DIFFEOMORPHISMS, ANALYTIC TORSION AND NONCOMMUTATIVE GEOMETRY

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ABSTRACT. We prove an index theorem concerning the pushforward of flat $\mathfrak{B}$-vector bundles, where $\mathfrak{B}$ is an appropriate algebra. We construct an associated analytic torsion form $\mathcal{T}$. If Z is a smooth closed aspherical manifold, we show that $\mathcal{T}$ gives invariants of $\pi_*(\mathrm{Diff}(Z))$.

1. INTRODUCTION

Let Z be a smooth connected closed n-dimensional $K(\Gamma,1)$-manifold. Let Diff(Z) be the group of diffeomorphisms of Z, with its natural smooth topology [28]. What are the rational homotopy groups of Diff(Z)? Farrell and Hsiang made the following conjecture :

Conjecture [13] : $\pi_1(\mathrm{Diff}(Z)) \otimes_{\mathbf{Z}} \mathbb{Q} = \mathrm{center}(\Gamma) \otimes_{\mathbf{Z}} \mathbb{Q}$

and if $i > 1$ is sufficiently small compared to n,

$$\pi_i(\mathrm{Diff}(Z)) \otimes_{\mathbf{Z}} \mathbb{Q} = \begin{cases} \bigoplus_{j=1}^{\infty} \mathrm{H}_{i+1-4j}(\Gamma;\mathbb{Q}) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{1.1}$$

It follows from the work of Farrell and Jones [14] that the conjecture is true when $n > 10$, $i < \frac{n-7}{3}$ and Γ is a discrete cocompact subgroup of a Lie group with a finite number of connected components. (For example, it is true when Z is a torus, something which was already shown in [13].) The π_1-result is what one would expect from homotopy theory. However, (1.1) is peculiar to the fact that we are looking at diffeomorphisms; the analogous rational homotopy groups of Homeo(Z) vanish. In the cases when the conjecture has been proven, the proofs are very impressive but rather indirect, using a great deal of topological machinery.

From a constructive viewpoint, suppose that we are given a smooth based map $\alpha : S^i \to \mathrm{Diff}(Z)$. How could we compute the corresponding rational homotopy class $[\alpha]_{\mathbb{Q}} \in \pi_i(\mathrm{Diff}(Z)) \otimes_{\mathbf{Z}} \mathbb{Q}$? First, let us make an auxiliary fiber bundle. Using α, we can glue two copies of $D^{i+1} \times Z$ along their boundaries to obtain a smooth manifold M which fibers over S^{i+1}, with fiber Z. Any (smooth) topological invariant of fiber bundles will give an invariant of $\pi_i(\mathrm{Diff}(Z))$.

Wagoner suggested [33] that the relevant invariant is a fiber-bundle extension of the Ray-Singer analytic torsion [32]. In [2], J.-M. Bismut and the author constructed a certain extension of the Ray-Singer analytic torsion which does give some information about $\pi_*(\mathrm{Diff}(Z))$. However, that extension is inadequate to capture all of the information in (1.1). In this

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paper, using ideas from noncommutative geometry, we will construct a "higher" analytic torsion which does potentially detect the right-hand-side of (1.1).

One can think of the analytic torsion as arising from the transgression of certain index theorems. We describe the relevant index theorems. Suppose that $M \xrightarrow{\pi} B$ is a smooth fiber bundle with connected closed fibers Z. Let E be a flat complex vector bundle on M. In [2], certain characteristic classes $c(E) \in \mathrm{H}^{odd}(M;\mathbb{R})$ were defined. The pushforward of E is defined to be

$$\pi_*(E) = \sum_{p=0}^{dim(Z)} (-1)^p \, \mathrm{H}^p(Z; E\big|_Z), \tag{1.2}$$

a formal alternating sum of flat vector bundles on B constructed from the cohomology groups of the fibers. Let $e(TZ) \in \mathrm{H}^{dim(Z)}(M; o(TZ))$ be the Euler class of the vertical tangent bundle TZ. The index theorem of [2, Theorem 0.1] stated

$$c(\pi_* E) = \int_Z e(TZ) \cup c(E) \qquad \text{in } \mathrm{H}^{odd}(B;\mathbb{R}). \tag{1.3}$$

In its proof, which was analytic in nature, a certain differential form $\mathcal{T} \in \Omega^{even}(B)$ appeared, called the analytic torsion form.

This index theorem was reproved topologically and extended by Dwyer, Weiss and Williams [12]. Their setup was a fiber bundle as above, a ring $\mathfrak{B}$ and a local system $\mathcal{E}$ of finitely-generated projective $\mathfrak{B}$-modules on M. The local system defines a class $[\mathcal{E}] \in K_{\mathfrak{B}}^{alg}(M)$ in a generalized cohomology group of M. One again has local systems $\{\mathrm{H}^p(Z; \mathcal{E}\big|_Z)\}_{p=0}^{dim(Z)}$ of finitely-generated $\mathfrak{B}$-modules on B. Suppose that they are projective. Define $\pi_*(\mathcal{E})$ as in (1.2). Then [12, Equation (0-3)] stated

$$[\pi_*(\mathcal{E})] = \mathrm{tr}^*[\mathcal{E}] \qquad \text{in } K_{\mathfrak{B}}^{alg}(B), \tag{1.4}$$

where tr^* is the Becker-Gottlieb-Dold transfer. When $\mathfrak{B} = \mathbb{C}$, (1.3) is a consequence of applying the characteristic class c to both sides of (1.4).

In the present paper, we essentially give an analytic proof of (1.4). Provided that $\mathfrak{B}$ is an algebra over $\mathbb{C}$ which satisfies certain technical conditions, we define a characteristic class

$$[CS] : K_{\mathfrak{B}}^{alg}(M) \longrightarrow \bigoplus_{\substack{p>q \\ p+q \text{ odd}}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B})), \tag{1.5}$$

where $\overline{H}_*(\mathfrak{B})$ is the noncommutative de Rham cohomology of the algebra $\mathfrak{B}$. Using analytic methods, we prove the following theorem.

Theorem 1. *Let the fiber bundle $M \xrightarrow{\pi} B$ be as above. Let $\mathcal{E}$ be a local system of finitely-generated projective $\mathfrak{B}$-modules on M. Suppose that the fiberwise differentials $\overline{d^Z}$ have closed image. Then*

$$[CS(\pi_*\mathcal{E})] = \int_Z e(TZ) \cup [CS(\mathcal{E})] \quad \text{in} \bigoplus_{\substack{p>q \\ p+q \text{ odd}}} \mathrm{H}^p(B; \overline{H}_q(\mathfrak{B})). \tag{1.6}$$

The condition that $\overline{d^Z}$ have closed image guarantees that $\mathrm{H}^*(Z;\mathcal{E}|_Z)$ is a local system of projective $\mathfrak{B}$-modules. If $\mathfrak{B} = \mathbb{C}$ then

$$\overline{H}_q(\mathfrak{B}) = \begin{cases} \mathbb{C} & \text{if } q = 0, \\ 0 & \text{if } q > 0 \end{cases} \tag{1.7}$$

and so we recover (1.3).

The statement of Theorem 1 can also be obtained by applying the characteristic class $[CS]$ to both sides of (1.4). As in [2], the interest of the analytic proof is that it gives a more refined statement at the level of differential forms. With notation that will be explained later, a certain explicit differential form $\mathcal{T} \in \overline{\Omega}^{\prime\prime,even}(B,\mathfrak{B})$ appears naturally in the proof of Theorem 1. We call it the analytic torsion form.

Theorem 2. *With the hypotheses of Theorem 1,*

$$d\mathcal{T} = \int_Z e\left(TZ, \nabla^{TZ}\right) \wedge CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right) - CS\left(\nabla^{\pi_*\mathcal{E}}, h^{\pi_*\mathcal{E}}\right) \text{ in } \overline{\Omega}^{\prime\prime,odd}(B,\mathfrak{B}). \tag{1.8}$$

Here $h^{\mathcal{E}}$ is a $\mathfrak{B}$-valued Hermitian metric on $\mathcal{E}$ and $h^{\pi_*\mathcal{E}}$ is the induced $\mathfrak{B}$-valued Hermitian metric on $\pi_*\mathcal{E}$.

