ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 13.

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§2.0. Foreword.

All calculations below are made by hands.

Acknowledgments. I express my deepest thanks to Professors B.Z.Moroz, I.I. Piatetski-Shapiro, Yu.I. Manin, A.G.Aleksandrov, P.Bundshuh, S.G.Gindikin and W.W.Zudilin for help and support.

I express my deepest thanks to the Rector of the Moscow State Institut of Electronics and Mathematiks Professor D.V. Bykov for administrative support.

§12.1. Properties of the considered systems as abstract systems. The case $l = 2$, $z = 1$ (continuation).

Lemma 13.1.1. . *The equality*

$$r^{\wedge\vee}_{2,3}(\nu) = 0 \tag{1}$$

holds.

Proof. In view of (86), (121) – (123) and results of §12.1 in [16],

$$r^{\wedge\vee}_{2,3}(\nu) = (7\mu^3 + 14\mu^2 + 7\mu + 1)(-2\mu^3 + 3\mu^2) + \tag{2}$$

$$4(\mu^4 + 28\mu^3 + 76\mu^2 + 48\mu + 8)(1574\mu^2 - 140\mu - 11)+$$

$$16(\mu^3 + 7\mu^2 + 8\mu + 2)(360\mu^2 - 118\mu - 11)(4\mu + 1)+$$

$$32(-\mu^3-15\mu^2-18\mu-4)(226\mu^3+265\mu^2-32\mu-2)+$$
$$32(3\mu^2+10\mu+4)(-317\mu^2+28\mu+2)(4\mu+1)+$$
$$(-1)(2\mu^3+27\mu^2+18\mu+2)(7113\mu^3+2586\mu^2-183\mu-9)+$$
$$2(3\mu^2+7\mu+1)(-1023\mu^3-2898\mu^2+165\mu+9)(4\mu+1)+$$
$$16(3\mu^2+17\mu+5)(122\mu^4+644\mu^3+166\mu^2-12\mu-1)+$$
$$16(6\mu+5)(260\mu^3+184\mu^2-10\mu-1)(4\mu+1)+$$
$$4(6\mu^2+24\mu+1)(218\mu^4+280\mu^3+2\mu^2-28\mu-3)+$$
$$16(-3\mu)(-20\mu^4-168\mu^3-40\mu^2+22\mu+3)(4\mu+1)+$$
$$16(3\mu+2)(2\mu^5+\mu^4+194\mu^3+215\mu^2+70\mu+7)+$$
$$48(-7\mu^4+44\mu^3+117\mu^2+56\mu+7)(4\mu+1)+$$
$$8(2\mu-1)(86\mu^5+511\mu^4+830\mu^3+521\mu^2+138\mu+13)+$$
$$16(8\mu^5+127\mu^4+356\mu^3+323\mu^2+112\mu+13)(4\mu+1)+$$
$$16(-14\mu^6-352\mu^5-1325\mu^4-1700\mu^3-940\mu^2-232\mu-21).$$

Coefficient at μ^6 in the polynomial $r^{\wedge\vee}_{2,3}(\nu)$ is equal to

$$(3)\qquad -2\times 7+4\times 1574+16(360)(4)+32(-226)-2\times 7113+$$
$$2(3)(-1023)(4)+16(3)(122)+4(6)(218)+16(-3)(-20)(4)+16(3)(2)+$$
$$8(2)(86)+16(8)(4)+16(-14)=2(-7-7113)+8(787-3069)+$$
$$16(327)+32(183+43-7+3)+64(-113)+256(15)+512(45+1)=$$
$$-32\times 445-16\times 1141+16\times 327+32\times 222-64\times 113+256 times 15+1024\times 23=$$
$$32(-223-407)-64\times 113+256\times 15+1024\times 23=$$
$$-64(315+113)+256 times 15+1024\times 23=256(-107+15)+1024\times 23=$$
$$256(-92+23\times 4)=0.$$

Coefficient at μ^1 in the polynomial $r^{\wedge\vee}_{2,3}(\nu)$ is equal to

$$(4)\qquad -64\times 33-128\times 35-128\times 11-64\times 59-128\times 11+$$
$$128\times 9+4096+128\times 5+512\times 7+1024+2(81+183)$$
$$+2\times(63+165)+8\times 9-64\times 15-16\times 17-32\times(3+25)-64\times 5$$
$$-16\times 7-32\times 9-16\times 9+16\times 21+64\times 35+128\times 21+64\times 21+$$
$$16\times(13-69)+256\times 7+64\times 13-128\times 29=$$
$$16(33+33-17-7-9+21+13-69)-32\times 9+64(-33-59-15-5+35+21+13)+$$
$$128(-35-11-11+9+5-7+21-29)+256\times 7+512\times 7+1024+4096=$$

$$-64\times 5-64\times 43-256\times 29+256\times 7+512\times 7+1024+4096=$$

$$-1024\times 3-512\times 11+512\times 7+2048=2048-512\times 4=0.$$

Clearly,

(5) $$r_{2,1}^{\wedge\vee}(\nu)\big|_{\mu=0}=-32\times 11-32\times 11+256+256+18+$$

$$18-16\times 5-16\times 5-4\times 3+32\times 7+16\times 21-8\times 13+$$

$$16\times 13-16\times 21=-64\times 11+512+32\times 7-16\times 15+16\times 13=$$

$$-64\times 11+32\times 6+512=-64\times 8+512=0.$$

So $r_{2,3}^{\wedge\vee}(\nu)=\mu^2t_{2,3}^{\wedge\vee}(\mu)$, where $t_{2,3}^{\wedge\vee}(\mu) in\mathbb{Q}[\mu]$, coefficients of $t_{2,3}^{\wedge\vee}(\mu)$ at μ^k are equal to 0, if $k>3$, and to prove that $t_{2,3}^{\wedge\vee}(\mu)=0$, it is sufficient to prove that $t_{2,3}^{\wedge\vee}(\mu)\in(\mu^4-1)\mathbb{Q}[\mu]$. So we most prove that

$$r_{2,3}^{\wedge\vee}(\nu)\big|_{\mu=\pm1}=r_{2,3}^{\wedge\vee}(\nu)\big|_{\mu=i}=0.$$

We have

$$r_{2,3}^{\wedge\vee}(\nu)\big|_{\mu=-1}=5+4\times 9\times 1703+0+$$

$$0-32\times 9\times 343+9\times 4353-2\times 9\times 2031+16\times 9\times 345$$

$$-16\times 3\times 67+4\times 17\times 7\times 5-16\times 9\times 89+16\times 43+$$

$$(-16)\times 9\times 17+8\times 2716\times 3\times 13+-256.$$

Since

$$\omega_1^{\vee}:=5+4\times 17\times 7\times 5=5\times 477=5\times 9\times 53,$$

$$\omega_2^{\vee}:=\omega_1^{\vee}+9\times 4353=9\times 4618=2\times 9\times 2309,$$

$$\omega_3^{\vee}:=\omega_2^{\vee}-2\times 9\times 2031=$$

$$2\times 9\times 278=4\times 9\times 139,$$

$$\omega_4^{\vee}:=\omega_3^{\vee}+4\times 9\times 1703=8\times 9\times 921,$$

$$\omega_5^{\vee}:=\omega_4^{\vee}+8\times 9\times 3=$$

$$8\times 9\times 924=32\times 9\times 231,$$

$$\omega_6^{\vee}:=16\times 43-16\times 16=16\times 9\times 3,$$

$$\omega_7^{\vee}:=-16\times 3\times 67+16\times 3\times 13=-32\times 9\times 9$$

$$\omega_8^{\vee}:=\omega_6^{\vee}+16\times 9(345-89-17)=$$

$$16\times 9(3+345-89-17)=32\times 9\times 121,$$

$$\omega_9^{\vee}:=\omega_5^{\vee}+\omega 8^{\vee}+\omega^7\vee-32\times 9\times 343=$$

$$32\times 9\times(-343+231-9+121)=0,$$

it follows that

(6) $$r_{2,3}^{\wedge\vee}(\nu)\big|_{\mu=-1}=0.$$

In view of (2),

$$r^{\wedge\vee}_{2,3}(\nu)\big|_{\mu=1} = 29 + 4\times 7\times 23\times 1423+$$
$$32\times 27\times 77\times 5 - 64\times 19\times 457 - 32\times 5\times 17\times 7\times 41+$$
$$(-1)\times 49\times 3\times 3169 + 2\times 5\times 11\times 3\times(-1249)+$$
$$16\times 25\times 919 + 16\times 5\times 11\times 433 + 4\times 31\times 7\times 67+$$
$$16\times 3\times 5\times 7\times 29 + 16\times 5\times 3\times 163+$$
$$16\times 3\times 7\times 31+$$
$$8\times 2099 + 16\times 5\times 3\times 313 - 128\times 3\times 191$$
$$16(8\mu^5 + 127\mu^4 + 356\mu^3 + 323\mu^2 + 112\mu + 13)(4\mu+1)+$$
$$16(-14\mu^6 - 352\mu^5 - 1325\mu^4 - 1700\mu^3 - 940\mu^2 - 232\mu - 21).$$

