BIORTHOGONAL EXPANSION OF NON-SYMMETRIC JACK FUNCTIONS

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ABSTRACT. We find an biorthogonal expansion of the Cayley transform of the non-symmetric Jack functions in terms of the non-symmetric Jack polynomials, the coefficients being the MP-type polynomials. This is done by computing the Cherednik-Opdam transform of the non-symmetric Jack polynomials multiplied by the exponential function.

1. INTRODUCTION

In [9] Opdam studied the non-symmetric eigenfunctions of the Cherednik operators associated with any root system with general multiplicity and proved Plancherel formula for the corresponding Cherednik-Opdam transform. For the root system of type A the polynomial eigenfunctions are also called the non-symmetric Jack polynomials and they have been studied in extensively (see e.g. [10]). There are other related non-symmetric polynomials such as the Hermite and Laguerre polynomials which roughly speaking are their images under the Hankel transform, which is basically the Fourier transform on the underlying space. The non-symmetric Laguerre polynomials form an orthogonal basis for the L^2-space and it is thus a natural question to find their Cherednik-Opdam transforms. In this paper we prove that they are, apart from a factor of Gamma functions, the non-symmetric Meixner-Pollaczek (MP) polynomials and we find a formula for them in terms of binomial coefficients. As a corollary we find an biorthogonal expansion of the Cayley transform of the non-symmetric Jack function in terms of the non-symmetric Jack polynomials, the coefficients being the MP polynomials.

There are basically three important families of polynomials associated with the root system of type A , namely, the Jack type polynomials, the Laguerre polynomials, the MP type polynomials which are orthogonal with respect the Harish-Chandra measure $|c(\lambda)|^{-2}d\lambda$ multiplied with a certain Gamma factor, which is the spherical transform of an exponential function. Our results give then a somewhat unified picture of the relation between these polynomials, and provide a combinatorial formula for the MP type polynomials. In brief the Laplace transform maps the Laguerre polynomials into Jack polynomials, and the Cherednik-Opdam transform maps the Laguerre polynomials into MP type polynomials. In the case when the root system corresponds to that of a symmetric cone [4] some results of this type has been studied in [4], [13] and [3].

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2. NON-SYMMETRIC JACK FUNCTIONS AND THE OPDAM-CHEREDNIK TRANSFORM

In this section we recall the non-symmetric Jack polynomials and functions and the Plancherel formula for the Opdam-Cherednik transform, developed in [9].

We consider the root system of type A_{r-1} on $\mathbb{R}^r$. For our purpose of studying Laplace transform we make a change of variables $x_j = e^{2t_j}$ where $t = (t_1, \cdots, t_r) \in \mathbb{R}^r$ and consider functions in $x \in \mathbb{R}^r_+$ instead. We fix an ordering of the roots so that (with some abuse of the notion of roots) the positive roots are $x_2 - x_1, x_3 - x_2, \cdots, x_r - x_{r-1}$ with root multiplicity $\frac{2}{\alpha}$ and we will identify the roots as vectors in $\mathbb{R}^r$. Let ρ be the half sum of positive roots, so that $\rho = (\rho_1, \rho_2, \cdots, \rho_r) = \frac{1}{\alpha}(-r+1, -r+3, \cdots, r-1)$. We consider the measure

$$d\mu(x) = \frac{1}{2^r}(x_1 \cdots x_r)^{-\frac{1}{\alpha}(r-1)-1} \prod_{1 \le j < k \le r} |x_j - x_k|^a dx_1 \cdots dx_r$$

on $\mathbb{R}^r_+$ and the corresponding Hilbert space $L^2(\mathbb{R}^r_+, d\mu)$.