If $\dim(Z)$ is odd and $\mathrm{H}^*(Z;\mathcal{E}|_Z) = 0$ then the right-hand-side of (1.8) vanishes automatically, implying that $\mathcal{T}$ is closed.

Theorem 3. *With the hypotheses of Theorem 1, suppose in addition that* $\dim(Z)$ *is odd and* $\mathrm{H}^*(Z;\mathcal{E}|_Z) = 0$. *Then the cohomology class*

$$[\mathcal{T}] \in \bigoplus_{\substack{p>q \\ p+q \text{ even}}} \mathrm{H}^p(B;\overline{H}_q(\mathfrak{B})) \tag{1.9}$$

is a (smooth) topological invariant of the fiber bundle $M \xrightarrow{\pi} B$ *and the local system* $\mathcal{E}$.

In particular, $[\mathcal{T}]$ will give invariants of $\pi_*(\mathrm{Diff}(Z))$ when Z is a smooth closed aspherical manifold.

We now describe the contents of this paper. The local system $\mathcal{E}$ on M can be thought of as a flat $\mathfrak{B}$-vector bundle. The first order of business is to define the relevant characteristic classes of $\mathcal{E}$. Unlike in [2], it is not enough to just use the flat connection on $\mathcal{E}$. Instead, we will need a connection on $\mathcal{E}$ which also differentiates in the "noncommutative" directions. The correct notion, due to Karoubi, is that of a partially flat connection (called a "connexion à courbure plate" in [17]).

In Section 2 we briefly review the geometry of $\mathfrak{B}$-vector bundles. We define certain complexes of noncommutative differential forms and describe their cohomologies. We review the notion of a $\mathfrak{B}$-connection on $\mathcal{E}$ and its Chern character. We define the relative Chern-Simons class of two $\mathfrak{B}$-vector bundles which are topologically isomorphic, each having a partially flat connection. Because our connections are partially flat and not completely flat, the formalism involved is fundamentally different than that of [2].

In Section 3 we look at the case when Γ is a finitely-generated discrete group and $\mathfrak{B}$ lies between the group algebra $\mathbb{C}\Gamma$ and the group C^*-algebra $C^*_r\Gamma$. If M is a manifold with a normal Γ-covering $M' \to M$ then there is a canonical $\mathfrak{B}$-vector bundle $\mathcal{E} = \mathfrak{B} \times_\Gamma M'$ on M.

We describe an explicit partially flat connection on $\mathcal{E}$ and compute the pairing of its Chern character with the group cohomology of Γ. This computation is important for applications.

In Section 4 we define the notion of a $\mathfrak{B}$-valued Hermitian metric $h^{\mathcal{E}}$ on a $\mathfrak{B}$-vector bundle $\mathcal{E}$. With our assumptions on $\mathfrak{B}$, a Hermitian metric on $\mathcal{E}$ always exists and is unique up to isotopy. A Hermitian metric gives a topological isomorphism between $\mathcal{E}$ and its antidual bundle $\overline{\mathcal{E}}^*$. If $\mathcal{E}$ has a partially flat connection, we can use this isomorphism to define the relative Chern-Simons class $CS\left(\mathcal{E}, h^{\mathcal{E}}\right) \in \overline{\Omega}^{\prime\prime,odd}(M, \mathfrak{B})$ of $\mathcal{E}$ and $\overline{\mathcal{E}}^*$. Its cohomology class is independent of the choice of Hermitian metric, giving the characteristic class

$$[CS(\mathcal{E})] \in \bigoplus_{\substack{p>q \\ p+q \text{ odd}}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B})). \tag{1.10}$$

In Section 5 we generalize the preceding results from connections to superconnections. This generalization will be crucial for the fiber bundle results. We define the notion of a partially flat superconnection on M. We construct the relative Chern-Simons class and analytic torsion form. Using these constructions, we prove a finite-dimensional analog of (1.4). This analog is similar to [2, Theorem 2.19], but the large-time analysis requires new techniques. We then relate the finite-dimensional analytic torsion form to various versions of the Reidemeister torsion.

In Section 6 we extend the methods of Section 5 to the setting of a fiber bundle $Z \to M \stackrel{\pi}{\to} B$. First, we prove some basic facts about $\mathfrak{B}$-pseudodifferential operators. Using heat kernel techniques, we prove Theorem 1. We then define the analytic torsion form $\mathcal{T} \in \overline{\Omega}^{\prime\prime,even}(B, \mathfrak{B})$ and prove Theorems 2 and 3.

Relevant examples of the preceding formalism come from finitely-generated discrete groups Γ. Let us introduce a certain hypothesis on Γ :

Hypothesis 1. *There is a Fréchet locally m-convex algebra $\mathfrak{B}$ containing $\mathbb{C}\Gamma$ such that*
1. $\mathfrak{B}$ is dense in $C_r^*\Gamma$ and stable under the holomorphic functional calculus in $C_r^*\Gamma$.
2. For each $[\tau] \in \mathrm{H}^q(\Gamma;\mathbb{C})$, there is a representative cocycle $\tau \in Z^q(\Gamma;\mathbb{C})$ such that the ensuing cyclic cocycle $Z_\tau \in HC^q(\mathbb{C}\Gamma)$ extends to a continuous cyclic cocycle on $\mathfrak{B}$.

Hypothesis 1 arises in analytic proofs of the Novikov Conjecture. It is known to be satisfied by virtually nilpotent groups and Gromov-hyperbolic groups [9, Section 3.5]. Using the characteristic class $[CS]$, in Section 4 we give a simple proof that the algebraic K-theory assembly map is rationally injective for such groups.

Let Z be a smooth connected closed n-dimensional manifold with fundamental group Γ. If the hypotheses of the preceding theorems are satisfied, we can define invariants of $\pi_i(\mathrm{Diff}(Z))$, $i > 1$, by constructing the auxiliary fiber bundle mentioned at the beginning, computing its analytic torsion form $\mathcal{T}$ and integrating over $B = S^{i+1}$ to get

$$\int_B [\mathcal{T}] \in \bigoplus_{\substack{q<i+1 \\ q\equiv i+1 \mod 2}} \overline{H}_q(\mathfrak{B}). \tag{1.11}$$

By Hypothesis 1, $\int_B [\mathcal{T}]$ then pairs with $\mathrm{H}^q(\Gamma;\mathbb{C})$.

To make contact with (1.1), in Section 7 we assume that Z is a $K(\Gamma,1)$-manifold. In order to satisfy the hypotheses of the preceding theorems, we would have to know that the differential form Laplacian on the universal cover $\widetilde{Z}$ is invertible in all degrees, something which is probably never the case. We present two ways to get around this problem. First, we consider the case when $\Gamma = \mathbb{Z}^n$. In this case we can apply ordinary "commutative" analysis to study the problem. We show that we can define a pairing between $\int_B [\mathcal{T}]$ and $\mathrm{H}^q(\Gamma;\mathbb{C})$ provided that $q < \min(i+1, n)$. This pairing vanishes for trivial reasons unless n is odd and $q \equiv i+1 \mod 4$. Second, we consider general Γ satisfying Hypothesis 1. Using the fact that the auxiliary fiber bundle $M \xrightarrow{\pi} S^{i+1}$ is fiber-homotopically trivial, we construct a relative analytic torsion form $\mathcal{T}$ such that $\int_B [\mathcal{T}]$ pairs with $\mathrm{H}^q(\Gamma;\mathbb{C})$ provided that $q < i+1$. Again, the pairing vanishes for trivial reasons unless n is odd and $q \equiv i+1 \mod 4$.

Based on a comparison with (1.1), we expect that the pairing between $\int_B [\mathcal{T}]$ and $\mathrm{H}^q(\Gamma;\mathbb{C})$ will be nonzero if n is odd and $q \equiv i+1 \mod 4$, at least if i is sufficiently small with respect to n. To show this, one will probably have to make a direct link between the analytic constructions of the present paper and the topological machinery.

