LIMITS OF CHOW GROUPS, AND A NEW CONSTRUCTION OF CHERN-SCHWARTZ-MACPHERSON CLASSES

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Abstract. We define an 'enriched' notion of Chow groups for algebraic varieties, agreeing with the conventional notion for complete varieties, but enjoying a functorial push-forward for arbitrary maps. This tool allows us to glue intersection-theoretic information across elements of a stratification of a variety; we illustrate this operation by giving a direct construction of *Chern-Schwartz-MacPherson* classes of singular varieties, providing a new proof of an old (and long since settled) conjecture of Deligne and Grothendieck.

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1. Introduction

In the remarkable article [Mac74], Robert MacPherson settled affirmatively a conjecture of Pierre Deligne and Alexandre Grothendieck (see [Sul71] p. 168; and [Gro85], note (^{87_1}), for Grothendieck's own comments on the genesis of the original conjecture). MacPherson's theorem states that there is a unique natural transformation from a functor of constructible functions on compact complex algebraic varieties to homology, associating to the constant function $1\!\!1_V$ on a nonsingular variety V the (Poincaré dual of the) total Chern class of the tangent bundle TV of V. The class corresponding to the constant $1\!\!1_X$ for an *arbitrary* compact complex algebraic variety X is therefore a very natural candidate for a notion of *Chern class* of a possibly singular variety.

After MacPherson's work it was realized that these classes agree, up to Alexander duality, with classes defined earlier by Marie-Hélène Schwartz ([Sch65a], [Sch65b]; and [BS81]). It is common nowadays to name these classes *Chern-Schwartz-MacPherson* (CSM) classes.

MacPherson's construction may be used to lift the classes to the Chow group A_*X, cf. [Ful84], §19.1.7. Several other approaches to CSM classes are known: for example through local polar varieties ([LT81]); characteristic cycles and index formulas for holonomic $\mathcal{D}$-modules ([BDK81], [Sab85], [Gin86]); currents and curvature measures ([Fu94]). Some of these approaches may be used to extend the definition of CSM classes to varieties over arbitrary algebraically closed fields of characteristic zero (see [Ken90]), proving naturality at the level of Chow groups, under proper push-forward.

The main goal of this paper is to provide a new construction of CSM classes in this algebro-geometric setting, independent of previous approaches, and including a complete proof of the naturality mandated by the Deligne–Grothendieck conjecture. Our approach is very direct: we simply define an invariant for nonsingular (but possibly noncomplete) varieties, and obtain the CSM class of an arbitrary variety X as the sum of these invariants, over any decomposition of X as finite disjoint union of nonsingular subvarieties. The contribution for a nonsingular variety is obtained as Chern class of the dual of a bundle of differential forms with logarithmic poles.

The approach is particularly transparent, as the contribution of a nonsingular subvariety U to the class for X is independent of how singular X is along U. Auxiliary invariants, such as the *local Euler obstruction* or the *Chern-Mather* class, which are common to several of the approaches listed above, play no rôle in our construction.

Naturality is straightforward, modulo one technical lemma (Lemma 5.3; see also Claim 6.1) on the behavior of the contributions under push-forward. The classes we define must agree with 'standard' CSM classes, because both satisfy the Deligne–Grothendieck prescription.

The main new ingredient making our construction possible is the introduction of *'proChow groups'*, as inverse limits of ordinary Chow groups over the system of maps to complete varieties. Thus, an element of the proChow group $\widehat{A}_*U$ is a compatible choice of a class in each complete variety to which U maps. The proChow group agrees with the conventional Chow group for complete varieties, but is in general much larger for noncomplete varieties.

The key feature of proChow groups is that they are functorial with respect to arbitrary maps: this is what allows us to define a contribution in A_*X from (for example) an *open* stratum U of X. If $i_U : U \to X$ is the embedding, there is in general no push-forward $i_{U*} : A_*U \to A_*X$, while there *is* a push forward $i_{U*} : \widehat{A}_*U \to \widehat{A}_*X$ at the level of proChow groups.

Given then the choice of a distinguished element $\boxed{U}$ in the proChow group of every *nonsingular* variety U, satisfying suitable compatibility properties, we may define an element $\boxed{X} \in \widehat{A}_*X$ for *arbitrary* (that is, possibly singular) varieties by setting

$$\boxed{X} = \sum_U i_{U*}\boxed{U}$$

for any decomposition $X = \amalg_U U$ of X into disjoint nonsingular subvarieties U. If X is complete, this gives a distinguished element of the ordinary Chow group A_*X of X.

Describing this mechanism gluing *local* intersection-theoretic information into *global* one is the second main goal of this article. We show (Proposition 3.1) that the compatibility required for this definition reduces to a simple blow-up formula. We then prove (Proposition 4.3) that this blow-up formula is satisfied by the Chern class of the bundle of differential forms with logarithmic poles along a divisor at infinity. By the mechanism described above we get a distinguished element $\boxed{X} \in \widehat{A}_*X$ for any X, and this is our *pro*CSM class of a (possibly singular) variety.

We should point out that the 'good local data' arising from the bundle of differential forms with logarithmic poles is in fact the only nontrivial case we know satisfying the compatibility requirement of Proposition 3.1. It would be quite interesting to produce other such 'gluable' data; perhaps it would be even more interesting to prove that this is in fact essentially *the only* such example, as it would provide a further sense in which (pro)CSM classes are truly canonical.

The definition of proCSM classes extends immediately to *constructible functions* on X, yielding a transformation from the functor F of constructible functions to the proChow functor $\widehat{A}_*$. Both functors have a push-forward defined for arbitrary (regular) morphisms, and we prove (Theorem 5.2) that the transformation is natural with respect to them. That is: denoting by $\boxed{\varphi}$ the proCSM class of the constructible function φ, we prove that, for an *arbitrary* morphism of varieties $f : X \to Y$,

$$\boxed{f_*(\varphi)} = f_* \boxed{\varphi} \quad .$$

If in particular X and Y are complete, and f is proper, all reduces to the more conventional naturality statement and yields a new proof of (the Chow flavor of) MacPherson's theorem.

From a technical standpoint, our construction relies on factorization of birational maps ([AKMW02]); the fact that this powerful result is relatively recent is the likely reason why the construction presented here was not proposed a long time ago. Also, we rely on MacPherson's 'graph construction' in the proof of the key Lemma 5.3, similarly to MacPherson's own proof of naturality in [Mac74].

The relation between CSM classes and Chern classes of bundles of differential forms with logarithmic poles is not new: cf. Proposition 15.3 in [GP02] and Theorem 1 in [Alu99b]. In fact, this relation and MacPherson's theory may be used to shortcut the paper substantially. Indeed, our definition of 'good local data' may be recast as follows: for every nonsingular variety U, one may define the element $\boxed{U} \in \widehat{A}_* U$ by selecting, for each complete variety X containing U, the element

$$c_*(1\!\!1_U) \in A_* X$$

obtained by applying MacPherson's natural transformation to the function that is 1 over U, and 0 outside of U. Both the basic compatibility (Proposition 4.3) and the key Lemma 5.3 follow then from MacPherson's naturality theorem. Granting these two facts, the material in §5 upgrades MacPherson's natural transformation $F \to A_*$ (which is natural with respect to *proper* morphisms) to a transformation $F \to \widehat{A}_*$, natural with respect to arbitrary morphisms.

However, working independently of MacPherson's theorem allows us to discriminate carefully between parts of the construction which may extend in a straightforward way to a more general context, and parts which depend more crucially on (for instance) the characteristic of the ground field. For example, Proposition 4.3 turns out to be a purely formal computation, while Lemma 5.3 is much subtler, and in fact fails in positive characteristic—this distinction is lost if one chooses to take the shortcut sketched above.

Our construction depends on canonical resolution of singularities; at the time of this writing, this is only known to hold in characteristic zero. Should resolution of singularities be proved in a more general setting, our construction will extend to that setting. However, characteristic zero is employed more substantially (for example through generic smoothness) in the proof of naturality, and simple examples show that naturality can*not* be expected to hold in general in positive characteristic. In fact (as Jörg Schürmann pointed out to me), covariance of the push-forward for constructible functions (Theorem 5.1) already fails in positive characteristic.

