Whitehead groups of finite polyhedra with nonpositive curvature

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1. Introduction

Our result is on the Whitehead groups $Wh\Gamma = K_1(Z\Gamma)/H_1\Gamma \times Z_2$ of some groups Γ relating to geometry. The strategy of using control topology plus geometry to study $Wh\Gamma$ was previously used to prove $Wh\pi_1 M = 0$ for closed flat manifolds M by Farrell and Hsiang in [6]. The following more general result was proved using ideas that involved sphere bundles and geodesic flows.

1.1 Theorem([8] in first order). $Wh\pi_1 M = 0$ for any closed riemannian manifolds with nonpositive curvature.

In this paper we try to obtain a Whitehead group result concerning finite polyhedra of nonpositive curvature in two steps. The first step is to transform the the problem to one about closed manifolds, by applying the idea of hyperbolization. In the second step that the Whitehead group of a closed manifold with PL nonpositive curvature is zero is proved as a development from [7], [8] and [13]. The meaning of a polyhedron with negative or nonpositive curvature was defined in [12] by Gromov to study hyperbolic groups. The result of this paper is as follows. It covers a major class of semihyperbolic groups.

1.2 Theorem. WhΓ = 0, $\Gamma = \pi_1 K$ for any finite polyhedra with nonpositive curvature.

Note that this implies $K_1(Z\Gamma) = H_1\Gamma \times Z_2$, $\tilde{K}_0(Z\Gamma) = 0$, $K_i(Z\Gamma) = 0$, $i \le -1$. Previous result in this respect is the vanishing of Whitehead groups in the negative curvature case in [13]. One interest in extending that to the nonpositive curvature case is from the application of 1.2 to p-adic groups via their Euclidean buildings([19]) which are the most interesting known examples of polyhedron nonpositive curvature structures. In particular there is

1.3 Corollary. Let Q_p be the field of p-adic numbers. For any torsion-free and cocompact discrete subgroup $\Gamma \subset SL_n(Q_p)$, WhΓ = 0.

2. From manifolds to polyhedra

This section shows

2.1 Lemma: If the Whitehead groups of closed manifolds with PL nonpositive curvature are zero then the same is true for finite polyhedra.

It is hyperbolization, an idea due to Gromov([12]) that allows us to see this. Because we only need to gain nonpositive rather than negative curvature, we will avoid the complicated and somewhat unclear strict hyperbolization needed in [13], by using here a weak but transparent hyperbolization discovered in [5] by Davis and Januszkiewicz, which is like the untwist version of the first hyperbolization of [12], 3.4.

Let K be finite simplical complex, A be subcomplex. Do the following: Let $hK^1 = K^1$; Take $hK^1 \times (\pm 1)$ = two copies of hK^1. If $\Delta^2 \subset A$, then denote $h\Delta^2 = \Delta^2 \times (\pm 1)$. But if $\Delta^2 \not\subset A$, then define $h\Delta^2 = \partial\Delta^2 \times [-1,1]$. $hK^2 = K^1 \times (\pm 1) \cup$all $h\Delta^2$, and so on. The end result h(K,A), the hyperbolization of K relative to A, is what we want to use. To make things clear we give

2.2 There is unique construction h such that

(1) For any finite simplical complex K^n and subcomplex A, h(K,A) is finite simplical complex. If L^i is subcomplex of K then

$h(L,L\cap A)\times(\pm 1)^{n-i} \subset h(K,A)$

Here $h(L,L\cap A)\times(\pm 1)^{n-i}$ represents the disjoint union of 2^{n-i} copies of $h(L,L\cap A)$. Note that if L is a set of vertices then we should use $L\times(\pm 1)^{n-1}$ rather then $L\times(\pm 1)^n$ because the construction starts at dimension one, not zero.

(2) If K^i, L^j and A are subcomplexes of some finite simplical complex P^n, k = dim($K\cap L$), then

$h(K,K\cap A)\times(\pm 1)^{n-i} \cup h(L,L\cap A)\times(\pm 1)^{n-j} = h(K\cup L, (K\cup L)\cap A)\times(\pm 1)^{n-\max(i,j)}$

$h(K,K\cap A)\times(\pm 1)^{n-i} \cap h(L,L\cap A)\times(\pm 1)^{n-j} = h(K\cap L, K\cap L\cap A)\times(\pm 1)^{n-k}$

(3) For any $A \subset \Delta^1$, $h(\Delta^1,A) = \Delta^1$. For $n \ge 2$

$h(\Delta^n, \Delta^n) = \Delta^n \times (\pm 1)^{n-1}$

$h(\Delta^n, A) = h(\partial\Delta^n, A) \times [-1,1]$, for any $A \subset \partial\Delta^n$. And $\partial\Delta^n \subset \Delta^n$ induces

$h(\partial\Delta^n, A) \times (\pm 1) \subset h(\partial\Delta^n, A) \times [-1,1]$

Now assume that K^n is finite simplical complex and A is subcomplex such that $\Delta^i \cap A$ is a simplex for any Δ^i in K.