Since

$$\omega^{\wedge}_1 := 29 - 49\times 9507 = -29\times 9506 - 2\times 95070 =$$
$$2(-29\times 4753 - 95070) = -2\times(142590 + 95070 - 4753) =$$
$$-2\times(237660 - 4753) = -2(232907),$$
$$\omega^{\wedge}_2 := \omega^{\wedge}_1 - 2\times 165\times 1249 =$$
$$-2(232907 + 206085) = -2(438992) = -32(27437);$$
$$\omega^{\wedge}_3 := 4\times 7\times 23\times 1423 + 4\times 31\times 7\times 67) =$$
$$28(32729 + 2077) = 28\times 34806 = 56\times 17403,$$
$$\omega^{\wedge}_4 := \omega^{\wedge}_3 + 8\times 2099 = 8(121821 + 2099) =$$
$$8(121821 + 2099) = 8\times 123920 = 128\times 7745,$$
$$\omega^{\wedge}_5 :=$$
$$80(4595 + 4763 + 609 + 489 + 939 + 651) = 80\times 12046) = 160\times 6023,$$
$$\omega^{\wedge}_6 := \omega^{\wedge}_5 + 32\times 27\times 77\times 5 -$$
$$32\times 5\times 17\times 7\times 41 = 160(6023 + 2079 - 4879) = 32\times 16115.$$
$$\omega^{\wedge}_7 := \omega^{\wedge}_6 + \omega^{\wedge}_2 = 32(16115 - 27437) = 32\times$$
$$-32\times 11322 = -64\times 5661, \omega^{\wedge}_8 := -64\times 19\times 457+$$
$$\omega^{\wedge}_7 = -64(5661 + 8683) = -64(14344) = -128\times 7172,$$
$$\omega^{\wedge}_9 := \omega^{\wedge}_4 + \omega^{\wedge}_8 - 128\times 3\times 191 + \omega^{\wedge}_8 =$$
$$128(7745 - 7172 - 573) = 0,$$

it follows that

$$r^{\wedge\vee}_{2,3}(\nu)\big|_{\mu=1} = 0. \tag{7}$$

In view of (2)

$$r^{\wedge\vee}_{2,3}(\nu)\big|_{\mu=i} = 13(3-2i)+$$
$$4(-67+20i)(-1585-140i) + 16(-5+7i)(-371-118i)(1+4i)$$
$$32(11-17i)(-267-258i) + 32(1+10i)(319+28i)(1+4i)+$$

$$(25-16i)(-2595-7296i)+2(-2+7i)(2907+1188i)(1+4i)+$$
$$16(2+17i)(-47-656i)+16(5+6i)(-185-270i)(1+4i)+$$
$$4(-5+24i)(213-308i)+48(-i)(23+190i)(1+4i)+$$
$$16(2+3i)(-207-122i)+48(-117+12i)(1+4i)+$$
$$8(-1+2i)(3-606i)+16(-183-252i)(1+4i)+16(-392+1116i)=$$

Since

$$\omega_1^* := 39-26i+(25-16i)(-2595-7296i)=$$
$$39-26i-181611-140880i=2(-90786-70453i),$$
$$\omega_2^* := \omega_1^*+2(-2+7i)(2907+1188i)(1+4i)=$$
$$2(-90786-70453i)+2(-30-i)(2907+1188i)=$$
$$2(-90786-70453i-86022-38547i)=$$
$$-2(176808+109000i)=-16(22101+13625i),$$
$$\omega_3^* := 4(-67+20i)(-1585-140i)+4(-5+24i)(213-308i)=$$
$$4(108995-22320i+6327+6652i)=4(115322-15668i)=8(57661-7834i),$$
$$\omega_4^* := \omega_3^*+8(-1+2i)(3-606i)=$$
$$8(57661-7834i+1209+612i)=16(29435-3611i),$$
$$\omega_{51}^* := 16(2+17i)(-45-656i)+16(2+3i)(-207-122i)=$$
$$16(11062-2077i-48-865i)=32(5507-1471i),$$
$$\omega_{52}^* := 16(5+6i)(-185-270i)(1+4i)+16(-183-236i)(1+4i)=$$
$$16(1+4i)(695-2460i-183-236i)=128(1+4i)(64-337i)=128(1412-81i),$$
$$\omega_{53}^* := 48(190-23i-117+12i)(1+4i)=48(73-11i)(1+4i)=$$
$$48(117+281i)=16(351+843i)$$
$$\omega_{61}^* := 16(-5+7i)(-371-118i)(1+4i)+\omega_{53}^*=$$
$$16((-5+7i)(101-1602i)+351+843i)=$$
$$16(10709+8717i+351+843i)=16(11060+9560)=64(2765+2390i,$$
$$\omega_{62}^* := \omega_2^*+\omega_4^*=$$
$$16(-22101-13625i+29435-3611i)=16(7334-17236i)=32(3667-8618i),$$
$$\omega_{71}^* := \omega_6 2^*+32(1+10i)(319+28i)(1+4i)=$$
$$32(3667-8618i+(1+10i)(207+1304i))=32(3667-8618i-12833+3374i)=$$
$$64(-4583-2622i),$$
$$\omega_{72}^* := \omega_{51}^*+32(11-17i)(-267-258i)=$$
$$32(5507-1471i-7323+1701i)=32(-1816+230i)=64(-908+115i),$$
$$\omega_{81}^* := \omega_{72}^*+16(-392+1116i)=$$
$$64(-908+115i-98+279i)=128(-503+197i)$$
$$\omega_{82}^* := \omega_{61}^*+\omega_{71}^*$$

$$64(2765+2390i-4583-2622i)=64(-1818-232i)=128(-909-116i);$$

$$\omega_9^* := \omega_{81}^* + \omega_{52}^* =$$

$$128(-503+197i+1412-81i)128(909+116i)=-\omega_{82}^*,$$

it follows that

(8) $$r_{2,3}^{\wedge\vee}(\nu)\Big|_{\mu=i}=0.$$

■

Lemma 13.1.2. . *The equality*

(9) $$r_{2,3}^{\wedge\wedge}(\nu)=0$$

holds.

Proof. In view of (87), (121) – (123) and results of §12.1 in [16],

(10) $$r_{2,3}^{\wedge\wedge}(\nu)=-2(\mu^3+6\mu^2+5\mu+1)(-2\mu^3+3\mu^2)+$$

$$(-8)(\mu^3+7\mu^2+8\mu+2)(4)(1574\mu^2-140\mu-11)+$$
$$(\mu^4+28\mu^3+76\mu^2+48\mu+8)(-8)(360\mu^2-118\mu-11)+$$
$$8(3\mu^2+10\mu+4)8(226\mu^3+265\mu^2-32\mu-2)+$$
$$4(-\mu^3-15\mu^2-18\mu-4)16(-317\mu^2+28\mu+2)+$$
$$4(3\mu^2+7\mu+1)(7113\mu^3+2586\mu^2-183\mu-9)+$$
$$(-2)(2\mu^3+27\mu^2+18\mu+2)(-1023\mu^3-2898\mu^2+165\mu+9)$$
$$(-4)(6\mu+5)(8)(122\mu^4+644\mu^3+166\mu^2-12\mu-1)+$$
$$4(3\mu^2+17\mu+5)(-8)(260\mu^3+184\mu^2-10\mu-1)+$$
$$(-24\mu)4(218\mu^4+280\mu^3+2\mu^2-28\mu-3)+$$
$$(6\mu^2+24\mu+1)(8)(-20\mu^4-168\mu^3-40\mu^2+22\mu+3)+$$
$$(24)(-4)(2\mu^5+\mu^4+194\mu^3+215\mu^2+70\mu+7)+$$
$$(-4)(3\mu+2)(8)(-7\mu^4+44\mu^3+117\mu^2+56\mu+7)+$$
$$8(-4)(86\mu^5+511\mu^4+830\mu^3+521\mu^2+138\mu+13)+$$
$$(-2)(2\mu-1)(8)(8\mu^5+127\mu^4+356\mu^3+323\mu^2+112\mu+13)+$$
$$(4)(8)(50\mu^5+395\mu^4+792\mu^3+602\mu^2+190\mu+21).$$

Coefficient at μ^6 in the polynomial $r_{2,3}^{\wedge\wedge}(\nu)$ is equal to

(11) $$4-64\times 45+4\times 1023-64\times 15-256=$$

$$4096-256\times 15-256=2^12-2^12=0.$$

Coefficient at μ^5 in the polynomial $r_{2,3}^{\wedge\wedge}(\nu)$ is equal to

(12) $$18-64\times 787+16\times 59-256\times 315+128\times 339+$$

$$64\times 317+4\times 21339+8\times 1449+2\times 27621-128\times 183+$$
$$(-128)\times 195+(-64)\times 327+(-128)\times 63-256\times 15+(-64)\times 3+$$

$$32 \times 21 + (-64) \times 43 + (-32)(127) + 128 + 64 \times 25 =$$
$$4(13815) + 4 \times 21339 + 8 \times 1449 + 16 \times 59 +$$
$$32 \times (21 - 127) + 64 \times (-787 + 317 - 327 - 3 - 43 + 25) +$$
$$128(339 - 183 - 195 - 63 + 1) - (256)(315 + 15) =$$
$$8(17577) + 8(1449) + 16 \times 59 - 64 \times 53 +$$
$$(-64) \times 818 - 128 \times 101 - 512 \times 165 =$$
$$16(9513 + 59) - 64 times 53 - 128(409 + 101) - 512 \times 165 =$$
$$64(2393 - 53) - 256 \times 255 - 512 \times 165) =$$
$$256(585 - 255) - 256 \times 330 = 0.$$

Coefficient at μ^1 in the polynomial $r^{\wedge\wedge}_{2,3}(\nu)$ is equal to

(13) $$64(44 + 140) + 64(66 + 118) - 128(10 + 64) +$$

$$128(-18 - 56) + 4(-183 - 63) + 4(-81 - 165) + 32(6 + 60) + 32(50 + 17) +$$
$$32 \times 9 + 8(22 + 72) + (-64) \times 105 + 32(-112 - 21) +$$
$$(-64) \times 69 + 16(112 - 26) + 64 \times 95 =$$
$$512 \times 23 + 512 \times 23 - 256 - 256 \times 37 - 8 \times 123 - 8 \times 123 +$$
$$64 \times 33 + 32(67 + 9) + 16 \times 47 - 128 \times 87 - 32 \times 133 +$$
$$32 \times 43 + 64 \times 95 =$$
$$-16(123 - 47) + 32 \times 76 + 64 \times 33 - 64 \times 45 + 64 \times 95 +$$
$$128(-87) - 512 \times 37 + 1024 \times 23 =$$
$$64(19 + 33 + 50) + 128(-87) - 512 \times 37 + 1024 \times 23 =$$
$$128(51 - 87) - 512 \times 37 + 1024 \times 23 = -512 \times 46 + 1024 \times 23 = 0.$$