We find this state of affairs intriguing. The construction of (pro)CSM classes may well extend to positive characteristic, retaining its basic normalization and additivity properties; these 'only' depend on resolution of singularities, as is shown in this note. But the subtler naturality property of these classes cannot carry over, at least within the current understanding of the situation.

It is formally possible to extend MacPherson's construction of CSM classes to arbitrary characteristic, independently of resolution of singularities; for example, this is done in [NA83], §2.5. It would be interesting to establish whether proCSM classes agree with those defined by Navarro Aznar. In the presence of naturality it is easy to see that proCSM classes agree (for complete varieties, and in the Chow group with $\mathbb{Q}$ coefficients) with the classes discussed in [Alu], §5; but these latter also depend on resolution of singularities.

Similar comments apply to a class which may be defined in general for hypersurfaces as a twist of Fulton's Chern class, and is known to agree with the CSM class in characteristic zero (see [Alu99a]; both resolution of singularities and naturality are needed in the proof).

We believe the local-to-global formalism described in this note (maybe with different target functors rather than Chow) should have other applications, for example simplifying the treatment of other invariants of singular varieties; this will be explored elsewhere. Clearly *pro*-flavors of other functors may be constructed similarly to the proChow functor presented here; we chose to concentrate on this example in view of the immediate application to CSM classes. Jörg Schürmann has pointed out that it would be worth analyzing the construction studied here vis-a-vis the *relative Grothendieck group of varieties* as used in [BSY] (particularly in view of the parallel between the criterion for 'good local data', given here in Proposition 3.1, and Franziska Bittner's description of the relations defining the Grothendieck group, cf. [Bit04]). Also, Jean-Paul Brasselet has suggested that the construction of proCSM classes may provide an alternative proof of the equality of Schwartz and MacPherson classes, that is, the main result of [BS81].

A substantial part of this work was done while the author was visiting the *Institut de Mathématiques de Luminy* and the *Max-Planck-Institut für Mathematik* in Bonn, in the summer of 2005. I would especially like to thank Jean-Paul Brasselet and Matilde Marcolli, for the hospitality and for stimulating discussions. I also thank Jörg Schürmann for pointing out inaccuracies in an earlier version of this note, and for several clarifying remarks concerning the case of positive characteristic.

2. proChow groups

2.1. We work over an algebraically closed field k; further restrictions will come into play later (§2.5 and ff., §5.1 and ff.). *Schemes* will be understood to be of finite type over k. We say that X is *complete* if it is proper over k.

2.2. We denote by A_*X the conventional Chow group of X, as defined in [Ful84]; A_* is then a functor from the category of schemes to abelian groups, covariant with respect to *proper* maps. We begin by defining a functor $\widehat{A}_*$ from schemes to abelian groups, covariant with respect to arbitrary (regular) morphisms.

For any scheme U consider the category $\mathcal{U}$ of maps

$$i : U \to X^i$$

with X^i complete, and morphisms $i \to j$ given by commutative diagrams

$$\begin{tikzcd} & X^i \arrow[d, "\pi"] \\ U \arrow[ur, "i"] \arrow[r, "j"'] & X^j \end{tikzcd}$$

with π a proper morphism. Taking (conventional) Chow groups gives an inverse system $\{A_*X^i\}_i$ under (proper) push-forward.

Definition 2.1. The *proChow* group of U is the inverse limit of this system:

$$\widehat{A}_*U := \varprojlim_i A_*X^i \quad .$$

Thus, an element $\alpha \in \widehat{A}_*U$ is the choice of a class α^i in the Chow group of every complete variety X^i to which U maps, compatibly with proper push-forward.

Remark 2.2. In particular, elements $\alpha \in \widehat{A}_*U$ have a well-defined *degree*

$$\int \alpha \in \mathbb{Z} \quad :$$

the structure map $U \to \operatorname{Spec} k$ is a map to a complete variety, so α determines an element $\int \alpha \in A_* \operatorname{Spec} k = \mathbb{Z}$.

2.3. Any morphism $f : U \to V$ realizes the category $\mathcal{V}$ corresponding to V as a subcategory of $\mathcal{U}$; thus, a compatible assignment of classes in A_*X^i for all i in $\mathcal{U}$ determines in particular a compatible assignment for i in $\mathcal{V}$. That is, f induces a homomorphism

$$f_* : \widehat{A}_*U \to \widehat{A}_*V \quad .$$

The following remarks should be clear.

Lemma 2.3. *With notation as above:*

- $(f \circ g)_* = f_* \circ g_*$*: that is,* $\widehat{A}_*$ *is a covariant functor from the category of algebraic varieties, with morphisms, to abelian groups.*
- *If* X *is complete, then there is a canonical isomorphism* $\widehat{A}_*X \cong A_*X$*.*
- *If* $f : X \to Y$ *is a proper map of complete varieties, the induced homomorphism*
$$f_* : A_*X \cong \widehat{A}_*X \to \widehat{A}_*Y \cong A_*Y$$
is the conventional proper push-forward for Chow groups.

2.4. Our aim is not the computation of groups $\widehat{A}_*U$, but it will be necessary to develop tools to define and manipulate elements of these groups. To this effect, the following variation on the definition and the one proposed in §2.5 are very convenient.

We will say that an embedding $i : U \hookrightarrow X^i$ is a *closure* of U if X^i is complete and U is a dense open set of X^i; recall that every scheme U has closures ([Nag63]). Define

$$\overline{A}_*U := \varprojlim_{i \text{ closure of } U} A_*X^i \quad .$$

There trivially is a natural map

$$\widehat{A}_*U \to \overline{A}_*U \quad .$$

Lemma 2.4. *This map is an isomorphism.*

Proof. Let $\alpha \in \widehat{A}_*(U)$, and let $i : U \to X^i$ be any map from U to a complete variety. Let $j : U \hookrightarrow \overline{U}$ be any fixed closure of U. Then i extends to a *rational* map $\overline{i} : \overline{U} \dashrightarrow X^i$, resolving the indeterminacies of which produces a diagram

$$\begin{tikzcd}
 & \hat{U} \arrow[d] \arrow[dr, "\hat{\imath}"] & \\
U \arrow[ur, hook, "\hat{\jmath}"] \arrow[r, hook, "j"] \arrow[rr, bend right, "i"'] & \overline{U} \arrow[r, dashed, "\overline{i}"] & X^i
\end{tikzcd}$$

with $\hat{\jmath}$ also a closure of U: since i is defined on the whole of U, resolving its indeterminacies may be achieved by blowing up a locus in the complement of U, so the inclusion $U \subset \overline{U}$ lifts to an inclusion $U \subset \hat{U}$. The proper map $\hat{\imath}$ gives a morphism from the inclusion $\hat{\jmath}$ to i, hence $\alpha^i = \hat{\imath}_*(\alpha^{\hat{\jmath}})$. This assignment satisfies the necessary compatibilities, yielding a homomorphism from the inverse limit $\overline{A}_*(U)$ of Chow groups of closures to $\widehat{A}_*(U)$. This homomorphism is manifestly the inverse of the natural map $\widehat{A}_*(U) \to \overline{A}_*(U)$, verifying the statement. $\square$

Therefore, in order to define an element of $\widehat{A}_*U$ it suffices to define compatible classes in *closures* of U.

2.5. *We now assume that canonical resolution of singularities holds.*

If U is a *nonsingular variety,* then a further simplification applies. We say that an embedding $i : U \hookrightarrow X^i$ is a *good closure* of U if it is a closure in the sense of §2.4, and further

- X^i is nonsingular;
- the complement of U in X^i is a divisor with simple normal crossings (that is, normal crossings and nonsingular components).