2.3 Lemma. For any $A \subset L \subset K$

$h(L,A) \times (\pm 1)^{n-\dim L} \subset h(K,A)$

is π_1-injective. That means that the inclusion induces injections of fundamental groups at all connected components.

Proof. For a subcomplex P and an integer m we will denote $hPm = h(P, P\cap A) \times (\pm 1)^{m-\dim P}$. Let r be the number of simplices in K that are not in A. First add K^1 to L. Note that $hL\cup K^1 n$ is the union of hLn and an one dimensional complex, so 2.3 is true for them. Therefore we can assume that the dimensions of the simplices in K but not in L are ≥ 2. Reduce the problem to one about $hPn \subset h(P\cup\Delta^i)n$, $A \subset P \subset P\cup\Delta^i \subset K$, where $i \geq 2$, $\Delta^i \not\subset P$, $\partial\Delta^i \subset P$. Write $\dim P = d$, $\max(d,i) = m$. Note that

$h(P\cup\Delta^i, A) = hPm \cup h\Delta^i m$,

$hPm \cap h\Delta^i m = h\partial\Delta^i m$,

$h(\Delta^i, \Delta^i \cap A) = h(\partial\Delta^i, \partial\Delta^i \cap A) \times [-1,1]$.

2.3.1 Lemma. Let X, Y and $X\cap Y = Z$ be compact polyhedra. If $Z \subset X$ and $Z \subset Y$ are π_1-injective then $X \subset X\cup Y$ is π_1-injective.

Proof: Let X_0 be one connected component of X. $X_0 \cap Z = Z_0$. Let Y_0 be the union of those components of Y that have intersections with Z_0. Then the fundamental group of X_0 expands to that of $X_0 \cup Y_0$ by generalized free products and HNN extensions. Let Z_1 be the union of components of Z that are in Y_0 but are not in Z_0. Let X_1 be the union of components of X that intersect Z_1. Consider $X_0\cup Y_0 \subset X_0 \cup Y_0 \cup X_1$. Note that the process terminates at a component of $X\cup Y$.

According to this 2.3.1, and since $h(\partial\Delta^i, \partial\Delta^i \cap A) \times (\pm 1) \subset h(\partial\Delta^i, \partial\Delta^i \cap A) \times [-1,1]$ is π_1-injective, the problem is reduced to $h\partial\Delta^i d \subset h(P,A)$. Since $hAd \cap h\partial\Delta^i d = hA\cap\partial\Delta^i d$, $hAd = A \times (\pm 1)^{d-1}$, $hA\cap\partial\Delta^i d = (A\cap\partial\Delta^i) \times (\pm 1)^{d-1}$, and $\partial\Delta^i \cap A = \Delta^i \cap A$ = simplex, $h\partial\Delta^i d \subset hA\cup\partial\Delta^i d$ is π_1-injective. So the problem is reduced to $hA\cup\partial\Delta^i d \subset h(P,A)$. The number of simplices in P but not in A is $\leq r-1$. This completes the proof of 2.3. We can also see

2.4 Lemma. For $A \subset L \subset K$, $i \geq 2$, $\Delta^i \not\subset L$, $\partial\Delta^i \subset L$, $h(\partial\Delta^i, \partial\Delta^i \cap A) \times (\pm 1)^{\dim L - i + 1} \subset h(L,A)$ is π_1-injective.

Let K be finite simplical complex. Assume that each simplex of K is a simplex with flat geometry such that all these geometric simplices can fit together. Now assume that the geometry of K has nonpositive curvature, whose definition was made in [12], 4.2. A subdivision doesn't change this status. So put K as a subcomplex of a closed PL manifold M. Do a barycentric subdivision to make sure that $\Delta^i \cap K$ is a simplex for any Δ^i in M. The PL geometry on K can easily be extended to one on M. It is known that h(M,K) is closed PL manifold and has nonpositive curvature([5], [12], 3-4. One way of proving h(M,K) has nonpositive curvature is to show that the inclusion in 2.3 is totally geodesic so that everything in the following process of going from K to h(M,K) is totally geodesic(compare [13],9)).