Clearly,

(14) $$r^{\wedge\wedge}_{2,1}(\nu)\big|_{\mu=0} = 64(\times 11 + \times 11) +$$

$$(-512 - 512) + 4 \times (-9 - 9) + 32(5 + 5) + 0 + 8 \times 3 - 64 \times 7 - 32 \times 21 +$$
$$16 \times 13 - 32 + 32 \times 21 =$$
$$(8(3 - 9) + 16 \times 13 - 32 \times 13) + 64(5 - 7 + 22) - 1024 =$$
$$-256 + 256 \times 5 - 1024 = 0.$$

So $r^{\wedge\wedge}_{2,3}(\nu) = \mu^2 t^{\wedge\wedge}_{2,3}(\mu)$, where $t^{\wedge\wedge}_{2,3}(\mu) \in \mathbb{Q}[\mu]$ coefficients of $t^{\wedge\wedge}_{2,3}(\mu)$ at μ^k are equal to 0, if $k > 2$, and, to prove that $t^{\wedge\wedge}_{2,3}(\mu) = 0$, it is sufficient to prove that

$$t^{\wedge\wedge}_{2,3}(\mu) \in ((\mu^2 + 1)(\mu + 1))\mathbb{Q}[\mu].$$

So we most prove that

$$r^{\wedge\wedge}_{2,3}(\nu)\big|_{m=-1} = r^{\wedge\wedge}_{2,3}(\nu)\big|_{m=i} = 0.$$

We have

(15) $$r^{\wedge\wedge}_{2,1}(\nu)\big|_{\mu=-1} = -10 + 0 - 8\times 9\times 467 +$$

$$0 + 64(-3)\times 69 + (-2)\times 9\times(-2031) + 4(-3)(-4353)+$$
$$(-32)(-9)(-67) + (32)(-345) + 8(-17)(89) + (32)(3)(-35)$$
$$+32\times 17 - 32(3)(-43) + 16(3)(-13) + (-32)(-9) + 64(-7) =$$
$$2(18279-5) + 4\times 13059 + 8(-4203) + 8(-1513) + 16(-39)+$$
$$32(-603-345-105+17+129+9)+$$
$$64(-207-7) =$$
$$4(9137+13059) + 16(-39) + 8(-4203-1513)+$$
$$32(-898) - 128\times 107 =$$
$$16(5549-39) - 32\times 1429 - 64\times 449 - 1024\times 13 =$$
$$32(2755-1429) - 64\times 449 - 1024\times 13 =$$
$$64(663-449) - 128\times 107 = 0.$$

Let

$$\omega^{**}_0 = (-2)(-5+4i)(-3+2i) = 2(-7+22i),$$
$$\omega^{**}_{1a} = (-8)(-67+20i)(-371-118i) = 8(-27217-486i),$$
$$\omega^{**}_{1b} = (-32)(-5+7i)(-1585-140i) = 32(-8905+10395i),$$
$$\omega^{**}_{2a} = 64(11-17i)(319+28i) = 64(3985-5115i),$$
$$\omega^{**}_{2b} = 64(1+10i)(-267-258i) = 64(2313-2928i,$$
$$\omega^{**}_{3a} = (-2)(-25+16i)(2907+1188i) = 2(91683-16812i),$$
$$\omega^{**}_{3b} = 4(-2+7i)(-2595-7296i) = 4(56262-3573i),$$
$$\omega^{**}_{4a} = (-32)(2+17i)(-185-270i) = 32(-4220+3685i),$$
$$\omega^{**}_{4b} = (-32)(5+6i)(-45-656i) = 32(-3711+3550i),$$
$$\omega^{**}_{5a} = 8(-5+24i)(23+190i) = 8(-4675-398i),$$
$$\omega^{**}_{5b} = 32(-3i)(213-308i) = 32(-924-639i,$$
$$\omega^{**}_{6a} = (-32)(2+3i)(-117+12i) = 32(270+327i),$$
$$\omega^{**}_{6b} = 32(-3)(-207-122i) = 32(621+366i),$$
$$\omega^{**}_{7a} = 16(1-2i)(-183-236i) = 16(-655+130i),$$
$$\omega^{**}_{7b} = 32(-3+606i),$$
$$\omega^{**}_{8a} = 32(-186-552i) = 64(-93-276i).$$

Then

$$\omega^{**}_9 := \omega^{**}_0 + \omega^{**}_{3a} =$$
$$4(45838-8395i,\ \omega^{**}_{10} := \omega^{**}_9 + \omega^{**}_{3b} =$$
$$16(25525-2992i),\ \omega^{**}_{11} := \omega^{**}_{10} + \omega^{**}_{7a} =$$
$$16(25525-2992i) + 16(-655+130i) = 32(12435-1431i,\ \omega^{**}_{12} :=$$

$$\omega^{**}_{11}+\omega^{**}_{1b}=64(1765+4482i),$$
$$\omega^{**}_{13}:=\omega^{**}_{1a}+\omega^{**}_{5a}=$$
$$8(-27217-486i)+8(-4675-398i)=32(-7973-221i),\ \omega^{**}_{14}:=$$
$$\omega^{**}_{13}+\omega^{**}_{4a}+\omega^{**}_{4b}=$$
$$32(-7973-4220-3711+(-221+3685+3550)i)=$$
$$64(-7952+3507i),\ \omega^{**}_{15}:=\omega^{**}_{5b}+$$
$$\omega^{**}_{6a}=64(-327-156i),\ \omega^{**}_{16}:=$$
$$\omega^{**}_{6b}+\omega^{**}_{7b}=64(309+486i),$$
$$\omega^{**}_{17}:=\omega^{**}_{16}+\omega^{**}_{2b}=$$
$$64(309+486i)+64(2313-2928i)=128(1311-1221i),$$
$$\omega^{**}_{18}:=\omega^{**}_{2a}+\omega^{**}_{8a}=$$
$$64(3892-5391i),\ \omega^{**}_{19}:=$$
$$\omega^{**}_{12}+\omega^{**}_{15}=128(719+2163i),$$
$$\omega^{**}_{20}:=\omega^{**}_{18}+\omega^{**}_{14}:=$$
$$256(-1015-471i),\ \omega^{**}_{21}:=\omega^{**}_{17}+$$
$$\omega^{**}_{19}=256(1015+471i),$$

(16)
$$r^{\wedge\wedge}_{2,1}(\nu)\Big|_{\mu=i}=\omega^{**}_0+$$
$$\left(\sum_{i=1}^{8}\omega^{**}_{ia}\right)+\sum_{i=1}^{7}\omega^{**}_{ib}=$$
$$\omega^{**}_9+\omega^{**}_{13}+\omega^{**}_{15}+$$
$$\omega^{**}_{16}+\omega^{**}_{18}+\omega^{**}_{1b}+$$
$$\omega^{**}_{2b}+\omega^{**}_{3b}+\omega^{**}_{4a}+$$
$$\omega^{**}_{4b}+\omega^{**}_{7a}=\omega^{**}_{10}+$$
$$\omega^{**}_{14}+\omega^{**}_{15}+\omega^{**}_{17}+$$
$$\omega^{**}_{18}+\omega^{**}_{1b}+\omega^{**}_{7a}=$$
$$\omega^{**}_{19}+\omega^{**}_{20}+\omega^{**}_{17}=$$
$$\omega^{**}_{21}+\omega^{**}_{20}=0.$$

■

Lemma 13.1.3. . *The equality*

(17)
$$r^{\wedge\vee}_{2,4}(\nu)=0$$

holds.

Proof. In view of (86), (121) – (123) and results of §12.1 in [16],

(18)
$$r^{\wedge\vee}_{2,4}(\nu)=(7\mu^3+14\mu^2+7\mu+1)(6\mu^2-2\mu)+$$

$$16(\mu^4+28\mu^3+76\mu^2+48\mu+8)(267\mu^2-359\mu)+$$
$$(-8)(\mu^3+7\mu^2+8\mu+2)8(121\mu)(4\mu+1)+$$
$$64(\mu^3+15\mu^2+18\mu+4)(48\mu^2+145\mu)+$$
$$(-64)(3\mu^2+10\mu+4)(90\mu^2-145\mu)(4\mu+1)+$$
$$(-16)(2\mu^3+27\mu^2+18\mu+2)(226\mu^3-277\mu^2-160\mu)+$$
$$(-32)(3\mu^2+7\mu+1)(43\mu^2-160\mu)(4\mu+1)+$$
$$2(3\mu^2+17\mu+5)(2105\mu^3-3214\mu^2-1255\mu-1)+$$
$$2(6\mu+5)(1023\mu^3-710\mu^2-1253\mu-1)(4\mu+1)+$$
$$4(6\mu^2+24\mu+1)(122\mu^4+44\mu^3-234\mu^2-72\mu-1)+$$
$$16(3\mu)(100\mu^3-96\mu^2-70\mu-1)(4\mu+1)+$$
$$(-64)(3\mu+2)(19\mu^4-29\mu^3-47\mu^2-14\mu-1)+$$
$$(-192)(5\mu^4+\mu^3-25\mu^2-12\mu-1)(4\mu+1)+$$
$$8(2\mu-1)(2\mu^5+165\mu^4+470\mu^3+371\mu^2+110\mu+11)+$$
$$16(21\mu^4+168\mu^3+217\mu^2+88\mu+11)(4\mu+1)+$$
$$(-64)(28\mu^5+194\mu^4+334\mu^3+220\mu^2+61\mu+6).$$

Coefficient at μ^6 in the polynomial $r^{\wedge\vee}_{2,4}(\nu)$ is equal to

(19) $$16(267)+64(-113)+16(183)+32=$$
$$32(225+1)+64(-113)=0.$$

Let

$$c^{\vee}_{504}:=42,\ c^{\vee}_{51a14}:=16(-359),$$
$$c^{\vee}_{51a24}:=64\times 7\times 267=64\times 1869,$$
$$c^{\vee}_{51b4}:=-256\times 121,\ c^{\vee}_{52a4}:=1024\times 3,$$
$$c^{\vee}_{52b4}:=-512\times 135,\ c^{\vee}_{53a14}:=32\times 277,$$
$$c^{\vee}_{53a24}:=-32\times 27\times 113=-32\times 3051,$$
$$c^{\vee}_{53b4}:=-128\times 129,\ c^{\vee}_{54a4}:=2\times 6315,$$
$$c^{\vee}_{54b4}:=16\times 3069,\ c^{\vee}_{55a14}:=32\times 33,$$
$$c^{\vee}_{55a24}:=64\times 183,\ c^{\vee}_{55b4}:=256\times 75,$$
$$c^{\vee}_{56a4}:=-64\times 57,\ c^{\vee}_{56b4}:=-256\times 15,$$
$$c^{\vee}_{57a4}:=16(165-1)=64\times 41,\ c^{\vee}_{57b4}:=64\times 21,$$
$$c^{\vee}_{58a4}:=-256\times 7.$$