We note that we do not construct a "higher" analytic torsion of a single manifold, in the sense of Novikov's higher signatures. In the case of a single manifold, i.e. if the base B of the fiber bundle is a point, the analytic torsion that we construct in this paper lies in $\mathfrak{B}/\overline{[\mathfrak{B},\mathfrak{B}]}$, something which pairs with the zero-dimensional cyclic cohomology of $\mathfrak{B}$. The higher-dimensional cyclic cohomology of $\mathfrak{B}$ only enters when the base of the fiber bundle is also higher-dimensional.

So far, our topological applications of the analytic torsion form are to the rational homotopy of diffeomorphism groups of aspherical manifolds. There are also results in the literature about the rational homotopy of diffeomorphism groups of simply-connected manifolds [7, §4]. It would be interesting to see if there is an analog of the analytic torsion form in the simply-connected case.

Finally, let us remark that in [21], we constructed a higher eta-invariant of a manifold with virtually nilpotent fundamental group. Using the methods of Sections 5 and 6 of the present paper, one can relax this condition to allow for Gromov-hyperbolic fundamental groups.

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2. Noncommutative Bundle Theory

In this section we review some facts about $\mathfrak{B}$-vector bundles and their characteristic classes. The material in this section is taken from [17], along with [10] and [20].

2.1. Noncommutative Differential Forms. Let $\mathfrak{B}$ be a Fréchet locally m-convex algebra with unit, i.e. the projective limit of a sequence

$$\ldots \longrightarrow B_{j+1} \longrightarrow B_j \longrightarrow \ldots \longrightarrow B_0 \tag{2.1}$$

of Banach algebras with unit. (A relevant example is $\mathfrak{B} = C^\infty(S^1)$ and $B_j = C^j(S^1)$.) We recall some basic facts about such algebras [26]. For $j \geq 0$, let $i_j : \mathfrak{B} \rightarrow B_j$ be the obvious homomorphism. The Banach norm $|\cdot|_j$ on B_j induces a submultiplicative seminorm $\|\cdot\|_j$ on $\mathfrak{B}$ by $\| b \|_j = |i_j(b)|_j$. Given $b \in \mathfrak{B}$, its spectrum $\sigma(b) \subseteq \mathbb{C}$ is given by

$$\sigma(b) = \bigcup_{j=0}^{\infty} \sigma(i_j(b)). \tag{2.2}$$

As each Banach algebra B_j has a holomorphic functional calculus, it follows from (2.2) that $\mathfrak{B}$ also has a holomorphic functional calculus.

We assume that B_0 is a C^*-algebra Λ, that i_0 is injective with dense image and that $\mathfrak{B}$ is stable under the holomorphic functional calculus in Λ. A consequence is that the invertible elements $\mathrm{Inv}(\mathfrak{B})$ are open in $\mathfrak{B}$, as $\mathrm{Inv}(\Lambda)$ is open in Λ and $\mathrm{Inv}(\mathfrak{B}) = i_0^{-1}(\mathrm{Inv}(\Lambda))$. Furthermore, $\sigma(b) = \sigma(i_j(b))$ for all $j \geq 0$.

Let us ignore the topology of $\mathfrak{B}$ for a moment. The universal graded differential algebra (GDA) of $\mathfrak{B}$ is

$$\Omega_*(\mathfrak{B}) = \bigoplus_{k=0}^{\infty} \Omega_k(\mathfrak{B}) \tag{2.3}$$

where as a vector space, $\Omega_k(\mathfrak{B}) = \mathfrak{B} \otimes (\otimes^k(\mathfrak{B}/\mathbb{C}))$. As a GDA, $\Omega_*(\mathfrak{B})$ is generated by $\mathfrak{B} = \Omega_0(\mathfrak{B})$ and $d\mathfrak{B} \subset \Omega_1(\mathfrak{B})$ with the relations

$$d1 = 0, \quad d^2 = 0, \quad d(\omega_k\omega_l) = (d\omega_k)\omega_l + (-1)^k \omega_k d\omega_l \tag{2.4}$$

for $\omega_k \in \Omega_k(\mathfrak{B})$, $\omega_l \in \Omega_l(\mathfrak{B})$. It will be convenient to write an element ω_k of $\Omega_k(\mathfrak{B})$ as a finite sum $\sum b_0 db_1 \ldots db_k$. There is a differential complex

$$\overline{\Omega}_*(\mathfrak{B}) = \Omega_*(\mathfrak{B})/[\Omega_*(\mathfrak{B}), \Omega_*(\mathfrak{B})]. \tag{2.5}$$

Let $\overline{Z}_*(\mathfrak{B})$, $\overline{B}_*(\mathfrak{B})$ and $\overline{H}_*(\mathfrak{B})$ denote its cocycles, coboundaries and cohomology, respectively. The latter is given by

$$\overline{H}_*(\mathfrak{B}) = \begin{cases} \mathrm{Ker}\,(B : HC_0(\mathfrak{B}) \,(= \mathfrak{B}/[\mathfrak{B},\mathfrak{B}]) \longrightarrow H_1(\mathfrak{B},\mathfrak{B})) & \text{if } * = 0, \\ \mathrm{Ker}\,\big(B : \overline{HC}_*(\mathfrak{B}) \longrightarrow H_{*+1}(\mathfrak{B},\mathfrak{B})\big) & \text{if } * > 0. \end{cases} \tag{2.6}$$

Here $\overline{HC}_*(\mathfrak{B})$ is the reduced cyclic homology of $\mathfrak{B}$ and $H_*(\mathfrak{B},\mathfrak{B})$ is the Hochschild homology. In particular, there is a pairing between $\overline{H}_*(\mathfrak{B})$ and the (reduced) cyclic cohomology of $\mathfrak{B}$.

Taking the topology on $\mathfrak{B}$ into consideration, there is a Fréchet completion of $\Omega_*(\mathfrak{B})$, which we again denote by $\Omega_*(\mathfrak{B})$. Furthermore, there is a Fréchet space $\overline{\Omega}_*(\mathfrak{B})$ defined as in (2.5), except quotienting by the closure of the commutator. Hereafter, when we refer to spaces of differential forms we will always mean these Fréchet spaces. Furthermore, all tensor products of Fréchet spaces will implicitly be projective tensor products. We again denote the (separable) homology of $\overline{\Omega}_*(\mathfrak{B})$ by $\overline{H}_*(\mathfrak{B})$. It pairs with the (reduced) topological cyclic cohomology of $\mathfrak{B}$.

Let $\mathfrak{E}$ be a Fréchet left $\mathfrak{B}$-module, meaning a Fréchet space which is a continuous left $\mathfrak{B}$-module. Hereafter, we assume that $\mathfrak{E}$ is a finitely-generated projective $\mathfrak{B}$-module. If $\mathfrak{F}$ is a Fréchet right $\mathfrak{B}$-module then there is a Fréchet space $\mathfrak{F} \otimes_{\mathfrak{B}} \mathfrak{E}$. If $\mathfrak{F}$ is a Fréchet $\mathfrak{B}$-bimodule then there is a "trace map"

$$\mathrm{Tr} : \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}, \mathfrak{F} \otimes_{\mathfrak{B}} \mathfrak{E}) \longrightarrow \mathfrak{F}/\overline{[\mathfrak{B}, \mathfrak{F}]}. \tag{2.7}$$

(We quotient by the closure of $[\mathfrak{B}, \mathfrak{F}]$ to ensure that the result lies in a Fréchet space.) If $\mathfrak{F}$ is a Fréchet algebra containing $\mathfrak{B}$ then Tr gives a trace

$$\mathrm{Tr} : \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}, \mathfrak{F} \otimes_{\mathfrak{B}} \mathfrak{E}) \longrightarrow \mathfrak{F}/\overline{[\mathfrak{F}, \mathfrak{F}]}. \tag{2.8}$$

