ON THE CONNECTIVITY OF COMPLEX AFFINE HYPERSURFACES

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Let $X : f = 0$ be a (reduced, algebraic) hypersurface in $\mathbb{C}^{n+1}$, with $n \geq 1$. It is well–known that X has the homotopy type of a CW–complex of dimension n, see [5], [8].

General results on the connectivity of affine varieties were obtained by M. Kato [9], extending previous results due to A. Howard [7].

Let $\overline{X}$ be the closure of X in $\mathbb{P}^{n+1}$, H_∞ be the hyperplane at the infinity in $\mathbb{P}^{n+1}$ and set $X_\infty = \overline{X} \cap H_\infty$. For any algebraic variety Z we let $S(Z)$ denote its singular part and use the convention $\dim \phi = -1$. With these preliminaries, Kato's result can be stated in the hypersurface case as follows.

Theorem (M. Kato [9])

X is $(n - 2 - \dim (S(\overline{X}) \cup S(X_\infty)))$ – connected.

We prove here the next better (usually by one!) estimation on the connectivity of X.

Theorem 1

X is $(n - 2 - \dim(S(\overline{X}) \cap H_\infty))$ – connected.

Let $f = f_0 + f_1 + \dots + f_d$ be the decomposition of the polynomial f into homogeneous components, with $f_d \not\equiv 0$.

Note that the set $\Sigma(f) = \overline{S(X)} \cap H_\infty$ is given in $H_\infty \simeq \mathbb{P}^n$ by the equations

$$\Sigma(f) : \partial f_d = 0 \, , \; f_{d-1} = 0$$

where for any polynomial g we denote by ∂g its gradient, i.e.

$$\partial g = \left[\frac{\partial g}{\partial x_0}, \dots , \frac{\partial g}{\partial x_n} \right] .$$

Forget for a moment the hypersurface X and consider the polynomial $f \in \mathbb{C}[x_0, \dots , x_n]$ as a basic object. It may happen that $f_{d-1} = 0$ and so let e be the greatest integer such that $f_e \not\equiv 0$ and $e < d$.

Define the subset $S(f) \subset \mathbb{P}^n$ by the next similar equations

$$S(f) : \partial f_d = 0 \, , \; f_e = 0 \, .$$

We prove in fact the next stronger version of Theorem 1.

<u>Theorem 2</u>

Assume that $e > 0$. Then any fiber of the polynomial function $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ is

$$(n - 2 - \dim S(f)) - \text{connected} \, .$$

<u>Corollary 3</u>

(i) (Angermüller [1]) If the degree form f_d of f is square free, then all the fibers of f are connected.

(ii) If $e > 0$ and if there is no polynomial g such that g^2 divides f_d and g divides f_e, then all the fibers of f are connected.

Using the setting of Angermüller [1], we get also a new result on the connectivity of the diagonal $\Delta f = \{(x,y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \; ; \; f(x) = f(y)\}$. Consider the set in $\mathbb{P}^{2n+1} = \mathbb{P}(\mathbb{C}^{n+1} \times \mathbb{C}^{n+1})$

$$\delta f = \{(x,y) \in \mathbb{P}^{2n+1} \; ; \; \partial f_d(x) = 0 \, , \; \partial f_d(y) = 0 \, , \; f_e(x) = f_e(y)\} \, .$$

<u>Corollary 4</u>

If $e > 0$, then the diagonal Δf is

$$(2n - 1 - \dim \delta f) - \text{connected} \, .$$

In a recent paper [3], Broughton has considered polynomial functions $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ such that

(a) f has only isolated singularities;

(b) the set $\Sigma(f)$ is finite.

For such polynomials f, he has shown that the generic fiber $F_c = f^{-1}(c)$ has the homology of a bouquet of spheres of dimension n, see Theorem 5.2. in [3]. This statement can be improved as follows (compare with Proposition 8 below).

Corollary 5

For $n \neq 2$, the generic fiber F_c in the above conditions has the homotopy type of a bouquet of spheres of dimension n.

Proof

For $n \geq 3$, it follows from Theorem 1 that all the fibers of f are 1–connected. Combining this with the homological information we get the result in the usual way, e.g. see [10], p. 58. The case $n = 1$ is obviously also true, since any connected affine curve has the homotopy type of a bouquet of circles.

To prove our results, we recall first the definition and some basic properties of tame and quasitame polynomials.

Definition 6 (Broughton [2], [3])

A polynomial f is called tame if there is no sequence of points $z^k \in \mathbb{C}^{n+1}$ such that $|z^k| \longrightarrow \infty$ and $\partial f(z^k) \longrightarrow 0$ for $k \longrightarrow \infty$.