We consider the Dunkl operators

$$T_j = \partial_j + \frac{1}{\alpha} \sum_{i \ne j} \frac{1}{x_j - x_i}(1 - s_{ij}) \tag{2.1}$$

and the Cherednik operators

$$U_j = U_j^A = x_j \partial_j + \frac{1}{\alpha} \sum_{i<j} \frac{x_j}{x_j - x_i}(1 - s_{ij}) + \frac{1}{\alpha} \sum_{j<k} \frac{x_k}{x_j - x_k}(1 - s_{jk}) - \frac{1}{2}\rho_j \tag{2.2}$$

Here $\partial_j = \frac{\partial}{\partial x_j}$ and s_{ij} stands for the permutation (ij) acting on functions $f(x_1, \ldots, x_r)$ interchanging the variables x_i and x_j. The operators $\{U_j\}$ can be written in terms of $\{T_j\}$ and the multiplication operators $\{x_j\}$, but we will not need it here. Both families $\{T_j\}$ and $\{U_j\}$ are commuting.

The polynomial eigenfunctions of the operators $\{U_j\}$ are given by the so-called non-symmetric Jack polynomials $E_\eta(x) = E(\eta, x)$, with $\eta = (\eta_1, \ldots, \eta_r) \in \mathbb{N}^r$. They are characterized as eigen-polynomials of $\{U_j\}$ with leading coefficients $x^\eta = x_1^{\eta_1} \cdots x_r^{\eta_r}$ in the sense that

$$E_\eta = x^\eta + \sum_{\zeta < \eta} c_{\eta\zeta} x^\zeta.$$

We recall that $\zeta < \eta$ here stands for the partial ordering defined by

$$\zeta < \eta \qquad \text{iff} \quad \begin{cases} \zeta^+ < \eta^+, & \zeta^+ \ne \eta^+ \\ \zeta < \eta, & \zeta^+ = \eta^+ \end{cases}$$

where η^+ is the unique partition obtained by permuting the entries of ζ and $<$ stands for the natural dominance ordering: $\zeta < \eta$ iff $\sum_{j=1}^p (\zeta_j - \eta_j) \ge 0, 1 \le p \le r$.

The functions E_η has holomorphic extension in the variable η. More precisely, there exists a function real analytic function $G_\lambda(x) = G(\lambda, x)$ in $x \in \mathbb{R}^r_+$ and holomorphic in $\lambda \in \mathfrak{a}^{\mathbb{C}}$, such that $G(\lambda, 1^r) = 1$, with $1^r = (1, \ldots, 1)$, and

$$U_j G(\lambda, x) = \lambda_j G(\lambda, x).$$

The relation between $G_\lambda(x)$ and $E_\eta(x)$ is

$$G_{\eta+\rho}(x) = \mathcal{E}_\eta(x) := \frac{E_\eta(x)}{E_\eta(1^r)}, \tag{2.3}$$

The value $E_\eta(1^r)$ has been computed by Sahi [10],

$$E_\eta(1^r) = \frac{e_\eta}{d_\eta};$$

where e_η and d_η are defined in the next section. (See also [9] for general root systems.) Define

$$\mathcal{F}_w[f](\lambda) = \int_{\mathfrak{a}} f(x)G(-\lambda, w^{-1}x)d\mu(x). \tag{2.4}$$

Then we have

$$\int_{\mathfrak{a}} |f(x)|^2 d\mu(x) = \sum_w \int_{i\mathfrak{a}^*_+} \mathcal{F}_w[f](\lambda)\overline{\mathcal{F}_w(f)(\lambda)}d\tilde{\mu}(\lambda) \tag{2.5}$$

where

$$d\tilde{\mu}(\lambda) = \frac{(2\pi)^{-r}\tilde{c}^2_{w_0}(\rho(k), k)}{\tilde{c}(\lambda)c(w_0\lambda)}d\omega(\lambda)$$

is the Plancherel measure. Here $d\omega(\lambda)$ is the Euclidean measure.

