Homotopy minimal periods for solvmanifolds maps *†

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Abstract

A natural number m is called the homotopy minimal period of a map $f : X \to X$ if it is a minimal period for every map g homotopic to f. In particular this invariant is stable for small perturbations of f . In this paper we give a complete description of the sets of homotopy minimal periods of a map of compact NR-solvmanifold. This shows that previous results for torus and compact nilmanifold extend to this case. The proof follows the earlier approaches.

1 Introduction

In this paper we study the homotopy minimal periods of self-map $f : X \to X$ i.e. these periods which are also minimal periods for every map g homotopic to f. Since the homotopy minimal period of a manifold map f preserves under a small perturbation, one can say that homotopy minimal periods give an information about rigid dynamics of f. The aim is to give a complete description of the set $\mathrm{HPer}(f)$ of all homotopy minimal periods in terms of the homological information on f.

It has been already done for the case of maps of the circle ([5]), in the second instance maps of two-dimensional torus ($X = T^2$) in [1], next in the work of Jiang and Llibre [19], and Jezierski and Marzantowicz [14] correspondingly, the problem was successfully studied for maps of d-dimensional torus, with arbitrary $d \geq 3$, and compact nilmanifold respectively.

It is done by the usage of the Nielsen theory, which for the torus maps has very nice algebraic description ([4]) and prepossessing geometric properties ([16], [17],[7], [11], [20], [32], [33]). Making use of these geometric properties Jiang and Llibre [19] applied the Anosov and summation formulas

(1) $$N(f) = |L(f)|\,,$$

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$$(2) \qquad N(f^m) \;=\; \sum_{k|m} NP_k(f)\,,$$

to describe the set $\mathrm{HPer}(f)$. Here $N(f)$ is the Nielsen number, $NP_k(f)$ is the k-periodic Nielsen number, and $L(f)$ is the Lefschetz number of f (see also [20]). The proof employs very subtle and difficult combinatorial, and number theory arguments, which carry over the case of a nilmanifold map ([14]).

It is worth of pointing out that the Anosov and summation formulas do not hold for all solvmanifolds (cf. [20]). However Keppelmann and McCord observed that the notion of linearization still is defined and the above formulas remain true for a map of a so called NR-solvmanifold. This class of solvmanifolds contains exponential solvmanifolds i.e. quotient homogenous spaces G/Γ with the covering simply-connected Lie group G for which the exponent map is onto ([20]), thus also nilmanifolds. It allows to formulate an analogous theorem (Theorem 3.1) to the main theorems of [19] and [14] for maps of compact NR-solvmanifolds.

For three-dimensional completely solvable solvmanifolds we give a detailed description of all sets of homotopy minimal periods, as a list (Theorem 4.4). As for the three-dimensional nilpotent manifolds ([15] Corollary 3.9) this classification theorem leads to a kind of Šarkovski type theorem (Corollary 4.6).

In more detail, let $f : X \to X$ be a map of compact exponential solvmanifold X, and A be the integral matrix which corresponds to the given map f by the Fadell-Husseini and Mostow fibration (see [20] or [14] for a definition in the nilponent case, and [20] for a definition for the exponential solvmanifold). Set:

Definition 1.1 *Let A be a sqare matrix with integer entries. Then*

$$T_A \;:=\; \{m \in \mathbb{N};\ \det(\mathrm{I} - A^m) \neq 0\}\,.$$

is said to be the set of algebraic periods of A.

Our main result says (Theorem 3.1):

i) if $N(f) = 0$ then $\mathrm{HPer}(f) = \emptyset$,

ii) if $N(f) \neq 0$ and the sequence $\{N(f^m)\}_1^\infty$ is bounded then $\mathrm{HPer}(f)$ is finite and its cardinality depends on the size of A (=dimension of X) only,

iii) if $\{N(f^m)\}_1^\infty$ is unbounded then T_A is infinite, $\mathrm{HPer}(f) \subset T_A$, the set $T_A \setminus \mathrm{HPer}(f)$ is finite and its cardinality depends on the size of A (= dimension of X) only.

As well as in [14] we have to complete the topological part of the Jiang and Llibre proof, namely exclude from the set $\mathrm{HPer}(f)$ all $n \in \mathbb{N}$ with $NP_n(f) = 0$. To do we prove Theorem 3.2 which says that a self-map f of a compact exponential solvmanifold is homotopic to a map with no points of minimal period n iff $NP_n(f) = 0$. It is a partial extension of the You theorem [32], [33]. The analogous theorem (Theorem B of [14]) was the main point of the proof for the nilmanifold case. Here we prove it adapting the argument of [14].

The paper can be outlined as follows. In Section 2 we remind the information about the topology and Nielsen theory of nil and solvmanifolds presenting and explaining different definitions of the linearization of a map. In Section 3 we show our main theorem. The final discussion and examples are contained in Section 4.

2 Lefschetz and Nielsen numbers for solvmanifold maps

In this section we give an overlook on basic notions and definitions which are necessary to formulate and prove the main result. The part of our presentation concerning nil- and solvmanifolds follows mainly [8, 31]. The most of references are taken from these books.

2.1 Nil- and solvmanifolds

A homogeneous space of a nilpotent or solvable group is to be said a **nilmanifold** or a **solvmanifold**, respectively. Of course, every nilmanifold is also a solvmanifold since obviously every nilpotent group is solvable [8].

Recall that a discrete subgroup $\Gamma \subset G$ of a group Lie G is called a **lattice** it there exists a finite G-invariant measure [28]. A closed subgroup $H \subset G$ is called a uniform subgroup if G/H is compact. Due to the Mostow theorem, a closed subgroup $H \subset G$ of a solvable group G is uniform if and only if there exists a finite G-invariant measure on G/H (cf. [28] Theorem 3.1). Consequently, a discrete subgroup $\Gamma \subset G$ of a solvable group G is a lattice if it is uniform.

It is known [8] that every compact nilmanifold is diffeomorphic to a nilmanifold of the form N/Γ, where N is a simply connected nilpotent Lie group and $\Gamma \subset N$ is a lattice. This is not true for solvmanifolds. For example the Klein Bottle is a compact solvmanifold [1] but it is not of the form G/Γ, for simply connected solvable Lie group G and a lattice $\Gamma \subset G$. If it were it would be parallelizable which is obviously not true.

