Chern Classes and Extraspecial Groups

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Abstract

The mod-p cohomology ring of the extraspecial p-group of exponent p is studied for odd p. We investigate the subquotient $ch(G)$ generated by Chern classes modulo the nilradical. The subring of $ch(G)$ generated by Chern classes of one-dimensional representations was studied by Tezuka and Yagita. The subring generated by the Chern classes of the faithful irreducible representations is a polynomial algebra. We study the interplay between these two families of generators, and obtain some relations between them.

Introduction One of the major outstanding problems in the cohomology of finite groups is the determination of the cohomology rings of the extraspecial groups. The case of extraspecial 2-groups was solved elegantly and completely by Quillen (see [9]), and there has been much work since then on the extraspecial p-groups for odd p. On the one hand, the cohomology ring of the extraspecial p-groups of order p^3 and exponent p has been determined by Lewis for integral coefficients and by the second named author for mod-p coefficients (see [7], [5]). On the other hand, there have been major advances in the general problem, which have concentrated on calculating the cohomology ring modulo its nilradical. Tezuka and Yagita calculated this up to inseparable isogeny by a generalization of Quillen's methods (see [11]),

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and Benson and Carlson have collected and refined the knowledge to date in their expository paper [3].

In this paper we consider the problem of determining the cohomology ring modulo its nilradical exactly, rather than up to inseparable isogeny. We shall attempt this by studying Chern classes of the irreducible representations of the extraspecial group p_+^{1+2n} which are obtained by inducing from maximal abelian subgroups. Chern classes will not in general generate the cohomology ring modulo its nilradical, even for p-groups (see the paper [6] of Yagita and the second named author, where examples of order p^4 and rank 2 are given). However, this approach does indeed give us new cohomology classes.

This is clearly seen by considering the group of order p^3. The cohomology ring modulo its radical of the extraspecial group p_+^{1+2} is the quotient of the polynomial ring $\mathbb{F}_p[\alpha, \beta, \kappa_0, \zeta]$ by the relations $\alpha^p\beta - \alpha\beta^p = 0$, $\alpha\kappa_0 = -\alpha^p$, $\beta\kappa_0 = -\beta^p$ and $\kappa_0^2 = \alpha^{2p-2} - \alpha^{p-1}\beta^{p-1} + \beta^{2p-2}$. Here α and β are first Chern classes of degree 1 representations, whereas κ_0 and ζ are Chern classes of a degree p irreducible representation. However, the best result known to date for general extraspecial groups only says that this is up to inseparable isogeny the quotient of $\mathbb{F}_p[\alpha, \beta, \zeta]$ by the one relation $\alpha^p\beta - \alpha\beta^p = 0$.

It is very probable that the full description of the mod-p cohomology ring is exceedingly complicated: one just needs to look at the result for the group of order p^3 to get an idea of this. It is not even certain that it is practical to calculate the whole of the cohomology ring modulo its radical. But the example of the p^3 case suggests that understanding the Chern classes will be a major step in the right direction. When investigating the cohomology ring of a p-group, the Chern subring modulo nilradical is therefore a worthy and interesting object of study.

The outline of the paper is as follows. Let p be an odd prime, and $n \geq 1$. After recalling necessary information about group cohomology, Chern classes, extraspecial p-groups and Dickson invariants, we obtain generators for the Chern subring modulo nilradical in Proposition 7. For p_+^{1+2n}, there are $3n+1$ generators: α_i and β_i for $1 \leq i \leq n$; κ_r for $0 \leq r \leq n - 1$; and ζ. The κ_r are the new generators: the other elements generate the subring studied by Tezuka and Yagita. The κ_r are Chern classes of a degree p^n faithful irreducible representation of p_+^{1+2n}, and restrict to maximal elementary abelian subgroups as Dickson invariants.

Our aim is to understand the relationship between the new generators and the old. In Proposition 15, we obtain an elegant alternating sum formula expressing $\kappa_0^{2^n}$ as a polynomial in the α_i and β_i. Theorem 20, the main result of the paper, generalises this by showing that $\kappa_r^{2^{n-r}p^r}$ also lies in the subring $\mathcal{T}$ generated by the α_i and β_i. The idea behind both these results is as follows. Pick a maximal elementary abelian subgroup M of p_+^{1+2n}; find some expression (typically a Dickson invariant) in the α_i and β_i whose restriction to M equals that of κ_r; patch these approximations together; and then appeal to Quillen's theorem, which states that the maximal elementary

abelian subgroups detect non-nilpotent elements.

