Discrete groups actions and some properties of corresponding modules

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Abstract

In a general formulation the problem to be discussed here is the following: what are the interrelations between the properties of an action of a discrete group Γ on a compact Hausdorff space X and algebraic and analytical properties of the module of all continuous functions $C(X)$ over the algebra of invariant continuous functions $C_\Gamma(X)$? The present paper is the next to our joint paper with M. Frank and V. Manuilov. Now we prove some statements inverse to obtained in the mentioned paper: from some properties of modules we deduce some properties of actions. In particular, it is proved that if for uniformly continuous action the module $C(X)$ is finitely generated projective over $C_\Gamma(X)$, then the cardinality of orbits is finite and fixed. Some sufficient conditions for existence of natural conditional expectations $C(X) \to C_\Gamma(X)$ are obtained.

1 Introduction

Suppose, Γ is a discrete countable group and X is a compact Hausdorff Γ-space. With this situation the following objects can be associated.

(1) The orbits, their cardinality and other dynamical properties.

(2) In some cases using invariant means on functions on Γ it is possible to define so-called conditional expectations $E_\Gamma : C(X) \to C_\Gamma(X)$ and study their properties.

(3) It is possible to consider properties of the module of all continuous functions $C(X)$ over the algebra of invariant continuous functions $C_\Gamma(X)$. In some cases with the help of E_Γ it can be equipped with an $C_\Gamma(X)$-valued C*-inner product.

The study of the relations between these properties was started systematically in [5] (for preliminary and related research see also [1, 2, 4, 7, 8, 9, 12, 13, 16] and the overview in [5]).

We prove in the present paper some statements inverse to obtained in the mentioned paper: from some properties of modules (3) we deduce some properties of actions (1). In particular, it is proved that if for uniformly continuous action the module $C(X)$ is finitely generated projective over $C_\Gamma(X)$, then the cardinality of orbits is finite and fixed.

Some sufficient conditions of the Lyapunov type for existence of natural conditional expectations $C(X) \to C_\Gamma(X)$ are obtained.

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2 Preliminaries and reminding

The necessary information about C*-Hilbert modules can be found in [10] and [11].

Suppose, Γ is a discrete countable group and X is a locally compact Hausdorff Γ-space. Let us denote by $M_\Gamma(f)$ or simply by $M(f)$ the invariant mean of bounded $f:\Gamma\to\mathbb{C}$ if it exists (if Γ is amenable or f is almost periodic). Let $C_\Gamma(X)$ be the algebra of continuous invariant functions. The algebra of all continuous functions $C(X)$ is a module over $C_\Gamma(X)$. Let us denote by L_g the left translation on functions on Γ: $(L_g\varphi)(h):=\varphi(gh)$.

Let us extract and reformulate some definitions and statements from [5]. Actions will always be assumed to satisfy the following condition.

Definition 1 [5, Def. 2.1] An action of a topological group G on a locally compact Hausdorff space X is *uniformly continuous* if for every point $x\in X$ and every neighborhood U_x of x there exists a neighborhood V_x of x such that $g(V_x)\subseteq U_x$ for every $g\in G_x$.

Definition 2 (cf. [5, Def. 2.3]) Let us define a conditional expectation $E_\Gamma : C(X)\to C_\Gamma(X)$ by the formula

$$(E_\Gamma(\varphi))(x):=M_\Gamma(\varphi_x),\qquad\text{where}\qquad \varphi_x(g):=\varphi(gx).$$

Remark 3 Of course, this expectation E_Γ is not always defined. The obstructions are 1) existence of averaging at each point, 2) the resulting function $E_\Gamma(\varphi)$ can be discontinuous.

Remark 4 As it was demonstrated in [5, p. 840], in the case of uniform boundedness of the cardinalities of orbits the averaging over orbits gives the same map E_Γ.

Definition 5 If E_Γ is well defined one can equip $C(X)$ with the structure of a C*-pre-Hilbert module over $C_\Gamma(X)$ by the formula

$$\langle\varphi,\psi\rangle_E := E_\Gamma(\varphi^*\psi).$$

Remark 6 Let us remind that a conditional expectation $E:A\to B\subset A$ is said to be *faithful* if $E(x^*x)=0$ implies $x=0$ for $x\in A$. This holds always for E_Γ.