Define

$$\widetilde{A}_*U := \varprojlim_{i \text{ good closure of } U} A_*X^i \quad ;$$

Then again we have a natural map

$$\widehat{A}_*U \to \widetilde{A}_*U \quad ,$$

and the analog of Lemma 2.4 holds:

Lemma 2.5. *This map is an isomorphism.*

Proof. By resolution of singularities, every closure is dominated by a good closure; this yields $\overline{A}_*U \cong \widetilde{A}_*U$, and the statement then follows from Lemma 2.4. $\square$

Corollary 2.6. *Assume U is nonsingular.*

- *In order to define an element $\alpha \in \widehat{A}_*U$ it suffices to assign $\alpha^i \in A_*X^i$ for good closures $i : U \hookrightarrow X^i$, satisfying the following compatibility requirement: if $i : U \hookrightarrow X^i$, $j : U \hookrightarrow X^j$ are good closures, and $i \to j$:*

$$\begin{tikzcd} & X^i \arrow[d, "\pi"] \\ U \arrow[ur, "i"] \arrow[r, "j"'] & X^j \end{tikzcd}$$

  with π a blow-up of X^j along a smooth center meeting $X^j \smallsetminus U$ with normal crossings, *then $\alpha^j = \pi_*(\alpha^i)$.*
- *Two elements α, β in $\widehat{A}_*U$ are equal if and only if $\alpha^i = \beta^i$ for all good closures i.*

Proof. The second assertion is immediate from Lemma 2.5.

For the first assertion, by Lemma 2.5 it suffices to compatibly assign α^i for good closures i; the reduction to the case of a blow-up is a standard application of the factorization theorem for birational maps ([AKMW02]). $\square$

2.6. The reader will have no difficulties establishing other simple analogous results, if she or he so desires.

For example, for every scheme U there is a canonical surjection $\widehat{A}_*U \to A_*U$, compatible with proper push-forwards; this may be realized by mapping $\widehat{A}_*U$ to A_*X^i for any closure i, and following with the natural surjection $A_*X^i \to A_*U$ ([Ful84], §1.8).

If U is nonsingular and $i : U \hookrightarrow X^i$ is a good closure, then $\widehat{A}_*U \to A_*X^i$ is in fact already a surjection; for example, this implies that $\widehat{A}_*\mathbb{A}^2$ is not finitely generated. In this sense $\widehat{A}_*U$ is, in general, 'much larger' than A_*U.

3. Globalizing local data

3.1. Next we come to the question of defining global invariants on a variety X from local data: for example, from data given on (open) strata U of a stratification of X. The functorial proChow group offers a natural way to do this:

- Suppose a class $\boxed{U} \in \widehat{A}_*U$ is defined for every *nonsingular* irreducible variety U;
- then we will define $\boxed{X} \in \widehat{A}_*X$ by

$$\boxed{X} := \sum_U i_{U*}\boxed{U} \quad ,$$

  where X is the disjoint union of the varieties U, each U is nonsingular, irreducible, and $i_U : U \to X$ denotes the inclusion.

Of course one has to check that this operation is well-defined. We say that the assignment $U \mapsto \boxed{U}$ for U nonsingular is *good local data* if this is the case. In this section we identify a condition yielding good local data.

3.2. We denote by

$$U \mapsto \mathfrak{c}_U^{\overline{U}} \in A_*\overline{U}$$

the assignment of a class to each nonsingular variety U, in a good closure $\overline{U}$ of U. In our application in §4, $\mathfrak{c}_U^{\overline{U}}$ will be obtained as the Chern class of a suitable bundle; in this section we are simply interested in whether this assignment defines *good* local data. We will use the following notations:

- U: a nonsingular irreducible variety;
- $\overline{U}$: a good closure of U;
- $D = \overline{U} \smallsetminus U$, a divisor with simple normal crossings in $\overline{U}$;
- W: a nonsingular closed irreducible subvariety of $\overline{U}$ meeting D with normal crossings;
- Z: the intersection $W \cap U$; note that W is a good closure of Z (if $Z \neq \emptyset$);
- $\pi : \overline{V} \to \overline{U}$: the blow-up of $\overline{U}$ along W;
- F: the exceptional divisor $\pi^{-1}(W)$;
- V: the blow-up of U along Z, that is, $\pi^{-1}(U)$;
- $E = \pi^{-1}(Z) = F \cap V$; F is a good closure of E if $Z \neq \emptyset$.

These may be collected in the diagram:

$$\begin{tikzcd}
& V \arrow[rr, hook] \arrow[dd] & & \overline{V} \arrow[dd, "\pi"] \\
E \arrow[ur, hook] \arrow[rr, hook] \arrow[dd] & & F \arrow[ur, hook] \arrow[dd] & \\
& U \arrow[rr, hook] & & \overline{U} \\
Z \arrow[ur, hook] \arrow[rr, hook] & & W \arrow[ur, hook, "w"'] &
\end{tikzcd}$$

Proposition 3.1. *Assume that, for all choices of U, $\overline{U}$, etc. as above,*

$$\mathfrak{c}_U^{\overline{U}} = \pi_*\mathfrak{c}_{V\smallsetminus E}^{\overline{V}} + w_*\mathfrak{c}_Z^W \quad .$$

Then the assignments $\mathfrak{c}_U^{\overline{U}}$ determine good local data $\boxed{U} \in \widehat{A}_*U$ in the sense of §3.1.

For notational convenience we understand $\mathfrak{c}_Z^W = 0$ if $Z = \emptyset$. Also note that $\overline{U}$ is complete and w, π are proper maps; the push-forwards appearing in the statement are the ordinary proper push-forwards for Chow groups $w_* : A_*W \to A_*\overline{U}$, $\pi_* : A_*\overline{V} \to A_*X$. The assignments $U \to \mathfrak{c}_U^{\overline{U}}$ determine a class $\boxed{U}$ in the proChow group of U by virtue of Corollary 2.6: it is part of the statement of the proposition that the necessary compatibility is satisfied.

Proof. If W is disjoint from U then $Z = E = \emptyset$, and $V \cong U$; the assumption reduces to

$$\mathfrak{c}_U^{\overline{U}} = \pi_*\mathfrak{c}_U^{\overline{V}} \quad ,$$

that is, the compatibility requirement in Corollary 2.6. Thus the prescription $U \mapsto \mathfrak{c}_U^{\overline{U}}$ does define an element $\boxed{U}$ in $\widehat{A}_*U$.

For $Z \neq \emptyset$, the stated assumption implies that

$$(*) \qquad \boxed{U} = z_* \boxed{Z} + i_* \boxed{U \smallsetminus Z}$$

for all nonsingular varieties U and nonsingular closed subvarieties Z, where $z : Z \hookrightarrow U$ and $i : U \smallsetminus Z \hookrightarrow U$ are the embeddings and z_*, i_* are the push-forwards defined in §2.3. Indeed, by Corollary 2.6 it suffices to check this equality after specializing to any good closure of U; and we may dominate any good closure of U with a good closure $\overline{U}$ such that $W = \overline{Z}$ is a good closure of Z. This is the situation considered above; since $U \smallsetminus Z \cong V \smallsetminus E$, and $\overline{V}$ is a good closure of the latter, the formula in the statement of the proposition implies the claimed equality.

To prove that the assignment $U \mapsto \boxed{U}$ defines good local data, we have to show that if X is any variety, and

$$X = \amalg_\alpha U_\alpha$$

is a decomposition of X as a finite disjoint union of nonsingular subvarieties $U_\alpha \xhookrightarrow{i_{U_\alpha}} X$, then

$$\sum_\alpha i_{U_\alpha *} \boxed{U_\alpha}$$

is independent of the decomposition. Any two such decompositions admit a common refinement, hence we may assume that every element U of one decomposition is a finite disjoint union of elements from the other:

$$U = V_1 \amalg V_2 \amalg \cdots \amalg V_r$$

where U and all V_j are nonsingular; and we may assume V_j is closed in $V_1 \amalg \cdots \amalg V_j$. The required equality

$$\boxed{U} = i_{1*} \boxed{V_1} + \cdots + i_{r*} \boxed{V_r}$$

where $i_j : V_i \hookrightarrow U$ denotes the inclusion, is then an immediate consequence of (*), concluding the proof. $\square$

3.3. The definition of $\boxed{X}$ for arbitrary varieties X, extending good local data for nonsingular varieties as specified in §3.1, satisfies 'inclusion-exclusion'. More precisely:

Proposition 3.2. *If $X = \cup_{j \in J} X_j$ is a finite union of subvarieties, then*

$$\boxed{X} = \sum_{\emptyset \neq I \subset J} (-1)^{|I|+1} \nu_{I*} \boxed{X_I}$$

where $X_I = \cap_{i \in I} X_i$, and $\nu_I : X_I \hookrightarrow X$ is the inclusion.