In a separate paper [4], the first named author completes this project for $n = 2$ by obtaining a presentation for the Chern subring modulo nilradical of p_+^{1+4}. In particular, it is shown that for all $n \geq 1$, the Chern class κ_0 lies outside the subring studied by Tezuka and Yagita.

A theorem of Quillen In this paper we study cohomology rings modulo their nilradicals. The rationale for this is provided by the following theorem of Quillen.

Theorem 1 (Quillen [8],[10]) *Let G be a finite group, p a prime number, and k a field of characteristic p. Then a class $\xi \in \mathrm{H}^*(G, k)$ is nilpotent if and only if its restriction to every elementary abelian p-subgroup of G is nilpotent.* ∎

Let G be a finite group, and p a prime number. Define $h^*(G, \mathbb{F}_p)$ to be the quotient of the graded commutative ring $\mathrm{H}^*(G, \mathbb{F}_p)$ by its nilradical. If the value of p is clear from the context, we will just write $h^*(G)$.

Of course, $h^*(G)$ is strictly commutative. If $\phi \colon H \to G$ is a group homomorphism, then nilpotent classes in $\mathrm{H}^*(G, \mathbb{F}_p)$ are mapped under ϕ^* to nilpotent classes in $\mathrm{H}^*(H, \mathbb{F}_p)$: hence ϕ induces a well-defined ring homomorphism $\phi^* \colon h^*(G) \to h^*(H)$. In particular, there is a well-defined restriction homomorphism if H is a subgroup of G. The version of Quillen's theorem that we shall use in this paper is now a trivial corollary of Theorem 1.

Corollary 2 *Let G be a finite group; let p be a prime number; and let ξ be a class in $h^*(G)$. Then ξ is zero if and only if $\mathrm{Res}_A \xi = 0$ in $h^*(A)$ for every elementary abelian p-subgroup A of G.* ∎

Chern classes For a concise introduction to Chern classes of group representations, we refer the reader to the appendix of [1]. Although Chern classes strictly belong to $\mathrm{H}^*(G, \mathbb{Z})$, they can be considered as elements of $h^*(G)$ via the map $\mathrm{H}^*(G, \mathbb{Z}) \to \mathrm{H}^*(G, \mathbb{F}_p) \to h^*(G)$. Write $ch(G)$ for the subring of $h^*(G)$ generated by Chern classes. This algebra $ch(G)$ is a large subquotient of $\mathrm{H}^*(G, \mathbb{F}_p)$; any pair of elements may be compared in a straightforward way; and the Chern classes of the irreducible representations form a finite set of generators (by the Whitney sum formula). This makes the algebra $ch(G)$ an object worthy of study. The object of this paper is to investigate it for G extraspecial.

Because their restrictions can be calculated directly, Chern classes lend themselves particularily well to a study using Quillen's theorem. Taking the Chern classes of a representation commutes with taking its restriction to a subgroup, and the Whitney sum formula expresses the Chern classes of a direct sum of representations in terms of the Chern classes of the summands.

After restricting a representation to an abelian subgroup, the irreducible summands all have degree one; and the Chern classes of such representations are very well understood.

The mod-p cohomology rings of the elementary abelian p-groups are well known, and may easily be derived from the cohomology ring of the cyclic group using the Künneth Theorem. Let A be an elementary abelian p-group of p-rank m. Then A is also an m-dimensional $\mathbb{F}_p$-vector space. Embed the additive group of $\mathbb{F}_p$ in $\mathbb{C}^\times$ by sending 1 to $\exp(2\pi i/p)$. This induces an isomorphism between $\operatorname{Hom}(A, \mathbb{C}^\times)$, a group under tensor product, and the dual vector space A^*, a group under addition. There is therefore a map $c_1 \colon A^* \to h^2(A)$ which sends an element of A^* to the first Chern class of the corresponding one-dimensional representation. The induced map from the symmetric algebra $S(A^*)$ to $h^*(A)$ is an isomorphism of $\mathbb{F}_p$-algebras. So $h^*(A)$ is an integral domain, and in fact a polynomial algebra.