2.5 Denote dimM = n. For any subcomplex P let hP represents $h(P, P\cap K)$ in 2.5.

$$hM = (K\cup M^1)\times(\pm 1)^{n-1}\bigcup_{i\geq 2} h\Delta^i\times(\pm 1)^{n-i}$$

By 2.4 there is the following process of constructing the fundamental group of hM from that of K. $G_1 *_H G_2$ denotes a free product with amalgamation, $G*_H t$ denotes an HNN extension.

K ---> $\pi_1 K$

$K \cup M^1$ ---> $\pi_1 K * Z \ldots * Z$

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$L = K \times (\pm 1)^{i-2} \cup hM^{i-1}$ ---> $\pi_1 L$

Take $\Delta^i \not\subset K$,

$\tilde{L} = L \times (\pm 1) \cup h\Delta^i$ ---> $\pi_1 L *_{\pi_1 h\partial\Delta^i} \pi_1 L$

take another $\bar{\Delta}^i \not\subset K$,

$\tilde{L} \cup h\bar{\Delta}^i$ ---> $\pi_1 \tilde{L} *_{\pi_1 h\partial\bar{\Delta}^i} t$

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$K \times (\pm 1)^{i-1} \cup hM^i$

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hM ---> $\pi_1 hM$

Waldhausen's theorem in [20] says that the following two sequences are exact

$Wh(H) \to Wh(G_1) \oplus Wh(G_2) \to Wh(G_1 *_H G_2)$

$Wh(H) \to Wh(G) \to Wh(G*_H t)$.

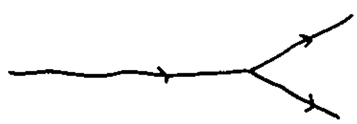
Since $h\partial\Delta^i$ are closed manifolds with PL nonpositive curvature, their Whitehead groups are zero by assumption. Then we get $Wh(\pi_1 K) \subset Wh(\pi_1 hM)$. The latter is zero again by assumption. This proves 2.1.

3. Proof of the manifold case

3.1 Section 3 will prove

3.1.1 Theorem: $Wh\pi_1 M = 0$ for any closed PL manifold M with nonpositive curvature.

M having nonpositive curvature means that each simplex in M is assigned a flat geometry of certain size and that any link of M is larger than or the same as a standard sphere([12], 4.2). The example of plane R^2 can be thought of as the composition of angles at the origin with total sum $\Sigma = 2\pi$. When one inserts more angles, say letting the sum become $\Sigma = 4\pi$, then the metric on R^2 is just the pullback, by $z^2 : R^2 \longrightarrow R^2$, of $ds^2 = dx^2 + dy^2$, which is $d\bar{s}^2 = 4(x^2+y^2)ds^2$, a riemannian metric with singularity. So M is like having a metric with various singularities, which maybe the background of the following geodesic singularity(a geodesic going into different directions), which is our main concern:



Three things are used to overcome this difficulty. They are the geodesic flow G which is the collection of parameterized geodesics, the sphere bundle $\vec{R}$ which is the collection of geodesic rays, and bundle S_T which is the collection of segments of length T.

3.1.2. Note that closed manifolds of almost nonpositive curvature(limits of riemannian manifolds of $K \le 0$, see [11]) are covered by 3.1.1 because their universal covers satisfy the property that the distance function of two geodesics are convex. In fact they are far less comlicated here in that there is no geodesic singularity. So Whitehead groups of the fundamental groups of them must also vanish.

3.2 The bundle S_T

Note that in section 3 we always assume M^n to be closed PL manifold with curvature ≤ 0, X its universal cover. First recall that the geodesic flow of M is G(M) = {all local isometries R ---> M}. There is a metric to G(M) that comes from one to G(X) by defining distance in G(M) to be the minimum of distances between elements in the inverse images in G(X). The metric on G(X) is

$$d(\alpha,\beta) = \int_{-\infty}^{+\infty} d[\alpha(t),\beta(t)] \cdot e^{-|t|}\, dt .$$

3.2.1 Lemma. For geodesics $\alpha(t)$, $\beta(t)$ in M and liftings $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$ in X,

$d(\alpha,\beta) \le d(\tilde{\alpha}, \tilde{\beta})$.

The sphere bundle of M is R(M) = {local isometries $[0,+\infty)$ ---> M}, which is fiber bundle over M with fiber the ideal boundary of X. Denote the ideal boundary as ∂X. It is homeomorphic to S^{n-1}.

