A Remark About Arithmetic Miyaoka–Yau Inequality

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It is well–known that a direct consequence of arithmetic Miyaoka–Yau inequality is Szpiro inequality about the minimal discriminant of elliptic curves [5]. As a byproduct, according to Frey [2], we will have the asymptotic version of Fermat last theorem. In this line, there are two different approaches: One is by Vojta [8] who deduces the problem to that only associated with so–called limited family, by using the famous Kodaira–Parshin construction. The other is by Frey and Kani [3], who deduce the problem to one only related with curves of genus 2, but with the inequality associated their Jacobians, by using the fact that elliptic curves may be covered by curves of genus 2 and the arithmetic properties of them.

On the other hand, to try to offer a proof of arithmetic Miyaoka–Yau inequalities seems to be very difficult as we really do not know how to "translate" the proof in the case of functional fields. But for some special kinds of surfaces, there is certain method to do so. In this note, we will offer a strategy to study hyperelliptic arithmetic surfaces. Note that in the functional field cases, the corresponding Miyaoka–Yau inequality can be obtained just by using some properties of double covering [4], [7], [9]. We hope that it can also be done for hyperelliptic arithmetic surfaces. In this case, our last form of conjectural arithmetic Miyaoka–Yau inequality takes the form as that of Parshin, but with the coefficients $a_1(g)$, $a_2(g)$, $a_3(g)$, the functions of genus. At last, it can be proved that for the application to Szpiro inequality, this form is enough.

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I. Geometric Situation

Let $f : X \longrightarrow Y$ be a relative minimal fibration of curves of genus g over a smooth projective curve Y with X also smooth.

For $y \in Y$, let

$$\delta_y := 2g-2 + \chi(X_y) \ .$$

Then the relative Noether formula becomes

$$\omega^2_{X/Y} + \sum_y \delta_y = 12 \deg(f_* \omega_{X/Y}) \ .$$

On the other hand, for every hyperelliptic fibration $f : X \longrightarrow Y$, there exists a $\mathbb{P}^1$–bundle

$$p : P := \mathbb{P}\mathrm{roj}(f_* \omega_{X/Y}) \longrightarrow Y$$

and a double covering

$$\Phi : \tilde{X} \longrightarrow P$$

such that $\tilde{X}$ is birational to X over Y. There exists an open set Y^0 of Y such that

i) $P^{-1}(Y^0)$ is isomorphic to $\mathbb{P}^1 \times Y^0$;

ii) $f^{-1}(Y^0)$ can be identified with the closure of

$$\{(x,t,z) \in \mathbb{C} \times Y \times \mathbb{C} \mid z^2 = \varphi(x,t)\} \quad ,$$

where x is an inhomogeneous coordinate of $\mathbb{P}^1$ and φ is a polynomial of x of degree 2g+1 or 2g+2 with coefficients in $K(Y^0)$.

<u>Proposition</u> [4]. Let $\Delta(\varphi)$ denote the discriminant of φ as a polynomial of x. Then

$$\Pi := \Delta(\varphi)^g \left(\frac{dx}{z} \wedge x\frac{dx}{z} \wedge \dots \wedge x^{g-1}\frac{dx}{z}\right)^{4(2g+1)}$$

defines a regular section of

$$(\Lambda^g f_* \omega_{X/Y})^{4(2g+1)} \quad ,$$

which depends only on $f: X \longrightarrow Y$.

Let

$$d_y := \frac{1}{4(2g+1)} \operatorname{ord}_y \Pi \quad .$$

Then we have the following

Theorem [4]: Let $f : X \longrightarrow Y$ be a relative minimal hyperelliptic fibration of genus g. For every closed point $y \in Y$,

$$\frac{g}{4(2g+1)}\,\delta_y \le d_y \le \frac{g^2 + \frac{1-(-1)^g}{2}}{4(2g+1)}\,\delta_y .$$

Remark: There are three steps to prove it. At first, by elementary transformation, we may assume that the branch locus has "good" singularities. Then we can estimate

$$\mathrm{ord}_y \Delta(\varphi)^g$$

and

$$\mathrm{ord}_y\left(\frac{dx}{z} \wedge x\frac{dx}{z} \wedge \ldots \wedge x^{g-1}\frac{dx}{z}\right)^{4(2g+1)}$$

respectively.