Then

$$c^{\vee}_{594}:=c^{\vee}_{504}+c^{\vee}_{54a4}=2(21+6315)=128,$$
$$c^{\vee}_{5104}:=c^{\vee}_{51a14}+c^{\vee}_{54b4}:=16(-359+3069)=32\times 1355,$$
$$c^{\vee}_{5114}:=c^{\vee}_{5104}+c^{\vee}_{55a14}=32(1355+33)=128\times 347,$$
$$c^{\vee}_{53a4}:=c^{\vee}_{53a14}+c^{\vee}_{53a24}:=-64\times 1387,$$

$$c^{\vee}_{5124} := c^{\vee}_{55a24} + c^{\vee}_{56a4} := 64(183 - 57) = 128 \times 63,$$
$$c^{\vee}_{5134} := c^{\vee}_{55b4} + c^{\vee}_{56b4} = 1024 \times 15,$$
$$c^{\vee}_{574} := c^{\vee}_{57a4} + c^{\vee}_{57b4} := 128 \times 31,$$
$$c^{\vee}_{5144} := c^{\vee}_{51a24} + c^{\vee}_{53a4} =$$
$$64(1869 - 1387) = 128 \times 241,$$
$$c^{\vee}_{5154} := c^{\vee}_{5144} + c^{\vee}_{53b4} = 128(241 - 129) = 2048 \times 7,$$
$$c^{\vee}_{5164} := c^{\vee}_{51b4} + c^{\vee}_{58a4} = -256 \times 128 = -2^{15},$$
$$c^{\vee}_{5174} := c^{\vee}_{594} + c^{\vee}_{5114} = 128(99 + 347) = 256 \times 223$$
$$c^{\vee}_{5184} := c^{\vee}_{5124} + c^{\vee}_{574} = 256 \times 47,$$
$$c^{\vee}_{5194} := c^{\vee}_{5174} + c^{\vee}_{5184} = 512 \times 135$$
$$c^{\vee}_{5204} := c^{\vee}_{52b4} + c^{\vee}_{5194} = 0,$$
$$c^{\vee}_{5214} := c^{\vee}_{52a4} + c^{\vee}_{5134} = 2048 \times 9,$$
$$c^{\vee}_{5224} := c^{\vee}_{5214} + c^{\vee}_{5154} = 2048 \times 16 = 2^{15},$$

and coefficient at μ^5 in the polynomial $r^{\wedge\vee}_{2,4}(\nu)$ is equal to

(20)
$$c^{\vee}_{504} + c^{\vee}_{51a14} + c^{\vee}_{51a24} + c^{\vee}_{51b4} + c^{\vee}_{52a4} +$$
$$c^{\vee}_{52b4} + c^{\vee}_{53a14} + c^{\vee}_{53a24} + c^{\vee}_{53b4} + c^{\vee}_{54a4}+$$
$$c^{\vee}_{54b4} + c^{\vee}_{55a14} + c^{\vee}_{55a24} + c^{\vee}_{55b4} + c^{\vee}_{56a4}+$$
$$c^{\vee}_{56b4} + c^{\vee}_{57a4} + c^{\vee}_{57b4} + c^{\vee}_{58a4} =$$
$$(c^{\vee}_{504} + c^{\vee}_{54a4}) + (c^{\vee}_{51a14} + c^{\vee}_{54b4}) + c^{\vee}_{55a14}+$$
$$(c^{\vee}_{53a14} + c^{\vee}_{53a24}) + (c^{\vee}_{55a24} + c^{\vee}_{56a4})+$$
$$(c^{\vee}_{55b4} + c^{\vee}_{56b4}) + (c^{\vee}_{57a4} + c^{\vee}_{57b4} + c^{\vee}_{51a24}$$
$$c^{\vee}_{53b4} + (c^{\vee}_{51b4} + c^{\vee}_{58a4}) + c^{\vee}_{52a4} + c^{\vee}_{52b4} =$$

$$c^{\vee}_{594} + c^{\vee}_{5104} + c^{\vee}_{55a14} + c^{\vee}_{53a4} + c^{\vee}_{5124}+$$
$$c^{\vee}_{5134} + c^{\vee}_{574} + c^{\vee}_{51a24} + c^{\vee}_{53b4} + c^{\vee}_{5164}+$$
$$c^{\vee}_{52a4} + c^{\vee}_{52b4} =$$
$$(c^{\vee}_{51a24} + c^{\vee}_{53a4}) + c^{\vee}_{594} + (c^{\vee}_{5104} + c^{\vee}_{55a14})+$$
$$(c^{\vee}_{5124} + c^{\vee}_{574}) + (c^{\vee}_{5134} + c^{\vee}_{52a4}) + c^{\vee}_{53b4}+$$
$$c^{\vee}_{52b4} + c^{\vee}_{5164} =$$

$$c^{\vee}_{5144} + c^{\vee}_{594} + c^{\vee}_{5114} + c^{\vee}_{5184} + c^{\vee}_{5214}+$$
$$c^{\vee}_{52b4} + c^{\vee}_{53b4} + c^{\vee}_{5164} =$$

$$(c^{\vee}_{5144} + c^{\vee}_{53b4}) + (c^{\vee}_{594} + c^{\vee}_{5114})+$$
$$c^{\vee}_{5184} + c^{\vee}_{5214} + c^{\vee}_{52b4} + c^{\vee}_{5164} =$$

$$c^{\vee}_{5154}+c^{\vee}_{5174}+$$
$$c^{\vee}_{5184}+c^{\vee}_{5214}+c^{\vee}_{52b4}+c^{\vee}_{5164}=$$

$$(c^{\vee}_{5154}+c^{\vee}_{5214})+(c^{\vee}_{5174}+c^{\vee}_{5184})+$$
$$c^{\vee}_{52b4}+c^{\vee}_{5164}=$$

$$c^{\vee}_{5224}+c^{\vee}_{5194}+c^{\vee}_{52b4}+c^{\vee}_{5164}=$$

$$(c^{\vee}_{52b4}+c^{\vee}_{5194})+(c^{\vee}_{5224}+c^{\vee}_{5164})=0.$$

Let

$$c^{\vee}_{404}:=2\times 35,\ c^{\vee}_{41a4}:=64(5073-2513)=2^15\times 5,$$
$$c^{\vee}_{41b14}:=-64\times 121,\ c^{\vee}_{41b24}:=-256\times 847,$$
$$c^{\vee}_{42a14}:=1024\times 45,\ c^{\vee}_{42a24}:=64\times 145,$$
$$c^{\vee}_{42b14}:=-128\times 135,\ c^{\vee}_{42b24}:=-1024\times 225,$$
$$c^{\vee}_{42b34}:=256\times 435,\ c^{\vee}_{43a14}:=1024\times 5,$$
$$c^{\vee}_{43a24}:=16\times 7479,\ c^{\vee}_{43a34}:=-64\times 1017,$$
$$c^{\vee}_{43b14}:=-32\times 129,\ c^{\vee}_{43b24}:=-128\times 301,$$
$$c^{\vee}_{43b34}:=4096\times 15,\ c^{\vee}_{44a14}:=-4\times 4821,$$
$$c^{\vee}_{44a24}:=2\times 35785,\ c^{\vee}_{44b14}:=4\times 3069,$$
$$c^{\vee}_{44b24}:=-32\times 1065,\ c^{\vee}_{44b34}=8\times 5115,$$
$$c^{\vee}_{45a14}:=-16\times 351,\ c^{\vee}_{45a24}:=128\times 33,$$
$$c^{\vee}_{45a34}:=8\times 61,\ c^{\vee}_{45b14}:=64\times 75,$$
$$c^{\vee}_{45b24}:=-2048\times 9,\ c^{\vee}_{46a14}:=64\times 87,$$
$$c^{\vee}_{46a24}:=-128\times 19,\ c^{\vee}_{46b14}:=-64\times 15,$$
$$c^{\vee}_{46b24}:=-256\times 3,\ c^{\vee}_{47a14}:=32\times 235,$$
$$c^{\vee}_{47a24}:=-8\times 165,\ c^{\vee}_{47b14}:=16\times 21,$$
$$c^{\vee}_{47b24}:=512\times 21,\ 9,2c^{\vee}_{48a4}:=-128\times 97.$$

