number (or ramification jump) of L/K and denoted by $h(L/K)$. In particular, $h(L/K)$ is always defined for a cyclic extension L/K of degree p.

§1. Properties of depth of ramifiction

The following properies are taken from the paper of Hyodo [H].

Proposition 1.1. *Let L/K be a finite extension, M a finite extension of L. Then*

$$d_K(M/L) = v_K(\mathcal{D}_{M/L}) + v_K(\pi_M) - v_K(\pi_L).$$

In particular, $d_K(M/L) = 0$ if and only if M/L is tame.

Proof. Immediately from the definition of the different. □

Proposition 1.2. *Let N be an intermediate field in M/L. Then*

$$d_K(M/L) = d_K(M/N) + d_K(N/L).$$

This is Lemma (2-4) in [H].

Corollary 1.3. *We have $d_K(M(\zeta_p)/L(\zeta_p)) = d_K(M/L)$.*

Proposition 1.4. *Let L/K be a cyclic extension of degree p. Then $d_L(L/K) = (p-1)h(L/K)$.*

This follows from Lemma (2-10) in [H].

Now we recall what are the possible values of $h(L/K)$ in this case. For details, see [FV, Prop. III.2.3 and Ex.1 in III.2].

Proposition 1.5. *1. Let L/K be wild, $h = h(L/K)$. Then either $0 < h < pe/(p-1)$, and $(p,h) = 1$, or $h = pe/(p-1)$. The last variant is possible only when $\operatorname{char} K = 0$ and $\zeta_p \in K$.*

2. Let L/K be ferocious, $h = h(L/K)$. Then $0 < h \leqslant e/(p-1)$.

Finally, we recall the Hyodo inequality, see [H, Lemma (4-1)] or [Z,§1]; see also Prop. 2.2 below.

Proposition 1.6. *Let M/K be a cyclic extension of degree p^2, L the intermediate field. Then*

$$d_K(M/L) \geqslant \min((p-1+p^{-1})d_K(L/K), e - p^{-1}e + p^{-1}d_K(L/K)).$$

§2. Some calculations

Proposition 2.1. *Let L/K be a cyclic extension of degree p, $d = d_K(L/K)$. Then*

$$v\left(\frac{N_{L/K}a}{a^p} - 1\right) \geqslant d$$

for any $a \in L^*$.

Proof. The assertion is trivial in the unramified case, and we may assume that L/K is either wild or ferocious.

Denote $v\left(\frac{N_{L/K}a}{a^p} - 1\right)$ by $j(a)$. Since $j(a^{-1}) = j(a)$, we have $\min_{a\in L^*} j(a) = \min_{a\in\mathcal{O}_L\setminus\{0\}} j(a)$. Next, for any $a, b \in \mathcal{O}_L$ we have

$$v(N_{L/K}(a+b) - N_{L/K}a - N_{L/K}b) \geqslant d + p\min(v(a), v(b)),$$

and

$$v((a+b)^p - a^p - b^p) \geqslant e + p\min(v(a), v(b)) \geqslant d + p\min(v(a), v(b)).$$

Suppose that $a, b \in \mathcal{O}_L$, $a \neq 0$, $b \neq 0$, $v(a+b) = \min(v(a), v(b))$. Then

$$\begin{aligned} j(a+b) &= v(N_{L/K}(a+b) - (a+b)^p) - p\min(v(a), v(b)) \\ &\geqslant \min(j(a), j(b), d). \end{aligned} \tag{1}$$

Next, any $a \in \mathcal{O}_L$ can be written as $\sum\limits_{i=0}^{p-1} \alpha_i x^i$, where $\alpha_i \in \mathcal{O}_K$. Here x is a prime in L, if L/K is wild, and $x \in \mathcal{O}_L$, $\overline{x} \notin \overline{K}$, if L/K is ferocious. It follows from (1) that

$$j(a) \geqslant \min(j(\alpha_0), j(\alpha_1 x), \ldots, j(\alpha_{p-1}x^{p-1}), d) = \min(j(x), d).$$

Let $\operatorname{char} K = 0$. Without loss of generality we may assume that $\zeta_p \in K$. Then $L = K(y)$, $y^p \in K$. If $d = e$, one may assume $x = y$, whence $j(x) = \infty$. Let $d < e$.

We may choose y so that $v(y-1) = \frac{1}{p-1}(e-d)$. For a suitable choice of x we have $x = \pi^{-n}(y-1)^i$, where π is a prime in K, $(i,p) = 1$. Hence $j(x) = j(y-1)$. It follows from $y^p \in K$ that

$$N_{L/K}(y-1) = N_{L/K}y + (-1)^p = (-1)^{p-1}y^p + (-1)^p.$$

We have

$$j(y-1) = v\left(\frac{y^p-1}{(y-1)^p} - 1\right) = v(y-1) + e - pv(y-1) = d,$$

and the assertion of Lemma is proved in this case.

Let now $\operatorname{char} K = p$. Then $L = K(y)$, $y^p - y \in K$, $v(y) = -\frac{1}{p-1}d$. For a suitable choice of x we have $x = \pi^n y^i$, where π, i, n are as above, and $j(x) = j(y)$. We have $N_{L/K}y = y^p - y$, whence $j(y) = v(-y^{1-p}) = d$. $\square$

Proposition 2.2. *Let* $\operatorname{char} K = 0$, $\zeta_p \in K$, M/K *a cyclic extension of degree* p^2, L *the intermediate field,* $d = d_K(L/K)$. *Suppose* $d > \frac{e}{p}$.

Let $L = K(x)$, $x^p = 1+\rho \in K$, $v(\rho) = \frac{p}{p-1}(e-d)$. *Then* $M = L(y)$, $y^p = b(1+\tau)$, $b \in K$, $\tau \in L$,

$$v(\tau^p - \rho) \geqslant \frac{p}{p-1}(e-d) + \min(pd-e, d).$$

Remark. We do not exclude the case $d = e$. In this case $v(\rho) = v(\tau) = 0$.

Proof. According to Lemma 1.3 in [BVZ], we have $M = L(\sqrt[p]{a})$, $a \in L$, $a^{\sigma-1} = c^p$, $c \in L^*$, $N_{L/K}c \neq 1$, where σ is a generator of $\operatorname{Gal}(L/K)$. Let $c = 1+\gamma$. We have

$$N_{L/K}c = 1 + N_{L/K}\gamma + \operatorname{Tr}_{L/K}\gamma', \tag{2}$$

where $v(\gamma' - \gamma) \geqslant 2v(\gamma)$. Since $N_{L/K}c$ is a primitive root of unity of degree p, we have $v(N_{L/K}\gamma + \operatorname{Tr}_{L/K}\gamma') = \frac{e}{p-1}$. On the other hand, $v(\operatorname{Tr}_{L/K}\gamma') \geqslant v(\gamma) + d$ by the definition of depth, whereas $v(N_{L/K}\gamma) = pv(\gamma)$. Since $d > \frac{e}{p}$, we obtain $v(\gamma) \leqslant \frac{e}{p(p-1)}$, whence $v(N_{L/K}\gamma) < v(\operatorname{Tr}_{L/K}\gamma')$, and finally $v(\gamma) = \frac{e}{p(p-1)}$.