Proposition 7 [5, Prop 1.1] *Let A be a C*-algebra and $E:A\to B\subseteq A$ be a conditional expectation with fixed point set B. There then exists a finite real number $K\geq 1$ such that the mapping $(K\cdot E-\mathrm{id}_A)$ is positive if and only if E is faithful and the (right) pre-Hilbert B-module $\{A, E(\langle\cdot,\cdot\rangle_A)\}$ is complete with respect to the norm $\|E(\langle\cdot,\cdot\rangle_A)\|_B^{\frac{1}{2}}$ (where $\langle a,b\rangle_A=a^*b$ for $a,b\in A$).*

Theorem 8 (combination of [5, Lemmas 2.9 and 2.11 and Theorem 2.12]) *Suppose a discrete group Γ acts on a locally compact Hausdorff space X in such a way that $k := \max\{\#(\Gamma x) : x \in X\} < +\infty$. Then E_Γ is well-defined.*

If X is a normal space, then $K(E_\Gamma) = k$. Hence, by Prop. 7, $C(X)$ is a Hilbert module over $C_\Gamma(X)$.

Theorem 9 [5, Theorem 3.6] *Suppose, X is a compact Hausdorff space, Γ is a discrete group acting in a uniformly continuous manner on X, and all orbits consist of the same finite number of points. Then the Hilbert $C_\Gamma(X)$-module $\{C(X), E_\Gamma(\langle\cdot,\cdot\rangle)\}$ is finitely generated and projective.*

A C*-Hilbert B-module M is called *auto-dual* or *self-dual* if there is natural isomorphism of M on the module M' of all anti-B-linear bounded functionals on M. For unital B any finitely generated projective module is self-dual. If $M'' = M$ the module M is called (B-)*reflexive.*

Theorem 10 [5, Theorem 3.9] *Let X be a compact Hausdorff space and Γ a discrete group acting in a uniformly continuous manner on X. If Γ acts in such a way that the lengths of all orbits are uniformly bounded by some number n and the number of points whose orbit is shorter than n is finite, then the Hilbert $C_\Gamma(X)$-module $\{C(X), E_\Gamma(\langle.,.\rangle)\}$ is $C_\Gamma(X)$-reflexive.*

The following final form of the previous theorem was obtained by V. Seregin.

Theorem 11 [15] *Let X be a compact Hausdorff space and Γ a discrete group acting in a uniformly continuous manner on X. If Γ acts in such a way that the cardinalities of all orbits are uniformly bounded by some number n, then the Hilbert $C_\Gamma(X)$-module $\{C(X), E_\Gamma(\langle.,.\rangle)\}$ is $C_\Gamma(X)$-reflexive.*

Thereafter X is **compact**, hence metrizable and normal.

3 Inverse theorems

In this section we will obtain some theorems, inverse to the main statements of the previous section.

Let us remind the following statement.

Theorem 12 *Let Z be an infinite compact Hausdorff space. There is no (complete) Hilbert space structure on $C(Z)$ with the corresponding norm satisfying $\|f\|_h \leq \|f\|_{C(Z)}$.*

Proof: By [3, Theorem II.2.5] the two norms have to be equivalent. But $C(Z)$ is reflexive if and only if Z is finite [3, IV.13.15]. □

Here it is reasonable to discuss the following example. The idea to study completions belongs to V. M. Manuilov.

Example 13 (revisited [5, Examples 2.4]) Let the discrete group $\mathbb{Z}$ act on the manifold S^1 by an irrational rotation. Then every orbit of a point $x \in S^1$ consists of infinitely many points, and the conditional expectation $E_\mathbb{Z}$ is well-defined. Namely, it can be represented by the integral formula. The positive linear functional $E_{\mathbb{Z}}$ turns out to be the standard integral of continuous functions over S^1.

Of course, the resulting module $C(S^1)$ over $C_{\mathbb{Z}}(S^1) = \mathbb{C}$ is not complete. This can be established directly or by the next theorem. Nevertheless its completion $L^2(S^1)$ is self-dual over $\mathbb{C}$ as an ordinary Hilbert space.

Theorem 14 *Let Γ be a discrete group acting uniformly continuously on the compact Hausdorff (hence, normal) topological space X. Then the following properties are equivalent*

(i) *cardinalities of all orbits are uniformly bounded;*

(ii) *E_Γ is well-defined and there exists a finite real number $K \geq 1$ such that the mapping $(K \cdot E - \mathrm{id}_A)$ is positive;*

(iii) *E_Γ is well-defined and the corresponding pre-Hilbert module is complete;*

(iv) *E_Γ is well-defined and the corresponding pre-Hilbert module is reflexive.*

Proof: (i)$\Rightarrow$(iv) is Theorem 11.