Then

$$c^{\vee}_{494}:=c^{\vee}_{404}+c^{\vee}_{44a24}=2\times 35820=8\times 8955,$$
$$c^{\vee}_{4104}:=c^{\vee}_{44a14}+c^{\vee}_{44b14}=-4(4821-3069)=-4\times 1752=$$
$$-32\times 219,\ c^{\vee}_{4114}:=c^{\vee}_{494}+c^{\vee}_{44b34}+c^{\vee}_{45a34}+$$
$$c^{\vee}_{47a24}=8(8955+5115+61-165)=8\times 13966=16\times 6983,$$
$$c^{\vee}_{4124}:=c^{\vee}_{4114}+c^{\vee}_{43a24}+c^{\vee}_{45a14}+c^{\vee}_{47b14}=$$
$$16(6983+7479-351+21)=16\times 14132=64\times 3533$$
$$c^{\vee}_{4134}:=c^{\vee}_{4104}+c^{\vee}_{43b14}+c^{\vee}_{44b24}+c^{\vee}_{46b24}=$$

$$32(-219-129-1065+235)=32\times 1178=-64\times 589$$

$$c^{\vee}_{4144}:=c^{\vee}_{4134}+c^{\vee}_{4124}+c^{\vee}_{41b14}+c^{\vee}_{42a24}+c^{\vee}_{43a34}+c^{\vee}_{45b14}+c^{\vee}_{46a14}+c^{\vee}_{46b14}=$$
$$64(-589+3533-121+145-1017+75+87-15)=64\times 2098=128\times 1049,$$

$$c^{\vee}_{4154}:=c^{\vee}_{4144}+c^{\vee}_{42b14}+c^{\vee}_{43b24}+$$
$$c^{\vee}_{45a24}+c_{46a24}+c^{\vee}_{48a4}=$$
$$128(2+1047-135-301+33-19-97)=128\times 528+256=2048\times 33+256,$$

$$c^{\vee}_{4164}:=c^{\vee}_{4154}+c^{\vee}_{45b24}+c^{\vee}_{41b24}+c^{\vee}_{42b34}+c^{\vee}_{46b24}=2^{14}\times 3+256(1-847+435-3)=$$
$$2^{14}\times 3-512\times 207$$

$$c_{4174}:=c^{\vee}_{4164}+c^{\vee}_{47b24}=$$
$$2^{14}\times 3-1024\times 93$$

$$c_{4184}:=c_{4174}+c^{\vee}_{42a14}+c^{\vee}_{42b24}+c^{\vee}_{43a14}=$$
$$2^{14}\times 3+1024(-93+45-225+5)=2^{14}\times 3-4096\times 67,$$

$$c_{4194}:=c_{4184}+c^{\vee}_{43b34}=4096\times 15,\ 2^{14}\times 3-2^{12}(67-15)=2^{14}(3-13)=-2^{15}\times 5,$$

and coefficient at μ^4 in the polynomial $r^{\wedge\vee}_{2,4}(\nu)$ is equal to

$$c^{\vee}_{41a4}+c^{\vee}_{4194}=0. \tag{21}$$

Let

$$c^{\vee}_{104}:=-2,\ c^{\vee}_{11a4}:=-2^7\times 359,$$
$$c^{\vee}_{11b4}:=-2^7\times 121,\ c^{\vee}_{12a4}:=2^8\times 145,$$
$$c^{\vee}_{12b4}:=2^8\times 145,\ c^{\vee}_{13a4}:=1024\times 5,$$
$$c^{\vee}_{13b4}:=1024\times 5,\ c^{\vee}_{14a4}:=-8\times 1573,$$
$$c^{\vee}_{14b14}:=-4\times 3,\ c^{\vee}_{44b24}:=-2\times 6265,$$
$$c^{\vee}_{14b34}=-8\times 5,$$
$$c^{\vee}_{15a4}:=-128\times 3,\ c^{\vee}_{15b4}:=-16\times 3,$$
$$c^{\vee}_{16a4}:=64\times 31,\ c^{\vee}_{16b4}:=1024\times 3,$$
$$c^{\vee}_{17a4}:=-64\times 11,\ c^{\vee}_{17b4}:=64\times 33,$$
$$c^{\vee}_{18a4}:=-64\times 61.$$

Then

$$c^{\vee}_{194}:=c^{\vee}_{104}+c^{\vee}_{44b24}=-4\times 3133,$$
$$c^{\vee}_{1104}:=c^{\vee}_{194}+c^{\vee}_{14b14}:=-256\times 49,$$
$$c^{\vee}_{1114}:=c^{\vee}_{13b4}+c^{\vee}_{14b34}=-16\times 789,$$
$$c^{\vee}_{1124}:=c^{\vee}_{1114}+c^{\vee}_{17b4}=-128\times 99,$$
$$c^{\vee}_{1134}:=c^{\vee}_{1124}+c^{\vee}_{15a4}=-256\times 51,$$
$$c^{\vee}_{1144}:=c^{\vee}_{16a4}+c^{\vee}_{17a4}+c^{\vee}_{17b4}+c^{\vee}_{18a4}=$$
$$64(31-11+33-61)=-512,$$
$$c^{\vee}_{1164}:=c^{\vee}_{1104}+c^{\vee}_{1134}=-2^{10}25,$$

$$c^{\vee}_{1174} := c^{\vee}_{1144} + c^{\vee}_{12a4} + c^{\vee}_{12b4} = 2^{13} \times 9,$$
$$c^{\vee}_{1184} := c^{\vee}_{11a4} + c^{\vee}_{11b4} = -2^{7}(359 + 121) = -2^{12} \times 15,$$
$$c^{\vee}_{1194} := c^{\vee}_{1164} + c^{\vee}_{13a4} + c^{\vee}_{13b4} + c^{\vee}_{16b4} =$$
$$2^{10}(-25 + 5 + 5 + 3) = -2^{12} \times 3,$$
$$c^{\vee}_{1204} := c^{\vee}_{1194} + c^{\vee}_{1184} = -2^{13} \times 9,$$

and coefficient at μ in the polynomial $r^{\wedge\vee}_{2,4}(\nu)$ is equal to

$$c^{\vee}_{1204} + c^{\vee}_{1174} = 0. \tag{22}$$

Clearly,

$$r^{\wedge\vee}_{2,4}(\nu)\big|_{\mu=0} = -2 \times 5 - 2 \times 5 + \tag{23}$$
$$(-4) + 128 + 192 - 88 + 176 - 128 \times 3 = 0$$

So $r^{\wedge\vee}_{2,4}(\nu) = \mu^2 t^{\wedge\vee}_{2,4}(\mu)$, where $t^{\wedge\vee}_{2,4}(\mu) \in \mathbb{Q}[\mu]$, coefficients of $t^{\wedge\vee}_{2,4}(\mu)$ at μ^k are equal to 0, if $k > 1$, and, to prove that $t^{\wedge\vee}_{2,4}(\mu) = 0$, it is sufficient to prove that $t^{\wedge\vee}_{2,4}(\mu) \in (\mu^2 - 1)\mathbb{Q}[\mu]$. Since $\mu\mathbb{Q}[\mu] + (\mu^2 - 1)\mathbb{Q}[\mu] = \mathbb{Q}[\mu]$, it follows that we must to prove that $r^{\wedge\vee}_{2,4}(\nu) \in (\mu^2 - 1)\mathbb{Q}[\mu]$ Let

$$\omega^{\wedge\vee}_{2,4,0}(\mu) :=$$
$$2(14\mu + 15)(-\mu + 3) \equiv 2(27\mu + 31)(\mod (\mu^2 - 1)),$$
$$\omega^{\wedge\vee}_{2,4,1a}(\mu) :=$$
$$16(76\mu + 85)(-359\mu + 267) \equiv 16(-10223\mu - 4589)(\mod (\mu^2 - 1)),$$
$$\omega^{\wedge\vee}_{2,4,1b}(\mu) :=$$
$$64(-121)9(\mu + 1)(4\mu + 1) \equiv 64 \times (-5445)(\mu + 1)(\mod (\mu^2 - 1)),$$
$$\omega^{\wedge\vee}_{2,4,2a}(\mu) :=$$
$$64 \times 19(\mu + 1)(145\mu + 48) \equiv 64 \times 3667(\mu + 1)(\mod (\mu^2 - 1)),$$
$$\omega^{\wedge\vee}_{2,4,2b}(\mu) :=$$
$$(-64)(10\mu + 7)(-145\mu + 90)(4\mu + 1) \equiv 64(38\mu + 47)(145\mu - 90) \equiv$$
$$64(3395\mu + 1280)(\mod (\mu^2 - 1)),$$
$$\omega^{\wedge\vee}_{2,4,3a}(\mu) :=$$
$$(-16)(20\mu + 29)(66\mu - 277) \equiv 16(3626\mu + 6713)(\mod (\mu^2 - 1)),$$
$$\omega^{\wedge\vee}_{2,4,3b}(\mu) :=$$
$$32(7\mu + 4)(160\mu - 43)(4\mu + 1) \equiv 32(339\mu + 948)(4\mu + 1) \equiv$$
$$32(4131\mu + 2304)(\mod (\mu^2 - 1)),$$
$$\omega^{\wedge\vee}_{2,4,4a}(\mu) :=$$
$$2(17\mu + 8)(850\mu - 3215) \equiv 2(-47855\mu - 11270)(\mod (\mu^2 - 1)),$$
$$\omega^{\wedge\vee}_{2,4,4b}(\mu) :=$$
$$2(6\mu + 5)(-230\mu - 711)(4\mu + 1) \equiv 2(-5416\mu - 4935)(4\mu + 1) \equiv$$

$$2(-25156\mu - 26599) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,5a}(\mu) :=$$

$$4(24\mu + 7)(-28\mu - 113) \equiv 4(-2908\mu - 1463) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,5b}(\mu) :=$$

$$16(3)(-97\mu + 30)(4\mu + 1) \equiv 48(23\mu - 358) \equiv$$

$$16(69\mu - 1074) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,6a}(\mu) :=$$

$$64(3\mu + 2)(43\mu + 29) \equiv 64(173\mu + 187) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,6b}(\mu) :=$$

$$(-192)(-11\mu - 21)(4\mu + 1) \equiv 192(95\mu + 65) \equiv$$

$$64(285\mu + 195) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,7a}(\mu) :=$$

$$8(2\mu - 1)(582\mu + 547) \equiv 8(512\mu + 617) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,7b}(\mu) :=$$

$$16(256\mu + 249)(4\mu + 1) \equiv 16(1252\mu + 1273) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,8a}(\mu) := 64(-423\mu - 420).$$