Proof. This follows immediately from the case in which $|J| = 2$, that is (omitting push-forwards):

$$\boxed{X \cup Y} = \boxed{X} + \boxed{Y} - \boxed{X \cap Y} \quad ,$$

which is straightforward from the definition. $\square$

The same applies *a fortiori* to the degrees $\int \boxed{X}$ (as defined in Remark 2.2).

3.4. The group of *constructible functions* of a variety X is the group $\mathcal{F}(X)$ of finite integer linear combinations of functions $1\!\!1_Z$, for Z subvarieties of X, where

$$1\!\!1_Z(p) = \begin{cases} 1 & p \in Z \\ 0 & p \notin Z \end{cases} \quad .$$

For $\varphi \in \mathcal{F}(X)$, we may define an element

$$\boxed{\varphi} \in \widehat{A}_* X$$

as follows: if $\varphi = \sum_Z n_Z 1\!\!1_Z$, we set

$$\boxed{\varphi} = \sum_Z n_Z \nu_{Z*} \boxed{Z} \quad ,$$

where $\nu_Z : Z \hookrightarrow X$ is the inclusion. Inclusion-exclusion implies that this definition is independent of the decomposition of φ (if $\boxed{\cdot}$ arises from good local data).

In good situations, the choice of good local data determines a *push-forward* for constructible functions, as follows: if $f : X \to Y$ is a morphism of algebraic varieties, define

$$f_* : F(X) \to F(Y)$$

by setting, for $\varphi = \sum_Z n_Z 1\!\!1_Z$ as above, and $p \in Y$,

$$f_*(\varphi)(p) = \sum_Z n_Z \int \boxed{f^{-1}(p) \cap Z} \quad ;$$

again, the independence on the choices follows from inclusion-exclusion.

For the local data to be defined in the next section, and in characteristic zero, the right-hand-side in this prescription is easily seen to be constructible as needed (by Lemma 5.8); and we will show that the resulting push-forward is covariant with respect to regular maps. The assignment $\varphi \mapsto \boxed{\varphi}$ will then give a transformation of functors

$$\mathcal{F} \rightsquigarrow \widehat{A}_* \quad ,$$

and the main result in the rest of the paper will be that, for that choice of local data, this is a *natural* transformation.

Specializing to complete varieties, ordinary Chow groups, and proper maps, will recover the standard algebraic version of MacPherson's natural transformation.

4. proCSM classes

4.1. We next choose specific local data, by using Proposition 3.1.

Let U be a nonsingular variety, and let $i : U \hookrightarrow \overline{U}$ be a good closure of U. In particular, $\overline{U} \smallsetminus U$ is a divisor D with normal crossings and nonsingular components D_i, $i = 1, \dots, r$, in $\overline{U}$.

Definition 4.1. Set

$$\mathfrak{c}_U^{\overline{U}} := c(\Omega^1_{\overline{U}}(\log D)^\vee) \cap [\overline{U}] \in A_* \overline{U} \quad .$$

Here $\Omega^1_{\overline{U}}(\log D)$ denotes the bundle of differential 1-forms with logarithmic poles along D.

As we will prove, this assignment specifies *good* local data. We will use the following immediate lemma:

Lemma 4.2. *Let $\rho^\circ : E \to Z$ be a proper, smooth, surjective map of nonsingular varieties. Let F, W resp. be good closure of E, W, and assume that ρ° is the restriction of a proper, smooth, surjective map $\rho : F \to W$.*

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "\rho^\circ"'] & F \arrow[d, "\rho"] \\ Z \arrow[r, hook] & W \end{tikzcd}$$

Then

$$\rho_* \mathfrak{c}_E^F = \chi \cdot \mathfrak{c}_Z^W \quad ,$$

where χ is the degree of the top Chern class of the tangent bundle to any fiber of ρ.

Proof. Let $D = W \smallsetminus Z$, a divisor with normal crossings and nonsingular components D_i by assumption; $F \smallsetminus E = \rho^{-1}(D)$ is also a divisor with simple normal crossings, and components $\rho^{-1}(D_i)$.

In this situation, the exact sequence of differentials on F:

$$0 \longrightarrow \rho^*\Omega^1_W \longrightarrow \Omega^1_F \longrightarrow \Omega^1_{F|W} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \rho^*\Omega^1_W(\log D) \longrightarrow \Omega^1_F(\log \rho^{-1}(D)) \longrightarrow \Omega^1_{F|W} \longrightarrow 0$$

The statement follows immediately from this sequence and the projection formula, since

$$\rho_* \left(c({\Omega^1_{F|W}}^\vee) \cap [F] \right) = \chi \cdot [W] \quad .$$

$\square$

4.2. Verifying that the assignment specified above defines good local data is now a straightforward (but somewhat involved) computation.

Proposition 4.3. *The classes $\mathfrak{c}_U^{\overline{U}}$ satisfy the conditions specified in Proposition 3.1; therefore, they define good local data in the sense of §3.1.*

Proof. We adopt the notation in §3.2, and in particular the blow-up diagram

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d, "\rho"'] & \overline{V} \arrow[d, "\pi"] \\ W \arrow[r, "w"'] & \overline{U} \end{tikzcd}$$

and we have to verify that

$$\mathfrak{c}_U^{\overline{U}} = \pi_* \mathfrak{c}_{V \smallsetminus E}^{\overline{V}} + w_* \mathfrak{c}_Z^W \quad .$$

Also recall that

$$c(\Omega^1_{\overline{U}}(\log D)^\vee) = \frac{c(T\overline{U})}{\prod_i (1 + D_i)}$$

(as follows from a residue exact sequence, cf. [Sil96], 3.1).

First assume that $Z = \emptyset$, that is, W is contained in D. Denote by $\widetilde{D}_i$ the proper transforms of the components D_i. Then it is easily checked (cf. Lemma 2.4 in [Alu04]) that the exceptional divisor F and the hypersurfaces $\widetilde{D}_i$ together form a divisor with simple normal crossings, and their union is the complement of $V \cong U$ in $\overline{V}$. The needed statement then becomes

$$\pi_* \frac{c(T\overline{V})}{(1+F)\prod_i(1+\widetilde{D}_i)} \cap [\overline{V}] = \frac{c(T\overline{U})}{\prod_i(1+D_i)} \cap [\overline{U}] \quad ,$$

under the assumption that W is contained in at least one component of D. This is Lemma 3.8, part (5), in [Alu].

If $Z \neq \emptyset$, that is, W is *not* contained in any component of D, then the proper transforms $\widetilde{D}_i$ of D_i agree with the inverse images $\pi^{-1}(D_i)$, for all i. The blow-up V of U along Z admits $\overline{V}$ as good closure, with complement $\overline{V} \smallsetminus V = \cup \widetilde{D}_i$. The projection formula gives

$$\pi_* \left(\frac{c(T\overline{V})}{\prod_i(1+\widetilde{D}_i)} \cap [\overline{V}] \right) = \frac{1}{\prod_i(1+D_i)} \cap \pi_*(c(T\overline{V}) \cap [\overline{V}])$$

since the class of $\widetilde{D}_i$ is $\pi^*(D_i)$. By part (2) of Lemma 3.8 in [Alu], this equals

$$\frac{1}{\prod_i(1+D_i)} \cap (c(T\overline{U}) \cap [\overline{U}] + (d-1) w_* c(TW) \cap [\overline{W}]) \quad ,$$

where d denotes the codimension of W in $\overline{U}$ and $w : W \hookrightarrow \overline{U}$ is the embedding.