A direct consequence of the above theorem is the following

Proposition:

1. $\omega^2_{X/Y} = \sum_y (12d_y - \delta_y)$

2. $\frac{g-1}{2g+1}\sum_y \delta_y \le \omega^2_{X/Y} \le \frac{6g^2-4g+1-3(-1)^g}{2(2g+1)}\sum_y \delta_y$

Remark: With respect to Miyaoka–Yau inequality, this result is bad enough. But we argue that

1. locally, the above bound is the sharp one;

2. the proof of this rough result may be translated to arithmetic case.

Also for some certain purposes,the above result is very useful. For example, if $g = 2$ we have

Theorem [9]: $K_X^2 \leq 8\,\chi(O_X)$.

In fact, with more detailed discussion, we can have an even better global result.

II. Arithmetic Situation

At first let us recall the arithmetic Noether formula. Let S be the spectrum of a henselian discrete valuation ring with algebraically closed residue field and consider a proper flat regular S–scheme $f : X \longrightarrow S$ with smooth generic fibre. The Artin conductor of X over S is the integer defined by

$$\mathrm{Art}(x/s) = \chi(X_{\bar{\eta}}) - \chi(X_s) + \mathrm{Sw}_s H^*(X_{\bar{\eta}}, \mathbb{Q}_\ell)$$

where $\mathrm{Sw}_s H^*(x_{\bar{\eta}}, \mathbb{Q}_\ell)$ is the alternating sum of the Swan conductor.

Now with the help of the following

Theorem (Mumford, Deligne) For any arbitrary proper smooth geometrically connected curve $f : X \longrightarrow Y$, there exists a canonical unique (up to sign) isomorphism

$$\Delta : \det(Rf_*\omega^2_{X/Y}) \xrightarrow{\sim} (\det(Rf_*\omega_{X/Y}))^{13} .$$

One can prove that

Theorem (T. Saito) Let S be the spectrum of a henselian discrete valuation ring with algebraically closed residue field and $f : X \longrightarrow S$ be a proper regular geometrically connected S–curve with smooth generic fibre. Then

$$-\operatorname{Art}(X/S) = \operatorname{ord} \Delta .$$

Here we think Δ as a canonical nonzero rational section of an invertible O_S–module

$$\operatorname{Hom}_{O_S}(\det(R\, f_*\omega^2_{X/S}), (\det(R\, f_*\omega_{X/S}))^{13})$$

defined by the above theorem and the order of a nonzero rational section ℓ of an invertible O_S–module L is a unique integer n such that

$$O_s \cdot \ell = p^n L$$

with p the maximal ideal of O_S .

From here, if we globalize it, we can easily have the following

Arithmetic Noether Formula (Faltings, T. Saito) For an arithmetic surface $f : X \longrightarrow Y$

$$13 \deg R f_* \omega_{X/Y} - \deg R f_*(\omega^2{}_{X/Y})$$

$$= - \sum_{v \in y_{fin}} \mathrm{Art}_v(X/Y) \log q_v + \sum_{\sigma \in y_\infty} \delta'(X_\sigma)^{\epsilon_\sigma} .$$

Imitating the above process, for hyperelliptic curve, we also have the following canonical nonzero rational section

$$\Delta' = \Delta'_{X/S}$$

of an invertible O_S–module

$$\mathrm{Hom}_{O_S}(O_S, (\det R f_* \omega_{X/S})^{4(2g+1)}) .$$

Thus we can introduce the <u>hyperelliptic discriminant</u> as the order of Δ' divided by $4(2g+1)$.

<u>Theorem</u> For an arithmetic surface $f : X \longrightarrow Y$ with generic fibre a hyperelliptic curve,

$$4(2g+1) \deg(R f_* \omega_{X/Y}, h_Q)$$

$$= \sum_{v \in Y_{fin}} \mathrm{ord}_v \Delta' \log q_v + \sum_{\sigma \in Y_\infty} \delta_1(x_\sigma) \epsilon_\sigma$$

where $\delta_1(X_\sigma) := -4(2g+1) \log \|\Delta'\|_{h_Q}$.