By Proposition 2.1, $v(N_{L/K}\gamma - \gamma^p) \geqslant \frac{e}{p-1} + d$. On the other hand, $v(\operatorname{Tr}_{L/K}\gamma') \geqslant \frac{e}{p(p-1)} + d$. Therefore, (2) implies

$$N_{L/K}c \equiv 1 + \gamma^p \mod \mathfrak{p}\left(\frac{e}{p(p-1)} + d\right)\mathcal{O}_L,$$

and, furthermore,

$$N_{L/K}c \equiv (1+\gamma)^p \mod \mathfrak{p}\left(\min\left(\frac{e}{p(p-1)} + d, e\right)\right)\mathcal{O}_L.$$

We have $(1+\gamma)^p = c^p = a^{\sigma-1}$. The element $x^{\sigma-1}$ is also a primitive root of unity of degree p, whence $x^{\sigma-1} = (N_{L/K}c)^i$ for some $1 \leqslant i \leqslant p-1$. We obtain

$$x^{\sigma-1} \equiv (a^i)^{\sigma-1} \mod \mathfrak{p}\left(\min\left(\frac{e}{p(p-1)} + d, e\right)\right)\mathcal{O}_L.$$

Write x as $a^i b_1(1+\tau_1)$ with $b_1 \in K$ and maximal possible $v(\tau_1)$. We have

$$v\left(\frac{1+\tau_1^\sigma}{1+\tau_1}-1\right) \geqslant \min\left(\frac{e}{p(p-1)}+d, e\right)\mathcal{O}_L.$$

On the other hand, the choice of τ_1 implies that $\tau_1 \notin K^* \cdot U_{1,L}$, whence $v(\tau_1^{\sigma-1}-1) = \frac{d}{p-1}$. Thus, $v(\tau_1) \geqslant \min\left(\frac{e}{p(p-1)}+d, e\right) - \frac{d}{p-1}$.

Put $y = \sqrt[p]{a^i}$. We have $M = L(\sqrt[p]{a}) = L(y)$,

$$y^p = a^i = b_1^{-1}x(1+\tau_1)^{-1} = b(1+\tau),$$

where $b = b_1^{-1}$, $\tau = x(1+\tau_1)^{-1}-1$. To estimate $v(\tau^p-\rho)$, notice that $x = (1+\tau)(1+\tau_1)$, whence $1+\rho = x^p = (1+\tau)^p(1+\tau_1)^p$, and $v(\tau^p-\rho) \geqslant \min(v(\tau_1^p), v(p\tau), v(p\tau_1))$. Since $v(x-1) = \frac{1}{p-1}(e-d)$, and $d > \frac{e}{p}$, we have $v(x-1) < v(\tau_1)$, whence $v(\tau) = v(x-1) = \frac{1}{p-1}(e-d)$, and

$$\begin{aligned} v(\tau^p-\rho) &\geqslant \min(v(\tau_1^p), v(p\tau)) \\ &= \min\left(pv(\tau_1), \frac{pe}{p-1} - \frac{d}{p-1}\right) \\ &= \min\left(\frac{e}{p-1} + pd - \frac{pd}{p-1}, pe\frac{pd}{p-1}, \frac{pe}{p-1} - \frac{d}{p-1}\right) \\ &= \frac{p}{p-1}(e-d) + \min(pd-e, d). \end{aligned}$$

In the last equality we have used the condition $p > 2$. $\square$

Remark. It is seen from the proof that $v(b) = v(a^i)$, $1 \leqslant i \leqslant p-1$. On the other hand, one may assume that either $p \nmid v_L(a)$, or $v_L(a) = 0$. Therefore, either $p \nmid v_L(b)$, or $v_L(b) = 0$.

§3. Independently ramified extensions

Proposition 3.1. *Let L/K and K'/K be any finite separable extensions, $L' = K'L$. Then $d_K(L'/K') \leqslant d_K(L/K)$.*

Proof. Let first L/K be a cyclic extension of degree p. Suppose $\operatorname{char} K = 0$. In view of Corollary 1.3, we may assume $\zeta_p \in K$. Then $L = K(x)$, $x^p = a \in K$, $v(a-1) = \frac{p}{p-1}(e - d_K(L/K))$. Since $L' = K'(x)$, we obtain

$$v(a-1) \leqslant \frac{p}{p-1}(e - d_K(L'/K')),$$

whence $d_K(L'/K') \leqslant d_K(L/K)$. The case $\operatorname{char} K = p$ is analogous.

Next, let L/K be any wf-extension of degree p. Denote the normal closure of L/K by M/K and the ramification subfield in M/K by E. Then M/E is cyclic of degree p, and we have

$$d_K(L'/K') = d_K(M'/K') = d_K(M'/E') \leqslant d_K(M/E) = d_K(M/K) = d_K(L/K).$$

Indeed, the extensions M/L, M'/L', E/K, E'/K' are tame.

The general L/K can be decomposed into a tower $K \subset K_1 \subset \cdots \subset K_n = L$, where K_1/K is tame, and K_{i+1}/K_i are wf-extensions of degree p, $i = 1, \dots, n-1$. $\square$

Corollary 3.2. *Let L_1/K, L_2/K be finite separable extensions. Then*

$$d_K(L_1L_2/K) \leqslant d_K(L_1/K) + d_K(L_2/K).$$

Proof. Combine Proposition 3.1 with Proposition 1.2. □

Definition. Finite separable extensions $L_1/K, L_2/K, \dots$, are said to be independently ramified (ind-ram), if $d_K(L_1 \dots L_r/K) = d_K(L_1/K) + \dots + d_K(L_r/K)$.

Proposition 3.3. *1. Let $K_1/K, \dots, K_r/K$ be subextensions of $L_1/K, \dots, L_r/K$ respectively. If $L_1/K, \dots, L_r/K$ are ind-ram, then $K_1/K, \dots, K_r/K$ are ind-ram as well.*

2. Let $r \geqslant 3$. Then the extensions $L_1/K, \dots, L_r/K$ are ind-ram if and only if $L_1/K, \dots, L_{r-1}/K$ are ind-ram, and L_r/K is ind-ram with $L_1 \dots L_{r-1}/K$.

Proof. Immediately from the definition. □

Remark 1. In the "classical case" (when the residue field of K is perfect) any two extensions cannot be ind-ram unless one of them is tame. Indeed, it is easy to observe that we have strict inequalities in the proof of Prop. 3.1.

Remark 2. Let L_1/K, L_2/K be finite separable extensions, L_1/K wild, L_2/K ferocious. Then L_1/K and L_2/K are ind-ram. Indeed, one can follow the proof of Prop. 3.1 and observe that we obtain equalities instead of inequalities.

Proposition 3.4. *Let $\operatorname{char} K = 0$, $\zeta_p \in K$, $L_1 = K(x_1)$, $L_2 = K(x_2)$, $x_1^p = a_1 \in K$, $x_2^p = a_2 \in K$. For every $i = 1, 2$ suppose that either $a_i = \rho_i$ or $a_i = 1 + \rho_i$, where $v(\rho_i) \geqslant 0$, $\rho_i \notin (K^*)^p \cdot U_{v_K(\rho_i)+1,K}$. Then the following two conditions are equivalent.*

1) L_1/K, L_2/K are ind-ram.

2) $\forall i = 1, 2, \dots, p-1$: $\rho_1/\rho_2^i \notin (K^*)^p \cdot U_{1,K}$.

Remark. We shall give the proof only for extensions with $d_K(L_1/K) = d_K(L_2/K)$. The proposition will be used only for extensions satisfying this condition.

Proof. Case 1: L_1/K is ferocious, L_2/K is wild. Then L_1/K and L_2/K are ind-ram by Remark 2 above. We have $p | v(\rho_1)$, $p \nmid v(\rho_2)$. Therefore, $p \nmid v(\rho_1/\rho_2^i)$ for any $i = 1, \dots, p-1$, whence $\rho_1/\rho_2^i \notin (K^*)^p \cdot U_{1,K}$.