Extraspecial Groups From now on we fix an odd prime p, and denote by G the extraspecial p-group of order p^{2n+1} and exponent p. There is a central extension $1 \to N \to G \xrightarrow{\psi} E \to 1$ with E an elementary abelian p-group of p-rank $2n$, and N cyclic of order p. Identify N with $\mathbb{F}_p$, and view E as a $2n$-dimensional $\mathbb{F}_p$-vector space. There is a well-defined nondegenerate symplectic bilinear form $b \colon E \times E \to N$ given by $b(x_1, x_2) = [\tilde{x}_1, \tilde{x}_2]$ for any $\tilde{x}_1, \tilde{x}_2 \in G$ such that $\psi(\tilde{x}_i) = x_i$, where $[a, b] = aba^{-1}b^{-1}$. There is a natural bijection between the set of maximal elementary abelian subgroups M of G and the set of maximal totally isotropic subspaces I of E, given by $I = M/N$ and $M = \psi^{-1}(I)$. Every M has p-rank $n + 1$, and every I has dimension n.

To determine the irreducible characters of G, embed $\mathbb{F}_p$ in $\mathbb{C}^\times$ as before. Let $\hat{\chi}$ be a nontrivial linear character of N, and define $\hat{\chi}_j = \hat{\chi}^{\otimes j}$ for $1 \le j \le p - 1$.

Lemma 3 *There are p^{2n} linear characters of G, and all factor through ψ. These correspond to the elements of E^*. The $p - 1$ remaining irreducible characters all have degree p^n and are induced from any maximal elementary abelian subgroup of G. They may be labelled $\chi_1, \dots, \chi_{p-1}$ such that for every $1 \le j \le p - 1$ and every $g \in G$,*

$$\chi_j(g) = \begin{cases} p^n \hat{\chi}_j(g) & \text{if } g \in N, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It follows that, for any maximal elementary abelian subgroup M of G, the restriction $\operatorname{Res}_M(\chi_j)$ is the sum of all those linear characters of M whose restriction to N is $\hat{\chi}_j$.

Proof: Choose a maximal elementary abelian subgroup M of G. For each $1 \le j \le p - 1$, pick a linear character $\hat{\chi}'_j$ of M whose restriction to N is $\hat{\chi}_j$.

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Define the character χ_j of G to be $\text{Ind}^G(\check{\chi}'_j)$. The character formula for induced characters then shows that (1) is satisfied. Using the orthogonality relations, and summing squares of degrees, it is seen that we have all irreducible characters. ∎

Dickson invariants We will see in Proposition 7 that the Chern classes of the induced representations restrict to maximal elementary abelian subgroups as Dickson invariants. We now recall the salient facts about these invariants. For a proof, see Benson's book [2].

Theorem 4 (Dickson) *Let V be a finite dimensional $\mathbb{F}_p$-vector space, and let $m = \dim(V)$.*

1. *There exist Dickson invariants $D_0(V), \ldots, D_{m-1}(V)$ in the symmetric algebra $S(V)$, with $D_r(V)$ in $S^{p^m - p^r}(V)$ for each $0 \leq r \leq m - 1$, such that*

$$\prod_{v \in V}(X - v) = X^{p^m} + \sum_{r=0}^{m-1}(-1)^{m-r}D_r(V)X^{p^r} . \tag{2}$$

2. *The Dickson invariants are algebraically independent. The ring of invariants $S(V)^{GL(V)}$ is the polynomial algebra $\mathbb{F}_p[D_0(V), \ldots, D_{m-1}(V)]$.* ∎

In the literature, $D_r(V)$ is usually denoted $c_{m,r}$. New notation is introduced here in order to identify the vector space V explicitly, and to avoid a clash with the notation $c_r(\rho)$ for Chern classes.

We now describe the relationship with Dickson invariants of quotient spaces. This takes a particularily elegant form when we work with dual spaces.

Lemma 5 *Let V be an m-dimensional $\mathbb{F}_p$-vector space, and U an ℓ-codimensional subspace. The inclusion of U in V induces a restriction map $S(V^*) \rightarrow S(U^*)$ and, for every $0 \leq r \leq m - 1$,*

$$\text{Res}_U(D_r(V^*)) = \begin{cases} D_{r-\ell}(U^*)^{p^\ell} & \text{if } \ell \leq r, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Proof: Obvious from the definition of the Dickson invariants. ∎

The top Dickson invariant $D_0(V)$ is, up to a sign, the product of all nonzero elements of V. A generalisation (due to Macdonald) of this interpretation will play an important role in this paper.