Since W meets D with normal crossings, the divisors D_i cut out on W nonsingular divisors, meeting with normal crossings in W, and whose union is the complement of Z in W. In other words W is a good closure of Z, and the computation given above yields

$$\pi_* \mathfrak{c}_V^{\overline{V}} = \mathfrak{c}_U^{\overline{U}} + (d-1) w_* \mathfrak{c}_Z^W \quad .$$

On the other hand, $\overline{V}$ is a good closure of $V \smallsetminus E$, with complement the normal crossing divisor consisting of the exceptional divisor F and the components $\widetilde{D}_i$. Thus

$$\begin{aligned} \mathfrak{c}_{V\smallsetminus E}^{\overline{V}} &= \frac{c(T\overline{V})}{(1+F)\prod_i(1+\widetilde{D}_i)} \cap [\overline{V}] = \frac{c(T\overline{V})}{\prod_i(1+\widetilde{D}_i)} \cap [\overline{V}] - \frac{c(T\overline{V})}{\prod_i(1+\widetilde{D}_i)} \cap [F] \\ &= \frac{c(T\overline{V})}{\prod_i(1+\widetilde{D}_i)} \cap [\overline{V}] - j_* \frac{c(TF)}{\prod_i(1+j^*\widetilde{D}_i)} \cap [F] \quad ; \end{aligned}$$

since F is a good closure of E, with complement given by the union of the intersections $\widetilde{D}_i \cap F$ (of class $j^*\widetilde{D}_i$), this shows

$$\mathfrak{c}_{V\smallsetminus E}^{\overline{V}} = \mathfrak{c}_V^{\overline{V}} - j_* \mathfrak{c}_E^F \quad .$$

Combining with the formula obtained above, we get

$$\mathfrak{c}_U^{\overline{U}} = \pi_* \mathfrak{c}_{V\smallsetminus E}^{\overline{V}} + w_* \left(\rho_* \mathfrak{c}_E^F - (d-1) \mathfrak{c}_Z^W \right) \quad .$$

Now $\rho : F \to W$ is a projective bundle, hence smooth and proper, with fibers $\mathbb{P}^{d-1}$; it restricts to the projective bundle $E \to Z$. Applying Lemma 4.2, with $\chi = \int c(T\mathbb{P}^{d-1}) \cap [\mathbb{P}^{d-1}] = d$, gives

$$\rho_* \mathfrak{c}_E^F = d\, \mathfrak{c}_Z^W \quad ,$$

concluding the proof. □

4.3. We are ready to define *proCSM classes,* and the corresponding transformation $F \rightsquigarrow \widehat{A}_*$.

Definition 4.4. The *proCSM class* of a (possibly singular) variety X is the class

$$\boxed{X} \in \widehat{A}_* X$$

in the proChow group of X, defined by patching the local data defined in §4.1, as explained in §3.

Explicitly, write $X = \amalg_\alpha U_\alpha$ in any way as a finite disjoint union of nonsingular subvarieties $U_\alpha \overset{i_{U_\alpha}}{\hookrightarrow} X$; then $\boxed{X} := \sum_\alpha i_{U_\alpha *}\boxed{U_\alpha}$ is independent of the decomposition, by Proposition 4.3.

The following 'normalization' properties are easy consequences of the definition.

Proposition 4.5.
- *If X is complete and nonsingular, then*

$$\boxed{X} = c(TX) \cap [X] \quad .$$

- *If X is a compact complex algebraic variety, then*

$$\int \boxed{X} = \chi_{top}(X) \quad ,$$

the topological Euler characteristic of X.

Proof. The first statement is immediate, as X is a good closure of itself and $\widehat{A}_* X = A_* X$ if X is complete, and $\mathfrak{c}_X^X = c(TX) \cap [X]$.

For the second statement, since both $\int \boxed{\cdot}$ and χ_{top} satisfy inclusion-exclusion we only need to check this equality for X compact *and nonsingular.* By the first statement (and the Poincaré-Hopf theorem)

$$\int \boxed{X} = \int c(TX) \cap [X] = \chi_{\text{top}}(X)$$

in this case, as needed. □

4.4. The proCSM class *of a constructible function* φ on a variety X, $\boxed{\varphi}$, and the push-forward of constructible functions f_* may now be defined as in §3.4.

The second statement in Proposition 4.5 implies that, for compact complex algebraic varieties and f proper, this definition of push-forward for constructible functions agrees with the conventional one (as given in [Ful84], §19.1.7).

However, covariance properties of the more general push-forward introduced here, and the naturality of the corresponding transformation $F \rightsquigarrow \widehat{A}_*$, do not appear to be immediate. We will address these questions in the next section.

5. The natural transformation $F \rightsquigarrow \widehat{A}_*$

5.1. To summarize, we have now defined a notion of *proCSM class* for possibly noncomplete, possibly singular varieties X, in the proChow group of X:

$$\boxed{X} \in \widehat{A}_* X \quad .$$

This definition relies on the local-to-global machinery of §3. We have used resolution of singularities, and the factorization theorem of [AKMW02], and the definition of proCSM class can be given in any context in which these tools apply.

By contrast, the statement to be proved in this section will use characteristic zero more crucially. *Therefore, in the rest of the paper we work over an algebraically closed field of characteristic zero.*

In §3.4 we have also proposed a notion of push-forward of constructible functions

$$f_* : F(X) \to F(Y)$$

for any morphism $f : X \to Y$, and a transformation

$$F \rightsquigarrow \widehat{A}_* \quad .$$

This depends on $f_*(1\!\!1_Z)$ being a constructible function on Y, for any subvariety Z of X; this is easily seen to be the case for the definition obtained from the data defined in §4, at least in characteristic zero.

Theorem 5.1 (Covariance)**.** *Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic varieties. Then*

$$(g \circ f)_* = g_* \circ f_*$$

as push-forwards $F(X) \to F(Z)$.

Theorem 5.2 (Naturality)**.** *The transformation $F \rightsquigarrow \widehat{A}_*$ is a* natural *transformation of covariant functors from the category of algebraic varieties over an algebraically closed field of characteristic zero, with morphisms, to the category of abelian groups.*

In view of Proposition 4.5, Theorem 5.2 shows that there is a natural transformation from F to $\widehat{A}_*$, and hence to homology, specializing to the total Chern class of the tangent bundle on nonsingular varieties. This recovers Theorem 1 in [Mac74]. The uniqueness of the natural transformation is immediate from resolution of singularities, and it follows that the (image in homology of the) proCSM classes defined in §4 agree with the classes constructed by MacPherson[1].

In order to provide a self-contained treatment of (pro)CSM classes, MacPherson's theorem will not be used in the proof of Theorems 5.1 and 5.2 (MacPherson's *graph construction* will be an ingredient in the proof of Lemma 5.3).

5.2. It is natural to wonder whether the characteristic zero restriction in Theorems 5.1 and 5.2 is crucial, or whether it is a technical requirement for our approach to the proof. The restriction is in fact necessary: as Jörg Schürmann pointed out to me, the presence in characteristic $p > 0$ of étale self-covers of $\mathbb{A}^1$, such as the Artin-Schreier map $x \mapsto x^p - x$, gives a counterexample to covariance. Other simple examples (such as the Frobenius map) indicate that the difficulty cannot be circumvented by naive modifications to the definition of push-forward of constructible functions.

We do not know, even at a conjectural level, how the formalism could be modified in order to avoid the characteristic zero requirement. In our argument the condition will enter in the proof of Lemma 5.3 (specifically, in the key Lemma 6.6), and through 'generic gentleness'. Schürmann's example shows that Lemma 5.3 fails in positive characteristic.

[1]This also follows more directly from MacPherson's theorem, arguing essentially as in the proof of Proposition 4.5.

5.3. The proofs of Theorems 5.1 and 5.2 depend on the following lemma, which at first sight would appear to be a harmless generalization of the trivial Lemma 4.2. Its proof is on the contrary rather technical, and we postpone it to §6. We state the lemma here, and use it to prove Theorems 5.1 and 5.2 in the rest of this section.

Lemma 5.3. *Let $f : U \to V$ be a proper, smooth, surjective map of nonsingular varieties over an algebraically closed field of characteristic zero. Then*

$$f_*\boxed{U} = \chi_f \cdot \boxed{V} \quad ,$$

where $\chi_f = \int \boxed{f^{-1}(p)}$ (for any $p \in V$).