(iv)$\Rightarrow$(iii) is evident.

(ii)$\Leftrightarrow$(iii) is Prop. 7.

To prove (ii)$\Rightarrow$(i) let us suppose the opposite.

Let us remark, that if $Y \subset X$ is a closed invariant subspace, then the pre-Hilbert $C_\Gamma(Y)$-module $C(Y)$ is also complete (see Lemma 20).

First, suppose, there is an infinite orbit Γx. Taking $Y = \overline{\Gamma x}$ and applying the remark above we obtain that $C(Y)$ could be equipped with the Hilbert space structure. Contradiction with Theorem 12.

The other case is the case of finite orbits of arbitrary cardinality. Here we can argue as in [5, Theorem 2.12]. By Prop. 7 there exists a real number $K \geq 1$ such that $K \cdot E_\Gamma(\varphi) \geq \varphi$ for any positive function $\varphi : X \to \mathbb{C}$. Choose a point $x \in X$ with $k_x := \#(\Gamma x) > K$. Then we can obtain a neighborhood U_x of x such that $g_i U_x \cap g_j U_x = \emptyset$ for $i \neq j$ and $\{g_1 = 1, g_2, \ldots, g_m\} = T_x := \Gamma / \Gamma_x$, where Γ_x is the isotropy subgroup. For a continuous non-negative function f with support inside U_x (cf. [14, Th. V.17.4] for the existence) we have the inequality

$$K \cdot E_\Gamma(f)(x) = K \frac{1}{k_x} \cdot \sum_{g_a \in T_x} f(g_a x) < f(x)\,.$$

This contradicts the definition of K. $\square$

Theorem 15 *Let Γ be a discrete group acting uniformly continuously on the compact Hausdorff connected topological space X. Suppose the averaging over orbits defines a structure of an auto-dual Hilbert module on $C(X)$ over the algebra of invariant continuous functions $C_\Gamma(X)$. Suppose the length of orbits is bounded. Then it is the same for all orbits.*

Proof: Let $W \subset X$ be the open subset formed by all points from orbits of the maximal length k. Suppose $W \neq X$ and take $x \in \overline{W} \cap (X \setminus W)$ (since X is connected, it is possible).

Then for any neighborhood U of x there exist $y \in U \cap W$ and $g \in \Gamma_x$ such that $gy \neq y$, where Γ_x is the stabilizer of x. Indeed, suppose the opposite: there exists a neighborhood U such that any element $y \in U \cap W$ is Γ_x invariant, i.e. $\Gamma_x \subset \Gamma_y$. Then $\#(\Gamma x) \geq \#(\Gamma y)$, we have a contradiction.

So, we can choose a sequences of elements $y_n \in W$ of different orbits (due to finiteness of orbits) and $g_n \in \Gamma_x$ such that $y_n \to x$ $(n \to \infty)$ and $y'_n := g_n y_n \neq y_n$. Since X is normal, we can choose (passing to the subsequence, if it is necessary) two sequences of open neighborhoods $V_n \ni y_n$ and $V'_n \ni y'_n$ non-intersecting with each other and with x and satisfying the following condition: for any neighborhood V of x the sets V_n and V'_n are inside V for all large n. Take continuous functions

$$\varphi_n : X \to [0,1], \qquad \varphi_n(y_n) = 1, \qquad \operatorname{supp} \varphi_n \subset V_n,$$

$$\varphi'_n : X \to [-1,0], \qquad \varphi'_n(y'_n) = -1, \qquad \operatorname{supp} \varphi'_n \subset V'_n.$$

Define function $\varphi : X \to [-1,1]$ be equal to φ_n on V_n, to φ'_n on V'_n, and vanish in other points. It is continuous except of x. Then $F(f) := E_\Gamma(\varphi\, f)$ is a $C_\Gamma(X)$ functional on $C(X)$. It can not be presented as $E_\Gamma(g\, f)$ with $g \in C(X)$. Contradiction with autoduality. $\square$

Lemma 16 *Suppose X is a compact normal space and $C(X)$ is a finitely generated module over $C_\Gamma(X)$. Then the cardinality of each orbit is finite.*

Proof: Suppose an orbit is infinite. Then the dimension (algebraic) of the space of continuous functions on it (or its closure) is infinite. Hence it can not be finitely generated over constants (invariant functions). By Tietze theorem all of these functions are the restrictions of some continuous functions on the closure of the orbit. Hence the same statement is true for all functions. $\square$

Definition 17 A C*-algebra A is called *module-infinite* (MI) if each Hilbert A-module is projective finitely generated if and only if it is auto-dual.