Theorem 6 (Macdonald) *Let V be an m-dimensional $\mathbb{F}_p$-vector space, and let $0 \leq r \leq m - 1$. Denote by $\mathcal{Y}_r$ the set of all r-dimensional subspaces of V. For any $Y \in \mathcal{Y}_r$, write $P_{V,Y}$ for the product $\prod_{v \in V \setminus Y} v$ in the symmetric algebra $S(V)$. Then*

$$D_r(V) = (-1)^{m-r} \sum_{Y \in \mathcal{Y}_r} P_{V,Y} . \tag{4}$$

Proof: (cf. [2], p. 91) The right hand side of (4) is clearly an invariant of $GL(V)$, and so must be a scalar multiple of the left hand side. Pick any $Y \in \mathcal{Y}_r$, and project down onto $S(U)$, where $U = V/Y$. This sends the left hand side to $D_0(U)^{p^r}$. On the right hand side, all summands are sent to zero, except that $P_{V,Y}$ is sent to $P_{U,0}^{p^r}$. Therefore we need only establish the case $r = 0$: but this is an immediate consequence of Theorem 4. ∎

Chern classes for extraspecial groups We start by considering the degree one representations. Choose a symplectic basis $A_1, \ldots, A_n, B_1, \ldots, B_n$ for E. That is, $A_i \perp A_j$, $B_i \perp B_j$ and $b(A_i, B_j) = \delta_{ij}$. Take the corresponding dual basis $A_1^*, \ldots, B_n^*$ for E^*, and recall that one-dimensional representations of G are identified with elements of E^*. For $1 \leq i \leq n$, define $\alpha_i = c_1(A_i^*)$ and $\beta_i = c_1(B_i^*)$. Equivalently, consider the A_i^* and B_i^* as elements of $h^2(E)$ via the isomorphism $h^*(E) \cong S(E)$, and define α_i, β_i to be the inflations $\psi^*(A_i^*)$, $\psi^*(B_i^*)$ respectively.

Let ρ_1 be a representation of G affording the induced character χ_1. Define $\kappa_r = (-1)^{n-r} c_{p^n - p^r}(\rho_1)$ for $0 \leq r \leq n - 1$, and $\zeta = c_{p^n}(\rho_1)$. Let γ be the element of N^* corresponding to the nontrivial linear character $\hat{\chi}$ of N.

Proposition 7 *The Chern subring of $h^*(p_+^{1+2n})$ is generated by $\kappa_0, \ldots, \kappa_{n-1}, \zeta, \alpha_1, \beta_1, \ldots, \alpha_n$ and β_n. Let M be a maximal elementary abelian subgroup of p_+^{1+2n}, and $I = M/N$ the corresponding maximal totally isotropic subspace of E. Notice that $S(M^*) \cong S(I^*) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\gamma]$. We have*

$$\mathrm{Res}_M \, \kappa_r = D_r(I^*) \quad \text{for } 0 \leq r \leq n - 1, \text{ and} \quad \mathrm{Res}_N \, \zeta = \gamma^{p^n} . \tag{5}$$

Proof: The α_j and β_j arise from a basis for E^*, and so the first Chern class of every degree one representation is in their span.

The inflation map $h^2(I) \to h^2(M)$ is the inclusion of I^* in M^*, and the restriction map $h^2(M) \to h^2(N)$ is the projection of M^* onto N^* with kernel I^*. Let $\hat{\gamma} \in M^*$ be a representative of the coset whose restriction to N^* is γ. It follows from Lemma 3 that, for every $1 \leq j \leq p - 1$, the induced character χ_j restricts to M as the direct sum of the elements of the coset $j\hat{\gamma} + I^*$, considered as linear characters of M. Let ρ_j be a representation of G

affording χ_j. Using the Whitney sum formula, the total Chern class of ρ_j is

$$\mathrm{Res}_M\, c(\rho_j) = \prod_{v \in I^*}(1 + j\hat{\gamma} + v)$$

$$= 1 + \sum_{r=0}^{n-1}(-1)^{n-r}D_r(I^*) + j\left(\hat{\gamma}^{p^n} + \sum_{r=0}^{n-1}(-1)^{n-r}D_r(I^*)\hat{\gamma}^{p^r}\right).$$

Therefore $c(\rho_j) = 1 + \kappa_0 + \cdots + \kappa_{n-1} + j\zeta$, and the restrictions are as claimed.