Definition 2.1 *A solvmanifold of the form G/Γ, where G is simply connected and Γ is a lattice is called special solvmanifold.*

Note that every solvmanifold is finitely covered by a special solvmanifold [2, 8]. This implies that solvmanifolds are aspherical, that is $\pi_i(X) = 0$ for $i \geq 2$ for any solvmanifold X. This is due to the fact that simply connected solvable Lie groups are diffeomorphic to Euclidean spaces.

[1] To see this, just represent the Klein Bottle as the set of all straight lines on the plane modulo the relation which identifies two parallel lines whose distance one from the other is an integer. On this set acts transitively the group of rigid motions of the plane which is solvable.

Let $\mathcal{G} \simeq T_e G$ be the Lie algebra of a given Lie group G. Now we designate a few classes of solvmanifolds that distinguish with respect to the spectrum of adjoint operator $\mathrm{ad}_X : \mathcal{G} \to \mathcal{G}$ (cf. [8]). Since some authors define these classes in terms of the operator $\mathrm{Ad}_X : G \to \mathcal{G}$, (cf. [20]) we recall the following commutative diagram which allows to show that these two definitions are equivalent

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\mathrm{ad}} & \mathrm{End}\,(T_e) \\ {\scriptstyle \exp}\big\downarrow & & {\scriptstyle \exp}\big\downarrow \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}\,(T_e) \end{array}$$

2.1.1 Imaginary or of the type (I)

Let $\mathcal{G}$ denote the Lie algebra of a solvable Lie group G. If for any $X \in \mathcal{G}$ all eigenvalues of the operator ad_X are pure imaginary then the group G and its Lie algebra $\mathcal{G}$ as well as every solvmanifold G/Γ is called imaginary or of the type (I). Notice that every nilmanifold is imaginary because in this case all eigenvalues are zero. On the other hand, if Γ is a lattice in an imaginary solvable Lie group G, then it is virtually nilpotent (i.e. it contains a nilpotent subgroup of finite index) [3].

Example 2.2 *Let $G = \mathbb{R} \times_\phi \mathbb{R}^2$ be a skew product, where $\phi : \mathbb{R} \to \mathrm{Aut}\,(\mathbb{R}^2)$ is given by*

$$\phi(t) = e^{\frac{\pi i t}{2}}.$$

It is easy to see that $\Gamma = \mathbb{Z} \times_\phi \mathbb{Z}^2$ is a lattice in G and $\mathbb{Z}^3 \subset \Gamma$ is a subgroup of index 4.

2.1.2 Completely solvable

If for any $X \in \mathcal{G}$ all eigenvalues of the operator ad_X are real then the group G and the Lie algebra $\mathcal{G}$ as well as every solvmanifold G/Γ is called completely solvable. Note that every nilmanifold belongs to this class.

The above remark says that every nilmanifold is completely solvable. The main property of completely solvable groups is the following **rigidity of lattices** [31, 29] property:

Theorem 2.3 *Let G and G' be simply connected completely solvable Lie groups and $\Gamma \subset G$ be a lattice. Then every homomorphism $f_* : \Gamma \to G'$ can be extended to a homomorphism $F : G \to G'$.* □

For our purposes, there will be relevant the following immediate consequence of Theorem 2.3 and the fact that G/Γ is a $K(\pi, 1)$-space with $\pi = \Gamma$.

Corollary 2.4 *Every continuous map $G_1/\Gamma_1 \to G_2/\Gamma_2$ between compact special completely solvable solvmanifolds is homotopic to the map induced by a homomorphism $G_1 \to G_2$.* □

An example of completely solvable Lie group G of dimension 3 which is neither abelian nor nilpotent is given in Proposition 4.1. There is infinitely many not isomorphic completely solvable compact solvmanifold of dimension 3 which are not diffeomorphic to torus or a nilmanifold (Example 4.7).

2.1.3 Exponential

If for any $X \in \mathcal{G}$ there is no pure imaginary eigenvalue of the operator ad_X then the group G and the Lie algebra $\mathcal{G}$ as well as every solvmanifold G/Γ is called exponential. This is equivalent to the fact that the exponential map $exp : \mathcal{G} \to G$ is a diffeomorphism, provided that G is simply connected. Notice that every completely solvable group is exponential. [2].

2.1.4 NR-manifolds

NR-solvmanifolds were introduced by Keppelmann and McCord, see [20] for the definition. They contain the class of special exponential solvmanifolds. The main property of NR-solvmanifolds is that the Anosov theorem can be generalized to them.

Let $N(f)$ and $L(f)$ denote the Nielsen and Lefschetz numbers of the map f, respectively. The Nielsen number gives an estimate of the number of fixed points of f but it is hard to derive in comparison with the Lefschetz number. Both numbers are homotopical invariants. We shall use in sequel the extension of the Anosov theorem for NR-solvmanifold thus also exponential and completely solvable solvmanifold due to Keppelmann and McCord [20].

Theorem 2.5 *If $f : X \to X$ is a self map of a compact NR-solvmanifold, $X = G/\Gamma$ then $N(f) = |L(f)|$.*

□

2.2 Linearization

Having proved the Anosov theorem, we would like to know the method which allows to compute the Lefschetz number for a self-map of manifolds discussed above. By definition, the Lefschetz formula is equal to the alternating sum of traces of the maps induced by f on (co-) homology:

$$L(f) = \sum_{k=0}^{k=d} (-1)^k \operatorname{tr} H^k(f)$$

The aim of this part is to give an effective method of computing the Lefschetz numbers of iterations of the mapping f between solvmanifolds in terms of products of determinants of matrices naturally induced by f. We start with the following standard observation.

[2] In fact, if a Lie group is exponential then it is solvable [31].