3. Laplace transform

Adapting the notation in [2] we denote $q := 1 + \frac{1}{\alpha}(r-1)$. Let $\binom{\kappa}{\sigma}$ be the binomial coefficients for two tuples of nonnegative integer $\kappa, \sigma \in Z^r_{\geq 0}$, see e.g. [11] and [2]. Recall further that the non-symmetric Laplace transform is defined by

$$\mathcal{L}[f](t) = \int_{[0,\infty)^r} \mathcal{K}_A(-t, x)f(x)(\prod_{j=1}^r x_j)^q d\mu(x),$$

where

$$\mathcal{K}_A(t, y) = \sum_\eta \alpha^{|\eta|}\frac{1}{d'_\eta}\mathcal{E}_\eta(t)E_\eta(y)$$

is the non-symmetric analogue of the hypergeometric ${}_0F_0$ function. (Note that our measure $d\mu$ differs a factor $(\prod_{j=1}^r x_j)^{-q}$ from that in [2, (3.67)].) Here each tuple η will be identified with a diagram of nodes $s = (i, j)$, $1 \leq j \leq \eta_i$, $d_\eta = \prod_{s\in\eta} d(s)$, $d'_\eta = \prod_{s\in\eta} d'(s)$, $e'_\eta = \prod_{s\in\eta} e(s)$ with

$$d'(s) = \alpha(a(s)+1) + l(s), \quad d(s) = d'(s) + 1, \quad e(s) = \alpha(a'(s)+1) + r - l'(s))$$

and $a(s) = \eta_i - j$, $\quad a'(s) = j - 1$ being the arm length, arm colength and

$$l(s) = \#\{k > i : j \leq \eta_k \leq \eta_i\} + \#\{k < i : j \leq \eta_k + 1 \leq \eta_i\},$$
$$l'(s) = \#\{k > i : \eta_k > \eta_i\} + \#\{k < i : \eta_k \leq \eta_i\},$$

the leg length and leg colength. See [10] and [5].

The function $\mathcal{K}_A(x, y)$ generalizes the exponential function in the sense that

$$T_i^{(x)}\mathcal{K}_A(x, y) = y_i\mathcal{K}_A(x, y), \tag{3.1}$$

where $T_i^{(x)}$ is the Dunkl operator acting on the variable x.

Recall further the definition of generalized Gamma function

$$\Gamma_\alpha(\kappa) = \prod_{j=1}^r \Gamma(\kappa_j - \frac{1}{\alpha}(j-1))$$

and the Pochammer symbol

$$[\nu]_\kappa = \frac{\Gamma_\alpha(\nu+\kappa)}{\Gamma_\alpha(\nu)}.$$

for $\nu, \kappa \in \mathbb{C}^r$, whenever it makes sense. A scalar $c \in \mathbb{C}$ will also be identified with $(c, \cdots, c) \in \mathbb{C}$ in the text below. We will also use the abbreviation $x^c = x_1^c \cdots x_r^c$ and $1 + x = (1 + x_1, \cdots, 1 + x_r)$ etc.

We recall further the binomial coefficient $\binom{\eta}{\nu}$ for $\eta, \nu \in \mathbb{N}^r$ defined by the expansion

$$\mathcal{E}_\eta(1+t) = \sum_\nu \binom{\eta}{\nu} \mathcal{E}_\nu(t). \tag{3.2}$$

See also [11] and [12]. We make the following generalization.

Definition 3.1. The binomial coefficients $\binom{\eta}{\nu}$ for any $\eta \in \mathbb{C}^r$ and $\nu \in \mathbb{N}^r$ are defined by

$$G_{\eta+\rho}(1+t) = \sum_{\nu \in \mathbb{N}^r} \binom{\eta}{\nu} E_\nu(t). \tag{3.3}$$

Since $G_\lambda(1 + t)$ is an analytic function near 1 and $E_\nu(t)$ form a basis for all polynomials the above definition makes sense, and it agrees with (3.2) in view of the relation (2.3). In particular $\binom{\eta}{\nu}$ is a polynomial of $\eta \in \mathbb{C}^r$. It follows from the definition and [2, Proposition 3.18] that

$$E_\nu(T) G_{\eta+\rho}|_{t=1} = \frac{d'_\nu}{\alpha^{|\nu|}} \binom{\eta}{\nu}. \tag{3.4}$$

The following lemma is proved in [2, (4.38)]. (Note that there is an error or misprint there: $E_\eta^{(L)}(\frac{1}{x})$ should be replaced by $E_\eta(\frac{1}{x})$. For symmetric case it was a conjecture of Macdonald [8] proved by e.g. in [1, (6.1)-(6.3)].)