∎

Remark: An argument involving the Adams–Frobenius operations may be used to show that the equalities $c_s(\rho_j) = j^s c_s(\rho_1)$ for all $s \geq 1$ hold even in $\mathrm{H}^*(G, \mathbb{Z})$.

Important subrings There are two important subrings of $ch(G)$. Both of these have the same Krull dimension as $ch(G)$ itself, and together they generate $ch(G)$. The structure of each of these subrings is known; but understanding how elements of one ring relate to elements of the other is more complicated. The core of this paper is an attempt to start understanding this relationship.

The first subring is generated by ζ, the α_i and the β_i, and was studied by Tezuka and Yagita. The Chern classes which generate $ch(G)$ were originally defined in $\mathrm{H}^*(G, \mathbb{Z})$, and therefore correspond to well-defined elements of $\mathrm{H}^*(G, \mathbb{F}_p)$. Let $\mathcal{T}$ denote the subring of $h^*(G)$ generated by the α_i and β_i. For $r \geq 1$, let $R_r = \alpha_1\beta_1^{p^r} - \alpha_1^{p^r}\beta_1 + \cdots + \alpha_n\beta_n^{p^r} - \alpha_n^{p^r}\beta_n$.

Theorem 8 (Tezuka–Yagita [11]) *The subalgebra $\mathcal{T}$ of $h^*(G)$ is the quotient of the polynomial algebra on the α_i and β_i by the ideal generated by the R_r for $1 \leq r \leq n$. Moreover, $R_r = 0$ in $h^*(G)$ for all $r \geq 1$. The ring $\mathcal{T} \otimes_{\mathbb{F}_p} \mathbb{F}_p[\zeta]$ is in fact a subalgebra of $\mathrm{H}^*(G, \mathbb{F}_p)$, and every element of $\mathrm{H}^*(G, \mathbb{F}_p)$ has some power lying in this subalgebra.* ∎

The second large subring is generated by ζ and the κ_r.

Proposition 9 *The Chern classes $\kappa_0, \ldots, \kappa_{n-1}$ and ζ are algebraically independent over $\mathbb{F}_p$. Moreover, no polynomial in these elements is a zero divisor in $h^*(G)$. The $\mathbb{F}_p$-algebra $h^*(G)$ is finite over the subalgebra generated by the κ_r and ζ.*

Proof: Algebraic independence is a result of the algebraic independence of the Dickson invariants. By Quillen's Theorem, every non-zero element of $h^*(G)$ has non-zero restriction to $h^*(M)$ for some maximal elementary abelian subgroup M. By Proposition 7, restriction to M is an injection on the subring we consider. But $h^*(M)$ is an integral domain. For the last part we appeal to Venkov's proof of the Evens–Venkov theorem, since ρ_1 is faithful. ∎

That the first of these subrings is not contained in the second is clear by degree considerations. Conversely, κ_0 lies in the Tezuka–Yagita subring for no value of n. For $n = 1$ this can be seen from Lewis' paper [7], and it is proved in [4] for general n.

We start our investigation of the relationship between these two subrings by establishing one simple identity.

Lemma 10 *Let x be one of the α_i or β_i; more generally, let x be the first Chern class of a one-dimensional representation of G. Then*

$$x^{p^n} - x^{p^{n-1}}\kappa_{n-1} + \cdots + (-1)^r x^{p^{n-r}}\kappa_{n-r} + \cdots + (-1)^n x\kappa_0 = 0 . \quad (6)$$

Proof: Let M be a maximal elementary abelian subgroup of G, and I the corresponding maximal totally isotropic subspace of E. Then $\mathrm{Res}_M(x) \in I^*$, and so $\prod_{v \in I^*}(\mathrm{Res}_M(x) - v) = 0$. But, from the definition of the κ_r, this product equals the restriction to M of the left hand side of (6). The result follows by Quillen's Theorem. ∎

We would like to obtain all the relations between the generators of $ch(G)$. In this paper we begin this task by investigating which powers of κ_r lie in the Tezuka–Yagita ring.

Bases In this section, we establish a result about the restrictions of the A_j^* and B_j^* to the dual space I^*, for any maximal totally isotropic subspace I of E. To this end, we shall introduce a symplectic form b_L on E^*, which will also play an important role in subsequent sections. For every $0 \le r \le n$, let $\mathcal{Z}_r$ denote the set of all r-dimensional subspaces V of E^* which have a basis $y_1, \ldots, y_r$ in which each y_i is either A_i^* or B_i^*.