Case 2: L_1/K is wild, L_2/K is ferocious. Similar to the Case 1.

Case 3: both L_1/K and L_2/K are wild. In the case $d_K(L_1/K) = d_K(L_2/K) = e$ we may assume that $v(\rho_1) = v(\rho_2) = 1$. Then $x_1 = \tau_1$ is prime in L_1. In the case $d_K(L_1/K) = d_K(L_2/K) = e - \frac{p-1}{p} j$, $j \geqslant 1$, we have $x_1 = 1 + \tau_1$, $v_{L_1}(\tau_1) = j$.

Suppose $\rho_1/\rho_2^i \notin (K^*)^p \cdot U_{1,K}$, $i = 1, \dots, p-1$. Then $\rho_2 = \rho_1 \cdot t$, $t \in U_K$, $\bar{t} \notin \overline{K}^p$. This means that in L_1 we have either $a_2 = \tau_1^p t + \dots$, or $a_2 = 1 + \tau_1^p t + \dots$, where dots denote terms of higher order. In both cases we see immediately that $d_K(L_1L_2/L_1) = d_K(L_2/K)$ ($= e$ or $e - \frac{p-1}{p} j$ respectively).

Suppose $\rho_1/\rho_2^i \in (K^*)^p \cdot U_{1,K}$; clearly, this can be only when $i = 1$. We get either $a_2 = \tau_2^p b^p + \dots$, or $a_2 = 1 + \tau_2^p b^p + \dots$ with some $b \in U_K$, whence $d_K(L_1L_2/L_1) < d_K(L_2/K)$.

Case 4: both L_1/K and L_2/K are ferocious. Let first $d_K(L_1/K) = d_K(L_2/K) = e$. Then $a_1 = \rho_1 \in U_K$, $a_2 = \rho_2 \in U_K$, $\overline{\rho_1}, \overline{\rho_2} \notin \overline{K}^p$. We have

$$\begin{aligned}\frac{\rho_1}{\rho_2^i} \notin (K^*)^p \cdot U_{1,K}, \quad i = 1, \dots, p-1 &\iff \frac{\overline{\rho_1}}{\overline{\rho_2}^i} \notin \overline{K}^p, \quad i = 1, \dots, p-1 \\ &\iff \overline{L_1} \cap \overline{L_2} = \overline{K} \\ &\iff \overline{\rho_2} \notin \overline{L_1}^p \\ &\iff d_K(L_1L_2/L_1) = e = d_K(L_2/K) \\ &\iff L_1/K \text{ and } L_2/K \text{ are ind-ram.}\end{aligned}$$

Next, let $d_K(L_1/K) = d_K(L_2/K) < e$. Then $a_1 = 1 + \pi^{pn} b_1$, $a_2 = 1 + \pi^{pn} b_2$, $b_1, b_2 \in U_K$, $\overline{b_1}, \overline{b_2} \notin \overline{K}^p$. Then

$$\frac{\rho_1}{\rho_2^i} \notin (K^*)^p \cdot U_{1,K}, \quad i = 1, \dots, p-1 \iff \frac{\overline{b_1}}{\overline{b_2}^i} \notin \overline{K}^p, \quad i = 1, \dots, p-1,$$

and we proceed as above. $\square$

Corollary 3.5. *Let* $\operatorname{char} K = 0$, $\zeta_p \in K$, $L_i = K(x_i)$, $x_i^p = 1 + \pi^n b_i \in K$, $v(b_i) = 0$, $n = \frac{p-1}{p}(e - d_K(L_i/K))$, $i = 1, \dots, m$, π *a prime in* K.

1. Let $p|n$. *In this case* $L_1/K, \dots, L_m/K$ *are ind-ram if and only if* $d\overline{b_1}, \dots, d\overline{b_m} \in \Omega_{\overline{K}/\overline{K}^p}$ *are linearly independent over* $\overline{K}$.

2. Let $p \nmid n$. *In this case* $L_1/K, \dots, L_m/K$ *are ind-ram if and only if* $d(\overline{b_2/b_1}), \dots, d(\overline{b_m/b_1}) \in \Omega_{\overline{K}/\overline{K}^p}$ *are linearly independent over* $\overline{K}$.

Proof. The first assertion is proved by induction on m. The second assertion follows from the first one applied to $L_2L_1/L_1, \dots, L_mL_1/L_1$. $\square$

The following two corollaries will not be used in the rest of the paper, and we omit their proofs.

Corollary 3.6. *Suppose that* L_1/K *and* L_2/K *are ind-ram cyclic extensions of degree* p, *and* $d(L_1/K) = d(L_2/K)$. *Let* M *be an intermediate field in* L_1L_2/K, $[M : K] = p$. *Then* $d(M/K) = d(L_1/K)$.

Corollary 3.7. *Suppose that* L_1/K *and* L_2/K *are ind-ram cyclic extensions of degree* p. *Denote by* $L_3, \dots, L_{p+1}$ *the intermediate fields in* L_1L_2/K *other than* K, L_1, L_2, L_1L_2. *Assume that* $d(L_1/K) \leqslant d(L_2/K)$. *Then* L_1/K *and* L_i/K *are ind-ram for* $i = 3, \dots, p+1$. *In particular, if* $d(L_1/K) = d(L_2/K)$, *then* L_i/K *and* L_j/K *are ind-ram for all* $i \neq j$.

Partial inversion of Corollary 3.6 looks as follows.

Proposition 3.8. *Let* $L_1/K, \dots, L_r/K$ *be cyclic extensions of degree* p *such that* $[L_1 \dots L_r : K] = p^r$. *Suppose that* $d(M/K) = e$ *for any* $M \subset L_1 \dots L_r$ *such that* $[M : K] = p$. *Then* $L_1/K, \dots, L_r/K$ *are ind-ram.*

Proof. We apply induction on r. Let first $r = 2$. Withous loss of generality we may assume $\zeta_p \in K$. Let $L_i = K(x_i)$, $x_i^p = a_i \in K$, $i = 1, 2$. By Kummer theory, various $M \subset L_1L_2$ correspond to $a_1, a_2, a_1a_2, a_1a_2^2, \dots, a_1a_2^{p-1}$. Since $d_K(M/K) =$

e for any M, we conclude that $a_1, a_2, a_1a_2, a_1a_2^2, \dots, a_1a_2^{p-1} \notin (K^*)^p \cdot U_{1,K}$. By Proposition 3.4, L_1/K and L_2/K are ind-ram.

Let now $r \geqslant 3$. Suppose that $L_1/K, \dots, L_r/K$ are not ind-ram. Applying the case $r = 2$, we obtain that $d_K(L_2L_1/L_1) = \dots = d_K(L_rL_1/L_1) = e$. Since $d_K(L_1/K) = e$, the extensions $L_2L_1/L_1, \dots, L_rL_1/L_1$ cannot be ind-ram. By the assumption of induction, there exists some M'/L_1 in $L_1 \dots L_r/L_1$ of degree p such that $d_K(M'/L_1) < e$. We have $M' = M_1L_1$, $[M_1 : K] = p$. Since $M_1 \subset L_1 \dots L_r$, $d_K(M_1/K) < e$, whence L_1/K and M_1/K are not ind-ram. Applying the case $r = 2$ once more, we come to the existence of $M \subset L_1M_1$ with $[M : K] = p$, and $d_K(M/K) < e$. □

The following observations will be used in §4 and §7 respectively.