Theorem 18 *Let Γ be a discrete group acting uniformly continuously on a compact Hausdorff connected topological space X. Suppose the averaging over orbits defines a structure of an auto-dual Hilbert module on $C(X)$ over the algebra $C_\Gamma(X)$. Suppose $C_\Gamma(X)$ is* MI. *Then all orbits have the same cardinality.*

Proof: By Lemma 16 the orbits are finite. Suppose there are orbits of length k and some less. Let $W \subset X$ be the open subset formed by all points from orbits of the cardinality $\geq k$. Then $W \neq X$ and take $x \in \overline{W} \cap (X \setminus W)$ (since X is connected, it is possible). So $\#(\Gamma x) = m < k$. The proof could be completed as for Theorem 15. $\square$

4 Localization and globalization

It turns so that we can deduce the facts about cardinality of orbits from a purely algebraic data.

Let us start from the following evident lemma, which formulates an idea from a proof from the previous section in a more general way.

Lemma 19 *Let $C(X)$ be m-generated over $C_\Gamma(X)$. Suppose, Y is a closed Γ-invariant subspace of X. Then $C(Y)$ is m-generated over $C_\Gamma(Y)$.*

Lemma 20 *Suppose, E_Γ is well-defined and the corresponding pre-Hilbert module is complete. If $Y \subset X$ is a closed invariant subspace, then the pre-Hilbert $C_\Gamma(Y)$-module $C(Y)$ is also complete.*

Proof: By Prop. 7 there exists a real number $K \geq 1$ such that $K \cdot E_\Gamma(\varphi) \geq \varphi$ for any positive function $\varphi : X \to \mathbb{C}$. Let $\widehat{\varphi} : Y \to \mathbb{C}$ be a positive function. By Tietze theorem, there is a positive continuous extension of $\widehat{\varphi}$ up to $\varphi : X \to \mathbb{C}$. We have

$$K \cdot E_\Gamma(\varphi) \geq \varphi, \qquad K \cdot E_\Gamma(\varphi)|_Y \geq \varphi|_Y, \qquad K \cdot E_\Gamma(\widehat{\varphi}) \geq \widehat{\varphi}.$$

Applying Prop. 7 in the opposite direction we complete the proof. □

Lemma 21 *Let an orbit $Y = \Gamma x$ consist exactly of k points. Then $C(Y)$ is the free k-generated module over $C_\Gamma(Y)$.*

Proof: $C_\Gamma(Y) \cong \mathbb{C}$ and $C(Y) \cong \mathbb{C}^k$ as a $\mathbb{C}$-module. □

Lemma 22 *Let an orbit $Y = \Gamma x$ be infinite. Then $C(\overline{Y})$ is not a finitely generated module over $C_\Gamma(\overline{Y})$.*

Proof: $C_\Gamma(\overline{Y}) \cong \mathbb{C}$ and the algebra of continuous functions on an infinite compact Hausdorff space is $\mathbb{C}$-infinite. □

We need the following statement.

Proposition 23 *Let $C(X)$ be m-generated over $C_\Gamma(X)$. Then cardinality of each orbit does not exceed m.*

Proof: The statement is a combination of Lemmas 19, 21, and 22. □

Theorem 24 *Suppose, an action of Γ on X is uniformly continuous. Let $C(X)$ be a finitely generated projective module over $C_\Gamma(X)$ Then all orbits have the same finite cardinality.*

Proof: By Prop. 23 the cardinality of orbits is bounded by m. Hence, by Theorem 8 E_Γ is well-defined and the pre-Hilbert module $C(X)$ over $C_\Gamma(X)$ is complete. Since it is finitely generated projective, it ia auto-dual. So, we have both properties and we do not need MI to complete the proof as in Theorem 18. □

5 Existence theorem

Starting from here we will concentrate ourself on some specific properties of action (infinite) discrete groups on compact Hausdorff spaces. Since these spaces are metrizable, let us denote by ρ some distance on X. Of course, the considerations can be formulated using topology and neighborhoods instead of the distance and ε-neighborhoods.