Then

$$\omega^{\wedge\vee}_{2,4,9}(\mu) := \omega^{\wedge\vee}_{2,4,0}(\mu)+$$

$$\omega^{\wedge\vee}_{2,4,4a}(\mu) + \omega^{\wedge\vee}_{2,4,4b}(\mu) \equiv$$

$$2(27\mu + 31) + 2(-47855\mu - 11270) + 2(-25156\mu - 26599) \equiv$$

$$4(-36492\mu - 18919) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,10}(\mu) := \omega^{\wedge\vee}_{2,4,9}(\mu)+$$

$$\omega^{\wedge\vee}_{2,4,5a}(\mu) \equiv 4(-36492\mu - 18919)+$$

$$4(-2908\mu - 1463) \equiv 8(-19700\mu - 10191) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,11}(\mu) := \omega^{\wedge\vee}_{2,4,10}(\mu)+$$

$$\omega^{\wedge\vee}_{2,4,7a}(\mu) \equiv 8(-19700\mu - 10191)+$$

$$8(512\mu + 617) \equiv 16(-9594\mu - 4787) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,12}(\mu) := \omega^{\wedge\vee}_{2,4,11}(\mu)+$$

$$\omega^{\wedge\vee}_{2,4,1a}(\mu) + \omega^{\wedge\vee}_{2,4,3a}(\mu)+$$

$$\omega^{\wedge\vee}_{2,4,5b}(\mu) + \omega^{\wedge\vee}_{2,4,7b}(\mu) \equiv$$

$$16(-9594\mu - 4787) + 16(-10223\mu - 4589) + 16(3626\mu + 6713) + 16(69\mu - 1074)+$$

$$16(1252\mu + 1273) \equiv 32(-7435\mu - 1232) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,13}(\mu) := \omega^{\wedge\vee}_{2,4,12}(\mu)+$$

$$\omega^{\wedge\vee}_{2,4,3b}(\mu) \equiv 32(-7435\mu - 1232)+$$

$$32(4131\mu + 2304) \equiv 2^8(-413\mu + 134) \; (\mod (\mu^2 - 1)),$$

$$\omega^{\wedge\vee}_{2,4,14}(\mu) := \omega^{\wedge\vee}_{2,4,1b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,2a}(\mu) + \omega^{\wedge\vee}_{2,4,2b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,6a}(\mu) + \omega^{\wedge\vee}_{2,4,6b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,8a}(\mu) \equiv$$
$$64 \times (-5445)(\mu+1) + 64 \times 3667(\mu+1) + 64(3395\mu + 1280)+$$
$$64(173\mu + 187) + 64(285\mu + 195) + 64(-423\mu - 420) \equiv$$
$$64(1652\mu - 536 \equiv 2^8(413\mu - 134) \,(\mod (\mu^2 - 1)).$$

In view of (18),

$$r^{\wedge\vee}_{2,4}(\nu) \equiv \omega^{\wedge\vee}_{2,4,0}(\mu) + \tag{24}$$
$$\omega^{\wedge\vee}_{2,4,1a}(\mu) + \omega^{\wedge\vee}_{2,4,1b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,2a}(\mu) + \omega^{\wedge\vee}_{2,4,2b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,3a}(\mu) + \omega^{\wedge\vee}_{2,4,3b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,4a}(\mu) + \omega^{\wedge\vee}_{2,4,4b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,5a}(\mu) + \omega^{\wedge\vee}_{2,4,5b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,6a}(\mu) + \omega^{\wedge\vee}_{2,4,6b}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,7a}(\mu) + \omega^{\wedge\vee}_{2,4,7b}(\mu)$$
$$+\omega^{\wedge\vee}_{2,4,8a}(\mu) \equiv \omega^{\wedge\vee}_{2,4,13}(\mu)+$$
$$\omega^{\wedge\vee}_{2,4,14}(\mu) \equiv 0 \,(\mod (\mu^2 - 1)).$$

■

Lemma 13.1.3. . *The equality*

$$r^{\wedge\wedge}_{2,4}(\nu) = 0 \tag{25}$$

holds.

Proof. In view of (86), (121) – (123) and results of §12.1 in [16],

$$r^{\wedge\wedge}_{2,4}(\nu) = -4(\mu^3 + 6\mu^2 + 5\mu + 1)(3\mu^2 - \mu) + \tag{26}$$
$$(\mu^4 + 28\mu^3 + 76\mu^2 + 48\mu + 8)32(121\mu)+$$
$$(-8)(\mu^3 + 7\mu^2 + 8\mu + 2)16(267\mu^2 - 359\mu)+$$
$$(-4)(\mu^3 + 15\mu^2 + 18\mu + 4)(-32)(90\mu^2 - 145\mu)+$$
$$8(3\mu^2 + 10\mu + 4)(-16)(48\mu^2 + 145\mu)+$$
$$(-4)(\mu^3 + 15\mu^2 + 18\mu + 4)(-32)(90\mu^2 - 145\mu)+$$
$$8(3\mu^2 + 10\mu + 4)(-16)(48\mu^2 + 145\mu)+$$
$$(-4)(\mu^3 + 15\mu^2 + 18\mu + 4)(-32)(90\mu^2 - 145\mu)+$$
$$8(3\mu^2 + 10\mu + 4)(-32)(48\mu^2 + 145\mu)+$$
$$(-2)(2\mu^3 + 27\mu^2 + 18\mu + 2)(-16)(43\mu^2 - 160\mu)+$$
$$8(3\mu^2 + 7\mu + 1)8(226\mu^3 - 277\mu^2 - 160\mu)+$$

$$(6\mu^2+24\mu+1)(-8)(100\mu^3-96\mu^2-70\mu-1)+$$
$$(-8)(3\mu)4(122\mu^4+44\mu^3-234\mu^2-72\mu-1)+$$
$$(-4)(3\mu+2)(-32)(5\mu^4+\mu^3-25\mu^2-12\mu-1)+$$
$$(24)16(19\mu^4-29\mu^3-47\mu^2-14\mu-1)+$$
$$(-2)(2\mu-1)8(21\mu^4+168\mu^3+217\mu^2+88\mu+11)+$$
$$8(-4)(2\mu^5+165\mu^4+470\mu^3+371\mu^2+110\mu+11)+$$
$$4(32)(2\mu^5+46\mu^4+140\mu^3+134\mu^2+49\mu+6).$$

Let

$$c^{\wedge}_{504}:=-4\times 3,\ c^{\wedge}_{51a4}:=32\times 121,$$
$$c^{\wedge}_{51b4}:=-2^7\times 267,\ c^{\wedge}_{52a4}:=2^8\times 45,$$
$$c^{\wedge}_{52b4}:=0,\ c^{\vee}_{53a4}:=2^6\times 43,$$
$$c^{\wedge}_{53b4}:=2^7\times 339,\ c^{\wedge}_{54a4}:=-4\times 3069,$$
$$c^{\wedge}_{54b4}:=0,\ c^{\wedge}_{55a4}:=-2^6\times 75,$$

,

$$c^{\wedge}_{55b4}:=-2^6\times 183,\ c^{\wedge}_{56a4}:=2^7\times 15,$$
$$c^{\wedge}_{56b4}:=0,\ c^{\wedge}_{57a4}:=-32\times 21,$$
$$c^{\wedge}_{57b4}:=-64,\ c^{\wedge}_{58a4}:=256.$$

Then

$$c^{\wedge}_{594}:=c^{\wedge}_{504}+c^{\wedge}_{54a4}=$$
$$-4\times 3072=-2^{12}\times 3,$$
$$c^{\wedge}_{5104}:=c^{\wedge}_{51a4}+c^{\wedge}_{57a4}=128\times 25,$$
$$c^{\wedge}_{5114}:=c^{\wedge}_{53a4}+c^{\wedge}_{55a4}=-2^{11},$$
$$c^{\wedge}_{5124}:=c^{\wedge}_{55b4}+c^{\wedge}_{57b4}=-2^9\times 23,$$
$$c^{\wedge}_{5134}:=c^{\wedge}_{5104}+c^{\wedge}_{51b4}+c^{\wedge}_{53b4}+$$
$$c^{\wedge}_{56a4}=2^7(-267+25+339+15)=2^{11}\times 7,$$
$$c^{\wedge}_{5144}:=c^{\wedge}_{52a4}+c^{\wedge}_{52a4}=2^9\times 23$$
$$c^{\wedge}_{5154}:=c^{\wedge}_{5114}+c^{\wedge}_{5134}2^{12}\times 3,$$

and coefficient at μ^5 in the polynomial $r^{\wedge\wedge}_{2,4}(\nu)$ is equal to

(27)
$$c^{\wedge}_{504}+c^{\wedge}_{51a14}+c^{\wedge}_{51a24}+c^{\wedge}_{51b4}+c^{\wedge}_{52a4}+$$
$$c^{\wedge}_{52b4}+c^{\wedge}_{53a14}+c^{\wedge}_{53a24}+c^{\wedge}_{53b4}+c^{\wedge}_{54a4}+$$
$$c^{\wedge}_{54b4}+c^{\wedge}_{55a14}+c^{\wedge}_{55a24}+c^{\wedge}_{55b4}+c^{\wedge}_{56a4}+$$
$$c^{\wedge}_{56b4}+c^{\wedge}_{57a4}+c^{\wedge}_{57b4}+c^{\wedge}_{58a4}=$$
$$c^{\wedge}_{594}+c^{\wedge}_{5124}+c^{\wedge}_{5144}+c^{\wedge}_{5154}=0.$$

Let

$$c^{\wedge}_{404}:=-4\times 17,$$

$$c^{\wedge}_{41a4} := 2^7 \times 847,\ c^{\wedge}_{41b4} = -2^7(7 \times 267 - 359) =$$
$$-2^8 \times 755,\ c^{\wedge}_{42a14} := -2^7 \times 145,$$
$$c^{\wedge}_{42a24} := 2^8 \times 675,\ c^{\wedge}_{42b4} := -2^{11} \times 9,$$
$$c^{\wedge}_{43a14} := -2^{11} \times 5,\ c^{\wedge}_{43a24} := 32 \times 1161,$$
$$c^{\wedge}_{43b14} := -2^6 \times 831,\ c^{\wedge}_{43b24} := 2^7 \times 791,$$
$$c^{\wedge}_{44a14} := 8 \times 1065,\ c^{\wedge}_{44a24} := -4 \times 17391,$$
$$c^{\wedge}_{44b4} := -8 \times 6315,\ c^{\wedge}_{45a14} := -2^8 \times 75,$$
$$c^{\wedge}_{45a24} := 2^9 \times 9,\ c^{\wedge}_{45b4} := -2^7 \times 33,$$
$$c^{\wedge}_{46a14} := 2^7 \times 3,\ c^{\wedge}_{46a24} := 2^8 \times 5,$$
$$c^{\wedge}_{46b4} := 2^7 \times 57,\ c^{\wedge}_{47a14} := -2^8 \times 21,$$
$$c^{\wedge}_{47a24} := 2^4 \times 21,\ c^{\wedge}_{47b4} := -32 \times 165,$$
$$c^{\wedge}_{48a4} := 2^8 \times 23.$$