Lemma 3.9. *Assume $p - 1 | e$. Let L/K and K'/K be wild cyclic extensions of degree p, which are not ind-ram. Suppose that $d_K(L/K) = e$, and $d_K(K'/K) \geqslant \frac{p-1}{p}$. Then $d_{K'}(LK'/K') \leqslant e_{K'} - (p-1)$. In particular, LK'/K' is not AMR.*

Proof. Direct calculation of Kummer equation yielding $LK'(\zeta_p)/K'(\zeta_p)$. □

Lemma 3.10. *Let L/K and K'/K be ind-ram Galois extensions, L/K ferocious. Then LK'/K' is ferocious.*

Proof. The assertion can be easily reduced to the case of cyclic L/K, $[L : K] = p$, $\zeta_p \in K$. In this case Lemma is obvious by Kummer theory. □

§4. General remarks on AMR extensions

In the section $\operatorname{char} K = 0$; K_n/K is a Galois wf-extension, $[K_n : K] = p^n$. Denote $p^s = e(K_n/K)$.

First we discuss the case $n = 1$.

Proposition 4.1. *Let L/K be a cyclic extension of degree p.*

1. Let L/K be wild, $\zeta_p \in K$. In this case L/K is AMR if and only if $h(L/K) \geqslant \frac{pe}{p-1} - 1$.

2. Let L/K be wild, $\zeta_p \notin K$. In this case L/K is AMR if and only if $h(L/K) = \left[\frac{pe-1}{p-1}\right]$.

3. Let L/K be ferocious. In this case L/K is AMR if and only if $h(L/K) = \frac{e}{p-1}$.

In general, L/K is AMR if and only if $d_K(L/K) \geqslant e - \frac{p-1}{p}$.

Proof. For a cyclic extension L/K of degree p the AMR condition is equivalent to $d_K(L/K) - v_K(\pi_L) + v_K(\pi_K) \geqslant e$, see Prop. 1.1. In the ferocious case this means that $d_L(L/K) = d_K(L/K) \geqslant e$, i. e., $h(L/K) = e/(p-1)$.

In the wild case the AMR condition is equivalent to $\frac{p-1}{p}h(L/K) - \frac{1}{p} + 1 \geqslant e$, i. e., $h(L/K) \geqslant \frac{pe}{p-1} - 1$. In the case $\zeta_p \notin K$ notice that $h(L/K) < \frac{pe}{p-1}$ and the only possible value for the ramification jump in an AMR extension is $\left[\frac{pe-1}{p-1}\right]$. □

Remark. The extensions of type 3 can exist only if $K(\zeta_p)/K$ is unramified. Therefore, if $e(K(\zeta_p)/K) \neq 1$, there exist no MR extensions of K.

Now we pass to the general case and start with almost obvious

Lemma 4.2. *Let K_n/K be AMR, M any intermediate field. Then M/K is AMR.*

Proof. Let $[M : K] = p^m$, $e(M/K) = p^r$. Obviously, $r \leqslant s$. Then

$$d_K(M/K) = d_K(K_n/K) - d_K(K_n/M) \geqslant ne - 1 + \frac{1}{p^s} - (n-m)e = me - 1 + \frac{1}{p^s}.$$

Since $p^r d_K(M/K) = d_M(M/K)$ should be an integer, we conclude that

$$d_K(M/K) \geqslant me - 1 + \frac{1}{p^r},$$

i. e., M/K is AMR. $\square$

Remark. K_n/M need not be in general AMR.

Now we construct certain subfields $K_0 = K, K_1, \dots, K_{n-1}$ in K_n/K such that $[K_i : K] = p^i$ by the following inductive procedure. Let G_h be the last non-trivial ramification subgroup in $\mathrm{Gal}(K_n/K)$; it is an elementary abelian p-group. Let H be any subgroup of order p in G_h; denote by K_{n-1} the corresponding intermediate field in K_n/K. Obviously, K_{n-1}/K is a Galois extension of degree p^{n-1}. Then this step is repeated with respect to K_{n-1}/K.

Notice that $d_K(K_n/K_{n-1}) = (p-1)\frac{h}{p^s}$, and $d_K(K_n/L) \leqslant (p-1)\frac{h}{p^s}$, where L is arbitrary intermediate field in K_n/K such that $[K_n : L] = p$. Therefore, we have the following

Lemma 4.3. *1. K_i/K are normal extensions of degree p^i, $i = 1, \dots, n$.*

2. If M is an intermediate field in K_i/K_{i-2}, then $d_K(K_i/M) \leqslant d_K(K_i/K_{i-1})$, and, therefore, $d_K(M/K_{i-2}) \geqslant d_K(K_{i-1}/K_{i-2})$, $i = 2, \dots, n$.

Theorem 4.4. *Assume $p-1|e$. The extension K_n/K is AMR if and only if there exists a normal subextension L/K of degree p^l, $0 \leqslant l \leqslant n$, with the following properties.*

1. L/K is cyclic wild, and $d_K(K_i/K_{i-1}) = e - 1/p^{i-1} + 1/p^i$, $i = 1, \dots, l$, K_i/K denotes the subextension of L/K of degree p^i.

2. K_n/L is MR.

Proof. The existence of L with two properties immediately implies that L/K is AMR; we are to show another implication. If $d_K(K_i/K_{i-1}) = e$ for any i, then we are done with $L = K$. Otherwise, consider the maximal l such that $d_K(K_l/K_{l-1}) < e$. Let $L = K_l$. Obviously, $d_K(K_n/L) = (n-l)e$, and it remains to prove the first property.

Suppose that $d_K(K_i/K_{i-1}) = e$ for some $i < l$; we may assume that i is maximal possible here. Consider K_{i+1}/K_{i-1} which should be a Galois extension of degree p^2. If it is cyclic, then $d_K(K_{i+1}/K_i) = e$ by Prop. 1.6, a contradiction. If it is elementary abelian, then for any intermediate field M in K_{i+1}/K_{i-1} we have $d_K(M/K_{i-1}) = e$ in view of Lemma 4.3. By Prop. 3.8, K_i/K_{i-1} is ind-ram with any such M/K_{i-1} distinct from K_i/K_{i-1}, and we conclude that in this case still $d_K(K_{i+1}/K_i) = e$.

Thus, $d_K(K_i/K_{i-1}) < e$ for all $i \leqslant l$.

Now we use induction on i to prove the following clame. For all $i = 1, \dots, l$: K_i/K_{i-1} is wild, $d_K(K_i/K_{i-1}) = e - 1/p^{i-1} + 1/p^i$, K_n/K_i is AMR, and K_i/K_{i-2} is cyclic when $i \geqslant 2$.

Let $i = 1$. If K_1/K were ferocious, then $d_K(K_1/K)$ would be an integer, whence $d_K(K_1/K) = e$, a contradiction. Thus, K_1/K is wild. Next, $pd_K(K_1/K)$ should be an integer, whence $d_K(K_1/K) \geqslant e - 1 + 1/p$ by Lemma 4.2. Notice that $p-1$ divides both e and $pd_K(K_1/K)$ by Prop. 1.4. Since $d_K(K_1/K) < e$, we conclude $d_K(K_1/K) = e - 1 + 1/p$. We have $d_{K_1}(K_n/K_1) \geqslant (n-1)e_{K_1} - 1 + \frac{1}{p^{s-1}}$, and K_n/K_1 is AMR.