In the case that $\mathfrak{E}$ is $\mathbb{Z}_2$-graded by an operator $\Gamma_{\mathfrak{E}} \in \mathrm{End}_{\mathfrak{B}}(\mathfrak{E})$ satisfying $\Gamma_{\mathfrak{E}}^2 = 1$, we can extend Tr to a supertrace by

$$\mathrm{Tr}_s(T) = \mathrm{Tr}(\Gamma_{\mathfrak{E}} T). \tag{2.9}$$

Let M be a smooth connected manifold. Put

$$\Omega^{p,q}(M,\mathfrak{B}) = \Omega^p(M;\Omega_q(\mathfrak{B})), \qquad \overline{\Omega}^{p,q}(M,\mathfrak{B}) = \Omega^p(M;\overline{\Omega}_q(\mathfrak{B}))$$
$$\Omega^k(M,\mathfrak{B}) = \bigoplus_{p+q=k} \Omega^{p,q}(M,\mathfrak{B}), \qquad \overline{\Omega}^k(M,\mathfrak{B}) = \bigoplus_{p+q=k} \overline{\Omega}^{p,q}(M,\mathfrak{B}).$$

We also write $C^\infty(M;\mathfrak{B})$ for $\Omega^0(M,\mathfrak{B})$. There is a total differential d on $\overline{\Omega}^*(M,\mathfrak{B})$ which decomposes as the sum of two differentials $d = d^{1,0} + d^{0,1}$. Put

$$\overline{\Omega}^{\prime,2k}(M,\mathfrak{B}) = Z^k(M;\overline{\Omega}_k(\mathfrak{B})) \oplus \left(\bigoplus_{\substack{p+q=2k \\ p<q}} \overline{\Omega}^{p,q}(M,\mathfrak{B}) \right), \tag{2.10}$$
$$\overline{\Omega}^{\prime,2k+1}(M,\mathfrak{B}) = \bigoplus_{\substack{p+q=2k+1 \\ p<q}} \overline{\Omega}^{p,q}(M,\mathfrak{B}),$$
$$\overline{\Omega}^{\prime\prime,*}(M,\mathfrak{B}) = \overline{\Omega}^*(M,\mathfrak{B})/\overline{\Omega}^{\prime,*}(M,\mathfrak{B}).$$

Then $\overline{\Omega}^{\prime,*}(M,\mathfrak{B})$ and $\overline{\Omega}^{\prime\prime,*}(M,\mathfrak{B})$ are also differential complexes. Let $H^*_{\mathfrak{B}}(M)$, $H'^{,*}_{\mathfrak{B}}(M)$ and $H''^{,*}_{\mathfrak{B}}(M)$ denote the cohomology groups of $\overline{\Omega}^*(M,\mathfrak{B})$, $\overline{\Omega}^{\prime,*}(M,\mathfrak{B})$ and $\overline{\Omega}^{\prime\prime,*}(M,\mathfrak{B})$, respectively. Then

$$H^k_{\mathfrak{B}}(M) \cong \bigoplus_{p+q=k} \mathrm{H}^p\left(M;\overline{H}_q(\mathfrak{B})\right), \tag{2.11}$$
$$H'^{,2k}_{\mathfrak{B}}(M) \cong \mathrm{H}^k(M;\overline{Z}_k(\mathfrak{B})) \oplus \left(\bigoplus_{\substack{p+q=2k \\ p<q}} \mathrm{H}^p(M;\overline{H}_q(\mathfrak{B})) \right),$$
$$H'^{,2k+1}_{\mathfrak{B}}(M) \cong \bigoplus_{\substack{p+q=2k+1 \\ p<q}} \mathrm{H}^p(M;\overline{H}_q(\mathfrak{B})),$$
$$H''^{,2k}_{\mathfrak{B}}(M) \cong \bigoplus_{\substack{p+q=2k \\ p>q}} \mathrm{H}^p(M;\overline{H}_q(\mathfrak{B})),$$
$$H''^{,2k+1}_{\mathfrak{B}}(M) \cong \mathrm{H}^{k+1}\left(M;\frac{\overline{\Omega}_k(\mathfrak{B})}{\overline{B}_k(\mathfrak{B})}\right) \oplus \left(\bigoplus_{\substack{p+q=2k+1 \\ p>q+1}} \mathrm{H}^p(M;\overline{H}_q(\mathfrak{B})) \right).$$

To realize the first isomorphism in (2.11) explicitly, if $\omega \in \overline{\Omega}^k(M,\mathfrak{B})$ is d-closed and $z \in Z_p(M;\mathbb{C})$ then $\int_z \omega \in \overline{Z}_{k-p}(\mathfrak{B})$. The other isomorphisms can be realized similarly.

2.2. **Noncommutative Connections and Chern Character.** Let $\mathcal{E}$ be a smooth $\mathfrak{B}$-vector bundle on M with fibers isomorphic to $\mathfrak{E}$. This means that if $\mathcal{E}$ is defined using charts $\{U_\alpha\}$ then a transition function is a smooth map $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}_{\mathfrak{B}}(\mathfrak{E})$. There is a corresponding element $[\mathcal{E}]$ in the topological K-group $K_{\mathfrak{B}}^{top}(M) = [M, K_0(\mathfrak{B}) \times BGL(\mathfrak{B})]$.

We will denote the fiber of $\mathcal{E}$ over $m \in M$ by $\mathcal{E}_m$. If $\mathfrak{F}$ is a Fréchet $\mathfrak{B}$-bimodule, let $\mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E}$ denote the $\mathfrak{B}$-vector bundle on M with fibers $(\mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E})_m = \mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E}_m$ and transition functions $\mathrm{Id}_{\mathfrak{F}} \otimes_{\mathfrak{B}} \phi_{\alpha\beta}$.

Let $C^\infty(M; \mathcal{E})$ denote the left $\mathfrak{B}$-module of smooth sections of $\mathcal{E}$ and let $\Omega(M; \mathcal{E})$ denote the left $\mathfrak{B}$-module of smooth sections of $\Lambda(T^*M) \otimes \mathcal{E}$. We put

$$\Omega^{p,q}(M, \mathfrak{B}; \mathcal{E}) = \Omega^p\left(M; \Omega_q(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E}\right) \tag{2.12}$$

and

$$\Omega^k(M, \mathfrak{B}; \mathcal{E}) = \bigoplus_{p+q=k} \Omega^{p,q}(M, \mathfrak{B}; \mathcal{E}). \tag{2.13}$$

Definition 1. *A connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map*

$$\nabla^{\mathcal{E}} : C^\infty(M; \mathcal{E}) \to \Omega^1(M, \mathfrak{B}; \mathcal{E}) \tag{2.14}$$

such that for all $f \in C^\infty(M; \mathfrak{B})$ and $s \in C^\infty(M; \mathcal{E})$,

$$\nabla^{\mathcal{E}}(fs) = f\, \nabla^{\mathcal{E}} s + df \otimes_{C^\infty(M;\mathfrak{B})} s. \tag{2.15}$$

We can decompose $\nabla^{\mathcal{E}}$ as

$$\nabla^{\mathcal{E}} = \nabla^{\mathcal{E},1,0} \oplus \nabla^{\mathcal{E},0,1}, \tag{2.16}$$

where

$$\nabla^{\mathcal{E},1,0} : C^\infty(M; \mathcal{E}) \to \Omega^1(M; \mathcal{E}) \tag{2.17}$$

is a connection on $\mathcal{E}$ in the usual sense which happens to be $\mathfrak{B}$-linear, and

$$\nabla^{\mathcal{E},0,1} : C^\infty(M; \mathcal{E}) \to C^\infty(M; \Omega_1(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E}) \tag{2.18}$$

is a $C^\infty(M)$-linear map which comes from a $\mathbb{C}$-linear bundle homomorphism

$$\partial^{\mathcal{E}} : \mathcal{E} \to \Omega_1(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E} \tag{2.19}$$

satisfying

$$\partial^{\mathcal{E}}(bs_m) = b\, \partial^{\mathcal{E}} s_m + db \otimes_{\mathfrak{B}} s_m \tag{2.20}$$

for all $m \in M$, $s_m \in \mathcal{E}_m$ and $b \in \mathfrak{B}$. One can consider $\nabla^{\mathcal{E},0,1}$ to be the part of $\nabla^{\mathcal{E}}$ which involves differentiation in the "noncommutative" direction.