3.2.2 Theorem(see[13], §4): The canonical map G(M) ---> R(M) can be approximated by homeomorphisms.

For T > 0, let $S_T(M)$ be the set of all parameterized geodesic segments of length T in M. Its topology is from $S_T(X)$, in which two elements are close if and only if they are close pointwise.

3.2.3 Lemma: $S_T(M)$ ---> M is a fiber bundle.

This assertion is equivalent to that, for a < b, the homeomorphism approximations of $S_b(x)$ ---> $S_a(x)$ can depend continuously on x in X. Davis and Januszkiewicz in [5] proved that $S_b(x)$ ---> $S_a(x)$ can be approximated by homeomorphisms, where $S_r(x)$ is the sphere of radius r with center x in X. In particular, any $S_r(x)$ is homeomorphic to S^{n-1}.

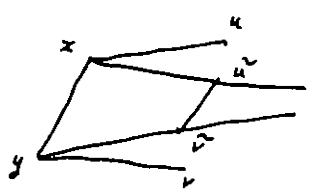
Let A and B be locally compact, separable and metric spaces, f: A ----> B proper and surjective. f being completely regular means that for each y_0 in B and $\varepsilon > 0$, there is neighborhood U for y_0 such that: for each y in U, there is homeomorphism h: $f^{-1}(y)$ ----> $f^{-1}(y_0)$ which is ε-close to Id_A.

3.2.4 Theorem(Dyer-Harmstrom [4]): If f: A ---> B is completely regular, B is locally finite-dimensional, and the point-inverses $f^{-1}(y)$ have locally contractible homeomorphism groups, then f is fiber bundle.

Proof of 3.2.3: We will just show that $S_T(X)$ is fiber bundle. In view of the above criterion one should show $S_T(X)$ ---> X to be completely regular. Note that Homeo(S^{n-1}) being locally contractible is known as a result of the Cernavskii theorem. Give $S_T(X)$ the following metric

$d(\alpha,\beta) = d[\alpha(0), \beta(0)] + d[\alpha(T), \beta(T)]$

Assume that $\varepsilon > 0$, x, y are in X, $d(x,y) \le \varepsilon/4$. Take homeomorphism f_x: ∂X ---> $S_T(x)$ that is $\varepsilon/4$-close to the canonical map. A similar f_y: ∂X ---> $S_T(y)$ is taken. Consider



where $\tilde{u} = (f_x^{-1}u)(T)$, $\tilde{v} = (f_y^{-1}v)(T)$.

$d(x,y) + d(u,v)$

$\le d(x,y) + d(u,\tilde{u}) + d(\tilde{u},\tilde{v}) + d(\tilde{v},v) \le d(x,y) + d(u,\tilde{u}) + d(x,y) + d(\tilde{v},v)$

$\le 4 \cdot \varepsilon/4 = \varepsilon$,

where $d(\tilde{u},\tilde{v}) \le d(x,y)$ is because the distance function of two asymptotic geodesics is a decreasing one. This means that $f_y\ f_x^{-1}$: $S_T(x)$ ---> $S_T(y)$ is ε-close to Id in $S_T(X)$. #

3.2.5 Corollary. There is continuous class of bundle equivalences h_t: R(M) × [0,1) ---> $S_T(M)$ such that h_1 = the canonical map.

Proof. For any ray α(t), t ∈ [0,+∞), one gets segment α(t), t ∈ [0,T]. This is the canonical map, which is cell-like by argument similar to that of 3.2.2. Since $S_T(M)$ is indeed a manifold by 3.2.3, [18] can imply h_t.

3.3 Technical estimates

Let α(t), 0 ≤ t ≤ 1 be curve in M, γ(s), 0 ≤ s < +∞ be geodesic ray with γ(0) = α(0). Assume that the diameter of α is ≤ d, T > 0. Lift α and γ to X to be $\tilde{\alpha}$ and $\tilde{\gamma}$ such that $\tilde{\alpha}(0) = \tilde{\gamma}(0)$. For each t in [0,1], draw the geodesic segment from $\tilde{\alpha}(t)$ to $\tilde{\gamma}(T+d)$. Since the length of this segment is ≥ T+d-d = T, a smaller segment of length T, denoted $\tilde{\alpha}(t)*\tilde{\gamma}$, is available. Map it down to M. The result, written as α(t) *γ, is independent of ways of lifting and is a curve in $S_T(M)$.

Assume that W is h-cobordism over M, p_t, q_t, 0 ≤ t ≤ 1, :W×[0,1] ---> W are deformations of W to M and to another boundary. The lifting of W to R(M) is

$\hat{W} = R(M)\times_M W = \{(\gamma, x) \in R(M)\times W : \gamma(0) = p_1(x)\}$.