For $i > 1$ we argue similarly. Suppose that K_i/K_{i-2} is not cyclic. The it is elementary abelian, i. e., a composite of K_{i-1}/K_{i-2} and some cyclic M/K_{i-2}. Since $d_K(K_i/K_{i-1}) > d_K(K_{i-1}/K_{i-2})$, we should have $d_K(M/K_{i-2}) > d_K(K_{i-1}/K_{i-2})$, whence $d_K(M/K_{i-2}) = e$. The extensions M/K_{i-2} and K_{i-1}/K_{i-2} cannot be ind-ram, since $d_K(K_i/K_{i-1}) < e$. By Lemma 3.9, $d_K(K_i/K_{i-1}) \leqslant e - \frac{p-1}{p^{i-1}}$, a contradiction.

We have proven the claim, and the cyclicity of L/K follows by group-theoretical argument. $\square$

Corollary 4.5. *Assume $p-1|e$. Let K_n/K be AMR. Then there exists at most one subextension M/K of degree p in K_n/K such that $d_K(M/K) < e$.*

Proof. Suppose there exists M/K such that $d_K(M/K) < e$, and $M \neq K_1$. Since L/K is cyclic, $M \not\subset L$. Then $d_K(LM/L) < e$, a contradiction with $d_K(K_n/L) = (n-l)e$. $\square$

Corollary 4.6. *Assume $p-1|e$. Suppose that K_n/K is a composite of two linearly disjoint extensions M/K and M'/K, and K_n/K is AMR. Then at least one of M/K, M'/K is MR.*

Proof. Suppose that K_n/K is AMR. At least one of M/K, M'/K is disjoint with L/K. Assume that it is M'/K. Since LM'/L is a subextension in K_n/L, it is MR, and, therefore, M'/K is MR. $\square$

§5. Cyclic AMR extensions

In this section we retain tne notation of §4 and assume additionally that K_n/K is cyclic.

Proposition 5.1. *Let M/K be a cyclic extension of degree p^2, L the intermediate field.*

1. Suppose that L/K is wild, $h(L/K) = \frac{pe}{p-1} - i$, $0 < i < \frac{\min(e_{K(\zeta_p)}, p)}{e(K(\zeta_p)/K)}$. Then either M/L is wild with $h(M/L) = \frac{p^2e}{p-1} - i$, or M/L is ferocious with $h(M/L) = \frac{pe}{p-1}$.

2. Suppose that L/K is ferocious, $h(L/K) = \frac{e}{p-1}$. Then either M/L is ferocious with $h(M/L) = \frac{e}{p-1}$, or M/L is wild with $h(M/L) = \frac{pe}{p-1}$.

3. Suppose that L/K is wild, $h(L/K) = \frac{pe}{p-1}$. Then M/L is wild with $h(M/L) = \frac{p^2e}{p-1}$.

Proof. Without loss of generality, we may assume that $\zeta_p \in K$, cf. Corollary 1.3.

1. Since

$$d_K(L/K) = \frac{p-1}{p}h(L/K) > e - \frac{p-1}{p}\min(e,p) \geqslant \frac{e}{p},$$

Proposition 2.2 is applicable. We have $v_K(\rho) = i$, whence $v_L(\tau) = i$. Write $b = \pi_K^j u$, where $u \in U_K$, $\pi_K = N_{L/K}\pi_L$, π_L is a prime in L; we may assume $0 \leqslant j \leqslant p-1$. By Proposition 2.1, $v_L(\pi_K/\pi_L^p - 1) \geqslant p\big(e - \frac{p-1}{p}i\big) > i$. Therefore, $b(1+\tau) \equiv \pi_L^{pj} u(1+\tau) \mod \mathfrak{p}_L(i+1)$. If $\overline{u} \notin (\overline{K})^p$, then M/L is ferocious, and $h(M/L) = \frac{pe}{p-1}$. Otherwise, $u \in (K^*)^p \cdot U_{1,K} \subset (K^*)^p \cdot U_{i+1,L}$. Therefore, M/L is wild, and $h(M/L) = \frac{p}{p-1}e_L - v_L(\tau) = \frac{p^2 e}{p-1} - i$.

2. Here the use of Proposition 2.2 yields $v(1+\tau) = 0$, and $\overline{1+\tau} \notin \overline{L}^p$. Now if $p | v_K(b)$, then M/L is ferocious and $h(M/L) = \frac{e}{p-1}$. Otherwise, M/L is wild with $h(M/L) = \frac{pe}{p-1}$.

3. Assuming that $1+\rho$ is a prime in K, we see from Prop. 2.2 that $1+\tau$ is a prime in L. It follows that $v_L(b(1+\tau)) \equiv 1 \mod p$, whence M/L is wild, and $h(M/L) = \frac{p^2 e}{p-1}$. □

Corollary 5.2. *Suppose that K_1/K is wild, and $h(K_1/K) = \frac{pe}{p-1} - i$, $0 < i < \frac{\min(e_{K(\zeta_p)}, p)}{e(K(\zeta_p)/K)}$. Then there exist integers l and m, $1 \leqslant l \leqslant m \leqslant n$, such that:*

1) for $1 \leqslant j \leqslant l$, K_j/K_{j-1} is wild, and $h(K_j/K_{j-1}) = \frac{p^j e}{p-1} - i$;

2) for $l+1 \leqslant j \leqslant m$, K_j/K_{j-1} is ferocious, and $h(K_j/K_{j-1}) = \frac{p^l e}{p-1}$;

3) for $m+1 \leqslant j \leqslant n$, K_j/K_{j-1} is wild, and $h(K_j/K_{j-1}) = \frac{p^{l+j-m} e}{p-1}$.

Moreover, $l < m$ unless $l = m = n$. In the case $\zeta_p \notin K$ we have $m = n$.

Corollary 5.3. *Suppose that K_1/K is MR. Then there exists an integer m, $0 \leqslant m \leqslant n$ such that:*

1) for $1 \leqslant j \leqslant m$, K_j/K_{j-1} is ferocious, and $h(K_j/K_{j-1}) = \frac{e}{p-1}$;

2) for $m+1 \leqslant j \leqslant n$, K_j/K_{j-1} is wild, and $h(K_j/K_{j-1}) = \frac{p^{j-m} e}{p-1}$.

In the case $\zeta_p \notin K$ we have $m = n$.

Corollary 5.4. *Suppose that K_1/K is AMR. Then there exist integers l and m, $1 \leqslant l \leqslant m \leqslant n$ such that:*

1) for $1 \leqslant j \leqslant l$, K_j/K_{j-1} is wild, and $h(K_j/K_{j-1}) = \big[\frac{p^j e - 1}{p-1}\big]$;

2) for $l+1 \leqslant j \leqslant m$, K_j/K_{j-1} is ferocious, and $h(K_j/K_{j-1}) = \frac{p^l e}{p-1}$;

3) for $m+1 \leqslant j \leqslant n$, K_j/K_{j-1} is wild, and $h(K_j/K_{j-1}) = \frac{p^{l+j-m} e}{p-1}$.

Moreover, $l < m$ unless $l = m = n$. In the case $\zeta_p \notin K$ we have $m = n$.

In particular, any K_j/K_{j-1} is AMR, and K_n/K is AMR. Furthermore,

$$d_K(K_n/K) = ne - \left(1 - \frac{1}{p^l}\right)i,$$

where $i = \frac{pe}{p-1} - \big[\frac{pe-1}{p-1}\big]$.