Extend $\nabla^{\mathcal{E}}$ to a $\mathbb{C}$-linear map

$$\nabla^{\mathcal{E}} : \Omega^*(M, \mathfrak{B}; \mathcal{E}) \to \Omega^{*+1}(M, \mathfrak{B}; \mathcal{E}) \tag{2.21}$$

by requiring that for all $\omega \in \Omega^k(M, \mathfrak{B})$ and $s \in \Omega^l(M, \mathfrak{B}; \mathcal{E})$,

$$\nabla^{\mathcal{E}}(\omega s) = (-1)^k\, \omega \wedge \nabla^{\mathcal{E}} s + d\omega \otimes_{C^\infty(M;\mathfrak{B})} s. \tag{2.22}$$

Similarly, extend $\nabla^{\mathcal{E},1,0}$ to

$$\nabla^{\mathcal{E},1,0} : \Omega^*(M; \mathcal{E}) \to \Omega^{*+1}(M; \mathcal{E}). \tag{2.23}$$

Now $(\nabla^{\mathcal{E}})^2$ is multiplication by an element of

$$\bigoplus_{p+q=2} \Omega^p(M; \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}, \Omega_q(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E})),$$

which we also denote by $(\nabla^{\mathcal{E}})^2$.

Definition 2. *The Chern character of* $\nabla^{\mathcal{E}}$ *is*

$$\mathrm{ch}(\nabla^{\mathcal{E}}) = \mathrm{Tr}\left(e^{-(\nabla^{\mathcal{E}})^2}\right) \in \overline{\Omega}^{even}(M, \mathfrak{B}). \tag{2.24}$$

As usual, $\mathrm{ch}(\nabla^{\mathcal{E}})$ is d-closed and its cohomology class $[\mathrm{ch}(\nabla^{\mathcal{E}})] \in H^{even}_{\mathfrak{B}}(M)$ only depends on $[\mathcal{E}] \in K^{top}_{\mathfrak{B}}(M)$. If $\mathcal{E}$ and $\mathcal{E}'$ are $\mathfrak{B}$ and $\mathfrak{B}'$-vector bundles on M, respectively, then

$$\mathrm{ch}(\nabla^{\mathcal{E}\otimes_{\mathbb{C}}\mathcal{E}'}) = \mathrm{ch}(\nabla^{\mathcal{E}}) \cdot \mathrm{ch}(\nabla^{\mathcal{E}'}) \in \overline{\Omega}^{even}(M, \mathfrak{B} \otimes_{\mathbb{C}} \mathfrak{B}'). \tag{2.25}$$

Definition 3. *A connection* $\nabla^{\mathcal{E}}$ *is partially flat if its component* $\nabla^{\mathcal{E},1,0}$ *is flat, meaning* $(\nabla^{\mathcal{E},1,0})^2 = 0$.

Definition 4. *A flat structure on* $\mathcal{E}$ *is given by a connection*

$$\nabla^{\mathcal{E},flat} : C^\infty(M; \mathcal{E}) \to \Omega^1(M; \mathcal{E}) \tag{2.26}$$

which is $\mathfrak{B}$*-linear and whose extension to* $\Omega^*(M; \mathcal{E})$ *satisfies* $(\nabla^{\mathcal{E},flat})^2 = 0$.

Clearly a partially flat connection on $\mathcal{E}$ determines a flat structure on $\mathcal{E}$ through its $(1,0)$-part. Conversely, given a flat structure on $\mathcal{E}$, there is a partially flat connection on $\mathcal{E}$ which is compatible with the flat structure, although generally not a unique one.

The flat structure $(\mathcal{E}, \nabla^{\mathcal{E},flat})$ is classified by a map $M \to B\mathrm{Aut}_{\mathfrak{B}}(\mathfrak{C})_\delta$, where δ denotes the discrete topology. Then there is a composite map

$$M \to B\mathrm{Aut}_{\mathfrak{B}}(\mathfrak{C})_\delta \to BGL(\mathfrak{B})_\delta \to BGL(\mathfrak{B})^+_\delta, \tag{2.27}$$

where $+$ denotes Quillen's plus construction. Thus the pair $(\mathcal{E}, \nabla^{\mathcal{E},flat})$ gives an element $[\mathcal{E}, \nabla^{\mathcal{E},flat}] \in K^{alg}_{\mathfrak{B}}(M) = [M, K_0(\mathfrak{B}) \times BGL(\mathfrak{B})^+_\delta]$, the $K_0(\mathfrak{B})$ factor simply representing the K-theory class of the fiber $\mathfrak{C}$.

If $\nabla^{\mathcal{E}}$ is partially flat then

$$(\nabla^{\mathcal{E}})^2 \in \Omega^1(M; \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}, \Omega_1(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E})) \oplus \Omega^0(M; \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}, \Omega_2(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E})). \tag{2.28}$$

Thus $\mathrm{ch}(\nabla^{\mathcal{E}}) \in \bigoplus_{\substack{p\leq q\\ p+q\ even}} \overline{\Omega}^{p,q}(M, \mathfrak{B})$. As $\mathrm{ch}(\nabla^{\mathcal{E}})$ is d-closed, its (p,p)-component $\mathrm{ch}^{p,p}$ must satisfy $d^{1,0}\mathrm{ch}^{p,p} = 0$. Hence $\mathrm{ch}(\nabla^{\mathcal{E}}) \in \overline{\Omega}^{\prime,even}(M, \mathfrak{B})$ and $[\mathrm{ch}(\nabla^{\mathcal{E}})] \in H^{\prime,even}_{\mathfrak{B}}(M)$. There is a commutative diagram

$$\begin{array}{ccc} K^{alg}_{\mathfrak{B}}(M) & \longrightarrow & K^{top}_{\mathfrak{B}}(M) \\ \mathrm{ch}\downarrow & & \mathrm{ch}\downarrow \\ H^{\prime,even}_{\mathfrak{B}}(M) & \longrightarrow & H^{even}_{\mathfrak{B}}(M). \end{array} \tag{2.29}$$

Example 1 : If $\mathfrak{B} = \mathbb{C}$ then $\Omega_0(\mathfrak{B}) = \overline{Z}_0(\mathfrak{B}) = \overline{H}_0(\mathfrak{B}) = \mathbb{C}$ and $\Omega_*(\mathfrak{B}) = \overline{Z}_*(\mathfrak{B}) = \overline{H}_*(\mathfrak{B}) = 0$ for $* > 0$. Then if $\mathcal{E}$ has a flat structure, by (2.11) we have that $[\mathrm{ch}(\nabla^{\mathcal{E}})]$ lies in $\mathrm{H}^0(M; \mathbb{C})$ and simply represents $\mathrm{rk}(\mathcal{E})$. On the other hand, $K^{alg}_{\mathbb{C}}(M)$ can be very rich.

Thus the Chern character does not see the interesting part of $K_{\mathbb{C}}^{alg}$. We now give another construction which will be used in Section 4 to see more of $K_{\mathfrak{B}}^{alg}$.