Assume that the maximum of the diameters of curves(called associated curves of the h-cobordism) p_1p_tx, p_1q_tx, x ∈ W, is d, which by definition is the diameter of W. Take T > 0. Use d and T to obtain, for any (γ,x) in R(M)×W with γ(0) = $p_1(x)$, a curve $p_1p_tx * \gamma$ in $S_T(M)$. Take the h_t of 3.2.5. Consider

$\{(h_0^{-1}h_t\gamma, x),\ 0 \le t \le 1\} \bigcup \{(h_0^{-1}p_1p_tx * \gamma, p_tx),\ 0 \le t \le 1\} \bigcup \{(h_0^{-1}h_{1-t}\gamma, p_1x),\ 0 \le t \le 1\}$

where notation ⋃ means the three curves are wedged together. This is a curve in $\hat{W}$. Let i: R(M) ---> $\hat{W}$ be the inclusion, j: $\hat{W}$ ---> R(M) be (γ,x) ---> γ. Then ji =Id; Id is homotopic to ij: (γ,x) ---> (γ,p_1x) via the collection of the above expressed curves. Therefore the associated curves in R(M) are

$(h_0^{-1}h_t\gamma, 0 \le t \le 1) \bigcup (h_0^{-1}[p_1p_tx * \gamma],\ 0 \le t \le 1)$

$\bigcup (h_0^{-1}h_{1-t}\gamma,\ 0 \le t \le 1)$, (γ, x) in $R(M)\times_M W$.

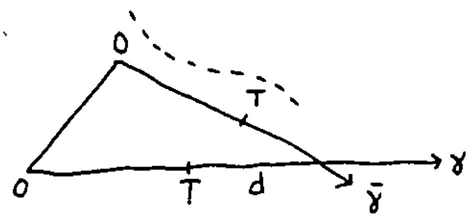
Consider the other boundary in the same way. One sees

3.3.1 Lemma. Let W be h-cobordism over M with diameter d, T > 0. Then there are homotopies(weak deformations) of $\hat{W}$ = R(M)$\times_M$W with its boundaries, and homeomorphism h_0: R(M) ---> $S_T(M)$ such that associated curves of $h_0(\hat{W})$ in $S_T(M)$ are arbitrarily close to the following curves

$p_1p_tx * \gamma,\ 0 \le t \le 1,\ p_1q_tx * \gamma,\ 0 \le t \le 1:\ (\gamma, x) \in R(M)\times_M W$.

We now prepare to change the above curves. For any ε > 0, take h_0, choose homeomorphism g_0: G(M) ---> $S_T(M)$ which is very close to the canonical map denoted f. Consider $g_0^{-1}h_0(\hat{W})$. Any associated curve of it, restricted from G(M) to S_T(M), can be ε-close to a curve of the form α(t)*γ, α ⊂ M, diam(α) ≤ d, γ is in R(M), γ(0) = α(0). Give S_T(X) the metric d(α,β) = d[α(0), β(0)] + d[α(T),β(T)], which is invariant under isometries and induces a metric on S_T(M). Consider any curve V in G(M) such that d(fV,α*γ) ≤ε. There must be lifting $\tilde{fV}$ of fV to S_T(X) such that d($\tilde{fV}$, $\widetilde{\alpha*\gamma}$ = d(fV,α*γ) ≤ε. V and $\tilde{fV}$ determine $\tilde{V}$ which is lifting of V and f$\tilde{V}$ = $\tilde{fV}$. d(f$\tilde{V}$,$\widetilde{\alpha*\gamma}$) ≤ε. With lemma 3.2.1, we can simply consider

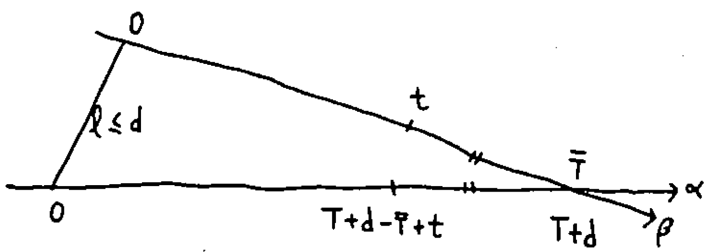
Fixed d > 0, any ε > 0 , T > 0, and a collection Σ(ε,T) of curves in G(X) such that for any of its curve, there is geodesic segment γ[0, T+d] such that for any point in the curve, expressed as geodesic α(t), t ∈ R, there is the following triangle in X



such that d[α(0), $\bar{\gamma}$(0)] ≤ε, d[α(T),$\bar{\gamma}(T)$] ≤ε. The purpose is to make Σ close to leaves of G(X). The following three points are needed.