Then

$$c^{\wedge}_{494} := c^{\wedge}_{404} + c^{\wedge}_{44a24} = -4 \times 17408 =$$
$$-4 \times 17 \times 1024 = -2^{12} \times 17,$$
$$c^{\wedge}_{4104} := c^{\wedge}_{44a14} + c^{\wedge}_{44b4} =$$
$$8 \times 1065 - 8 \times 6315 = -16 \times 2625,\ c^{\wedge}_{4114} := c^{\wedge}_{4104} +$$
$$c^{\wedge}_{47a24} = -16(2625 - 21) = -64 \times 651,$$

$$c^{\wedge}_{4124} := c^{\wedge}_{43a24} + c^{\wedge}_{47b4} =$$
$$32(1161 - 165) = 128 \times 249,$$
$$c^{\wedge}_{4134} := c^{\wedge}_{4114} + c^{\wedge}_{43b14} =$$
$$-64(651 + 831) = -128 \times 741$$
$$c^{\wedge}_{4144} := c^{\wedge}_{4124} + c^{\wedge}_{42a14} +$$
$$c^{\wedge}_{45b4} + c^{\wedge}_{46b4} = 2^7(249 - 145 - 33 + 57) = 2^{14},$$
$$c^{\wedge}_{4154} := c^{\wedge}_{4134} + c^{\wedge}_{41a4} + c^{\wedge}_{43b24} +$$
$$c^{\wedge}_{46a14} = 2^9 \times 225,$$
$$c^{\wedge}_{4164} := c^{\wedge}_{41b41} + c^{\wedge}_{42a24} + c^{\wedge}_{45a14} +$$
$$c^{\wedge}_{46a42} + c^{\wedge}_{47a14} + c^{\wedge}_{48a4} =$$
$$2^8(-755 + 675 - 75 + 5 - 21 + 23) = -2^{10} \times 37,\ c^{\wedge}_{4174} :=$$
$$c^{\wedge}_{4154} + c^{\wedge}_{45a42} = 2^9(225 + 9) = 2^{10} \times 117,$$
$$c^{\wedge}_{4184} := c^{\wedge}_{4174} + c^{\wedge}_{4164} =$$
$$2^10 \times 80 = 2^{14} \times 5$$
$$c^{\wedge}_{4194} := c^{\wedge}_{42b4} + c^{\vee}_{43a14} = -2^{12} times 7,$$
$$c^{\wedge}_{4204} := c^{\wedge}_{4194} + c^{\wedge}_{494} = -2^{15} times 3,$$

$$c^{\wedge}_{4214} := c^{\wedge}_{4184} + c^{\wedge}_{4144} = 2^{15} times 3,$$

and coefficient at μ^4 in the polynomial $r^{\wedge\wedge}_{2,4}(\nu)$ is equal to

(28)
$$c^{\wedge}_{404} + c^{\wedge}_{41a4} + c^{\wedge}_{41b4} + c^{\wedge}_{42a14} + c^{\wedge}_{42a24} +$$
$$c^{\wedge}_{42b4} + c^{\wedge}_{43a14} + c^{\wedge}_{43a24} + c^{\wedge}_{431b4} + c^{\wedge}_{43b24} +$$
$$c^{\wedge}_{44a14} + c^{\wedge}_{44a24} + c^{\wedge}_{44b4} + c^{\wedge}_{45a14} + c^{\wedge}_{45a24} +$$
$$c^{\wedge}_{45b4} + c^{\wedge}_{46a14} + c^{\wedge}_{46a24} + c^{\wedge}_{46b4} + c^{\wedge}_{47a14} +$$
$$c^{\wedge}_{47a24} + c^{\wedge}_{47b4} + c^{\wedge}_{48a4} = c^{\wedge}_{4204} + c^{\wedge}_{4214} = 0.$$

Let

$$c^{\wedge}_{104} := 4,$$
$$c^{\wedge}_{11a4} := 2^8 \times 121,\ c^{\wedge}_{41b4} = 2^8 \times 359),$$
$$c^{\wedge}_{12a4} := -2^9 \times 145,\ c^{\wedge}_{12b4} := -2^9 \times 145,$$
$$c^{\wedge}_{13a4} := -2^{11} \times 5,\ c^{\wedge}_{13b4} := -2^{11} \times 5,$$
$$c^{\wedge}_{14a4} := (-4)(-6265 - 17) = 8 \times 3141,$$
$$c^{\wedge}_{14b14} := 8 \times 3,\ c^{\wedge}_{14b24} := 4 \times 6275,$$
$$c^{\wedge}_{15a14} := 2^6 \times 3,\ c^{\wedge}_{15a24} := 2^4 \times 35,$$

,

$$c^{\wedge}_{15b4} := 2^5 \times 3,\ c^{\wedge}_{46a14} := -2^7 \times 3,$$
$$c^{\wedge}_{16a24} := -2^{10} \times 3,\ c^{\wedge}_{16b4} := -2^8 \times 21,$$
$$c^{\wedge}_{17a14} := -2^5 \times 11,\ c^{\wedge}_{17a24} := 2^7 \times 11,$$
$$c^{\wedge}_{17b4} := -2^6 \times 55,\ c^{\wedge}_{18a4} := 2^7 \times 49.$$

Then

$$c^{\wedge}_{194} := c^{\wedge}_{104} + c^{\wedge}_{14b24} = 2^4 \times 1569,$$
$$c^{\wedge}_{1104} := c^{\wedge}_{14a4} + c^{\wedge}_{14b14} = 2^6 \times 393,$$
$$c^{\wedge}_{1114} := c^{\wedge}_{194} + c^{\wedge}_{15a24} := 2^6 \times 401,$$
$$c^{\wedge}_{1124} := c^{\wedge}_{15b4} + c^{\wedge}_{17a14} = -2^8,$$
$$c^{\wedge}_{1134} := c^{\wedge}_{1104} + c^{\wedge}_{1114} + c^{\wedge}_{15a14} +$$
$$c^{\wedge}_{17b4} = 2^6(393 + 401 + 3 - 55) = 2^7 \times 371,\ c^{\wedge}_{1144} :=$$
$$c^{\wedge}_{1134} + c^{\wedge}_{46a14} + c^{\wedge}_{17a24} +$$
$$c^{\wedge}_{18a4} = 2^7(371 - 3 + 11 + 49) = 2^9 \times 107,\ c^{\wedge}_{1154} :=$$
$$c^{\wedge}_{1124} + c^{\wedge}_{11a4} + c^{\wedge}_{41b4} + c^{\wedge}_{16b4} =$$
$$2^8(-1 + 121 + 359 - 21) = 2^9 \times 229,\ c^{\wedge}_{1164} :=$$
$$c^{\wedge}_{1154} + c^{\wedge}_{1144} + c^{\wedge}_{12a4} + c^{\wedge}_{12b44} =$$
$$2^9(107 + 229 - 145 - 145) = 2^{10} \times 23,\ c^{\wedge}_{1174} :=$$
$$c^{\wedge}_{1164} + c^{\wedge}_{16a24} = 2^{12} \times 5,$$

and coefficient at μ in the polynomial $r^{\wedge\wedge}_{2,4}(\nu)$ is equal to

(29)
$$c^{\wedge}_{104}+c^{\wedge}_{11a4}+c^{\wedge}_{11b4}+c^{\wedge}_{12a4}+c^{\wedge}_{12b4}+$$
$$c^{\wedge}_{12b4}+c^{\wedge}_{13a4}+c^{\wedge}_{13b4}+c^{\wedge}_{14a4}+$$
$$c^{\wedge}_{14b14}+c^{\wedge}_{14b24}+c^{\wedge}_{15a14}+c^{\wedge}_{15a24}+$$
$$c^{\wedge}_{15b4}+c^{\wedge}_{16a14}+c^{\wedge}_{16a24}+c^{\wedge}_{16b4}+$$
$$c^{\wedge}_{17a14}+c^{\wedge}_{47a24}+c^{\wedge}_{17b4}+c^{\wedge}_{18a4}=$$
$$c^{\wedge}_{1164}+c^{\wedge}_{13a4}+c^{\wedge}_{13b4}=0.$$

Clearly,

(30)
$$r^{\wedge\wedge}_{2,1}(\nu)\big|_{\mu=0}=$$
$$20+20+8-256-384+176-352+768=0.$$

So $r^{\wedge\wedge}_{2,4}(\nu)=\mu^2 t^{\wedge\wedge}_{2,4}(\mu)$, where $t^{\wedge\wedge}_{2,4}(\mu)\in\mathbb{Q}[\mu]$, coefficients of $t^{\wedge\wedge}_{2,4}(\mu)$ at μ^k are equal to 0, if $k>1$, and, to prove that $t^{\wedge\vee}_{2,3}(\mu)=0$, it is sufficient to prove that $t^{\wedge\vee}_{2,3}(\mu)\in(\mu^2-1)\mathbb{Q}[\mu]$. Since $\mu\mathbb{Q}[\mu]+(\mu^2-1)\mathbb{Q}[\mu]=\mathbb{Q}[\mu]$, it follows that we must to prove that $r^{\wedge\wedge}_{2,4}(\nu)\in(\mu^2-1)\mathbb{Q}[\mu]$. Let

$$\omega^{\wedge\wedge}_{2,4,0,\pm1}(\mu):=$$
$$-4(6\mu+7)(-\mu+3)\equiv-4(11\mu+15)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,1a,\pm1}(\mu):=$$
$$(76\mu+85)32(121\mu)\equiv 32(10285\mu+9196)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\vee}_{2,4,1b,\pm1}(\mu):=$$
$$(-128)(9)(\mu+1)(-359\mu+267)\equiv 2^9\times 207(mu+1)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\vee}_{2,4,2a,\pm1}(\mu):=-128\times 19\times 55(\mu+1)\equiv$$
$$-128\times 1045(\mu+1)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,2b,\pm1}(\mu):=-128(10\mu+7)(145\mu+48)\equiv$$
$$-128(1495\mu+1786)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,3a,\pm1}(\mu):=$$
$$32(20\mu+29)(-160\mu+43)\equiv 32(-3780\mu-1953)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,3b,\pm1}(\mu):=$$
$$64(7\mu+4)(66\mu-277)\equiv 64(-1675\mu-646)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,4a,\pm1}(\mu):=$$
$$-4(17\mu+8)(-230\mu-711)\equiv 4(13927\mu+9598)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,4b,\pm1}(\mu):=$$
$$-4(6\mu+5)(850\mu-3215)\equiv 4(15040\mu+10975)\ (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,5a,\pm1}(\mu):=$$