Proposition 11 *Let $1 \le r \le n$, and let I be a maximal totally isotropic subspace of E. Let $V \in \mathcal{Z}_{r-1}$, and suppose that the restriction of V to I^* also has dimension $r - 1$. Then there exists an element of $\mathcal{Z}_r$ which contains V, and whose restriction to I^* has dimension r.*

Hence, for every $0 \le r \le n$ and for every I, there is at least one $V \in \mathcal{Z}_r$ whose restriction to I^* has dimension r.

The nondegenerate symplectic form b on E induces an $\mathbb{F}_p$-vector space isomorphism $L \colon E \to E^*$ as follows: for all e, $e' \in E$, $L(e)(e') = b(e, e')$. There is then a unique symplectic form b_L on E^* such that $b_L(L(e), L(e')) = b(e, e')$, for all e, $e' \in E$. Since b is nondegenerate, so is b_L.

Lemma 12 *Let U and V be subspaces of E^*, and let I be a totally isotropic subspace of E. If $U \perp V$, if V is totally isotropic, and if $U \subseteq V + L(I)$, then U is totally isotropic.*

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Proof: Let u, $u' \in U$. Since $U \subseteq V + L(I)$, there exist v, $v' \in V$ and i, $i' \in I$ such that $u = v + L(i)$ and $u' = v' + L(i')$. Then $b_L(u, u') = b_L(v, u') + b_L(u, v') + b_L(L(i), L(i')) - b_L(v, v')$, and each of these terms is zero by assumption. ∎

Proof of Proposition 11: Let $U = \mathrm{span}(A_r^*, B_r^*)$. Then V is totally isotropic and $U \perp V$, but U is not totally isotropic. Hence, by the Lemma, U is not contained in $V + L(I)$. In particular, at least one of A_r^*, B_r^* does not lie in $V + L(I)$. But $u \in V + L(I)$ if and only if the restriction of u to I^* lies in the restriction of V. The last part follows by induction on r. ∎

Characteristic functions In this section we show that $\kappa_0^{2^n}$ lies in the Tezuka–Yagita subring $\mathcal{T}$. This is a special case of Theorem 20, and the results of this section are not necessary to prove that theorem. However, the methods we use are more transparent here than in the general case, and we also succeed in establishing an elegant formula for $\kappa_0^{2^n}$.

We shall prove that $\kappa_0^{2^n} \in \mathcal{T}$ using characteristic functions, analogously to the alternating sum formula for the measure of a finite union. Of fundamental importance is the following special case of Lemma 5. Let V be an n-dimensional subspace of E^*, let M be a maximal elementary abelian subgroup of G, and let I be the corresponding maximal totally isotropic subspace of E. Then

$$\mathrm{Res}_M(D_0(V)) = \begin{cases} D_0(I^*) & \text{if } \mathrm{Res}_I(V) \text{ is the whole of } I^*, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Recall that $\mathcal{Z}_n$ denotes the set of all n-dimensional subspaces of E^* which have a basis of the form $y_1, \ldots, y_n$ such that each y_i is either A_i^* or B_i^*. In this section we will write $\mathcal{Z}$ for $\mathcal{Z}_n$. Note that $\mathcal{Z}$ has cardinality 2^n.

We now introduce some more notation for this section. Let $\mathcal{I}$ denote the set of all maximal totally isotropic subspaces of E. For each subset T of $\mathcal{Z}$, define $I(T)$ to be the set of all $I \in \mathcal{I}$ such that $\mathrm{Res}_I(V) = I^*$ for every $V \in T$. If $V \in \mathcal{Z}$, write $I(V)$ for $I(\{V\})$.

For any subset T of $\mathcal{Z}$, define $\chi_T: \mathcal{I} \to \{0, 1\}$ to be the characteristic function of $I(T)$: that is, for $I \in \mathcal{I}$,

$$\chi_T(I) = \begin{cases} 1 \text{ if } I \in I(T), \text{ and} \\ 0 \text{ otherwise.} \end{cases} \tag{8}$$

For $V \in \mathcal{Z}$, write χ_V for $\chi_{\{V\}}$. The following result is now a consequence of Quillen's Theorem.