2.3. **Chern-Simons Classes of Partially Flat Connections.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be smooth $\mathfrak{B}$-vector bundles on M with flat structures. Suppose that there is a smooth isomorphism $\alpha : \mathcal{E}_1 \to \mathcal{E}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ as topological $\mathfrak{B}$-vector bundles. The triple $(\mathcal{E}_1, \mathcal{E}_2, \alpha)$ defines an element of Karoubi's relative K-group $K_{\mathfrak{B}}^{rel}(M)$, which fits into an exact sequence

$$K_{\mathfrak{B}}^{alg,-1}(M) \longrightarrow K_{\mathfrak{B}}^{top,-1}(M) \longrightarrow K_{\mathfrak{B}}^{rel}(M) \longrightarrow K_{\mathfrak{B}}^{alg}(M) \longrightarrow K_{\mathfrak{B}}^{top}(M). \tag{2.30}$$

Choose partially flat connections $\nabla^{\mathcal{E}_1}$, $\nabla^{\mathcal{E}_2}$ which are compatible with the flat structures. For $u \in [0,1]$, put $\nabla^{\mathcal{E}}(u) = u\nabla^{\mathcal{E}_1} + (1-u)\,\alpha^*\nabla^{\mathcal{E}_2}$. Note that for $u \in (0,1)$, $\nabla^{\mathcal{E}}(u)$ may not be partially flat on $\mathcal{E}_1$.

Definition 5. *The relative Chern-Simons class* $CS\left(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2}\right) \in \overline{\Omega}^{\prime\prime,odd}(M, \mathfrak{B})$ *is*

$$CS\left(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2}\right) = -\int_0^1 \mathrm{Tr}\left(\left(\partial_u \nabla^{\mathcal{E}}(u)\right) e^{-\left(\nabla^{\mathcal{E}}(u)\right)^2}\right) du. \tag{2.31}$$

By construction,

$$dCS\left(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2}\right) = \mathrm{ch}(\nabla^{\mathcal{E}_1}) - \mathrm{ch}(\nabla^{\mathcal{E}_2}) \tag{2.32}$$

vanishes in $\overline{\Omega}^{\prime\prime,even}(M, \mathfrak{B})$. Thus there is a class $\left[CS\left(\nabla^{\mathcal{E}_1,flat}, \nabla^{\mathcal{E}_2,flat}\right)\right] \in H_{\mathfrak{B}}^{\prime\prime,odd}(M)$ which turns out to only depend on $[\mathcal{E}_1, \mathcal{E}_2, \alpha] \in K_{\mathfrak{B}}^{rel}(M)$. In particular, $\left[CS\left(\nabla^{\mathcal{E}_1,flat}, \nabla^{\mathcal{E}_2,flat}\right)\right]$ is independent of the choice of the partially flat connections $\nabla^{\mathcal{E}_1}$, $\nabla^{\mathcal{E}_2}$ and only depends on α through its isotopy class; this will also follow from Proposition 9.

From (2.11),

$$\left[CS\left(\nabla^{\mathcal{E}_1,flat}, \nabla^{\mathcal{E}_2,flat}\right)\right] \in \left(\bigoplus_p \mathrm{H}^{p+1}\left(M; \frac{\overline{\Omega}_p(\mathfrak{B})}{\overline{B}_p(\mathfrak{B})}\right)\right) \oplus \left(\bigoplus_{\substack{p>q+1 \\ p+q\ odd}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B}))\right). \tag{2.33}$$

The next proposition is implicitly contained in [10, p. 444-448]. We give a simpler proof.

Proposition 1. $\left[CS\left(\nabla^{\mathcal{E}_1,flat}, \nabla^{\mathcal{E}_2,flat}\right)\right]$ *actually lies in* $\bigoplus_{\substack{p>q \\ p+q\ odd}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B}))$.

Proof. Let $CS^{p+1,p} \in \overline{\Omega}^{p+1,p}(M, \mathfrak{B})$ denote the $(p+1,p)$-component of the explicit differential form in (2.31). From (2.32), $d^{1,0}\, CS^{p+1,p} = 0$ and so $CS^{p+1,p}$ defines an element of $\mathrm{H}^{p+1}\left(M; \frac{\overline{\Omega}_p(\mathfrak{B})}{\overline{B}_p(\mathfrak{B})}\right)$. We show that this element lies in $\mathrm{H}^{p+1}(M, \overline{H}_p(\mathfrak{B}))$. For $i \in \{1,2\}$, define

$$\partial^{\mathcal{E}_i} : \mathcal{E}_i \to \Omega_1(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E}_i \tag{2.34}$$

as in (2.19). Put

$$g(x) = \frac{e^{-x^2} - 1}{x}. \tag{2.35}$$

If $z \in Z_{p+1}(M;\mathbb{C})$ then in $\overline{\Omega}_{p+1}(\mathfrak{B})$,

$$\begin{aligned} d^{\mathfrak{B}} \int_z CS^{p+1,p} &= \int_z d^{0,1}\, CS^{p+1,p} = \int_z \left(d^{1,0}\, CS^{p,p+1} + d^{0,1}\, CS^{p+1,p}\right) \\ &= \int_z \left[\mathrm{Tr}\left(e^{-(\nabla^{\mathcal{E}_1,1,0}\partial^{\mathcal{E}_1})^2}\right) - \mathrm{Tr}\left(e^{-(\nabla^{\mathcal{E}_2,1,0}\partial^{\mathcal{E}_2})^2}\right)\right] \\ &= \int_z \left[\mathrm{Tr}\left((\nabla^{\mathcal{E}_1,1,0}\partial^{\mathcal{E}_1})\, g(\nabla^{\mathcal{E}_1,1,0}\partial^{\mathcal{E}_1})\right)\right. \\ &\qquad \left. -\mathrm{Tr}\left((\nabla^{\mathcal{E}_2,1,0}\partial^{\mathcal{E}_2})\, g(\nabla^{\mathcal{E}_2,1,0}\partial^{\mathcal{E}_2})\right)\right] \\ &= \int_z d^{1,0}\left[\mathrm{Tr}\left(\partial^{\mathcal{E}_1} g(\nabla^{\mathcal{E}_1,1,0}\partial^{\mathcal{E}_1})\right) - \mathrm{Tr}\left(\partial^{\mathcal{E}_2} g(\nabla^{\mathcal{E}_2,1,0}\partial^{\mathcal{E}_2})\right)\right] = 0. \end{aligned} \tag{2.36}$$

The proposition follows. □

If $\mathcal{E}'$ is a $\mathfrak{B}'$-vector bundle with a flat structure and a partially flat connection $\nabla^{\mathcal{E}'}$ then

$$CS\left(\nabla^{\mathcal{E}_1 \otimes_{\mathbb{C}} \mathcal{E}'}, \nabla^{\mathcal{E}_2 \otimes_{\mathbb{C}} \mathcal{E}'}\right) = CS\left(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2}\right) \cdot \mathrm{ch}\left(\nabla^{\mathcal{E}'}\right) \in \overline{\Omega}^{\prime\prime,odd}(M, \mathfrak{B} \otimes_{\mathbb{C}} \mathfrak{B}'). \tag{2.37}$$

Finally, for future reference we define a trace on an algebra of integral operators on $\mathcal{E}$. Suppose that M is compact and Riemannian. Let $\mathfrak{F}$ be a Fréchet algebra containing $\mathfrak{B}$. Let $\mathrm{Hom}^{\infty}_{\mathfrak{B}}(\mathcal{E}, \mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E})$ be the algebra of integral operators

$$T : C^\infty(M;\mathcal{E}) \to C^\infty(M;\mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E})$$

with smooth kernels $T(m_1, m_2) \in \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}_{m_2}, \mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E}_{m_1})$. That is, for $s \in C^\infty(M;\mathcal{E})$,

$$(Ts)(m_1) = \int_M T(m_1, m_2) s(m_2)\, d\mathrm{vol}(m_2) \in \mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E}_{m_1}. \tag{2.38}$$

Put

$$\mathrm{TR}(T) = \int_M \mathrm{Tr}(T(m,m))\, d\mathrm{vol}(m) \in \mathfrak{F}/\overline{[\mathfrak{F},\mathfrak{F}]}. \tag{2.39}$$

Then TR is a trace on $\mathrm{Hom}^{\infty}_{\mathfrak{B}}(\mathcal{E}, \mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E})$. If $\mathcal{E}$ is $\mathbb{Z}_2$-graded then there is a supertrace TR_s on $\mathrm{Hom}^{\infty}_{\mathfrak{B}}(\mathcal{E}, \mathfrak{F} \otimes_{\mathfrak{B}} \mathcal{E})$.