(1) For any triangle (l,a,b) in X, with $l \le d$, then |a-b| ≤ $l \le d$. By the metric formula in 3.2, segment of length d in X means segment of length 2d in G(X).

(2) Consider two geodesics α(t) and β(t) in X in the following situation



Since X has curvature ≤ 0,

$x(t) \le \frac{T+d-t}{T} 2d$

For t in [(1-2ε)T,T],

$x(t) \le 2d^2/T + 4d\varepsilon$.

Denote $\tau = (1-\varepsilon)T$. We estimate

$$d[\tau\beta, (T+d-\bar{T}+\tau)\alpha] = \int_{-\infty}^{+\infty} d[\beta(\tau+t), \alpha(T+d-\bar{T}+\tau+t)]e^{-|t|}dt = \int_{-\infty}^{-\varepsilon T} + \int_{-\varepsilon T}^{\varepsilon T} + \int_{\varepsilon T}^{+\infty}$$

$$\int_{-\epsilon T}^{\epsilon T} = \int_{-\epsilon T}^{\epsilon T} x(\tau+t)\, e^{-|t|}dt$$

$$\leq 4d^2/T + 8d\varepsilon.$$

$$\int_{-\infty}^{-\epsilon T} d[\beta(\tau+t), \alpha(T+d-\bar{T}+\tau+t)]\, e^{-|t|}dt$$

$$\leq \int_{-\infty}^{-\epsilon T} [2(-\epsilon T-t)+\frac{2d^2}{T}+4d\varepsilon]\, e^{-|t|}dt$$

$$\leq 2e^{-\epsilon T} + \frac{2d^2}{T} + 4d\varepsilon.$$

The same is true for the integration from ϵT to $+\infty$. So

$$d[\tau\beta, (T+d-\bar{T}+\tau)\alpha] \leq 4e^{-\epsilon T} + \frac{8d^2}{T} + 16d\varepsilon.$$

(3) If there are two geodesics $\alpha(t)$ and $\beta(t)$ in X such that $d[\alpha(0),\beta(0)] \leq \varepsilon$, $d[\alpha(T),\beta(T)] \leq \varepsilon$, $\tau = (1-\epsilon)T$, then $d(\tau\alpha, \tau\beta) \leq 4e^{-\tau} + 4\varepsilon$. These three points together imply that $\tau\cdot\Sigma(\varepsilon, T)$ is foliated controlled by the following bound. Note that for a class of curves in an one dimensional foliation we say it is (u,v)-controlled, or its diameter is $\leq$ (u,v), if any curve in the class is in a v-neighborhood of some leaf segment whose length is $\leq$ u.

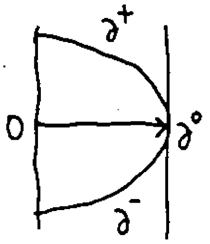
$$(2d, 4e^{-\epsilon T}+\frac{8d^2}{T} + 16d\varepsilon + 4e^{-(1-\epsilon)T} + 4\varepsilon)$$

If $T=1/\varepsilon^2$, ε ---> 0, then the second term goes to zero. This gives

3.3.2 Proposition. Let W be h-cobordism over M with diameter d, $\hat{W}$ be the lifting of W to R(M). Then for any $\delta > 0$ there is homeomorphism $g = \tau g_0^{-1} h_0$: R(M) ---> G(M) such that $g(\hat{W})$ is $(2d, \delta)$-controlled.

3.4 The proof

We now proof theorem 3.1.1, i.e., $Wh\pi_1 M = 0$. It is o.k. to consider $M\times S^1$ instead of M because $Wh\pi_1 M \subset Wh(\pi_1 M\times Z)$. Orient S^1. Then there is natural decomposition $S_T(M\times S^1) = S_T^+ \cup S_T^0 \cup S_T^-$, which comes from a decomposition of $S_T(X\times R)$. Because if we take γ in $S_T(X)$, then $(S_T(\gamma\times R))_O$, the union of which for all γ being $S_T(X\times R)$, has the following natural decomposition



$$\left(S_T(\gamma\times \mathbb{R})\right)_0 = \partial^+ \cup \partial^\circ \cup \partial^-$$

Apparently there are similar decompositions $G(M\times S^1) = G^+ \cup G^0 \cup G^-$, $R(M\times S^1) = R^+ \cup R^0 \cup R^-$.