In the present section we will obtain some sufficient conditions for E_Γ to be well defined.

Let us start from a couple of examples.

Example 25 Let $X \subset \mathbb{R}^3$ consist of 2 circles

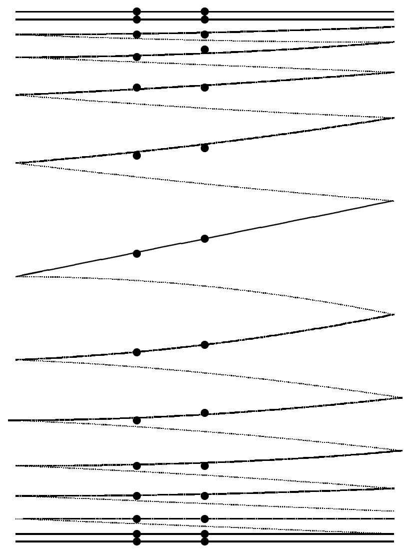
$$S_\pm : \qquad \begin{cases} x &= \cos 2\pi t \\ y &= \sin 2\pi t \\ z &= \pm 1, \end{cases} \qquad t \in [0,1],$$

and non-uniform spiral

$$\Sigma : \qquad \begin{cases} x &= \cos 2\pi \tau \\ y &= \sin 2\pi \tau \\ z &= \frac{2}{\pi} \cdot \arctan \tau, \end{cases} \qquad \tau \in [0,1].$$

Let the generator g of $\Gamma = \mathbb{Z}$ act on all three components by

$$t \mapsto t + \frac{1}{n}, \qquad \tau \mapsto \tau + \frac{1}{n}.$$



Since $\mathbb{Z}$ is amenable, the pointwise value of E_Γ is defined. Suppose, $\varphi : X \to \mathbb{R}_+ \subset \mathbb{C}$ is non-negative continuous function with

$$\varphi|_{S_+} = 0, \qquad \varphi|_\Sigma = 1 \quad \text{for} \quad t \leq 0, \qquad \varphi|_{S_-} = 1.$$

Then $E_\Gamma(\varphi)|_{S_+} = 0$ and $E_\Gamma(\varphi)|_\Sigma > 0$. So, the function $E_\Gamma(\varphi)$ is discontinuous.

In this example the action is not uniformly continuous and we always emphasized that these are bad ones. Unfortunately, the next example shows that the restriction for action to be uniformly continuous does not help.

Example 26 Let X and Γ be as in the previous example, but the action be defined by

$$t \mapsto t+\alpha, \qquad \tau \mapsto \tau+\alpha,$$

where α is a positive irrational number. Then the isotropy group of each point of X is trivial. Hence, the condition of uniform continuity holds automatically. Nevertheless, a discontinuous $E_\Gamma(\varphi)$ corresponds to the same function φ.

Remark 27 Let in any of these examples $x \in \Sigma$. Then for the mentioned φ the function $\varphi_x : \mathbb{Z} \to \mathbb{C}$ is not almost periodic. Of course, for amenable $\Gamma = \mathbb{Z}$ this is not an obstruction, but a similar phenomenon can occur for non-amenable groups too.

Now we introduce a condition which is sufficient to overcome these difficulties.

Definition 28 An action of a group Γ on a metric space X is called (Lyapunov) *stable* if for any $\varepsilon > 0$ and any $x \in X$ there exist $\delta > 0$ such that

$$\rho(gx, gy) < \varepsilon \quad \text{for any} \quad g \in \Gamma \quad \text{if} \quad \rho(x,y) < \delta.$$

The following statement is evident.

Proposition 29 *If an action is stable, then it is uniformly continuous.*

Proposition 30 *Let a discrete group Γ act on a compact Hausdorff space X in a stable way. Suppose, $\varphi : X \to \mathbb{C}$ is a continuous function. Then for any $x \in X$ the function $\varphi_x : \Gamma \to \mathbb{C}$, $\varphi_x(g) := \varphi(gx)$, is almost periodic. Hence its mean $M(\varphi_x)$ is defined.*

Proof: Let $\varepsilon > 0$ be arbitrary. Since X is compact and φ is continuous, it is uniform continuous, in particular, there exists $\widetilde{\varepsilon}$ such that

$$|\varphi(y) - \varphi(z)| < \varepsilon \qquad \text{if} \quad \rho(z,y) < \widetilde{\varepsilon}. \tag{1}$$