Lemma 3.2. *Suppose $c > -1$. The Laplace transform of the functions $x^c E_\eta(x)$ is given by*

$$\mathcal{L}[x^a E_\eta(x)](t) = \mathcal{N}_0^{(L)}[c+q]_\eta (\prod_{j=1}^r t_j^{-(c+q)}) E_\eta(\frac{1}{t}).$$

The normalization constant $\mathcal{N}_0^{(L)}$ is computed in [2].

We fix in the rest of the text an even integer $b > 2q + 2$. Let

$$E_\kappa^{(L)}(x) = E_\kappa^{(L,b)}(x) = \frac{(-1)^{|\kappa|}[b]_\kappa e_\kappa}{d_\kappa} \sum_\sigma \frac{(-1)^{|\sigma|}}{[b]_\sigma} \binom{\kappa}{\sigma} \mathcal{E}_\sigma(x) \tag{3.5}$$

be the non-symmetric Laguerre polynomial. The next Lemma follows from [2, Proposition 4.35] after a change of variable $2x = t^2$. Our parameter b is their $a + q$.

Lemma 3.3. *Let $p_1(x) = \sum_{j=1}^r x_j$. The Laguerre functions*

$$l_\kappa(x) := l_\kappa^b(x) := E_\kappa^L(2x) e^{-p_1(x)} (2x)^{\frac{b}{2}}$$

form an orthogonal basis for the space $L^2(\mathbb{R}_+^r, d\mu)$.

The norm $\|l_\kappa\|^2$ has also been explicitly evaluated in [2].

We can now compute the Cherednik-Opdam transform of the Laguerre function. By the Plancherel formula (2.5) they are then orthogonal functions in λ.

Proposition 3.4. *Suppose $b > (q-1) = \frac{1}{\alpha}(r-1)$. The Cherednik-Opdam transform $\mathcal{F}_w[l_\kappa]$ of the Laguerre function $l_\kappa(x)$, for $w \in S_r$ being the identity e, is*

$$\mathcal{F}_e[l_\kappa](\lambda) = 2^{\frac{rb}{2}}\mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}-\rho-\lambda)}{\Gamma_\alpha(\frac{b}{2})}M_\kappa(\lambda)$$

where

$$M_\kappa(\lambda) = \frac{(-1)^{|\kappa|}[b]_\kappa e_\kappa}{d_\kappa}\sum_\sigma \frac{1}{[b]_\sigma}\frac{d'_\sigma d_\sigma}{\alpha^{|\sigma|}e_\sigma}(-2)^{|\sigma|}\binom{\kappa}{\sigma}\binom{-\frac{b}{2}+\lambda^*+\rho^*}{\sigma}$$

Proof. The function l_κ is a linear combinations of the functions $e^{-p_1(x)}E_\sigma(x)$ and we compute the Cherednik-Opdam transform of these functions. We write Lemma 3.2 as

$$\int_{\mathbb{R}_+^r}\mathcal{K}_A(-t,x)x^{\frac{b}{2}}\mathcal{E}_\eta(x)d\mu(x) = \mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}+\eta)}{\Gamma_\alpha(\frac{b}{2})}t^{-\frac{b}{2}}\mathcal{E}_{-\eta^*}(t) = \mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}+\eta)}{\Gamma_\alpha(\frac{b}{2})}\mathcal{E}_{-\frac{b}{2}-\eta^*}(t),$$

with $\eta^* = (\eta_r, \eta_{r-1}, \cdots, \eta_1)$. Let the operator $E_\sigma(T)$ act on it evaluated as $t = 1^r = (1, \dots, 1)$. The resulting equality, by (3.1) and the fact that $\mathcal{K}_A(x, 1^r) = e^{p_1(x)}$, and (3.4), is,

$$\int_{\mathbb{R}_+^r} e^{-p_1(x)}\mathcal{E}_\sigma(-x)x^{\frac{b}{2}}\mathcal{E}_\eta(x)d\mu(x) = \mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}+\eta)}{\Gamma_\alpha(\frac{b}{2})}\frac{d'_\sigma d_\sigma}{\alpha^{|\sigma|}e_\sigma}\binom{-\frac{b}{2}-\eta^*}{\sigma}.$$