Example 2.6 Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a self map of a torus. Then its cohomology is isomorphic to the exterior algebra on n generators. Thus we have that $H^k(f) = \Lambda^k H^1(f)$. In other words, all maps on cohomology are determined by the map of the first cohomology. By a direct computation we obtain that

$$L(f) = \sum_{k=0}^{k=d}(-1)^k \mathrm{tr}\, H^*(f) = \sum_{k=0}^{k=d}(-1)^k \mathrm{tr}\,(\Lambda^k H^1(f)) = \det(\mathbb{I} - H^1(f)).$$

Abbreviating $H^1(f) = A$, where A denotes an $n \times n$-matrix, we get also the formula for the Lefschetz numbers of the iteration of f:

$$L(f^m) = \det(\mathbb{I} - A^m).$$

2.2.1 Linearization for maps of completely solvable solvmanifolds

Now we want to give an analogous formula to the one presented in the above example for completely solvable solvmanifolds. We start with the following result of Hattori [10] that generalized a previous result of Nomizu [27].

Theorem 2.7 *Let $(\Lambda^* \mathcal{G}^*, \delta)$ denotes the Chevalley-Eilenberg complex associated to the Lie algebra $\mathcal{G}$ of a simply connected completely solvable Lie group G. If $\Gamma \subset G$ is a lattice, then*

$$H^*(G/\Gamma; \mathbb{R}) \cong H^*(\Lambda^* \mathcal{G}^*, \delta).$$

□

This result together with the Hopf lemma leads to the following

Proposition 2.8 *Let $f : G/\Gamma \to G/\Gamma$ be a self map of a compact special completely solvable solvmanifold of dimension d. Then there exists an $d \times d$ integral matrix A, called linearization of f, such that for every $m \in \mathbb{N}$*

$$L(f^m) = \det(\mathbb{I} - A^m).$$

Proof: Let $f_* : \Gamma \to \Gamma$ be a homomorphism induced by f on the fundamental group of G/Γ. Then, according to Corollary 2.4, it can be extended to a homomorphism $F : G \to G$. Let A denote a matrix representing a map $F^* : \mathcal{G}^* \to \mathcal{G}^*$ induced by F, i.e $DF(e)$. Then we have

$$L(f^m) = \sum_{k=0}^{k=d}(-1)^k \mathrm{tr}\, H^k(f^m) = \sum_{k=0}^{k=d}(-1)^k \mathrm{tr}\, \Lambda^k (F^m)^* = \det(\mathbb{I} - A^m),$$

where the maps $\Lambda^k(F^m)^* : \Lambda^k \mathcal{G}^* \to \Lambda^k \mathcal{G}^*$ are induced by f^* on the Chevalley-Eilenberg complex of $\mathcal{G}$. The second equality follows from the Hopf lemma. □

Remark 2.9 *For a self-map f of a compact nilmanifold $X = G/\Gamma$ the linearization matrix can be also obtained by an inductive procedure, the central tower of G, and the fibre maps technics as in [7] (see [14], or [20] for detailed description of it). In general using that construction way we get another integral matrix than that of 2.8 but with the same property $L(f^m) = \det(\mathbb{I} - A^m)$. To get the Fadell-Husseini linearization we have to reduce the above matrix $A = DF(e)$ taking only its blocks near the diagonal that correspond to the filtration $\mathcal{G}_i := [\mathcal{G}_i, \mathcal{G}]$ of a nilpotent algebra $\mathcal{G}$ of G.*

2.2.2 Linearization for maps of NR-solvmanifold and the Mostow fibration.

In this subsection we describe a construction of the linearization matrix for a map of an NR-solvmanifold. It is essentially that of [20] but we make a use of "the differential" representative of this matrix given in 2.8.
Let $N \subset G$ denotes **nilradical**, i.e. the greatest nilpotent connected normal subgroup of G. The following theorem of Mostow ([27]) implies that every solvmanifold admits a fibration over torus with the fibre being a nilmanifold.

Theorem 2.10 *Let $\Gamma \subset G$ be a lattice in a solvable Lie group G. Then $\Gamma \cap N$ is a lattice in the nilradical N.* □

The fact that $\Gamma \cap N$ is a lattice in N is equivalent to closeness of ΓN in G. This, in turn, implies that there exists a smooth locally trivial fibration

$$\begin{array}{ccccc} \Gamma N/\Gamma & \hookrightarrow & G/\Gamma & \rightarrow & G/\Gamma N \\ || & & || & & || \\ N/\Gamma \cap N & \hookrightarrow & G/\Gamma & \rightarrow & (G/N)/(\Gamma/\Gamma \cap N), \end{array}$$

where $\Gamma/\Gamma \cap N \subset (G/N)$ is a lattice in the abelian group. Notice that the above statement is true for any analytic nilpotent subgroup $N \subset G$ containing $[G, G]$ (see [25]). These fibrations are usually called the **Mostow fibrations**. Now we are able to extend the result of Proposition 2.8 to the general case. First we recall the following result of McCord [25].

Theorem 2.11 *Let $f : G/\Gamma \to G/\Gamma$ be a continuous self map of solvmanifold. Then f is homotopic to a fiber-preserving map of any Mostow fibration associated with G/Γ, provided that $f_*(\Gamma \cap N) \subset \Gamma \cap N$.* □

It is the most convenient to choose the commutator subgroup $[G, G]$ as a nilpotent subgroup defining the Mostow fibration. Then the assumption of the above theorem is trivially satisfied. Notice also, that any map preserving the Mostow fibration is homotopic to the map which is linear on the base i.e. induced by a linear map of $\mathbb{R}^t$, $t = \dim T$, which preserves $\mathbb{Z}^t$.

Theorem 2.12 *Let $f : X \to X$ be a continuous map of an d-dimensional compact NR-solvmanifold $X = G/\Gamma$. Then there exists a $d \times d$ integral matrix A such that for every $m \in \mathbb{N}$*

$$N(f^m) = |L(f^m)| = |\det(\mathbb{I} - A^m)|\,.$$

Moreover, if X is special then this matrix can be derived from the map induced by f on the fundamental group in the following way.