Proof. Notice that $1 < \frac{\min(e_{K(\zeta_p)}, p)}{e(K(\zeta_p)/K)}$ unless $e = e_K = 1$. In the case $e = 1$ we still have

$$\frac{pe}{p-1} - \left[\frac{pe-1}{p-1}\right] = \frac{1}{p-1} < \frac{\min(e_{K(\zeta_p)}, p)}{e(K(\zeta_p)/K)}.$$

Therefore, any extension K_n/K with AMR K_1/K suits the conditions of either Corollary 5.2 or Corollary 5.3. Therefore, Corollary 5.2 with $i = \frac{pe}{p-1} - \big[\frac{pe-1}{p-1}\big]$

and Corollary 5.3 give a classification of all cyclic wf-extensions K_n/K with AMR K_1/K. (In the situation of Corollary 5.3 we have assumed $l = 0$.)

It remains to check that K_n/K is AMR. We obtain

$$d_K(K_n/K) = \sum_{j=1}^{n} d_K(K_j/K_{j-1}) = \sum_{j=1}^{l}\left(e - \frac{p-1}{p^j}i\right) + \sum_{j=l+1}^{n} e = ne - \left(1 - \frac{1}{p^l}\right)i,$$

where $i = \frac{pe}{p-1} - \left[\frac{pe-1}{p-1}\right] \leqslant 1$. It follows $v_K(\mathcal{D}_{K_n/K}) = d_K(K_n/K) + 1 - \frac{1}{p^{l+n-m}} \geqslant ne = v_K(p^n)$. $\square$

Theorem 5.5. *The following 3 assertions are equivalent.*

1. *K_n/K is AMR.*
2. *K_j/K_{j-1} is AMR for $j = 1, \dots, n$.*
3. *K_1/K is AMR.*

Proof. $1 \Longrightarrow 2$ by Theorem 4.4.

$2 \Longrightarrow 3$ trivially.

$3 \Longrightarrow 1$ by Corollary 5.4. $\square$

§6. Abelian AMR extensions, $p - 1|e$

Lemma 6.1. *Let L/K, K_1/K, ... , K_r/K be ind-ram cyclic extensions of degree p. Let M/K be a cyclic extension of degree p^2 such that $L \subset M$. Suppose that L/K is AMR, and K_1/K, ... , K_r/K are MR. Next, suppose that $d_K(M/K) \leqslant d_K(M'/K)$, where M'/K is any cyclic subextension of degree p^2 in $MK_1 \dots K_r/K$. Then M/L, LK_1/L, ... LK_r/L are ind-ram.*

Proof. Denote $K' = K_1 \dots K_r$.

Case 1: M/L is not MR. Since L/K is AMR, we can apply Cor. 5.4. We obtain that $d_K(L/K) = e - \frac{p-1}{p}i$, and $d_K(M/L) = e - \frac{p-1}{p^2}i$, where $i = \frac{pe}{p-1} - \left[\frac{pe-1}{p-1}\right]$. Therefore, $d_K(LK'/K') = e - \frac{p-1}{p}i$, and $d_K(MK'/LK') \leqslant e - \frac{p-1}{p^2}i$. On the other hand, $d_K(MK'/LK') \geqslant e - \frac{p-1}{p^2}i$ by Prop. 1.6. We conclude that M/L is ind-ram with LK'/L. It remains to notice that LK_1/L, ... , LK_r/L are obviously ind-ram and to apply Prop. 3.3.

Case 2: L/K is MR. Then $d_K(LK'/K') = e$, whence (e. g., by Prop. 1.6) $d_K(MK'/LK') = e$.

Case 3: L/K is not MR, whereas M/L is MR. Suppose that M/L is not ind-ram with LK'/L. By Proposition 3.8, there exists $M' \subset MK'$ such that $[M' : L] = p$, and $d_K(M'/L) < e$. Clearly, $M' \not\subset LK'$, whence M'/K is cyclic of degree p^2. We have

$$d_K(M'/K) = d_K(L/K) + d_K(M'/L) < d_K(L/K) + e = d_K(M/K),$$

a contradiction. $\square$

Theorem 6.2. *Assume $p - 1|e$. Let E/K be an abelian p-extension. Denote by E_1/K the maximal subextension of E/K of exponent p. Then the following conditions are equivalent.*

1. *E/K is AMR.*
2. *E^1/K is AMR.*

3. $E = MN$, where M/K and N/K are ind-ram, M/K is cyclic AMR extension, N/K is MR extension.

Proof. $1 \Longrightarrow 3$. Consider the subfield L as in Theorem 4.4. Denote by M/K the cyclic subextension of maximal degree containing L. Denote by N/K any complementary extension, i. e., any subextension of E/K such that M/K and N/K are linearly disjoint, and $MN = E$. Since E/L is maximally ramified, we have $d_K(E/M) = ne$, whence $d_K(N/K) = ne$, i. e., N/K is maximally ramified and ind-ram with M/K.

$3 \Longrightarrow 2$. We have $E_1 = M_1 N_1$, where M_1/K, N_1/K are maximal subextensions of exponent p in M/K, N/K respectively. Then M_1/K is AMR, E_1/M_1 is MR, whence E_1/K is AMR.

$2 \Longrightarrow 1$. In view of Th. 4.4, $E_1 = K_1 K'$, where K_1/K is of degree p and AMR, K'/K is of exponent p and MR, K_1/K and K'/K are ind-ram. Let M/K be a cyclic subextension in E/K of maximal degree such that $K_1 \subset M$, and M/N is not MR for any N, $K \subset N \subset M$. Let L_1/K be a cyclic subextension of E/K of maximal degree containing M/K. Next, choose arbitrary L_i/K, $i = 2, \dots, r$ so that L_i/K are cyclic, $L_1 \dots L_r = E$, $[L_1 : K] \dots [L_r : K] = [L_1 \dots L_r : K]$, and $L_i \cap K' = K_i \neq K$, $i = 2, \dots, r$.

Let L_1^i/K be the subextension of degree p^i in L_1/K. Applying Lemma 6.1, we obtain by induction on i, that L_1^{i+1}/L_1^i, $K_2 L_1^i/L_1^i$, ... , $K_r L_1^i/L_1^i$ are ind-ram. Thus, $K_2 L_1/L_1$, ... , $K_r L_1/L_1$ are MR and ind-ram.

Applying the same argument, we obtain that $K_3 L_1 L_2/L_1 L_2$, ... , $K_r L_1 L_2/L_1 L_2$ are MR and ind-ram. Continuing by induction, we obtain that E/L_1 is MR, whence E/K is AMR by Th. 5.5 and Th. 4.4. $\square$

Remark. In the classical case (when the residue fields are perfect), an AMR extension should be cyclic. In the general case it is easy to construct non-cyclic AMR, and even MR, extension. In fact, let $\zeta_p \in K$, $M = k(x)$, $N = K(y)$, where $x^p \in a \in \mathcal{O}_K$, $\overline{a} \notin \overline{K}^p$, $y^p = \pi$ is a prime in K. Then M/K and N/K are ind-ram by Prop. 3.4, and MN/K is MR by Th. 6.2.

§7. Abelian AMR extensions, $p - 1 \nmid e$

If $p - 1 \nmid e$, we cannot apply Th. 4.4 any more. However, in this case the ramification of composites of ind-ram cyclic AMR extensions has quite a regular behavior.

We say that a cyclic extension L/K is purely AMR if it is AMR, and for any intermediate M the extension L/M is not MR. It is clear that any purely AMR extension is wild, and that in the case $p-1 \nmid e$ all cyclic AMR extensions are purely AMR.

For a cyclic p-extension L/K we denote by L_j/K the subextension of degree p^j in L/K, where $p^j \leqslant [L : K]$.