$$-8(24\mu+7)(30\mu-97) \equiv -8(-2118\mu+41) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,5b,\pm 1}(\mu) :=$$
$$(-32)(3\mu)(-28\mu-113) \equiv 32 \times (339\mu+84) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,6a,\pm 1}(\mu) :=$$
$$128(3\mu+2)(-11\mu-21) \equiv 128(-85\mu-75) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,6b,\pm 1}(\mu) :=$$
$$128 \times 3(-43\mu-29) \equiv 128(-129\mu-87), \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,7a,\pm 1}(\mu) :=$$
$$-16(2\mu-1)(256\mu+249) \equiv 16(-242\mu-263) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,7b,\pm 1}(\mu) := -32(582\mu+547)$$
$$\omega^{\wedge\wedge}_{2,4,8a,\pm 1}(\mu) := 128(191\mu+186).$$

Then

$$\omega^{\wedge\wedge}_{2,4,9,\pm 1}(\mu) := \omega^{\wedge\wedge}_{2,4,0,\pm 1}(\mu) +$$
$$\omega^{\wedge\wedge}_{2,4,4a,\pm 1}(\mu) + \omega^{\wedge\wedge}_{2,4,4b,\pm 1}(\mu) \equiv$$
$$4(-11\mu-15+13927\mu+9598+15040\mu+10975) \equiv$$
$$8(14478\mu+10279) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,10,\pm 1}(\mu) := \omega^{\wedge\wedge}_{2,4,9,\pm 1}(\mu) +$$
$$\omega^{\wedge\wedge}_{2,4,5a,\pm 1}(\mu) \equiv 8(14478\mu+10279) +$$
$$8(2118\mu-41) \equiv 16(8298\mu+5119) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,11,\pm 1}(\mu) := \omega^{\wedge\wedge}_{2,4,10,\pm 1}(\mu) +$$
$$\omega^{\wedge\wedge}_{2,4,7a,\pm 1}(\mu) \equiv 16(8298\mu+5119) +$$
$$16(-242\mu-263) \equiv 2^7(1007\mu+607) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,12,\pm 1}(\mu) := \omega^{\wedge\wedge}_{2,4,1a,\pm 1}(\mu) +$$
$$\omega^{\wedge\wedge}_{2,4,5b,\pm 1}(\mu) \equiv 32(10285\mu+9196) +$$
$$32 \times (339\mu+84) \equiv 2^{11}(166\mu+145) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,13,\pm 1}(\mu) := \omega^{\wedge\wedge}_{2,4,1a,\pm 1}(\mu) +$$
$$\omega^{\wedge\wedge}_{2,4,7b,\pm 1}(\mu) \equiv 32(-3780\mu-1953) +$$
$$32(-582\mu-547) \equiv 64(-2181\mu-1250) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,14,\pm 1}(\mu) := \omega^{\wedge\wedge}_{2,4,13,\pm 1}(\mu) +$$
$$\omega^{\wedge\wedge}_{2,4,3b,\pm 1}(\mu) \equiv 64(-2181\mu-1250) +$$
$$64(-1675\mu-646) \equiv 2^9(-482\mu-237) \, (\mod (\mu^2-1)),$$
$$\omega^{\wedge\wedge}_{2,4,15,\pm 1}(\mu) := \omega^{\wedge\wedge}_{2,4,11,\pm 1}(\mu) +$$
$$\omega^{\wedge\vee}_{2,4,2a,\pm 1}(\mu) + \omega^{\wedge\wedge}_{2,4,2b,\pm 1}(\mu) +$$
$$\omega^{\wedge\wedge}_{2,4,6a,\pm 1}(\mu) + \omega^{\wedge\wedge}_{2,4,6b,\pm 1}(\mu) +$$

$$\omega^{\wedge\wedge}_{2,4,8a,\pm1}(\mu) \equiv 128(1007\mu + 607)+$$
$$128 \times (-1045\mu - 1045 - 1495\mu - 1786 - 85\mu - 75 - 129\mu - 87)+$$
$$128(191\mu + 186) \equiv 128(-1556\mu - 2200) \equiv 2^9(-389\mu - 550)$$
$$\omega^{\wedge\wedge}_{2,4,16,\pm1}(\mu) := \omega^{\wedge\wedge}_{2,4,13,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,15,\pm1}(\mu) + \omega^{\wedge\vee}_{2,4,1b,\pm1}(\mu) \equiv$$
$$2^9(-482\mu - 237) + 2^9(-389\mu - 550) + 2^9 \times 207(\mu + 1) \equiv$$
$$2^9(-664\mu - 580) \equiv 2^{11}(-166\mu - 145) \pmod{\mu^2 - 1},$$

and

$$r^{\wedge\wedge}_{2,4}(\nu) \equiv \omega^{\wedge\wedge}_{2,4,0,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,1a,\pm1}(\mu) + \omega^{\wedge\wedge}_{2,4,1b,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,2a,\pm1}(\mu) + \omega^{\wedge\wedge}_{2,4,2b,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,3a\pm1}(\mu) + \omega^{\wedge\wedge}_{2,4,3b,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,4a,\pm1}(\mu) + \omega^{\wedge\wedge}_{2,4,4b,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,5a,\pm1}(\mu) + \omega^{\wedge\wedge}_{2,4,5b,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,6a,\pm1}(\mu) + \omega^{\wedge\wedge}_{2,4,6b,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,7a,\pm1}(\mu) + \omega^{\wedge\wedge}_{2,4,7b,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,8a,\pm1}(\mu) \equiv \omega^{\wedge\wedge}_{2,4,13,\pm1}(\mu)+$$
$$\omega^{\wedge\wedge}_{2,4,15,\pm1}(\mu) \equiv 0 \pmod{\mu^2 - 1}.$$

■

§13.2. The corrections to the previous parts of this work.

The equality

$$r^{\vee\vee}_{2,3,3} = 7113\mu^3 + 2586\mu^2 - 183\mu - 9$$

must stand in the section §12.1 of [16] instead of

$$r^{\vee\vee}_{2,3,3} = 1953 - 7959\mu - 1302\mu^2 - 7113\mu^3.$$

The equality

$$r^{\vee\vee}_{2,3,4} = 3616\mu^3 - 4432\mu^2 - 2560\mu$$

must stand in the section §12.1 of [16] instead of

$$r^{\vee\vee}_{2,3,4} = -2688 - 7504\mu - 1072\mu^2 + 3616\mu^3.$$

The equality

$$v^{\vee\vee}_{2,3,4} = 3632\mu^3 - 4360\mu^2 - 2512\mu + 8$$

must stand in the section §12.1 of [16] instead of

$$v^{\vee\vee}_{2,3,4} = -2680 - 7456\mu - 1000\mu^2 + 3632\mu^3.$$

The equality
$$r^{\wedge\wedge}_{2,3,4} = -688\mu^2 + 2560\mu$$
must stand in the section §12.1 of [16] instead of
$$r^{\wedge\wedge}_{2,3,4} = 2688 + 2128\mu - 688\mu^2.$$
The equality
$$v^{\wedge\wedge}_{2,3,4} = -712\mu^2 + 2528\mu - 8$$
must stand in the section §12.1 of [16] instead of
$$v^{\wedge\wedge}_{2,3,4} = 2680 + 2096\mu - 712\mu^2.$$
The equalities
$$r^{\wedge\wedge}_{2,5,4} = r^{\wedge\wedge}_{2,5,4} = 976\mu^4 + 352\mu^3 - 1872\mu^2 - 576\mu - 8,$$
must stand in the section §12.1 of [16] instead of
$$v^{\wedge\wedge}_{2,5,4} = 976\mu^4 + 352\mu^3 - 1872\mu^2 - 576\mu - 8$$
$$v^{\wedge\wedge}_{2,5,4} = 976\mu^4 + 352\mu^3 - 1872\mu^2 - 576\mu - 8$$

References.

[1] R.Apéry, Interpolation des fractions continues
et irrationalite de certaines constantes,
Bulletin de la section des sciences du C.T.H., 1981, No 3, 37 – 53;

[2] L.A.Gutnik, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$
(the detailed version,part 3), Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2002, 57, 1 – 33;

[3] ___________, On the measure of nondiscreteness of some modules,
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2005, 32, 1 – 51. *

[4] ___________, On the Diophantine approximations
of logarithms in cylotomic fields.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2006, 147, 1 – 36.

[5] ___________, On some systems of difference equations. Part 1.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2006, 23, 1 – 37.

[6] ___________, On some systems of difference equations. Part 2.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2006, 49, 1 – 31.

[7] ___________, On some systems of difference equations. Part 3.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2006, 91, 1 – 52.

[8] ___________, On some systems of difference equations. Part 4.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2006, 101, 1 – 49.

[9] ___________, On some systems of difference equations. Part 5.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2006, 115, 1 – 9.

[10] ___________, On some systems of difference equations. Part 6.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2007, 16, 1 – 30.

[11] ___________, On some systems of difference equations. Part 7.

Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2007, 53, 1 − 40.

[12] ____________, On some systems of difference equations. Part 8.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2007, 64, 1 − 44.

[13] ____________, On some systems of difference equations. Part 9.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2007, 129, 1 − 36.

[14] ____________, On some systems of difference equations. Part 10.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2007, 131 , 1 − 33.

[15] ____________, On some systems of difference equations. Part 11.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2008, 38 , 1 − 45.

[16] ____________, On some systems of difference equations. Part 12.
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2008, 55, 1 − 45.

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