Let $f_* : \pi_1 = \Gamma \to \Gamma$ be the map induced on the fundamental group. Denote by $f_1 : [\Gamma, \Gamma] \to [\Gamma, \Gamma]$ its restriction and $\overline{f}_* : (\Gamma/[\Gamma, \Gamma] = \mathbb{Z}^t) \to (\Gamma/[\Gamma, \Gamma] = \mathbb{Z}^t)$ the induced map. Then $A = A_{\mathbb{T}} \oplus A_0$, where $A_{\mathbb{T}}$ is the matrix of homomorphism $\overline{f}_* : \mathbb{Z}^t \to \mathbb{Z}^t$. A_0 is equal to the derivative matrix $D\phi(e) : [\mathcal{G}, \mathcal{G}] \to [\mathcal{G}, \mathcal{G}]$ where $\phi : [G, G] \to [G, G]$ is a (unique) extension of the homomorphism f_1 given by Theorem 2.3.

Proof: Using the above argument we can assume that a self map $f : X = G/\Gamma \to X$ of a NR**-solvmanifold** X preserves the Mostow fibration and is a linear map on the base with a finite number of fixed points. Let's introduce the notation as in the following diagram.

$$\begin{tikzcd}
F_b \arrow[r] \arrow[d, "f_b"] & X \arrow[r] \arrow[d, "f"] & \mathbb{T} \arrow[d, "f_{\mathbb{T}}"] \\
F_b \arrow[r] & X \arrow[r] & \mathbb{T}
\end{tikzcd}$$

The fixed points of f are in the fibres over the fixed points of the map $f_{\mathbb{T}}$ induced on the base. Let's compute the absolute value of the Lefschetz number of f.

$$\begin{aligned}
|L(f)| &= |\sum_{b \in Fix(f_{\mathbb{T}})} L(f_b)| = |\sum_{b \in Fix(f_{\mathbb{T}})} \det(\mathbb{I} - A_b)| = \\
&= |L(f_{\mathbb{T}})|\,|\det(\mathbb{I} - A_0)| = |\det(\mathbb{I} - A_{\mathbb{T}}) \det(\mathbb{I} - A_0)|\,.
\end{aligned}$$

Here A_b denotes the matrix for the map of the fibre $f_b : F_b \to F_b$ over the point $b \in Fix(f_{\mathbb{T}})$. The first equality follows from the fact that the indices of the fixed points of the base map $f_{\mathbb{T}}$ (induced by a linear one) are all either $+1$ or all are -1 and from the Fixed Point Index Product Formula. The second one follows from Proposition 2.8. The third is the result of Keppelmann and McCord [20], which states that for every fixed point $b \in \mathbb{T}$ there exists an integral matrix A_0 such that $\det(\mathbb{I} - A_b) = \det(\mathbb{I} - A_0)$ provided that the solvmanifold X is an NR-solvmanifold.

If the solvmanifold X is special then to derive the matrices $A_{\mathbb{T}}$ and A_0 it sufficient to know the map $f_* : \Gamma \to \Gamma$ induced by f on the fundamental group. Indeed, if f_* is given then f_* restricted to the commutator subgroup $[\Gamma, \Gamma]$ defines the matrix A_0 given by Proposition 2.8, as follows from Theorem 2.3 and the fact that $|\det(\mathbb{I} - A_0)| = |L(f_b)|$. The matrix $A_{\mathbb{T}}$ is simply given by the map induced by the homomorphism f_* on the free abelian quotient group $\Gamma/[\Gamma,\Gamma]$. Moreover, the above formula holds also for every iteration of the map f.

Finally, we define a $d\times d$ integral matrix A by

$$A := \begin{bmatrix} A_0 & 0 \\ 0 & A_{\mathbb{T}} \end{bmatrix}$$

which, according to Theorem 2.5, finishes the proof. $\square$

2.2.3 Nielsen type numbers

We need the notion of the Nielsen type number of period n, denoted by $NP_n(f)$ of a self-map $f: X \to X$ (cf. [16] Definition III 4.7). It may be defined as $NP_n(f) := n \cdot O_n(f)$ where $O_n(f)$ is the number of irreducible essential periodic point orbits of f^n ([18] def. 2.1). Note that $NP_1(f) = N(f)$ and $NP_n(f) \le \#P_n(g)$ for every $g \simeq f$ where $\#S$ denotes the cardinality of a set S.

At the end we remind another notion of the homotopy theory of periodic points of a self-map $f: X \to X$ namely the Nielsen number $NF_n(f)$ also introduced in [16]. The main property of this homotopy invariant is the estimate

$$N(f^n) \ \le\ NF_n(f) \ \le\ \#\mathrm{Fix}\,(\mathrm{f^n})\,,$$

and the Halpern conjecture states that $NF_n(f)$ is the best homotopy invariant $\le \#\mathrm{Fix}\,(\mathrm{f^n})$ (cf. [16, 12, 13]).

In general there is not true that $NF_n(f) = N(f^n)$. In a naive approach, e.g. for a selfmap of a finite set, the Nielsen number $N(f^n)$ splits into the sum $NF_n(f) = N(f^n) = \sum_{k|n} NP_k(f)$ of k-periodic Nielsen numbers of f. As a matter of fact such a decomposition holds for NR-solvmanifolds, thus for compact nilmanifolds and exponential solvmanifolds as it was showed by Heath and Keppelmann ([11] Theorems 1.2 and 4.6).

Theorem 2.13 *Let $f: X \to X$ be a self-map of a compact NR-solvmanifold. If $N(f^n) \neq 0$, then*

$$N(f^n) \ =\ NF_n(f) = \sum_{k|n} NP_k(f) \qquad \textit{and consequently} \qquad NP_n(f) \ =\ \sum_{k|n} \mu(n)\, N(f^{n/k}),$$

where μ is the classical Möbius function. $\square$

Note that it is enough to show only the first equality, because the second is the Möbius inverse formula for two arithmetic functions satisfying the first. In the case of selfmap of a torus the above formula had been proved already by Halpern ([9]).

At the end we formulate a property of solvmanifolds we shall use in next.