Thus, using the relation (2.3) we see that

$$\begin{aligned}\mathcal{F}[l_\kappa](\lambda) &= \int_{[0,\infty)^r} l_\kappa(x)G(-\lambda,x)d\mu(x)\\ &= 2^{\frac{rb}{2}}\mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}-\rho-\lambda)}{\Gamma_\alpha(\frac{b}{2})}\frac{(-1)^{|\kappa|}[b]_\kappa e_\kappa}{d_\kappa}\sum_\sigma\frac{1}{[b]_\sigma}\frac{d'_\sigma d_\sigma}{\alpha^{|\sigma|}e_\sigma}(-2)^{|\sigma|}\binom{\kappa}{\sigma}\binom{-\frac{b}{2}+\lambda^*+\rho^*}{\sigma}.\end{aligned}$$

as claimed. □

By using the same argument we can also find a formula for the function $\mathcal{F}_w[l_\kappa](\lambda)$ for any $w \in S_r$. The Plancherel formula (2.5) then gives an orthogonality relation for the polynomials. In the next section we will find another formula for $M_\kappa(\lambda)$.

Remark 3.5. In the case of one variable $r = 1$ we have $E_l = x^l$, $d_l = d'_l e_l = l!$, our polynomial is

$$M_k(\lambda) = (b)_k(-1)^k\sum_l\frac{1}{(b)_l}l!(-2)^l\binom{k}{l}\binom{-\frac{b}{2}+\lambda}{l} = (b)_k(-1)^k{}_2F_1(-k,\frac{b}{2}-\lambda;b;2)$$

which is the MP polynomial $P_k^{(b/2)}(x;\frac{\pi}{2})$ [6, 1.7.1], more precisely

$$M_k(\lambda) = k!i^kP_k^{(b/2)}(-i\lambda;\frac{\pi}{2}).$$

The functions $\Gamma(\frac{b}{2}+i\lambda)M_k(i\lambda)$ are orthogonal in the space $L^2(\mathbb{R}, d\lambda)$, as a consequence of (2.5).

4. A BINOMIAL FORMULA FOR $M_\kappa(\lambda)$

We recall first the following result [2, Proposition 4.13].

Lemma 4.1. *The following expansion holds*

$$\prod_{j=1}^r (1-z_j)^{-b}\mathcal{K}_A(-x;\frac{z}{1-z}) = \sum_\eta (-\alpha)^{|\eta|}\frac{1}{d'_\eta}E_\eta^{(L)}(x)\mathcal{E}_\eta(z)$$

We need another expansion of the $\prod_{j=1}^r(1-z_j^2)^{-\frac{b}{2}}E_{\kappa-\frac{b}{2}}(\frac{1-z}{1+z})$ in terms of the polynomials $E_\eta(z)$.

Lemma 4.2. *Consider the following expansion*

$$\prod_{j=1}^r (1-z_j^2)^{-\frac{b}{2}}E_{\eta-\frac{b}{2}}(\frac{1-z}{1+z}) = \sum_\kappa \mathcal{C}_\kappa(\eta)\mathcal{E}_\kappa(z),$$

for $\eta\in\mathbb{N}^r$, $\eta_j\ge\frac{b}{2}$. The coefficients are then given by

$$\mathcal{C}_\kappa(\eta) = E_{\eta-b}(-1)\sum_\sigma \binom{\eta-b}{\sigma}(-2)^{|\sigma|}\binom{-\sigma^*-b}{\kappa} \tag{4.1}$$

and is a polynomial in η.