Any element of $Wh(\pi_1 M\times Z)$ is the Whitehead torsion $\tau(W)$ of an h-cobordism W over $M\times S^1$. Lift W to $\hat{W}$ over $R(M\times S^1)$. $\hat{W} = \hat{W}^+ \cup \hat{W}^0 \cup \hat{W}^-$. $\hat{W}^+ \cup \hat{W}^0$ is h-cobordism over $R^+ \cup R^0$, which is fiber bundle over $M\times S^1$, with disc E^{n+1} as fiber . So $\tau(\hat{W}^+\cup\hat{W}^0) = \tau(W)$. So consider $\hat{W}^+\cup\hat{W}^0$.

To apply 3.3.2 to change this h-cobordism, choose h_t and g_0 there to respect decompositions. A problem is that homotopies of $\hat{W}^+ \cup \hat{W}^0$ constructed at the beginning of 3.3 may go out of $\hat{W}^+ \cup \hat{W}^0$. Take neighborhood $\partial^0 \cup \partial^{-1/2}$ for $\partial^0 \subset \partial^0 \cup \partial^{-1}$. The obvious retraction $\partial^0 \cup \partial^{-1/2}$ ---> ∂^0 induces a map J: $S_T{}^+\cup S_T{}^0 \cup S_T{}^{-1/2}$ ---> $S_T{}^+\cup S_T{}^0$. Now change each $p_1p_t x*\gamma$, $\gamma \in R^+ \cup R^0$, $\gamma(0) = p_1x$, to $J(p_1p_t x*\gamma)$. This gives us a correct homotopy of $\hat{W}^+ \cup \hat{W}^0$. But then we expect $J(p_1p_t x*\gamma)$ to be very close to $p_1p_t x*\gamma$

This is true if projection of $p_1p_t x$ from $M\times S^1$ to S^1 is small. To gain that, take large k in Z, consider $Id\times z^k: M\times S^1$ ---> $M\times S^1$, substitute W by $W_k = (Id\times z^k)*W$. If $\tau(W_k) = 0$ is proved, then $k\cdot\tau(W) = 0$. Take another large l in Z, $(k,l)=1$. As $k\cdot\tau(W) = l\cdot\tau(W) = 0$, $\tau(W) = 0$.

So apply 3.3.2 to take homeomorphism g: $R^+ \cup R^0$ ---> $G^+ \cup G^0$ such that $g(\hat{W}^+ \cup \hat{W}^0)$ is $(2d,\delta)$-controlled, where δ can be arbitrarily small. Now it is better to add a trivial h-cobordism over $G^0 \cup G^-$(see lemma 3.8 of [7]) so that we need only consider an h-cobordism $\tilde{W}$ over G that is $(4d,\delta)$-controlled, where δ can be arbitrarily small. Now we turn to

3.4.1 Theorem. Assume that $m \geq 5$, G^m is manifold and 1-dimensional foliation. $A \subset G$ compact such that any leaf intersecting A has length > l. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that the following is true. For any h-cobordism H over G with diam(H) $\leq (l,\delta)$, there is handlebody structure for H such that there is no handle over A and that the diameter of the handlebody structure is $\leq (C(m)l,\varepsilon)$.

This is an adjustment of [13], 7.6 to the language used by Quinn([16] or [17]), to consider handlebody structures of h-cobordisms directly without having to mention the concept of products which won't be enough latter.

In our case let l = 4d. Let G_{4d} be the union of all closed orbits in G with periods $\leq 4d$. Then $\tilde{W}$ as well as its handlebody structure are (D,ε)-controlled and all handles are over a neighborhood of G_{4d}, where D = C(2n-1)4d depends on d and n only, ε can be arbitrarily small and the neighborhood can be arbitrarily close to G_{4d}. We now want to apply the thin h-cobordism to G_{4d} because the h-cobordism is very close to the circles in G_{4d}. One thing is that G_{4d} is not fibered by S^1 although [8] shows it can be filtered into a stratification of fiber bundles. But we will see that the local situation of G_{4d} is still within the ability of [16].

3.4.2 Definition.

(1) If $\alpha(t)$, $t \in R$ is closed geodesic with period(i.e. minimum

period) u, k ≥ 1, then we can have a map $S^1(ku)$ --> $[0,ku]/0=ku \xrightarrow{\alpha} M\times S^1$. Call this map, together with the orientation and the length of $S^1(ku)$ but dropping the reference point, a k-fold oriented geodesic circle from α. The period of this circle means u not ku.