Since by hypothesis the fibers $f^{-1}(p)$ are complete and nonsingular, χ_f equals the degree of the top Chern class of their tangent bundle (by Proposition 4.5).

5.4. First, we formulate an upgrade of Lemma 5.3 to a mildly larger class of maps.

Definition 5.4. A morphism $f : U \to V$ of nonsingular varieties is *gentle* if it is smooth and surjective, and further there is a variety $\underline{U}$ and a *proper,* smooth, surjective morphism $\underline{f} : \underline{U} \to V$ such that:

- U is an open dense subset of $\underline{U}$, and $\underline{f}|_U = f$;
- the complement $H = \underline{U} \smallsetminus U$ is a divisor with normal crossings and nonsingular components H_i;
- letting H_I denote the intersection $\cap_{i \in I} H_i$ (so $H_\emptyset = \underline{U}$, and each H_I is nonsingular), each restriction
$$\underline{f}|_{H_I} : H_I \to V$$
is proper, smooth, and surjective.

Lemma 5.5. *If $f : U \to V$ is gentle, then the number*

$$\chi_f := \int \boxed{f^{-1}(p)}$$

is independent of $p \in V$; further, in characteristic zero,

$$f_*\boxed{U} = \chi_f \cdot \boxed{V} \quad .$$

Proof. By inclusion-exclusion (Proposition 3.2):

$$\boxed{U} = \sum_I (-1)^{|I|+1} i_{H_I *} \boxed{H_I}$$

and

$$\int \boxed{f^{-1}(p)} = \sum_I (-1)^{|I|+1} \int \boxed{H_I \cap \underline{f}^{-1}(p)}$$

with notation as in Definition 5.4, and denoting by i_{H_I} the inclusion $H_I \hookrightarrow \underline{U}$.

Since each H_I maps properly, smoothly, and surjectively onto V, the statement follows from Lemma 5.3. □

5.5. We also single out the following easy properties of 'gentleness':

Lemma 5.6.
- *If $f: U \to V$ is gentle, and $W \subset V$ is a nonsingular subvariety, then the restriction*
$$f^{-1}(W) \to W$$
is gentle;
- *'Generic gentleness' holds in characteristic zero: if $f: X \to Y$ is any morphism of varieties, and X is nonsingular, then there exists a nonempty, nonsingular open subset $V \subset Y$ such that f restricts to a gentle map $U = f^{-1}(V) \to V$;*
- *If f, g, and $g \circ f$ are all gentle:*
$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & V \arrow[r, "g"] & W \end{tikzcd}$$
then $\chi_{g\circ f} = \chi_g \cdot \chi_f$.

Proof. The first statement is clear.

The second is straightforward from Nagata's theorem extending f to a proper map ([Nag63]), embedded resolution of singularities to guarantee the complement is a divisor H with simple normal crossings, and ordinary generic *smoothness* (Corollary 10.7 in [Har77]) applied to all intersections of components of H.

For the third, let p be any point of W; by the first statement, the restriction
$$f_p : (g \circ f)^{-1}(p) \to g^{-1}(p)$$
is gentle (as f is); therefore, by Lemma 5.5,
$$f_{p*}\boxed{(g\circ f)^{-1}(p)} = \chi_f \cdot \boxed{g^{-1}(p)} \quad .$$
Taking degrees gives the stated equality, since $g\circ f$ and g are gentle, and push-forwards preserve degrees. □

5.6. Generic gentleness lets us decompose *any* map into gentle ones (in characteristic zero):

Proposition 5.7. *Let $f : X \to Y$ be any morphism of varieties over an algebraically closed field of characteristic zero. Then there are decompositions*
$$X = \amalg_{\alpha,i} U_{\alpha i} \quad , \quad Y = \amalg_\alpha V_\alpha$$
into disjoint nonsingular irreducible subvarieties, such that, for all α and i, f restricts to a gentle map
$$f_{\alpha i} := f|_{U_{\alpha i}} : U_{\alpha i} \to V_\alpha \quad .$$

Proof. This follows immediately from generic gentleness, after decomposing X as a disjoint union of nonsingular irreducible subvarieties. □

5.7. The proofs of Theorem 5.1 and 5.2 are now straightforward. We begin with covariance.

Proof of Theorem 5.1. It suffices to show that the two push-forwards agree on the characteristic function of each subvariety of X, and by restricting f we are reduced to showing

$$(g \circ f)_*(1\!\!1_X) = g_* \circ f_*(1\!\!1_X) \quad .$$

Two applications of Proposition 5.7 yield decompositions

$$X = \amalg_{\alpha,i,j} U_{\alpha ij} \quad , \quad Y = \amalg_{\alpha,i} V_{\alpha i} \quad , \quad Z = \amalg_\alpha W_\alpha$$

such that all restrictions $U_{\alpha ij} \to V_{\alpha i}$, $V_{\alpha i} \to W_\alpha$ are gentle. Generic gentleness allows us to assume that the compositions $U_{\alpha ij} \to W_\alpha$ are also gentle. Denote by $\chi'_{\alpha ij}$, $\chi_{\alpha i}$, $\chi_{\alpha ij}$ the corresponding fiberwise degrees; by Lemma 5.6,

$$\chi_{\alpha ij} = \chi'_{\alpha ij} \cdot \chi_{\alpha i} \quad .$$

The computation is then completely straightforward:

$$g_*(f_*(1\!\!1_{U_{\alpha ij}})) = g_*(\chi'_{\alpha ij} 1\!\!1_{V_{\alpha i}}) = \chi'_{\alpha ij} \cdot \chi_{\alpha i} 1\!\!1_{W_\alpha} = \chi_{\alpha ij} 1\!\!1_{W_\alpha} = (g \circ f)_*(1\!\!1_{U_{\alpha ij}}) \quad ,$$

and $g_*(f_*(1\!\!1_X)) = (g \circ f)_*(1\!\!1_X)$ follows by linearity as $1\!\!1_X = \sum_{\alpha,i,j} 1\!\!1_{U_{\alpha ij}}$. □

5.8. For naturality, note that any splitting of a morphism $f : X \to Y$ into gentle maps gives a parallel splitting of both $f_*(1\!\!1_X)$ and $f_*\boxed{X}$:

Lemma 5.8. *Let $f : X \to Y$ be a morphism of varieties in characteristic zero, and let $X = \amalg U_{\alpha i}$, $Y = \amalg V_\alpha$ be decompositions as in Proposition 5.7. For each α, let $\chi_\alpha = \sum_i \chi_{f_{\alpha i}}$. Then*

$$f_*(1\!\!1_X) = \sum_{\alpha \in A} \chi_\alpha 1\!\!1_{V_\alpha} \quad ;$$

$$f_*\boxed{X} = \sum_{\alpha \in A} \chi_\alpha \cdot \boxed{V_\alpha} \quad .$$

Proof. Any given $p \in Y$ is in precisely one V_α; and then $f^{-1}(p)$ is the disjoint union of the fibers of $f_{\alpha i}$. Hence

$$\int \boxed{f^{-1}(p)} = \sum_i \int \boxed{f_{\alpha i}^{-1}(p)} = \sum_i \chi_{f_{\alpha i}} = \chi_\alpha \quad .$$

This gives the first formula, by definition of push-forward of constructible functions.

The second formula follows from Lemma 5.5:

$$f_*\boxed{X} = \sum_{\alpha,i} f_*\boxed{U_{\alpha i}} = \sum_\alpha \sum_i \chi_{f_{\alpha i}} \cdot \boxed{V_\alpha} = \sum_\alpha \chi_\alpha \cdot \boxed{V_\alpha} \quad ,$$

since each $f_{\alpha i} : U_{\alpha i} \to V_\alpha$ is gentle. □

Naturality follows immediately:

Proof of Theorem 5.2. We have to show that for any $f : X \to Y$, and any constructible function $\varphi \in F(X)$,

$$f_*\boxed{\varphi} = \boxed{f_*\varphi} \quad .$$

By linearity of (both) f_* it suffices to prove this equality for the characteristic function of any subvariety of X; by restricting f we may assume this subvariety is X. That is, it suffices to prove that

$$f_* \boxed{1\!\!1_X} = \boxed{f_*(1\!\!1_X)} \quad ;$$

and this follows immediately from Lemma 5.8. □

5.9. This concludes the verification of the Deligne-Grothendieck conjecture for the local data defined in §4. The only outstanding item is the proof of the key Lemma 5.3, with which we will close the paper.