Lemma 7.1. *Let $K^1/K, \dots, K^m/K$ be ind-ram purely AMR extensions of degree p. Then $K^1 \dots K^m/K^1$ is ferocious.*

Proof. The extensions $K^i K^1/K^1$, $i = 2, \dots, m$, are ind-ram and ferocious. Their composite is ferocious by Lemma 3.10. $\square$

Proposition 7.2. *Let $K^1/K,\dots,K^m/K$ be purely AMR cyclic extensions. Suppose that $K_1^1/K,\dots,K_1^m/K$ are ind-ram. Then:*
1. *$K^1/K,\dots,K^m/K$ are ind-ram;*
2. *$e(K^1\dots K^m/K)=\max([K^1:K],\dots,[K^m:K])$.*

Proof. Let $p^r=\max([K^1:K],\dots,[K^m:K])$; we apply induction on r.

If $r=1$, we have nothing to prove, see Lemma 7.1. Let $r\geqslant 2$. Denote $L=K_1^1\dots K_1^m$. Without loss of generality, we may assume that for some $l\leqslant m$ we have $[K^i:K]\geqslant p^2$, $i=1,\dots,l$, and $[K^{l+1}:K]=\dots=[K^n:K]=p$. Now it is sufficient to prove that $K^1L/L,\dots,K^lL/L$ are purely AMR, and $K_2^1L/L,\dots,K_2^lL/L$ are ind-ram. In fact, by the assumption of induction it follows that $K^1L/L,\dots,K^lL/L$ are ind-ram. Notice that L/K_1^i is ferocious by Lemma 7.1 and, therefore, ind-ram with K^i/K_1^i, $i=1,\dots,l$. We obtain

$$\begin{aligned}
d_K(K^1\dots K^m/K)&=d_K(K^1\dots K^m/L)+d_K(L/K)\\
&=d_K(K^1\dots K^lL/L)+d_K(L/K)\\
&=d_K(K^1L/L)+\dots+d_K(K^lL/L)+d_K(L/K)\\
&=d_K(K^1/K_1^1)+\dots+d_K(K^l/K_1^l)\\
&\quad+d_K(K_1^1/K)+\dots+d_K(K_1^m/K)\\
&=d_K(K^1/K)+\dots+d_K(K^m/K)
\end{aligned}$$

and

$$e(K^1\dots K^m/K)=e(K^1\dots K^m/L)+e(L/K)=r-1+1=r.$$

The fact that all K^iL/L are purely AMR is now clear since K^i/K_1^i are wild, and L/K_1^i are ferocious.

It remains to show that $K_2^1L/L,\dots,K_2^lL/L$ are ind-ram. This is equivalent to the fact that $K_2^1L(\zeta_p)/L(\zeta_p),\dots,K_2^lL(\zeta_p)/L(\zeta_p)$ are ind-ram.

For any i, the extension $K_1^i(\zeta_p)/K(\zeta_p)$ corresponds to Kummer equation $x_i^p=1+\pi^nu_i$, where $u_i\in U_K$, π is a prime in $K(\zeta_p)$, and $n=e(K(\zeta_p)/K)\left(\frac{pe}{p-1}-\left[\frac{pe-1}{p-1}\right]\right)\leqslant p-1$ is independent from i. Applying Prop. 2.2, we see that $K_2^i(\zeta_p)/K_1^i(\zeta_p)$ corresponds to Kummer equation $y_i^p=b_i(1+\tau_i)$, $v_{K(\zeta_p)}(\tau_i^p-\pi^nu_i)>n$, $b_i\in K(\zeta_p)$. Taking into account Remark after Prop 2.2, we may assume that $v(b)=0$. Since $K_2^i(\zeta_p)/K_1^i(\zeta_p)$ is wild, we see that $\overline{b_i}\in\overline{K(\zeta_p)}^p$, i. e., $b_i=\theta_i^p(1+\pi a_i)$, $\theta_i,a_i\in\mathcal{O}_{K(\zeta_p)}$. Denote $\tau_i'=(1+\tau_i)(1+\pi a_i)-1$ We conclude that $K_2^i(\zeta_p)=K_1^i(\zeta_p,y_i')$, where $y_i'^p=1+\tau_i'$, $v_{K(\zeta_p)}(\tau_i'^p-\pi^nu_i)>n$.

It follows immediately that $\tau_2'/\tau_1',\dots\tau_m'/\tau_1'$ are units in $L(\zeta_p)$, and $(\overline{\tau_i'/\tau_1'})^p=\overline{u_i/u_1}$, $i=2,\dots,m$. By Corollary 3.5, $d\left(\overline{u_i/u_1}\right)\in\Omega_{\overline{K(\zeta_p)}/\overline{K(\zeta_p)}^p}$, $i=2,\dots,m$ are linearly independent over $\overline{K(\zeta_p)}$. There is an injective map

$$\begin{gathered}\overline{L(\zeta_p)}\to\overline{K(\zeta_p)}\\ x\mapsto x^p.\end{gathered}$$

It follows that $d\left(\overline{\tau_i'/\tau_1'}\right)\in\Omega_{\overline{L(\zeta_p)}/\overline{L(\zeta_p)}^p}$, $i=2,\dots,m$ are linearly independent over $\overline{L(\zeta_p)}$. Applying Corollary 3.5 once more, we conclude that $K_2^1L(\zeta_p)/L(\zeta_p),\dots,K_2^lL(\zeta_p)/L(\zeta_p)$ are ind-ram. $\square$

In the setting of Prop. 7.2 it is easy to calculate the depth of ramification of $K^1 \dots K^m/K$. Let $\varepsilon = \frac{pe}{p-1} - \left[\frac{pe-1}{p-1}\right]$, and $p^{r_i} = [K^i : K]$, $i = 1, \dots, m$. We obtain

$$\begin{aligned} d_K(K^1 \dots K^m/K) &= \sum_{i=1}^{m} d_K(K^i/K) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{r_i} d_K(K^i_j/K^i_{j-1}) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{r_i} \left(e - \frac{p-1}{p^j}\varepsilon\right) \\ &= \sum_{i=1}^{m} \left(r_i e - \left(1 - \frac{1}{p^{r_i}}\right)\varepsilon\right). \end{aligned}$$

Proposition 7.3. *Let $K^1/K, \dots, K^m/K$ be linearly disjoint purely AMR cyclic extensions. Then $K^1 \dots K^m/K$ is AMR if and only if the following 2 conditions are satisfied.*

1. $K^1_1/K, \dots, K^m_1/K$ are ind-ram.

2. $\varepsilon \sum_{i=1}^{m} \left(1 - \frac{1}{p^{r_i}}\right) \leqslant 1 - \frac{1}{p^r}$, where $p^{r_i} = [K^i : K]$, $i = 1, \dots, m$; $r = \max(r_1, \dots, r_m)$; $\varepsilon = \frac{pe}{p-1} - \left[\frac{pe-1}{p-1}\right]$.

Proof. If $K^1_1/K, \dots, K^m_1/K$ are ind-ram, then, in view of the above calculation of $d_K(K^1 \dots K^m/K)$, the second condition is equivalent to the fact that $K^1 \dots K^m/K$ is AMR.

Suppose that $K^1_1/K, \dots, K^m_1/K$ are not ind-ram. Take the minimal j such that $K^1_1/K, \dots, K^j_1/K$ are not ind-ram. It can be shown in the spirit of Lemma 3.9 that

$$d_K(K^1_1 \dots K^j_1/K^1_1 \dots K^{j-1}_1) \leqslant e - \frac{p-1}{p}\varepsilon - \frac{p-1}{p}.$$

It follows that

$$d_K(K^1_1 \dots K^j_1/K) \leqslant j\left(e - \frac{p-1}{p}\varepsilon\right) - \frac{p-1}{p} < e - 1 + \frac{1}{p^2},$$

i. e., $K^1_1 \dots K^j_1/K$ is not AMR, and $K^1 \dots K^m/K$ is not AMR by Lemma 4.2. □

We obtain immediately the following classification of abelian AMR extensions in the case $p - 1 \nmid e$.