Proposition 2.14 *Let H be a connected subgroup of connected simply connected solvable group G. Then H is also simply connected.*

Proof. The space G/H is aspherical (i.e. $\pi_i(G/H) = 0$ for $i \ge 2$) as a solvmanifold [8]. Then the statement follows from the long exact sequence of homotopy groups of the fibration $H \to G \to G/H$. $\square$

3 The Main Theorem

Theorem 3.1 *Let $f : X \to X$ be a map of a compact special NR-solvmanifold X of dimension d, A a linearization $d \times d$ integral matrix associated with f and $T_A \subset \mathbb{N}$ the set of algebraic periods of A as in Definition 1.1.*

Then $\mathrm{HPer(f)} \subset \mathrm{T_A}$ *and it is in one of the following three (mutually exclusive) types*

(E) $\mathrm{HPer}(f)$ *is empty if and only if* $N(f) = L(f) = 0$;

(F) $\mathrm{HPer}(f)$ *is nonempty but finite if and only if all eigenvalues of A are either zero or roots of unity;*

(G) $\mathrm{HPer}(f)$ *is infinite and* $T_A \setminus \mathrm{HPer}(f)$ *is finite.*

Moreover, for every dimension d of X, there are finite sets $P(d)$, $Q(d)$ of integers such that $\mathrm{HPer}(f) \subset P(d)$ *in Type F and* $T_A \setminus \mathrm{HPer}(f) \subset Q(d)$ *in Type G.*

We remark that the letters E, F, and G are chosen to represent "empty", "finite" and "generic" case respectively.

Proof. To show statement it is enough to repeat all the combinatorial and number theory arguments of the proof of the corresponding theorem for the torus [19] or nilmanifold [14]. This can be done according to 2.13 and Theorems 2.5, 2.12. As in [14] we shall complete the proof if we prove the following counter-partner of Theorem B of [14]. □

Theorem 3.2 *Let $f : X \to X$ be a selfmap of a compact special NR-solvmanifold $X = G/\Gamma$ of dimension $d \geq 3$. If $NP_n(f) = 0$ then f is homotopic to a map $f' : X \to X$ such that $P_n(g) = \emptyset$.*

Proof. We divide the proof into two steps:

Step 1: for compact special completely solvable solvmanifolds of dimension at least 3

We repeat all arguments of the corresponding Theorem B of [14] emphasizing only these points at which the argumentation can not be literally taken from [14].

The first longer part of the proof of Theorem B in [14] contains a proof of Theorem B for the special case of a local homeomorphism f of a manifold of dimension ≥ 3. It is of general character i.e does not use the nilmanifold structure and the essential is additional condition on f (Ass. **A1-A3** of [14]). Consequently it is enough to repeat the second part of it i.e. to show that in the case of a map of completely solvable solvmanifold the general case reduce to the special case by a homotopy argument.

By Theorem 2.3 every homotopy class of selfmaps of a compact special completely solvable solvmanifold $X = G/\Gamma$ is represented by a homomorphism $F : G \to G$, with $F(\Gamma) \subset \Gamma$. By the above we can assume that our map $f : X \to X$ is induced by the homomorphism F. First we consider the case F is an isomorphism. We will show that f is homotopic to a map satisfying conditions **A1-3** from [14] i.e.

A1 *f is a finite covering*
A2 $\mathrm{Fix}\,(f^n)$ *is finite*
A3 *If $f^k(x_0) = x_0$ $(k|n,\ x_0 \in X)$ then there exists a Euclidean neighbourhood $V \ni x_0$ and its open subset V' such that: $x_0 \in V' \subset V$, $f^k(V') \subset V$ and $f^k : V' \to V = \mathbb{R}^m$ is the restriction of a linear map.*

Then our theorem will follow from Theorem 3.1 in [14]. If X is a nilmanifold then **A1-3** follow from [14]. Otherwise $f : X \to X$ is a fibre map with the respect to Mostow fibration i.e. the diagram

$$\begin{tikzcd}
X_b \arrow[r] \arrow[d, "f_b"] & X \arrow[r] \arrow[d, "f"] & \mathbb{T} \arrow[d, "\bar{f}"] \\
X_b \arrow[r] & X \arrow[r] & \mathbb{T}
\end{tikzcd}$$

commutes where X_b is a nilmanifold. Moreover the map of base spaces and the restriction to the fibres are also finite coverings.

The selfmap of the torus $\bar{f}$ is homotopic to a map satisfying **A1-3**. The homotopy lifts to the homotopy of X and the restrictions to the fibres are still finite coverings. Let us fix a point $b \in P_k(\bar{f}) \subset \mathrm{Fix}\,(\bar{f}^n)$ (the last set is finite by **A2**). Now the map $(f^k)_{|b} : X_b \to X_b$ is homotopic to a map satisfying **A1-3** i.e. there is a homotopy h'_t from $h'_0 = (f^k)_b$ to a map h'_1 satisfying **A1-3**. We will show that this yields a homotopy $h_t : X_b \to X_{\bar{f}b}$ satisfying $h_0 = f_b$, $f^{k-1}h_t = h'_t$. In fact the diagram

$$\begin{tikzcd}
& X_{\bar{f}(b)} \arrow[d, "f"] \\
& X_{\bar{f}^2(b)} \arrow[d, "f"] \\
& \vdots \arrow[d, "f"] \\
X_b \arrow[uuur, "f"] \arrow[r, "f_b^k"'] & X_{\bar{f}^k(b)=b}
\end{tikzcd}$$

commutes and the vertical maps are coverings hence the homotopy h'_t starting from $h'_0 = (f^k)_b$ lifts to a homotopy $h_t : X_b \to X_{\bar{f}b}$ satisfying $f^{k-1}h_t = h'_t$. This can be done to all orbits in $\mathrm{Fix}(f^n)$ simultaneously. The obtained homotopies h_t of fibres can be extended to a fibre homotopy $f_t : X \to X$ satisfying $pf_t = \bar{f}$, the restriction to the fibres are still coverings and $(f_1^k)_b : X_b \to X_b$ satisfies **A1-3** for $b \in P_k(\bar{f})$. Now by Corollary 3.2 in [14] f is homotopic to a map f' with $P_n(f') = \emptyset$.