6. Proof of Lemma 5.3

6.1. We have to prove that if U, V are nonsingular varieties over an algebraically closed field of characteristic zero, and $f : U \to V$ is proper, smooth, and surjective, then

$$f_* \boxed{U} = \chi_f \cdot \boxed{V} \quad ,$$

where χ_f is the degree of the proCSM class of the fibers of f; that is, χ_f equals the degree of the top Chern class of the tangent bundle of any fiber of f.

By Corollary 2.6 and the definition of push-forward on proChow groups, this amounts to verifying that, for any good closure $j : V \hookrightarrow Y$,

$$\boxed{U}^{j\circ f} = \chi_f \cdot \boxed{V}^{j} \quad .$$

Dominating $j \circ f$ by a good closure $i : U \hookrightarrow X$, we have the following fiber square:

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ V \arrow[r, "j"'] & Y \end{tikzcd}$$

with i and j good closures, f smooth, f and g proper and surjective, and we have to verify that

$$g_* \mathfrak{c}_U^X = \chi_f \cdot \mathfrak{c}_V^Y \quad .$$

With the local data chosen in §4.1, this amounts to the following claim:

Claim 6.1. *In a fiber square as above, denote by D, E the complements $X \smallsetminus U$, $Y \smallsetminus V$ (which are divisors with simple normal crossings by assumption). Then*

$$g_* \left(c(\Omega^1_X(\log D)^\vee) \cap [X] \right) = \chi_f \cdot c(\Omega^1_Y(\log E)^\vee) \cap [Y] \quad .$$

This is our objective. The reader should compare Claim 6.1 with Lemma 4.2: the given formula is immediate if g is a *smooth* map extending the smooth map f; while the general case of Claim 6.1 requires a bit of work (presented in the next several subsections) and relies more substantially on the hypothesis on the characteristic.

Claim 6.1 is in a sense equivalent to the naturality of (pro)CSM classes: Theorem 5.2 is proved in §5 as a consequence of Lemma 5.3 (and hence of Claim 6.1); conversely, Claim 6.1 could be proved as a corollary of MacPherson's naturality theorem (exercise for the reader!). In order to keep the paper self-contained, the proof given here does not assume the result of [Mac74].

6.2. We will prove Claim 6.1 by applying the *graph construction* (see [Ful84], Chapter 18 and especially Example 18.1.6) to the (logarithmic) differential map

$$dg : g^*\Omega^1_Y(\log E) \to \Omega^1_X(\log D) \quad .$$

As g is smooth along U, this map is injective over U. The graph construction produces a 'cycle at infinity' measuring the singularities of dg, which may be used to evaluate the difference in the Chern classes of Claim 6.1. We will show that all components of the cycle at infinity dominating loci within $D = X \smallsetminus U$ give vanishing contribution to this difference, and the claim will follow.

6.3. For notational convenience we switch to bundles of differential forms with logarithmic poles (rather than their duals). The formula in Claim 6.1 is equivalent to

$$g_*\left(c(\Omega^1_X(\log D)) \cap [X]\right) = \underline{\chi}_f \cdot c(\Omega^1_Y(\log E)) \cap [Y] \quad ,$$

where $\underline{\chi}_f$ is the degree of the top Chern class of the *co*tangent bundle of any fiber of f.

Denote by m, n resp. the dimensions of X, Y. The differential dg determines a *rational* map to the Grassmann bundle

$$\gamma : X \times \mathbb{P}^1 \dashrightarrow \mathbb{G} = \mathrm{Grass}_n(g^*\Omega^1_Y(\log E) \oplus \Omega^1_X(\log D))$$

restricting on $X \times \{(\lambda : 1)\}$ for $\lambda \neq 0$ to the map assigning to $x \in X$ the graph of $\frac{1}{\lambda}dg$ at x. Over $X \times \infty := X \times \{(1 : 0)\}$ (that is, for $\lambda \to \infty$) γ acts as the section corresponding to $g^*\Omega^1_Y(\log E) \oplus 0$.

6.4. The indeterminacies of γ are contained in $X \times 0 := X \times \{(0 : 1)\}$; and in fact in $D \times 0 \subset X \times 0$ (since dg is injective along U). Closing the graph of γ (or equivalently blowing up the ideal of indeterminacies) gives a variety $\widetilde{X \times \mathbb{P}^1}$ and a *regular* lift $\widetilde{\gamma}$ of γ:

$$\begin{tikzcd}
\widetilde{X \times \mathbb{P}^1} \arrow[d, "\pi"] \arrow[dr, "\widetilde{\gamma}"] \arrow[ddd, bend right, "p"'] & \\
X \times \mathbb{P}^1 \arrow[r, dashed, "\gamma"] \arrow[d, "\rho"] & \mathbb{G} \arrow[dl] \\
X \arrow[d, "g"] & \\
Y &
\end{tikzcd}$$

Now

$$[\pi^{-1}(X \times \infty)] = [\pi^{-1}(X \times 0)]$$

as rational equivalence classes of divisors. Note that $\pi^{-1}(X \times \infty)$ maps isomorphically to X. As for $\pi^{-1}(X \times 0)$, it consists of the proper transform $\widetilde{X}$ of $X \times 0$ and of the components Γ_i of the exceptional divisor, appearing with multiplicities r_i.

6.5. Let $\mathcal{Q}$ denote the *universal quotient bundle* over $\mathbb{G}$, a bundle of rank m. We have

$$c(\widetilde{\gamma}^*\mathcal{Q}) \cap [\pi^{-1}(X \times \infty)] = c(\widetilde{\gamma}^*\mathcal{Q}) \cap \left([\widetilde{X}] + \sum_i r_i [\Gamma_i] \right)$$

in $A_*(X \times \mathbb{P}^1)$.

Lemma 6.2. *The following equalities hold in Y:*

- $p_* \left(c(\widetilde{\gamma}^*\mathcal{Q}) \cap [\pi^{-1}(X \times \infty)] \right) = g_* \left(c(\Omega^1_X(\log D)) \cap [X] \right) \quad ;$
- $p_* \left(c(\widetilde{\gamma}^*\mathcal{Q}) \cap [\widetilde{X}] \right) = \underline{\chi}_f \cdot c(\Omega^1_Y(\log E)) \cap [Y] \quad .$

Proof. These are easy consequences of the basic set-up. For the second equality, the restriction $\widetilde{\gamma}'$ of $\widetilde{\gamma}$ to $\widetilde{X}$ factors through

$$\mathbb{G}' := \mathrm{Grass}_n(\Omega^1_X(\log D)) \cong \mathrm{Grass}_n(\Omega^1_X(\log D) \oplus 0) \subset \mathbb{G} \quad .$$

Over $\mathbb{G}'$, $\mathcal{Q}$ splits as the direct sum of the universal quotient bundle $\mathcal{Q}'$ of rank $(m-n)$ of $\mathbb{G}'$, and the pull-back of $g^*\Omega^1_Y(\log E)$; thus

$$c(\widetilde{\gamma}^*\mathcal{Q}) \cap [\widetilde{X}] = c(p^*\Omega^1_Y(\log E)) \cdot c(\widetilde{\gamma}'^*\mathcal{Q}') \cap [\widetilde{X}] \quad ,$$

and the given formula follows immediately, since $\mathcal{Q}'$ restricts to the cotangent bundle on fibers over points of V. □

6.6. As promised in the short summary in §6.2, Lemma 6.2 shows that the difference between the classes appearing in Claim 6.1 is controlled by components of the cycle at infinity in the graph construction. Explicitly:

Corollary 6.3.

$$g_* \left(c(\Omega^1_X(\log D)) \cap [X] \right) - \underline{\chi}_f \cdot c(\Omega^1_Y(\log E)) \cap [Y] = \sum_i r_i \cdot p_* \left(c(\mathcal{Q}) \cap [\Gamma_i] \right)$$

Therefore, the following claim will conclude the proof of Lemma 5.3:

Claim 6.4. *For any component Γ of the exceptional divisor in $\widetilde{X \times \mathbb{P}^1}$,*

$$p_* \left(c(\mathcal{Q}) \cap [\Gamma] \right) = 0 \quad .$$

Incidentally, up to this point the discussion could have been carried out for the ordinary differential of the map g; but Claim 6.4 fails for the ordinary differential (cf. [Ful84], Example 18.1.6 (f)). Claim 6.4 is a remarkable property of the *logarithmic* differential (in characteristic zero).