Proof. Note that $\mathcal{C}_\kappa(\eta)$ is a polynomial in η follows from the remark after Definition 3.1. Change variables $y_j=1+z_j$, $\frac{1-z_j}{1+z_j}=\frac{2}{y_j}-1$, $1-z_j^2=(1+z_j)^2\frac{1-z_j}{1+z_j}=y_j^2(\frac{2}{y}-1)$. We have

$$\begin{aligned}(1-z^2)^{-\frac{b}{2}}E_{\eta-\frac{b}{2}}(\frac{1-z}{1+z}) &= (y^2(\frac{2}{y}-1))^{-\frac{b}{2}}E_{\eta-\frac{b}{2}}(\frac{2}{y}-1)\\ &= y^{-b}E_{\eta-b}(\frac{2}{y}-1)\\ &= (-1)^{|\eta-b|}y^{-b}E_{\eta-b}(1-\frac{2}{y}).\end{aligned} \tag{4.2}$$

We expand $E_{\eta-b}(1-\frac{2}{y})$ by using the binomial formula,

$$\begin{aligned}y^{-b}E_{\eta-b}(1-\frac{2}{y}) &= y^{-b}E_{\eta-b}(1)\sum_\sigma\binom{\eta-b}{\sigma}\mathcal{E}_\sigma(-\frac{2}{y})\\ &= E_{\eta-b}(1)\sum_\sigma\binom{\eta-b}{\sigma}(-2)^{|\sigma|}\mathcal{E}_{\sigma^*-b}(y).\end{aligned}$$

Here we have used the relations $\mathcal{E}_\sigma(-\frac{2}{y})=(-2)^{|\sigma|}E_{-\sigma^*}(y)$ and $y^{-b}E_{-\sigma^*}(y)=E{-\sigma^*-b}(y)$. Now each term $\mathcal{E}_{-\sigma^*-b}(y)=\mathcal{E}_{-\sigma^*-b}(1+z)$ can again be expanded in terms of $\mathcal{E}_\kappa(z)$. Interchanging the order of the summations we find that (4.2) is

$$E_{\eta-b}(-1)\sum_\kappa\left(\sum_\sigma\binom{\eta-b}{\sigma}(-2)^{|\sigma|}\binom{-\sigma^*-b}{\kappa}\right)\mathcal{E}_\kappa(z).$$

This completes the proof. ∎

Theorem 4.3. *It holds that*

$$(-\alpha)^{|\kappa|}\frac{1}{d'_\kappa}M_\kappa(\lambda)=\mathcal{C}_\kappa(\lambda-\rho)$$

Proof. We replace x by $2x$ in Lemma 4.1,

$$\prod_{j=1}^{r}(1-z_j)^{-b}\mathcal{K}_A(-x;\frac{2z}{1-z})=\sum_\kappa(-\alpha)^{|\kappa|}\frac{1}{d'_\kappa}E_\kappa^{(L)}(2x)\mathcal{E}_\kappa(z),$$

where we have used the fact that $\mathcal{K}_A(2x;y)=\mathcal{K}_A(x;2y)$. Multiplying both sides by $\prod_{j=1}^{r}(2x_j)^{\frac{b}{2}}e^{-p_1(x)}$ and since $\mathcal{K}_A(-x;y+1^r)=\mathcal{K}_A(-x;y)e^{-p_1(x)}$ we get

$$\prod_{j=1}^{r}(1-z_j)^{-b}\mathcal{K}_A(-x;\frac{1+z}{1-z})\prod_{j=1}^{r}(2x_j)^{\frac{b}{2}}=\sum_\kappa(-\alpha)^{|\kappa|}\frac{1}{d'_\kappa}l_\kappa^{(\nu)}(x)\mathcal{E}_\kappa(z).$$

Multiplifying the both sides by $E_\eta(x)$, $\eta\in\mathbb{N}^r$, and integrating with respect to $d\mu(x)$ for z in neighborhood of $z=0$ (we omit the routine estimates guaranteeing that the interchanging of the integration and summation is valid), we get, by Lemma 3.1,

$$\mathcal{N}_0^{(L)}[\frac{b}{2}]_\eta\prod_{j=1}^{r}(1-z_j)^{-b}E_\eta(\frac{1-z}{1+z})=\mathcal{N}_0^{(L)}[\frac{b}{2}]_\eta\sum_\kappa(-\alpha)^{|\kappa|}M_\kappa(\eta+\rho)\frac{1}{d'_\kappa}\mathcal{E}_\kappa(z).$$