To prove the general case (the assumption that F is an isomorphism is dropped) we follow the second part of the proof of Theorem B at the end section 3 in [14]. In the inductive argument we need only to know that the homomorphism image of a connected, simple connected, completely solvable group into a connected, simple connected, completely solvable group is a group of the same property. It follows from the definition of completely solvable group and Proposition 2.14. The proof of Step 1 is complete.

Step 2: for compact special NR-solvmanifold of dimension at least 4

This part follows easily from the Wecken theorem for periodic points. The latter, called also the Halpern conjecture, was proved recently in [13] after a partial result of [12]. The only properties that are necessary to carry the proof are the summation formula 2 of [11] and the Anosov theorem [20]. We begin with a formulation of the Wecken theorem.

Theorem 3.3 *Let $f : X \to X$ be a map of connected PL-manifold of dimension $d \geq 4$, and $n \in \mathbb{N}$ a natural number. Then f is homotopic to a map $g : X \to X$ such that*

$$\#\mathrm{Fix}\,(g) \;=\; NF_n(f)\,,$$

where $NF_n(f)$ is another periodic Nielsen number that estimate $\#\mathrm{Fix}\,(f^n)$ from below (see [16],[11] for a definition). □

Assume that $NP_n(f) = 0$. To show that there exists a map $g \simeq f$ with $P_n(g) = \emptyset$ it is sufficient to find a map $g : X \to X,\;\; g \simeq f$ such that $\#\mathrm{Fix}\,(g^n) \;=\; \#\mathrm{Fix}\,(g^{n/p})$ for any prime $p|n$. Since $f : X \to X$ is a map of an NR-solvmanifold, from Theorem 1.2 of 2.13 it follows that

$$NF_n(f) = N(f^n) \;\text{ and }\; NP_n(f) \;=\; N(f^n) - N(f^{n/p})$$

by the Möbius summation formula. By assumption we have $N(f^n) \;=\; N(f^{n/p})$. Let $g \simeq f$ be the selfmap of X given by Theorem 3.3 i.e with $\#\mathrm{Fix}\,(g^n) = N(f^n)$. Of course $N(g^{n/p}) \;\leq\; \#\mathrm{Fix}\,(g^{n/p})$. Consequently we get

$$0 \;\leq\; \#\mathrm{Fix}\,(g^n) \;-\; \#\mathrm{Fix}\,(g^{n/p}) \;\leq\; \#\mathrm{Fix}\,(g^n) \;-\; N(f^{n/p}) \;=\; N(f^n) \;-\; N(f^{n/p}) \;=\; 0\,.$$

This proves the statement under the assumption $d \geq 4$.

Finally, notice that the only special solvmanifolds of dimension 3 are completely solvable [31]. This observation completes the proof of Theorem 3.2. □

4 Maps of completely solvable solvmanifolds of dimension 3.

In this section we give a complete description of all possible sets of homotopy minimal periods of maps of compact special completely solvable solvmanifolds of dimension three which are not isomorphic neither to the torus not to any other nilmanifold. It is a complement to previous descriptions given for the case of three torus ([19]) and the three-dimensional nilmanifold ([15]). We follow the approach of [15] giving a description of the all possible forms of linearization matrices, which we derive from the form of the homomorphism induced by a map on the fundamental group. Moreover the conditions on such a matrix are condition on a linear map to be a homomorphism of the Lie algebra corresponding to the simply connected solvable group covering such a manifold.

We begin with a brief summary of the information on three dimensional Lie groups and Lie algebras which is necessary for our consideration. All this material is contained in [21](Chapter I.6). As follows from the Cartan theorem, there is one-one correspondence between Lie algebras and connected simply connected Lie groups. Moreover any connected Lie group G is of the form G_0/Δ where G_0 is the simply connected group corresponding to a given Lie algebra $\mathcal{G}$ and Δ is a discrete normal subgroup.

It follows from the classification of Lie algebras of dimension 3 that all solvable 3-dimensional Lie algebras are of the following form:

- Abelian i.e. $[\mathcal{X},\mathcal{Y}]=0 \;\; \forall\, \mathcal{X},\, \mathcal{Y} \in \mathcal{G}$. It corresponds to $G=\mathbb{R}^3$ with the structure of linear space.

- There exists linearly independent vectors $\mathcal{X},\, Y,\, \mathcal{Z} \in \mathcal{G}$ such that $[\mathcal{X},\mathcal{Y}]=\mathcal{Z},\; [\mathcal{X},\mathcal{Z}]=[\mathcal{Y},\mathcal{Z}]=0$. The algebra is nilpotent and corresponds to the Lie group of matrices

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},$$

  where $x,\, y,\, z \in \mathbb{R}$. We have $[\mathcal{G},\mathcal{G}]=\langle \mathcal{Z}\rangle$, thus $\dim[\mathcal{G},\mathcal{G}]=1$ here.

- The last case consists of an infinite series of not isomorphic solvable Lie algebras for which $\dim[\mathcal{G},\mathcal{G}]=2$. Let $\dim[\mathcal{G},\mathcal{G}]=2$ and $\mathcal{Y},\, \mathcal{Z}$ be its generators. Any choice for $[\mathcal{X},\mathcal{Y}]=a_{11}\mathcal{Y}+a_{12}\mathcal{Z}$, $[\mathcal{X},\mathcal{Z}]=a_{12}\mathcal{Y}+a_{22}\mathcal{Z}$, or in other words any choice of 2×2 matrix $\mathcal{A}\in \mathrm{Aut}\,(\mathbb{R}^2)$ corresponding to the operator $\mathrm{ad}_{\mathcal{X}} : \mathcal{G}\to\mathcal{G}$ defines a solvable Lie algebra by the above formula. Moreover two matrices $\mathcal{A}$ $\mathcal{B}$ gives two isomorphic algebras iff there exists $0\neq c\in\mathbb{R}$ and $\mathcal{C}\in \mathrm{Aut}\,(\mathbb{R}^2)$ such that $c\mathcal{A}=\mathcal{C}\mathcal{B}\mathcal{C}^{-1}$. From it follows that $\mathcal{G}$ is determined uniquely by the the lines joining the origin in $\mathbb{C}$ with eigenvalues of $\mathcal{A}$. Every such algebra corresponds to a simply connected solvable Lie group of the form

$$G \;:=\; \mathbb{R}\times_{\phi(t)}\mathbb{R}^2\,,$$

  where $\phi(t):\mathbb{R}\,\to\,\mathrm{Aut}\,(\mathbb{R}^2)$ is one-parameter group given by

$$t\;\mapsto\; e^{t\mathcal{A}}\,.$$

  Furthermore if $\mathcal{A}$ has not purely imaginary eigenvalues then $Z(G)=\{e\}$ for such Lie group.