3. Groups and Covering Spaces

In this section we review the calculation of the cyclic cohomology of a group algebra. We then describe the relationship between analysis on a normal covering space $M' \to M$ and on a certain $\mathfrak{B}$-vector bundle $\mathcal{E}$ over M. We put an explicit partially flat connection on $\mathcal{E}$ and compute the pairing of its Chern character with the cohomology of the covering group.

3.1. **Cyclic Cohomology of Group Algebras.** Let Γ be a discrete group. Let $\mathbb{C}\Gamma$ be the group algebra of Γ. Let $\langle\Gamma\rangle$ denote the conjugacy classes of Γ, and $\langle\Gamma\rangle'$ (resp. $\langle\Gamma\rangle''$) those represented by elements of finite (resp. infinite) order. For $x \in \Gamma$, let Z_x denote its centralizer in Γ and put $N_x = Z_x/\{x\}$, the quotient of Z_x by the cyclic group generated by x. If x and x' are conjugate then N_x and $N_{x'}$ are isomorphic groups, and we will write $N_{\langle x\rangle}$

for their isomorphism class. Let $\mathbb{C}[z]$ be a polynomial ring in a variable z of degree 2. Then the cyclic cohomology of $\mathbb{C}\Gamma$ is given [6] by

$$HC^*(\mathbb{C}\Gamma) = \left(\bigoplus_{\langle x\rangle \in \langle \Gamma \rangle'} \mathrm{H}^*(N_{\langle x\rangle}; \mathbb{C}) \otimes \mathbb{C}[z] \right) \oplus \bigoplus_{\langle x\rangle \in \langle \Gamma \rangle''} \mathrm{H}^*(N_{\langle x\rangle}; \mathbb{C}). \tag{3.1}$$

We will need explicit cocycles for $HC^*(\mathbb{C}\Gamma)$. Fix a representative $x \in \langle x\rangle$. Put

$$\begin{aligned} C_x^k = \{\tau : &\Gamma^{k+1} \to \mathbb{C} : \tau \text{ is skew and for all } (\gamma_0, \ldots, \gamma_k) \in \Gamma^{k+1} \text{ and } z \in Z_x, \\ &\tau(\gamma_0 z, \gamma_1 z, \ldots, \gamma_k z) = \tau(\gamma_0, \gamma_1, \ldots, \gamma_k) \text{ and} \\ &\tau(\gamma_0 x, \gamma_1, \ldots, \gamma_k) = \tau(\gamma_0, \gamma_1, \ldots, \gamma_k)\}. \end{aligned} \tag{3.2}$$

Let δ be the usual coboundary operator :

$$(\delta\tau)(\gamma_0, \ldots, \gamma_{k+1}) = \sum_{j=0}^{k+1} (-1)^j \, \tau(\gamma_0, \ldots, \widehat{\gamma_j}, \ldots, \gamma_{k+1}). \tag{3.3}$$

Denote the resulting cohomology groups by H_x^k. Then H_x^k is isomorphic to $\mathrm{H}^k(N_{\langle x\rangle}; \mathbb{C})$ and for each cocycle $\tau \in Z_x^k$, there is a cyclic cocycle $Z_\tau \in ZC^k(\mathbb{C}\Gamma)$ given by

$$Z_\tau(\gamma_0, \gamma_1, \ldots, \gamma_k) = \begin{cases} 0 \text{ if } \gamma_k \ldots \gamma_0 \notin \langle x\rangle \\ \tau(\gamma_0 g, \gamma_1\gamma_0 g, \ldots, \gamma_k \ldots \gamma_0 g) \text{ if } \gamma_k \ldots \gamma_0 = gxg^{-1}. \end{cases} \tag{3.4}$$

For $k > 0$, these are in fact reduced cocycles. In particular, from (2.6), they pair with $\overline{H}_k(\mathbb{C}\Gamma)$.

3.2. **Noncommutative Geometry of Covering Spaces.** The material in this subsection is essentially taken from [20], with a change from right modules to left modules.

Let Γ be a finitely generated discrete group. Let $\| \circ \|$ be a right-invariant word-length metric on Γ. Put

$$\mathfrak{B}^\omega = \{b : \Gamma \to \mathbb{C} : \text{for all } q \in \mathbb{Z}, \sup_{g\in\Gamma} \left(e^{q\|g\|}|b(g)|\right) < \infty\}. \tag{3.5}$$

Then $\mathfrak{B}^\omega$ is independent of the choice of $\| \circ \|$ and is a Fréchet locally m-convex algebra with unit. Note that $\mathfrak{B}^\omega$ is generally not stable under the holomorphic functional calculus in the reduced group C^*-algebra $C_r^*\Gamma$. For this reason, we will eventually replace it with a larger algebra. But let us continue with $\mathfrak{B}^\omega$ for the moment.

Let M be a smooth connected compact Riemannian manifold. Let $\rho : \pi_1(M) \to \Gamma$ be a surjective homomorphism. There is an induced connected normal Γ-covering M' of M, on which $g \in \Gamma$ acts on the left by $L_g \in \mathrm{Diff}(M')$. Let $\pi : M' \to M$ be the projection map. Put

$$\mathcal{D}^\omega = \mathfrak{B}^\omega \times_\Gamma M'. \tag{3.6}$$

Then $\mathcal{D}^\omega$ is a $\mathfrak{B}^\omega$-vector bundle on M with a flat structure.

Let E be a complex vector bundle on M with connection ∇^E and let E' be the pulled-back vector bundle π^*E on M' with connection $\nabla^{E'} = \pi^*\nabla^E$. Define

$$\mathcal{E}^\omega = \mathcal{D}^\omega \otimes E, \tag{3.7}$$

a $\mathfrak{B}^\omega$-vector bundle on M. Fix a basepoint $x_0 \in M'$. There is an isomorphism between $C^\infty(M;\mathcal{E}^\omega)$ and

$$\{s \in C^\infty(M';E') : \text{for all } q \in \mathbb{Z} \text{ and multi-indices } \alpha, \quad \sup_{x\in M'} \left(e^{qd(x_0,x)}|\nabla^\alpha s(x)|\right) < \infty\}. \tag{3.8}$$

The action of $\mathfrak{B}^\omega$ on $C^\infty(M;\mathcal{E}^\omega)$ is given explicitly by saying that $b = \sum_{g\in\Gamma} b_g\, g \in \mathfrak{B}^\omega$ sends $s \in C^\infty(M';E')$ to

$$b \cdot s = \sum_{g\in\Gamma} b_g\, L^*_{g^{-1}} s. \tag{3.9}$$

We now construct an explicit partially flat connection $\nabla^{\mathcal{D}^\omega}$ on $\mathcal{D}^\omega$. The $(1,0)$-part $\nabla^{\mathcal{D}^\omega,1,0}$ is determined by the flat structure on $\mathcal{D}^\omega$. It remains to construct

$$\nabla^{\mathcal{D}^\omega,0,1} : C^\infty(M;\mathcal{D}^\omega) \to C^\infty(M;\Omega_1(\mathfrak{B}^\omega)\otimes_{\mathfrak{B}^\omega}\mathcal{D}^\omega). \tag{3.10}$$

Let $h \in C^\infty_0(M')$ be a real-valued function satisfying

$$\sum_{g\in\Gamma} L^*_g h = 1. \tag{3.11}$$

Given $s \in C^\infty(M;\mathcal{D}^\omega)$, considering it to be an element of $C^\infty(M')$ by (3.8), define its covariant derivative to be

$$\nabla_g s = h \cdot L^*_g s \in C^\infty(M'). \tag{3.12}$$

Proposition 2. [20, Prop. 9]

$$\nabla^{\mathcal{D}^\omega,0,1} s = \sum_{g\in\Gamma} dg \otimes_{C^\infty(M;\mathfrak{B}^\omega)} \nabla_g s \tag{3.13}$$

defines the $(0,1)$*-part of a partially flat connection on* $\mathcal{D}^\omega$.