Lemma 13 *Let $s \geq 1$; let T be a non-empty subset of $\mathcal{Z}$ such that $|T| \leq s$; and let $V_1, \ldots, V_s$ be a sequence of elements of T in which each element of T*

appears at least once. Define

$$D_{T,s} = \psi^* \left(\prod_{j=1}^{s} D_0(V_j) \right) \quad in \ h^{2s(p^n-1)}(G). \tag{9}$$

Let M be a maximal elementary abelian subgroup of G, and let I be the corresponding maximal totally isotropic subspace of E. Then

$$\mathrm{Res}_M(D_{T,s}) = \chi_T(I)D_0(I^*)^s \,, \tag{10}$$

and so $D_{T,s}$ is independent of the choice of the V_j, which justifies the notation. ∎

Lemma 14 *Let T_1 and T_2 be subsets of $\mathcal{Z}$. Then*

1. $\displaystyle\bigcup_{V \in \mathcal{Z}} I(V) = \mathcal{I}.$

2. $I(T_1) \cap I(T_2) = I(T_1 \cup T_2)$, *and therefore* $\chi_{T_1 \cup T_2} = \chi_{T_1}\chi_{T_2}$.

3. $\displaystyle\prod_{V \in \mathcal{Z}}(1 - \chi_V) = 0.$

Proof: Part 1 is a consequence of Proposition 11. Part 2 is an immediate consequence of the definition. Part 3 now follows from the formula for the characteristic function of a union. ∎

Proposition 15 *Let $s \geq 2^n$. Then*

$$\kappa_0^s = - \sum_{\emptyset \neq T \subseteq \mathcal{Z}} (-1)^{|T|} D_{T,s}. \tag{11}$$

Proof: We of course use Quillen's Theorem. Let M be a maximal element-ary abelian subgroup, and I the associated maximal totally isotropic sub-space. By Proposition 7, $\mathrm{Res}_M(\kappa_0^s) = \chi_\emptyset(I)D_0(I^*)^s$. Now, by Lemma 13 and Lemma 14,

$$\mathrm{Res}_M \left(\kappa_0 + \sum_{\emptyset \neq T \subseteq \mathcal{Z}} (-1)^{|T|} D_{T,s} \right) = \sum_{T \subseteq \mathcal{Z}} (-1)^{|T|} \chi_T(I)D_0(I^*)^s \tag{12}$$

$$= \left(\prod_{V \in \mathcal{Z}} (1 - \chi_V(I)) \right) D_0(I^*)^s \tag{13}$$

$$= 0 \,, \tag{14}$$

which proves the result. ∎

This result does not imply that $\kappa_0^s \notin \mathcal{T}$ whenever $s < 2^n$. In fact, it is proved in [4] that, for $n = 2$, $\kappa_0^s \in \mathcal{T}$ if and only if $s \geq 2$. However, the following lemma demonstrates that κ_0^s cannot be expressed in terms of the $D_{T,s}$ if $s < 2^n$.

Lemma 16 *Let $V \in \mathcal{Z}$, and let T_1 and T_2 be subsets of $\mathcal{Z}$. Then*

1. *$I(\mathcal{Z})$ is not empty.*

2. *$I(\mathcal{Z} \setminus V)$ is strictly larger than $I(\mathcal{Z})$.*

3. *$I(T_1) = I(T_2)$ if and only if $T_1 = T_2$.*

Proof: For each $1 \leq i \leq n$, define $X_i = B_i A_i \in E$; then $X_1, \ldots, X_n$ generate a maximal totally isotropic subspace which lies in $I(\mathcal{Z})$. The automorphism group of G acts transitively on the set of maximal elementary abelian subgroups, and so in part 2 we may assume without loss of generality that $V = \{A_1^*, \ldots, A_n^*\}$. Define elements $Y_1, \ldots, Y_n$ of E by $Y_r = B_1 \ldots B_r A_r A_{r+1}^{-1}$ if $r \leq n - 1$, and $Y_n = B_1^{2-n} \ldots B_r^{r+1-n} \ldots B_n A_n A_1^{-1}$. Then $Y_1, \ldots, Y_n$ commute with each other, and generate a maximal totally isotropic subspace which lies in $I(\mathcal{Z} \setminus V)$ but not in $I(\mathcal{Z})$. Finally, part 3 now follows. ■

Integrality Each κ_r is integral over $\mathcal{T}$, by the Tezuka–Yagita theorem. In this section, we obtain explicit monic polynomial equations satisfied by the κ_r, and prove that $\kappa_r^{tp^r}$ lies in $\mathcal{T}$ for sufficiently large t. Recall that E^* carries a nondegenerate symplectic form b_L.