6.7. The proof of Claim 6.4 will rely on the following general observation.

Lemma 6.5. *Let $p : \Gamma \to W$ be a proper morphism of schemes, and let $\mathcal{Q}_\Gamma$ be a vector bundle on Γ, of rank $\le \dim \Gamma$. Assume that there is a surjection $\mathcal{Q}_\Gamma \twoheadrightarrow p^*\mathcal{T}$ of coherent sheaves on Γ, where $\mathcal{T}$ is a coherent sheaf on W of rank $> \dim W$. Then*

$$p_* \left(c(\mathcal{Q}_\Gamma) \cap [\Gamma] \right) = 0 \quad .$$

Proof. There exists (see for example [NA83], §1.1) a proper birational morphism $\nu : \widetilde{W} \to W$ such that the pull-back $\nu^*(\mathcal{T})$ is locally free modulo torsion: in particular, there is a surjection $\nu^*\mathcal{T} \twoheadrightarrow \widehat{\mathcal{T}}$ with $\widehat{\mathcal{T}}$ locally free on $\widetilde{W}$, of rank equal to the rank of $\mathcal{T}$ (and hence $> \dim W$). Let $\widetilde{\Gamma}$ be the component dominating Γ in the fiber product:

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[r, "\widehat{\nu}"] \arrow[d, "\widetilde{p}"'] & \Gamma \arrow[d, "p"] \\ \widetilde{W} \arrow[r, "\nu"'] & W \end{tikzcd}$$

Then we have surjections

$$\widehat{\nu}^* \mathcal{Q}_\Gamma \twoheadrightarrow \widehat{\nu}^* p^* \mathcal{T} = \widetilde{p}^* \nu^* \mathcal{T} \twoheadrightarrow \widetilde{p}^* \widehat{\mathcal{T}}$$

with $\widetilde{p}^*\widehat{\mathcal{T}}$ locally free on $\widetilde{\Gamma}$, of rank $> \dim W$. Let $\mathcal{K}$ be the kernel, so we have the exact sequence of locally free sheaves on $\widetilde{\Gamma}$:

$$0 \longrightarrow \mathcal{K} \longrightarrow \widehat{\nu}^* \mathcal{Q}_\Gamma \longrightarrow \widetilde{p}^* \widehat{\mathcal{T}} \longrightarrow 0$$

and $\operatorname{rk} \mathcal{K} = \operatorname{rk} \mathcal{Q}_\Gamma - \operatorname{rk} \widehat{\mathcal{T}} < \dim \widetilde{\Gamma} - \dim \widetilde{W}$. It follows that $\widetilde{p}_*(c(\mathcal{K}) \cap [\widetilde{\Gamma}]) = 0$, and this implies the statement by the projection formula: $p_*(c(\mathcal{Q}) \cap [\Gamma])$ equals

$$p_* \widehat{\nu}_* (c(\widehat{\nu}^* \mathcal{Q}) \cap [\widetilde{\Gamma}]) = \nu_* \widetilde{p}_* \left(c(\widetilde{p}^* \widehat{\mathcal{T}}) \cdot c(\mathcal{K}) \cap [\widetilde{\Gamma}] \right) = \nu_* c(\widehat{\mathcal{T}}) \cap \widetilde{p}_* \left(c(\mathcal{K}) \cap [\widetilde{\Gamma}] \right) = 0$$

as needed. □

6.8. We choose a component Γ of the exceptional divisor, and restrict all relevant maps to it:

$$\begin{tikzcd} \Gamma \arrow[r, "\widetilde{\gamma}"] \arrow[d, "\sigma"] \arrow[dd, bend right, "p"'] & \mathbb{G} \arrow[d] \\ Z \arrow[r, hook] \arrow[d] & X \arrow[d, "g"] \\ W \arrow[r, hook] & Y \end{tikzcd}$$

Here Z and W are the images of Γ in X, Y respectively. Note that $Z \subset D$ and $W \subset E$; we assume $W \subset E_i$ for $i \le s$, $W \not\subset E_i$ for $i > s$, and we denote by $E_{\underline{s}}$ the intersection $E_1 \cap \dots \cap E_s$.

We let $\mathcal{S}_\Gamma$, $\mathcal{Q}_\Gamma$ be resp. the pull-back to Γ of the universal sub- and quotient bundle over $\mathbb{G}$; hence we have an exact sequence

$$0 \longrightarrow \mathcal{S}_\Gamma \longrightarrow \sigma^*(g^*\Omega^1_Y(\log E) \oplus \Omega^1_X(\log D))|_Z \longrightarrow \mathcal{Q}_\Gamma \longrightarrow 0 \quad .$$

6.9. The residue exact sequence for the bundle of logarithmic differential forms restricts to an exact sequence

$$0 \longrightarrow \Omega^1_{E_{\underline{s}}}|_W \longrightarrow \Omega^1_Y(\log E)|_W \longrightarrow \mathcal{O}_W^{\oplus s} \oplus (\oplus_{i>s} \mathcal{O}_{Z \cap E_i}) \longrightarrow 0 \quad .$$

In particular, $\Omega^1_Y(\log E)|_W$ contains a distinguished copy of the *conormal sheaf* $N^*_W E_{\underline{s}}$, that is, the kernel of the natural surjection $\Omega^1_{E_{\underline{s}}}|_W \to \Omega^1_W$. The quotient defines a coherent sheaf $\mathcal{T}$ on W, which by construction fits in an exact sequence

$$0 \longrightarrow \Omega^1_W \longrightarrow \mathcal{T} \longrightarrow \mathcal{O}_W^{\oplus s} \oplus (\oplus_{i>s}\mathcal{O}_{Z\cap E_i}) \longrightarrow 0$$

In particular, note that $\operatorname{rk}\mathcal{T} > \dim W$.

Lemma 6.6. *There is a surjection*

$$\mathcal{Q}_\Gamma \longrightarrow\!\!\!\!\!\rightarrow p^*\mathcal{T}$$

of coherent sheaves on Γ.

Proof. It suffices to show that the image of $\mathcal{S}_\Gamma$ in

$$\sigma^*(g^*\Omega^1_Y(\log E)\oplus 0)|_Z \cong p^*\Omega^1_Y(\log E)|_W$$

is contained in the image of $p^*N^*_W E_{\underline{s}}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}_\Gamma \arrow[r] & \sigma^*(g^*\Omega^1_Y(\log E)\oplus\Omega^1_X(\log D))|_Z \arrow[r] \arrow[d] & \mathcal{Q}_\Gamma \arrow[r] \arrow[d, dashed] & 0 \\ & p^*N^*_W E_{\underline{s}} \arrow[r] & p^*\Omega^1_Y(\log E)|_W \arrow[r] & p^*\mathcal{T} \arrow[r] & 0 \end{tikzcd}$$

and this may be verified by a computation in local coordinates. □

6.10. By Lemma 6.5, Lemma 6.6 implies the vanishing prescribed in Claim 6.4, concluding the proof of Lemma 5.3.

The coordinate computation in Lemma 6.6 uses characteristic zero: for example, in characteristic $p > 0$ problems arise if some component of D dominating a component of E appears with multiplicity equal to a multiple of p. This is in fact precisely what happens with the Artin-Schreier map, which gives (as mentioned in §5.2) a simple counterexample to Lemma 5.3 in positive characteristic.

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