Cancelling the common factor $\mathcal{N}_0^{(L)}[\frac{b}{2}]_\eta$ and rewriting then the left hand as

$$\prod_{j=1}^{r}(1-z_j)^{-b}\left(\prod_{j=1}^{r}\frac{1-z_j}{1+z_j}\right)^{\frac{b}{2}}E_{\eta-\frac{b}{2}}(\frac{1-z}{1+z})=\prod_{j=1}^{r}(1-z_j^2)^{-\frac{b}{2}}E_{\eta-\frac{b}{2}}(\frac{1-z}{1+z}),$$

we get

$$\prod_{j=1}^{r}(1-z_j^2)^{-\frac{b}{2}}E_{\eta-\frac{b}{2}}(\frac{1-z}{1+z})=\sum_\kappa(-\alpha)^{|\kappa|}M_\kappa(\eta+\rho)\frac{1}{d'_\kappa}\mathcal{E}_\kappa(z).$$

Comparing this with Lemma 4.1 proves our claim. □

Finally we can also consider the same problem for the symmetric Jack functions. The Laplace transform is the symmetric case has been studied earlier by Macdonald [8] and Baker-Forrester [1]. Using their results and by similar arguments as above we can the proposition below.

Let $\Omega_\kappa(x)=\Omega_\kappa^{(\alpha)}(x)$ for a partition $\kappa=(\kappa_1,\cdots,\kappa_r)$ be the Jack symmetric polynomials in r-variables, normalized so that $\Omega_\kappa(1^r)=1$; see [7]. The corresponding functions for $\kappa\in\mathbb{C}^r$, which we call the Jack function (and which for general root system is called the Heckman-Opdam hypergeometric function) will be denoted still by $\Omega_\kappa(x)$, $x\in(0,\infty)^r$.

Proposition 4.4. *Consider the following expansion*

$$\prod_{j=1}^{r}(1-z_j^2)^{-\frac{b}{2}}\Omega_{\eta-\frac{b}{2}}(\frac{1-z}{1+z})=\sum_\kappa Q_\kappa(\eta)\Omega_\kappa(z).$$

The coefficient $Q_\kappa(\eta)$ are polynomials in η up to a ρ shift, and a slight modification

$$f_\kappa(\lambda)=2^{rb/2}\mathcal{N}_0^{(L)}\frac{d'_\kappa}{(-\alpha)^{|\kappa|}}Q_\kappa(\lambda-\rho)$$

form an orthogonal basis in the space

$$L^2((0,\infty)^r, |\frac{\Gamma_\alpha(\frac{b}{2}-\rho-\lambda)}{\Gamma_\alpha(\frac{b}{2})}|^2|c(\lambda)|^{-2}ds)^W$$

of symmetric L^2-functions.

Note that the functions $\Gamma_\alpha(\frac{b}{2}-\rho-\lambda)Q_\kappa(\lambda-\rho)$ is up to a constant depending only on κ are the spherical transform of the Laguerre functions $e^{p_1(x)}L_\kappa(2x)(2x)^{b/2}$, where $L_\kappa(2x)(2x)^{b/2}$ is the symmetric Laguerre polynomials defined in [1], which then implies the orthogonality of the functions f_κ by the Plancherel formula (2.5) and the orthogonality of the Laguerre functions [1].

We observe the spherical transform of the functions $L_\kappa(2x)e^{p_1(x)}(2x)^{b/2}$ is a Weyl group invariant polynomial of that of $e^{p_1(x)}(2x)^{b/2}$, and as a consequence of our result we may get a Rodrigues type formula expressing the former as polynomials of the Cherednik operators $\{U_j\}$ acting on the latter. However it might be more interesting to reverse the procedure, namely to find the polynomials producing a Rodrigues type formula with the spherical transform as an immediate consequence; see e.g. [14] for the case of Wilson polynomials.

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