Lemma 17 *Let V be a subspace of E^*, and I a maximal totally isotropic subspace of E. Suppose that $\dim \mathrm{Res}_I(V) = \dim(V)$. Then $\mathrm{Res}_I(V^\perp) = I^*$.*

Proof: Let A be the subspace $L^{-1}(V)$ of E. Then $\dim \mathrm{Res}_I(V) = \dim(V)$ if and only if $I \cap A = 0$, and $\mathrm{Res}_I(V) = I^*$ if and only if $A + I = E$. So we must prove that $A^\perp + I = E$ if $I \cap A = 0$. Since b is nondegenerate and $I^\perp = I$, this is standard linear algebra. ■

Let V be a finite-dimensional $\mathbb{F}_p$-vector space. Choose one non-zero vector from each one-dimensional subspace of V, and define $\Delta(V) \in S(V)$ to be the product of all these subspace representatives. Then $\Delta(V)$ is well-defined up to multiplication by a scalar, and Macdonald's Theorem shows us that $\Delta(V)^{p-1} = D_0(V)$.

Recall that the inflation map ψ^* induces an $S(E^*)$-module structure on $h^*(G)$.

Proposition 18 *Let $0 \leq r \leq n - 1$; let $n - r \leq s \leq n$; and let V be an s-dimensional subspace of E^*. Then $\Delta(V)\left(\kappa_r^{p^{n-s}} - D_{n+r-s}(V^\perp)\right) = 0$ and*

$$D_0(V)\left(\kappa_r^{p^{n-s}} - D_{n+r-s}(V^\perp)\right) = 0 \ in \ h^*(p_+^{1+2n}).$$

Proof: By Quillen's Theorem, it suffices to prove these equalities after restriction to each maximal elementary abelian subgroup M of G. For such an M, let I be the corresponding maximal totally isotropic subspace I of E. If $\dim \mathrm{Res}_I(V) < \dim(V)$, then $D_0(V)$, and hence $\Delta(V)$, restrict to zero by Lemma 5. Otherwise $\dim \mathrm{Res}_I(V) = \dim(V)$, and so Lemma 17 tells us that $\mathrm{Res}_I(V^\perp) = I^*$. Applying Lemma 5 again, $\mathrm{Res}_M\left(D_{n+r-s}(V^\perp)\right) = D_r(I^*)^{p^{n-s}}$: but this is also $\mathrm{Res}_M(\kappa_r^{p^{n-s}})$, by Proposition 7. ∎

Corollary 19 *Let $0 \leq r \leq n - 1$; let V be a $(n + s)$-dimensional subspace of E^* for some $0 \leq s \leq r$; and let Y be a $2s$-dimensional subspace of V. For any complementary subspace W of Y in V, the equation $\kappa_r^{p^s} P_{V,Y} = D_{r+s}(W^\perp)P_{V,Y}$ holds in $h^*(G)$. Hence $\kappa_r^{tp^s} D_{2s}(V) \in \mathcal{T}$ for all $t \geq 1$.*

Proof: Observe that $D_0(W)$ divides $P_{V,Y}$. The last part follows by Macdonald's Theorem. ∎

Theorem 20 *For every $0 \leq r \leq n - 1$ and every $0 \leq s \leq n$,*

$$\prod_{V \in \mathcal{Z}_s} \left(\kappa_r^{p^{n-s}} - D_{n+r-s}(V^\perp)\right) = 0 \ . \tag{15}$$

In particular, $\kappa_r^{tp^r} \in \mathcal{T}$ for all $t \geq 2^{n-r}$.

Proof: We use Quillen's Theorem. Let M be a maximal elementary abelian subgroup of G, and I the corresponding maximal totally isotropic subspace of E. By Proposition 7, the restriction of κ_r to M is $D_r(I^*)$. By Proposition 11, there is some $V \in \mathcal{Z}_s$ whose restriction to I has dimension $s = \dim(V)$. Then by Lemma 17, the restriction of $V^\perp$ is I^*. Hence $\kappa_r^{p^{n-s}} - D_r(V^\perp)$ restricts to zero by Lemma 5. The last part follows by Corollary 19. ∎

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