Consequently we have the following.

Proposition 4.1 *There is only one, up to isomorphism, completely solvable simply connected Lie group admiting latices. Its Lie algebra corresponds is defined by the matrix*

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

or in other words defined by generators $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ *and commutators*

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{Y}, \quad [\mathcal{X}, \mathcal{Z}] = -\mathcal{Z}, \quad [\mathcal{Y}, \mathcal{Z}] = 0.$$

Furthermore this group has the following form

$$G := \mathbb{R} \times_{\phi(t)} \mathbb{R}^2, \quad \textit{where} \quad \phi(t) := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Moreover every three dimensional completely solvable solvmanifold is special, consequently it is, up to diffeomorphism, of the form G/Γ *where* $\Gamma \subset G$ *is a lattice.*

Proof. It is easy to see that the group G is the only completely solvable group which is unimodular. Hence the only group which can admit a latice. Examples of latices in G are given at the end of this section (Example 4.7). The last statement of the proposition is shown in [4]. □

To shorten notation, we denote by G the 3-dimensional simply connected completely solvable group of Proposition 4.1, and by $\mathcal{G}$ its Lie algebra. Now we apply Proposition 2.8 to give a description of the linearizations thus Lefschetz and Nielsen numbers of maps of 3-dimensional completely solvable solvmanifolds. The following lemma can be derived from the above relations by a direct computation.

Lemma 4.2 *Let* $A : \mathcal{G} \to \mathcal{G}$ *be any endomorphism of Lie algebra 4.1. Then it has the following form with respect to the basis* $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$

$$A = \begin{bmatrix} a & 0 & 0 \\ b & r & s \\ c & u & v \end{bmatrix},$$

where the coefficients satisfy the following conditions: either $r = v = s = u = 0$ *and* $a \in \mathbb{Z}$ *is an arbitrary integer,*
or there exists a coefficient r, u, s, v *different from* 0 *and* $a \in \{-1, 1\}$ *then. Moreover we have:*

1. *if* $a = -1$ *then* $r = v = 0$;
2. *if* $a = \ 1$ *then* $s = u = 0$.

Proof. Let $A(\mathcal{X}) = a\mathcal{X} + b\mathcal{Y} + cZ$. Since $[\mathcal{G},\, \mathcal{G}] = \langle \mathcal{Y},\, \mathcal{Z} \rangle$ and $A([\mathcal{G},\, \mathcal{G}]) \subset [\mathcal{G},\, \mathcal{G}]$ we have $A(\mathcal{Y}) = r\mathcal{Y} + u\mathcal{Z},\ A(\mathcal{Z}) = s\mathcal{Y} + v\mathcal{Z}$.

Using the relations $[\mathcal{X}, \mathcal{Y}] = \mathcal{Y}$, $[\mathcal{X}, \mathcal{Z}] = -\mathcal{Z}$ and the fact that A is an endomorphism of the Lie algebra we get the conditions 1–3. E.g. if $A(\mathcal{X}) = b\mathcal{Y} + c\mathcal{Z}$, then $A(\mathcal{Y}) = A([\mathcal{X}, \mathcal{Y}] = [A(\mathcal{X}),\, A(\mathcal{Y})] = 0$, analogously $A(\mathcal{Z}) = 0$. □

Lemma 4.2 allows us to derive the Lefschetz and Nielsen numbers of the mapping of 3-dimensional completely solvable solvmanifold which is not a nilmanifold. We summarize this in the following

Proposition 4.3 *Let $f\, :\, G/\Gamma \to G/\Gamma$ be a continuous map of a compact completely solvable solvmanifold of dimension 3 which is not diffeomorphic to a nilmanifold. Then its linearization matrix has the form*

$$(3) \qquad \textit{either} \quad 1^0) \ \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \textit{or} \quad 2^0) \ \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & v \end{bmatrix} \qquad \textit{or} \quad 3^0) \ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & s \\ 0 & u & 0 \end{bmatrix},$$

where $a,\, r,\, s,\, u,\, v \in \mathbb{Z}$. *In particular, the absolute value of the Lefschetz number, thus the Nielsen number, is either*

$1^0)$ $\quad |L(f)| = N(f) = |1 - a|$ *with any* $a \in \mathbb{Z}$, *or*

$2^0)$ $\quad N(f) = 0$, *or*

$3^0)$ $\quad N(f) = |2(1 - s\,u)|$ *with* $s \neq 0$, *or* $u \neq 0$, *respectively.*

□

We are in position to formulate a description of all possible sets HPer(f) for maps of three dimensional completely solvable solvmanifolds which is a counterpart of Theorem C of [19] and of Theorem 3.1 of [15].

Theorem 4.4 *Let $f\, :\, M \to M$ be a map of a compact three dimensional completely solvable solvmanifold which is not diffeomorphic to a nilmanifold. Let next $\mathcal{M}_{3\times 3}(\mathbb{Z})$ be the linearization of the form given in Proposition 4.2 (cf. Proposition 4.3). Then we have three mutually disjoint cases:*

(E) HPer(f) $= \emptyset$ *iff* $L(f) = 0$ *iff* $(a = 1$ *or* $(a = -1$ *and* $s\,u = 1))$.

(G) HPer(f) $= \mathbb{N}$ *iff* $a \neq \{-2, -1, 0 + 1\}$ *and* $r = s = u = v = 0$;
HPer(f) $= \mathbb{N} \setminus \{2\}$ *iff* $a = -2 \quad r = s = u = v = 0$;
HPer(f) $= \mathbb{N} \setminus 2\mathbb{N}$ *iff* $a = -1$, $|s\,u| \geq 2$ *and* $r = v = 0$.