<u>Remark</u>: The above theorem is in fact another version of Noether formula. Combinating

this with Deligne's Riemann–Roch theorem, we have the following

Theorem′. $$\omega^2_{X/Y} = \sum_{v \in Y_{fin}} \left(\frac{3\,\mathrm{ord}_v \Delta'}{2g+1} - \delta_v\right)\log q_v$$

$$+ \sum_{\sigma \in Y_\infty} \left(\frac{3\delta_1(x_\sigma)}{2g+1} - \delta'(X_\sigma)\right)\epsilon_\sigma$$

with $\delta'(X_\sigma)$ the modification of Faltings' δ–function by a constant depend only on genus g.

From here, with respect to the corresponding theorem in the first section, we have the following

Conjectural Arithmetic Miyaoka–Yau Inequality: There exist 3 effectively computable positive numbers $a_1(g)$, $a_2(g)$, $a_3(g)$ such that for all number fields K and all minimal arithmetic surfaces $f : X \longrightarrow Y = \mathrm{Spec}(O_K)$ of genus g,

$$\omega^2_{X/Y} \le a_1(g) \sum_{v \in Y} \delta_v \, \epsilon_v + a_2(g)\log|D_{K/\mathbb{Q}}| + a_3(g)[k : \mathbb{Q}]$$

with

$$\epsilon_v = \begin{cases} \log q_v, & v \text{ non-Archimedean} \\ 1, & v \text{ real} \\ 2, & v \text{ computes} \end{cases}$$

and

$$\delta_v = \begin{cases} -\text{Artin conductor} & v \text{ non-Archimedean} \\ \text{Faltings } \delta \text{ function} & \text{otherwise} \end{cases} .$$

Remark: We note that this kind of inequality is too wide than Parshin's original one. But in practice, it is hopeful that our formula can be obtained for some special arithmetic surfaces. And in the next section, we will see that for the application of Szpiro inequality it is enough.

III. Szpiro Inequality

The aim of this section is to emphasize the fact that the conjectural arithmetic Miyaoka–Yau inequality above also has Szpiro inequality as its direct consequence.

In fact if a_0 is an aboslute constant, $a_1(g) = (g-1) \cdot a_1'$ with a_1' another absolute constant, then Parshin offered a proof of the following

Theorem (Parshin) Suppose that the arithmetic Miyaoka–Yau inequality is true with above condition for all arithmetic surfaces with stable fibres and generic fibre of genus > 1. Then the following statements hold:

1. For every arithmetic surfaces X, there is an effectively computable constant $c(X)$ such that, for any section $i : Y \longrightarrow X$,

$$i(Y)\omega_{X/Y} \leq c(X) \ ;$$

2. There exist effectively computable constants c_1, c_2, c_3, such that

   a) for all elliptic curves E over $\mathbb{Q}$ with stable reduction

$$\Delta_{min}(E) \leq N(E)^{c_1}$$

where $\Delta_{min}(E)$ is the minimal discriminant and $N(E)$ is the conductor;

b) for all ellipric curves E over a number field K, on which they have stable reduction,

$$H(E) \leq c_3(D_{K/\mathbb{Q}}N(E))^{c_1}$$

where $H(E)$ is the canonical height of the curve E.

The basic tool in the proof is Kodaira–Parshin construction. After with some modifications of deep use of this method, Vojta also forms a conjecture of this style. All of them have the asymptotic Fermat last theorem as their consquence.

Note that Parshin's proof still holds if we use the conjecture stated in the last section, we have

Corollary. If the conjectural arithmetic Miyaoka–Yau inequality holds, the above Parshin's theorem is true without any condition about $a_1(g)$, $a_2(g)$, $a_3(g)$.

Remark: After I finish the above work, I suddenly find a simila modification of Parshin's arithmetic Miyaoka–Yau inequality in [10], with the help of Dr. U. Jannsen.

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