Definition 7 (Némethi [11], [12])

A polynomial f is called quasitame if there is no sequence of points $z^k \in \mathbb{C}^{n+1}$ such that $|z^k| \longrightarrow \infty$, $\partial f(z^k) \longrightarrow 0$ and the sequence $c^k = f(z^k) - \sum_{j=0,n} \frac{\partial f}{\partial x_j}(z^k) \cdot z_j^k$ has a finite limit for $k \longrightarrow \infty$.

The main properties of the tame polynomials and of the (more general) quasitame polynomials are the same, see [2], [3], [11], [12]. We need only the next

Proposition 8 (Némethi [11], [12]).

If f is a quasitame polynomial, then any of its fibers has the homotopy type of a bouquet of spheres of dimension n .

Our simple but key remark is the next.

Lemma 9

If either $\Sigma(f) = \phi$ or $S(f) = \phi$ and $e > 0$, then f is a quasitame polynomial.

Proof

We give the proof only in the case $\Sigma(f) = \phi$, the other case being completely similar.

Let z^k be a sequence in $\mathbb{C}^{n+1}$ such that $|z^k| \longrightarrow \infty$ and $\partial f(z^k) \longrightarrow 0$ for $k \longrightarrow \infty$. We can and do assume that the sequence $\bar{z}^k = z^k / |z^k|$ has a limit z^∞ on the unit sphere in $\mathbb{C}^{n+1}$.

Note that $\partial f(z^k) \longrightarrow 0$ implies $\partial f(z^\infty) = 0$ and also (via the Euler formula)

$$\frac{1}{|z^k|} \sum_{j=1,d} j f_j(z^k) \longrightarrow 0 .$$

Since $\Sigma(f) = \phi$, it follows that $f_{d-1}(z^{\infty}) \neq 0$. The assumption that the sequence c^k in Def. 7 has a finite limit (and Euler formula again!) gives

$$\frac{1}{|z^k|} \sum_{j=1,d} (j-1)\, f_j(z^k) \longrightarrow 0 .$$

By linearity we get

$$\frac{1}{|z^k|} \sum_{j=1,d} (d-j)\, f_j(z^k) = |z^k|^{d-2} f_{d-1}(\bar{z}^{\,k}) + \ldots \quad \longrightarrow 0$$

where the dots ... stand for lower order terms.

This is clearly in contradiction with $f_{d-1}(z^{\infty}) \neq 0$ and hence a sequence $\{z^k\}$ with the above properties does not exist.

Proof of Theorem 1

By induction on $s = \dim \Sigma(f) = \dim S(\bar{X}) \cap H_{\infty}$. When $\Sigma(f) = \phi$, it follows by Lemma 9 that f is a quasitame polynomial and then by Proposition 8 it follows that X is $(n-1)$ – connected. Hence Theorem 1 is true in this case.

Assume now $s \geq 0$ and that the Theorem is true for $s-1$. It is clear that for a generic hyperplane H in $\mathbb{P}^{n+1}$ one has

$$\dim(S(\bar{X} \cap H) \cap H_{\infty}) = s - 1 .$$

Then by the induction hypothesis it follows that the hypersurface $X \cap H$ is $(n-2-s)$ – connected.

On the other hand, using Theorem 2 in Hamm [6] (more precisely the version described in the remark following it) or using Lefschetz Theorem in Goresky–MacPherson book [4], p. 153 it follows that the pair $(X, X \cap H)$ is $(n-1)$ – connected. These two facts together imply that X is also at least $(n-s-2)$ – connected and hence Theorem 1 is proved.

Proof of Theorem 2

Exactly as the proof of Theorem 1, only more care should be taken in the choice of the generic hyperplane H in $\mathbb{C}^{n+1}$.

In order to apply induction, we should consider only hyperplanes H which pass through the origin of $\mathbb{C}^{n+1}$. The relation

$$S(f \mid H) = S(f) \cap \overline{H}$$

where $\overline{H}$ is the hyperplane in $\mathbb{P}^n$ associated with H, clearly holds for a generic H through 0.

Moreover, the connectivity of the pair $(X, X \cap H)$ is $(n-1)$ as soon as H is transversal to the strata of a regular stratification of X_∞, see Hamm's remark mentioned above or [4], p.. 154. (Here of course $X = f^{-1}(c)$ is an arbitrary fiber of f). Hence it is enough to take H such that $\overline{H}$ is transversal to the given stratification of X_∞.

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