(F) HPer(f) $= \{1\}$ *in the remaining cases.*

Proof. We recall three facts which will be the crucial arguments in the proof (cf. [19], [14], and also [15]):

i) $N(f) = L(f) = 0$ implies HPer(f) $= \emptyset$.

ii) If there exists a prime $k|n,\ k < n$, such that $N(f^k) = |L(f^k)| = |L(f^n)| = N(f^n)$ then $n \notin$ HPer(f) .

iii) If $|L(f^k)| < |L(f^n)|$ for all $k|n$, $k < n$ then $n \in$ HPer(f) .

Now we check case by case. We consider each of three matrices $1^0) - 3^0)$ of the linearization as in Proposition 4.3 separately.
1^0) Note then the characteristic polynomial is $\chi_{A^k}(t) = t^2(t - a^k)$.
Now $a = 1$ implies $L(f) = \chi_A(1) = 0$ hence HPer(f) $= \emptyset$, thus we have case (E).
$a = 0$ implies $L(f^k) = \chi_{A^k}(1) = 1$ hence $L(f^k) = L(f)$, which gives HPer(f) $= \{1\}$, the case(F).
If $a = -1$ then

$$L(f^k) = \chi_{A^k}(1) = \begin{cases} 0 & \text{for } k \text{ even} \\ 2 & \text{for } k \text{ odd} \end{cases}$$

Consequently no even number belongs to HPer(f). On the other hand for every odd k , $L(f^k) = L(f) = 2$ hence HPer(f) $= \{1\}$, and we have the case (F) then.
If $a = -2$ then $|L(f^k)| = |\chi_{A^k}(1)| = |1 - (-2)^k|$ hence $|L(f)| = |L(f^2)| = 3$ and the sequence is strictly growing for $k > 3$. Thus HPer(f) $= \mathbb{N} \setminus \{2\}$ which is a special case of (G).
If $a \notin \{-2, -1, 0, 1\}$ then the sequence $|L(f^k)| = |\chi_{A^k}(1)| = |1 - a^k|$ is strictly increasing hence HPer(f) $= \mathbb{N}$ and we have the case (G).
2^0) Then $L(f) = 0$, which implies HPer(f) $= \emptyset$, we have the case (E).
3^0) The characteristic polynomial $\chi_A(t) = (t+1)(t^2 - u\,s)$ then. One may check inductively that:
$L(f^k) = \chi_{A^k}(1) = 0$ implies $k \notin$ HPer(f) for k even
$\chi_{A^k}(t) = (t + 1)(t^2 - (s\,u)^k)$ for k odd

We consider subcases:
(a) $s\,u = 0$ then $L(f^k) = \chi_{A^k}(1) = 2$ for k odd, hence $L(f^k) = L(f)$ implies HPer(f) $= \{1\}$ which gives the case (F).
(b) $s\,u = 1$ then $L(f) = \chi_A(1) = 0$ implies HPer(f) $= \emptyset$ and gives the case (E).
(c) $s\,u = -1$ then (for k odd) $L(f^k) = \chi_{A^k}(1) = 2 \cdot 2 = 4$ hence HPer(f) $= \{1\}$, case (F) then.
(d) $|s\,u| \geq 2$ then the sequence $|L(f^k)| = |\chi_{A^k}(1)| = |2(1 - (s\,u)^k)|$ is strictly increasing, as k runs the set of odd natural numbers. Thus HPer(f) $= \mathbb{N} \setminus 2\mathbb{N}$, and we have the case (G). □

Corollary 4.5 *For any homeomorphism $f : M \to M$ of a compact special three dimensional completely solvable solvmanifold which is not diffeomorphic to a nilmanifold* HPer(f) *is either empty or consists of the single number* 1.

□

Corollary 4.6 *For a map as in Theorem 4.4 we have Šarkovski type implications:*

$$2 \in \mathrm{HPer(f)} \quad \textit{implies} \quad \mathrm{HPer(f)} = \mathbb{N},$$
$$\textit{If} \quad \mathrm{HPer(f)} \quad \textit{contains an even number then} \quad \mathbb{N} \setminus \{2\} \subset \mathrm{HPer(f)}$$
$$\textit{If} \quad \mathrm{HPer(f)} \quad \textit{contains at least two numbers then} \quad \mathbb{N} \setminus 2\mathbb{N} \subset \mathrm{HPer(f)}\,.$$

□

At the end we describe a family of 3-dimensional completely completely solvable solvmanifolds which are not diffeomorphic.

Example 4.7 *First for $a+a^{-1} = n$ and $2 \neq n \in \mathbb{N}$ we define a family of simply connected solvable Lie group by $G_n := \mathbb{R} \ltimes_{\phi_n} \mathbb{R}^2$, where*

$$\phi_n(t) = \begin{bmatrix} \frac{a^{t+1}-a^{-t-1}}{a-a^{-1}} & \frac{a^{-t}-a^{t}}{a-a^{-1}} \\ \frac{a^{t}-a^{-t}}{a-a^{-1}} & \frac{a^{1-t}-a^{t-1}}{a-a^{-1}} \end{bmatrix}$$

is a family of 1-parameter subgroups of $SL_2(\mathbb{R})$. Each group G_n is isomorphic to G_3 [31] and Proposition 4.1. It is easy to see that each of them contains a lattice of the form $\Gamma_n := \mathbb{Z} \ltimes_{\phi_n(1)} \mathbb{Z}^2$, where

$$\phi_n(1) = \begin{bmatrix} n & -1 \\ 1 & 0 \end{bmatrix}.$$

Thus we obtain a family of solvmanifolds $X_n := G_n/\Gamma_n$. They are not homeomorphic, since the groups $\Gamma_n = \pi_1(X_n)$ are pairwise non-isomorphic. Indeed, representation of Γ_n on its commutator subgroup factors through $\mathbb{Z}$. If we consider this representation tensored by $\mathbb{Q}$, then it is generated by the matrix $\phi_n(1)$. For the indices $n \neq m$ these representations are not equivalent because the traces of its generators are different.

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