Theorem 7.4. *Assume $p - 1 \nmid e$. Let L/K be an abelian p-extension. Write L as $K^1 \dots K^m$, where K^i/K are linearly disjoint cyclic, $i = 1, \dots, m$. Then L/K is AMR if and only if the following 3 conditions are satisfied.*

1. The subextension of degree p in K^i/K is AMR for all $i = 1, \dots, m$.

2. The subextensions of degree p in K^i/K are ind-ram, $i = 1, \dots, m$.

3. $\varepsilon \sum_{i=1}^{m} \left(1 - \frac{1}{p^{r_i}}\right) \leqslant 1 - \frac{1}{p^r}$, where $p^{r_i} = [K^i : K]$, $i = 1, \dots, m$; $r = \max(r_1, \dots, r_m)$; $\varepsilon = \frac{pe}{p-1} - \left[\frac{pe-1}{p-1}\right]$.

§8. More on composites of AMR extensions

We return to the case $e|p-1$. If we have two disjoint AMR extensions, none of which is MR, the composite cannot be AMR in view of Cor. 4.6. However, in the ind-ram case the ramification of the composite is rather regular.

In this section K_n/K is a cyclic extension of degree p^n, L_m/K is a cyclic extension of degree p^m, $n \geqslant m$. These extensions are always assumed to be AMR. Denote by K_i/K, L_i/K the subextensions of degree p^i in K_n/K and L_m/K respectively. We assume that K_1/K and L_1/K are ind-ram.

Lemma 8.1. *Let $n \geqslant 2$. Then K_2L_1/K_1L_1 is AMR.*

Proof. Let L_1/K be ferocious. Then

$$d_{L_1}(K_1L_1/L_1) = d_K(K_1L_1/L_1) = d_K(K_1/K) \geqslant e_K - \frac{p-1}{p} = e_{L_1} - \frac{p-1}{p},$$

whence K_1L_1/L_1 is AMR. By Theorem 5.5, K_2L_1/K_1L_1 is AMR.

Let K_1/K be ferocious. Then $d_K(K_1/K) = e$, K_1L_1/L_1 is MR, and, by Theorem 5.5, K_2L_1/K_1L_1 is AMR.

Finally, let both L_1/K and K_1/K be wild. Then it is clear that K_1L_1/L_1 is ferocious. We have $d_K(K_1L_1/L_1) = d_K(K_1/K) \geqslant e - \frac{p-1}{p}$, whence $d_{L_1}(K_1L_1/L_1) \geqslant e_{L_1} - (p-1)$, and, by Prop. 1.6, $d_{L_1}(K_2L_1/K_1L_1) \geqslant e_{L_1} - \frac{p-1}{p}$. Since $e(K_1L_1/L_1) = 1$, we obtain that K_2L_1/K_1L_1 is AMR. □

Remark. The following example shows that K_2/K_1 and L_1K_1/K_1 are in general not ind-ram. Take $K = \mathbb{Q}_p(\zeta_{p^2})\{\{t\}\}$, $\pi = \zeta_{p^2} - 1$, $K_2 = K\big(\sqrt[p^2]{t^p(1+\pi)}\big)$, $L_1 = K\big(\sqrt[p]{1+\pi t}\big)$.

Lemma 8.2. *In the situation of Lemma 8.1 suppose that neither of K_1/K, L_1/K, K_2L_1/K_1L_1 is MR. Then $K_1(\zeta_p) = K(\zeta_p, \sqrt[p]{1+\rho})$, $K_2L_1(\zeta_p) = K_1L_1(\zeta_p, \sqrt[p]{1+\tau})$, $v(\tau^p - \rho) > v(\rho) > 0$.*

Proof. Direct calculation using Prop. 2.2. □

Lemma 8.3. *Let $m \geqslant 2$. Then K_2L_2/K_1L_1 is a composite of two cyclic ind-ram AMR extensions of degree p.*

Proof. 1. Suppose first that at least one of K_1/K, L_1/K is MR. Then our assertion follows immediately from Theorem 6.2.

2. Now both K_1/K and L_1/K are not MR. Consider all subextensions of degree p in K_2L_2/K_1L_1: $M_1/K_1L_1, \dots, M_{p+1}/K_1L_1$. Suppose that all they are MR. Then any two them are ind-ram in view of Prop. 3.8.

3. Suppose that some of $M_1/K_1L_1, \dots, M_{p+1}/K_1L_1$ are MR, and some not; say, M_1/K_1L_1 is MR, and M_2/K_1L_1 is not MR. Changing K_2 and L_2 inside K_2L_2, we may assume without loss of generality that $M_1 = K_2L_1$, $M_2 = K_1L_2$. It follows that K_2/K_1 is MR. Applying results of §4, we conclude that $K_2(\zeta_p) = K_1(\zeta_p, \sqrt[p]{a})$, where $a \in \mathcal{O}_{K_1(\zeta_p)}$, $\overline{a} \notin \big(\overline{K_1(\zeta_p)}\big)^p$. It follows that K_2L_1/K_1L_1 is ferocious. On the other hand, K_1L_2/K_1L_1 is AMR (by Lemma 8.1) but not MR. Therefore, it is wild and ind-ram with K_2L_1/K_1L_1.

4. It remains to consider the case when all $M_1/K_1L_1, \dots, M_{p+1}/K_1L_1$ are not MR.

Let $K' = K(\zeta_p)$. We have $K_1K' = K'(\sqrt[p]{1+\rho_1})$, $L_1K' = K'(\sqrt[p]{1+\rho_2})$, where

$$v_{K'}(\rho_1) = v_{K'}(\rho_2) = i = \left(\frac{pe}{p-1} - \left[\frac{pe-1}{p-1}\right]\right)e(K'/K).$$

Since K_1/K and L_1/K are ind-ram, by Prop. 3.4, $\overline{\rho_1/\rho_2} \notin (\overline{K'})^p$. By Lemma 8.2, $K_2L_1K' = K_1L_1K'(\sqrt[p]{1+\tau_1})$, $K_1L_2K' = K_1L_1K'(\sqrt[p]{1+\tau_2})$, $v_{K'}(\tau_1^p - \rho_1) > i$, $v_{K'}(\tau_2^p - \rho_2) > i$. Therefore,

$$\overline{\left(\frac{\tau_1}{\tau_2}\right)} = \sqrt[p]{\frac{\rho_1}{\rho_2}} \notin (\overline{K_1L_1K'})^p.$$

Indeed, $(\overline{K_1L_1K'})^p \subset \overline{K'}$. By Prop. 3.4, we conclude that K_2L_1/K_1L_1 and K_1L_2/K_1L_1 are ind-ram. □

Theorem 8.4. *Let $n > m$. Then $K_nL_m/K_{n-1}L_m$ is AMR.*

Proof. Application of Lemma 8.3 decreases both m and n by 1. In the case $m = 1$ Theorem follows from Lemma 8.1. □

Theorem 8.5. *Let $n = m$. Then $K_nL_m/K_{n-1}L_{m-1}$ is a composite of two ind-ram AMR cyclic extensions of degree p.*

Proof. Apply n times Lemma 8.3. □

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