For each point gx of the orbit let us find an ε_g-neighborhood U_g with the following property:

$$\rho(hgx, hy) < \widetilde{\varepsilon} \qquad \text{for any } h \in \Gamma \text{ if } y \in U_g. \tag{2}$$

Let $U_{g_1}, \ldots, U_{g_m}$ is a finite subcovering of the (closure of) the orbit Γx. Then the functions $L_{g_s}\varphi_x$, $s = 1, \ldots, m$, form an ε-mesh for the set $L_g\varphi_x$, $g \in \Gamma$, with the respect to uniform norm. Indeed, for any $g \in \Gamma$ we can choose $s_0 \in \{1, \ldots, m\}$ such that $y = gx \in U_{g_{s_0}}$. Then by (1) and (2) we have

$$\sup_{h\in\Gamma} |(L_g\varphi_x)(h) - (L_{g_{s_0}}\varphi_x)(h)| = \sup_{h\in\Gamma} |\varphi(hgx) - \varphi(hg_{s_0}x)| < \varepsilon.$$

$\square$

Theorem 31 *Let a discrete group Γ act on a compact Hausdorff space X. If the action is stable, then E_Γ is well defined.*

Proof: By Prop. 30 we only need to verify continuity of mean $M(\varphi_x)$ in $x \in X$. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Let us remind (cf. [6, pp. 250–251]) that we can choose $h_1, \ldots, h_p \in \Gamma$ such that the uniform distance on $\Gamma \times \Gamma$ between the function

$$\frac{1}{p}\sum_{j=1}^{p} D_{h_j}\varphi_x : \Gamma \times \Gamma \to \mathbb{C} \qquad (\text{where} \quad (D_h\psi)(g_1, g_2) := \psi(g_1 h g_2))$$

and some constant is less then ε. In this case the uniform distance

$$\left\| M(\varphi_x) - \frac{1}{p}\sum_{j=1}^{p} D_{h_j}\varphi_x \right\|_u < 2\varepsilon.$$

Let us choose a δ-neighborhood U of x such that

$$|\varphi(gy) - \varphi(gx)| < \varepsilon, \qquad \text{for any } g \in \Gamma,\ y \in U.$$

This neighborhood can be found as in the proof of Prop. 30: $\varepsilon \to \widetilde{\varepsilon} \to \delta$. More precisely, first we can find $\widetilde{\varepsilon} > 0$ such that $|\varphi(y) - \varphi(z)| < \varepsilon$ whenever $\rho(y,z) < \widetilde{\varepsilon}$ (using compactness of X). Then, by stability of the action, we can find for our fixed x a number $\delta > 0$ such that $\rho(gy, gx) < \widetilde{\varepsilon}$ for any $g \in \Gamma$ whenever $\rho(y,x) < \delta$.

Then for any $y \in U$ one has

$$\begin{aligned}
\left(\frac{1}{p}\sum_{j=1}^{p} D_{h_j}\varphi_y\right)(g_1, g_2) &= \left(\frac{1}{p}\sum_{j=1}^{p} D_{h_j}\varphi_x\right)(g_1, g_2) + \frac{1}{p}\sum_{j=1}^{p}\left(\varphi_y(g_1 h_j g_2) - \varphi_x(g_1 h_j g_2)\right) = \\
&= \left(\frac{1}{p}\sum_{j=1}^{p} D_{h_j}\varphi_x\right)(g_1, g_2) + \frac{1}{p}\sum_{j=1}^{p}\left(\varphi(g_1 h_j g_2 y) - \varphi(g_1 h_j g_2 x)\right).
\end{aligned}$$

Each term of the second summand is less then ε. Hence, the second summand is less then ε. Thus,

$$\left\| M(\varphi_x) - \frac{1}{p}\sum_{j=1}^{p} D_{h_j}\varphi_x \right\|_u < 3\varepsilon.$$

Hence, regarding $M(\varphi_x)$ as an arbitrary constant, we have

$$\left\| M(\varphi_y) - \frac{1}{p}\sum_{j=1}^{p} D_{h_j}\varphi_x \right\|_u < 6\varepsilon.$$

Finally,

$$|M(\varphi_y) - M(\varphi_x)| < 9\varepsilon.$$

$\square$

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