(2) If $S^1(u)\times[a,b]$ --> $M\times S^1$ is a totally geodesic immersion such that the oriented geodesic circles at (a,b) are all one fold, then call it a primitive move from the one fold version of the circle at a to the one fold version of the circle at b. Call u the period of the primitive move and b-a its perpendicular distance. A move is a combination of several primitive moves. The perpendicular distance of a move means the sum of those of the primitives. A down move is a combination of primitive moves such that the period of any primitive move is equal to that of its beginning circle.

Let B denote the collection of one fold oriented geodesic circles of periods $\leq 4d$. Assume α is closed geodesic in $M\times S^1$. Take a regular neighborhood for it. Note that for any closed geodesic β of period v which is very close to α under the metric of $G(M\times S^1)$, it must be in the regular neighborhood, therefore its one fold oriented geodesic circle is homotopic to a certain fold circle of α. By an elementary application of the fact that the distance function of any two geodesics in the universal cover $X\times R$ is convex, we can produce a totally geodesic immersion $S^1(v)\times[a,b]$ --> $M\times S^1$ such that the circle at a is the one fold circle of β, that at b is a circle of α, and in fact the immersion is a down move. This observation suggests that from the point of view of topology we should divide B into disconnected subsets using the equivalence relation that two elements in B are equivalent if and only if between them there is a move consisting of primitive moves of periods $\leq 4d$. In a component define the distance between two elements to be the lower bound of the perpendicular distances of all the moves between them. This gives a metric to B.

The map f: G_{4d} --> B of taking a closed geodesic to its one fold circle is continuous. If α and β are elements in B , β is very close to α in B, then β must be in a regular neighborhood of $\alpha \subset M\times S^1$. Then we can get a unique down move from β to α. This shows that for any α in B there is r > 0 such that its closed ball E of radius r is contractible. And $f^{-1}(E)$ can be deformed to $f^{-1}\alpha$ which is homeomorphic to S^1. We now want to apply the following 3.4.3 to $\tilde{W}$.

Note that notations in this paragraph and 3.4.3 are independent of the preceding ones. Let M^n be closed manifold, n ≥ 5, X be compact subset in M, W be h-cobordism over M with deformations p_t and q_t, 0 ≤ t ≤1, to $\partial_-W = M$ and to ∂_+W. Recall that for a x ∈ W it has two associated curves p_1p_tx and

$p_1 q_t x$. For $\varepsilon > 0$, X^ε denotes the set of points that are ε-close to some points in X. Let δ be > 0, k ≥ 0. Say that W is (X,δ,k)-controlled if there is

$0 = \delta_0 < \cdots \delta_k < \delta_{k+1} = \delta$

such that the associated curves of $p_1^{-1}(X^{\delta_i})$ are in $X^{\delta_{i+1}}$, $0 \le i \le k$. A handlebody structure of W is over X^ε if all handles are inside $p_1^{-1}(X^\varepsilon)$. Let W_{-1} be the collar part of the handlebody structure with homeomorphism h_t: $M \times [0,1] \to W_{-1}$, $h_0 = Id_M$. Recall that for a x ∈ M its associated curve is $p_1 h_t x$. W_{-1} is (X,δ,k)-controlled if the associated curves of X^{δ_i} are in X^{δ_i+1}. Assume that U is neighborhood of X in M, u_t, $0 \le t \le 1$, is deformation of U to X, B is compact metric space and f: X --> B is continuous map. W is ε-controlled at B if the diameters of the images under fu_1 of the associated curves of W that are inside U are ≤ ε. W_{-1} being ε-controlled at B is understood in a similar way.

3.4.3 Theorem. Assume that X is locally contractible, B is locally 1-connected, and for any point in B and any sufficiently small r > 0 the closed r-ball E satisfies $Wh(\pi_1(f^{-1}E) \times Z^i) = 0$, $i \ge 0$. Then there are $\varepsilon_0 > 0$, $\delta_0 > 0$, and k_0 that depends only on n, such that for any $\varepsilon \le \varepsilon_0$, $\delta \le \delta_0$ and $k \ge k_0$ W is trivial.

This is a slight extension from 2.7, [16] (also see [2]) to treat a subset X of a manifold rather than the whole manifold M itself. The above arrangements are to ensure that controlled handle eliminations can be carried out near X. The reader can now see that §6, [16] works for 3.4.3. X being locally contractible makes sure that there are always neighborhood retractions.

Return to $\tilde{W}$ which is (D,ε)-controlled. When ε is small enough the theorem applies to $\tilde{W} \supset G \supset G_{4d} \xrightarrow{f} B$, in particular $Wh(\pi_1 S^1 \times Z^i) = 0$, $i \ge 0$, so that $\tilde{W}$ is trivial. This proves 3.1.1.

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