We will use the inclusion

$$\Omega^*(M,\mathfrak{B}^\omega;\mathcal{D}^\omega) \to \Omega_*(\mathfrak{B}^\omega)\otimes_{\mathfrak{B}^\omega}\Omega^*(M'). \tag{3.14}$$

The curvature of $\nabla^{\mathcal{D}^\omega}$, acting on $s \in C^\infty(M')$, is computed to be

$$\left(\nabla^{\mathcal{D}^\omega}\right)^2 s = -\sum_{g\in\Gamma} dg\left(d^{M'}h\right) L^*_g s + \sum_{g,g'\in\Gamma} dg\, dg'\, h\left(L^*_g h\right) L^*_{g'g} s. \tag{3.15}$$

Let $\tau \in Z^k_x$ be as in (3.2) and let $Z_\tau \in ZC^k(\mathbb{C}\Gamma)$ be the corresponding cyclic cocycle. Suppose that there are constants $C, D > 0$ such that for all $(\gamma_0,\ldots,\gamma_k) \in \Gamma^{k+1}$,

$$|Z_\tau(\gamma_0,\ldots,\gamma_k)| \le Ce^{D(\|\gamma_0\|+\ldots\|\gamma_k\|)}. \tag{3.16}$$

Then Z_τ extends to an element of $ZC^k(\mathfrak{B}^\omega)$.

The cover M' of M is classified by a map $\nu : M \to B\Gamma$, defined up to homotopy. If $x = e$, we can think of $[\tau]$ as an element of $\mathrm{H}^k(\Gamma;\mathbb{C}) \cong \mathrm{H}^k(B\Gamma;\mathbb{C})$. Recall that $\left[\mathrm{ch}\left(\nabla^{\mathcal{D}^\omega}\right)\right] \in H^{even}_{\mathfrak{B}^\omega}(M)$.

Proposition 3. *The pairing* $\langle Z_\tau, \mathrm{ch}(\nabla^{\mathcal{D}^\omega})\rangle \in \mathrm{H}^*(M;\mathbb{C})$ *is given by*

$$\langle Z_\tau, \mathrm{ch}(\nabla^{\mathcal{D}^\omega})\rangle = \begin{cases} 0 & \text{if } x \neq e \\ c_k\, \nu^*[\tau] & \text{if } x = e, \end{cases}$$

where c_k *is a nonzero constant which only depends on* k.

Proof. Let c_k denote a generic nonzero k-dependent constant. We use equation (3.15) for the curvature of $\nabla^{\mathcal{D}^\omega}$. Consider first the term in $\mathrm{ch}(\nabla^{\mathcal{D}^\omega})$ coming from $\left(-\sum_{g\in\Gamma} dg\left(d^{M'}h\right)L_g^*\right)^k$. For $s \in C^\infty(M')$, we have

$$\begin{aligned}
&\left(\sum_{g\in\Gamma} dg\left(d^{M'}h\right)L_g^*\right)^k s = &(3.17)\\
&\sum_{g_1\cdots g_k}\left[dg_1\left(d^{M'}h\right)L_{g_1}^*\right]\cdots\left[dg_k\left(d^{M'}h\right)L_{g_k}^*\right]s =\\
&c_k\sum_{g_1\cdots g_k} dg_1\dots dg_k\left(d^{M'}h\right)\left(L_{g_1}^*d^{M'}h\right)\cdots\left(L_{g_{k-1}\cdots g_1}^*d^{M'}h\right)L_{g_k\cdots g_1}^*s =\\
&c_k\sum_{g_1\cdots g_k} dg_1\dots dg_k\left(d^{M'}h\right)\left(L_{g_1}^*d^{M'}h\right)\cdots\left(L_{g_{k-1}\cdots g_1}^*d^{M'}h\right)(g_k\dots g_1)^{-1}\cdot s =\\
&c_k\sum_{g_1\cdots g_k} dg_1\dots dg_k(g_k\dots g_1)^{-1}L_{(g_k\cdots g_1)^{-1}}^*\left[\left(d^{M'}h\right)\left(L_{g_1}^*d^{M'}h\right)\cdots L_{g_{k-1}\cdots g_1}^*d^{M'}h\right]s =\\
&c_k\sum_{g_1\cdots g_k} dg_1\dots dg_k(g_k\dots g_1)^{-1}\left(L_{(g_k\cdots g_1)^{-1}}^*d^{M'}h\right)\cdots\left(L_{g_k^{-1}}^*d^{M'}h\right)s. &(3.18)
\end{aligned}$$

The contribution of this term to $\langle Z_\tau, \mathrm{ch}(\nabla^{\mathcal{D}^\omega})\rangle$, or more precisely the pullback of the contribution to M', is

$$\begin{aligned}
&c_k\sum_{g_1\cdots g_k} Z_\tau\left(dg_1\dots dg_k(g_k\dots g_1)^{-1}\right)\left(L_{(g_k\cdots g_1)^{-1}}^*d^{M'}h\right)\cdots\left(L_{g_k^{-1}}^*d^{M'}h\right) =\\
&c_k\sum_{g_1\cdots g_k} Z_\tau\left((g_k\dots g_1)^{-1}dg_1\dots dg_k\right)\left(L_{(g_k\cdots g_1)^{-1}}^*d^{M'}h\right)\cdots\left(L_{g_k^{-1}}^*d^{M'}h\right).
\end{aligned}$$

It is clear at this point that a nonzero contribution only arises when $x = e$, in which case we get

$$\begin{aligned}
&c_k\sum_{g_1\cdots g_k}\tau\left((g_k\dots g_1)^{-1},\dots,g_k^{-1},e\right)\left(L_{(g_k\cdots g_1)^{-1}}^*d^{M'}h\right)\cdots\left(L_{g_k^{-1}}^*d^{M'}h\right) = &(3.19)\\
&c_k\sum_{\gamma_1\cdots\gamma_k}\tau(\gamma_1,\dots,\gamma_k,e)\left(L_{\gamma_1}^*d^{M'}h\right)\cdots\left(L_{\gamma_k}^*d^{M'}h\right). &(3.20)
\end{aligned}$$

One can show [20, Lemma 3] that there is a closed form $\omega \in \Omega^k(M)$ such that

$$\sum_{\gamma_1\cdots\gamma_k}\tau(\gamma_1,\dots,\gamma_k,e)\left(L_{\gamma_1}^*d^{M'}h\right)\cdots\left(L_{\gamma_k}^*d^{M'}h\right) = \pi^*\omega. \tag{3.21}$$

Furthermore, the de Rham cohomology class $[\omega] \in \mathrm{H}^k(M;\mathbb{C})$ of ω satisfies [20, Prop. 14]

$$[\omega] = c_k\, \nu^*[\tau]. \tag{3.22}$$

We now argue that this is in fact the only nonzero contribution to $\left\langle Z_\tau, \mathrm{ch}\left(\nabla^{\mathcal{D}^\omega}\right)\right\rangle$. First, looking at the group element factors in $\left(\nabla^{\mathcal{D}^\omega}\right)^2$ and the structure of Z_τ, it is clear that $\left\langle Z_\tau, \mathrm{ch}\left(\nabla^{\mathcal{D}^\omega}\right)\right\rangle$ vanishes if $x \neq e$. Next, consider the possible contributions of the term $\sum_{g,g'\in\Gamma} dg\, dg'\, h\left(L_g^* h\right) L_{g'g}^*$ to $\left\langle Z_\tau, \mathrm{Tr}\left(\nabla^{\mathcal{D}^\omega}\right)^{2j}\right\rangle$. For example, consider the case $j=1$. Then for $s \in C^\infty(M')$,

$$\begin{aligned} &\sum_{g,g'\in\Gamma} dg\, dg'\, h\left(L_g^* h\right) L_{g'g}^* s = \\ &\sum_{g,g'\in\Gamma} dg\, dg'\, h\left(L_g^* h\right) (g'g)^{-1}\cdot s = \\ &\sum_{g,g'\in\Gamma} dg\, dg'(g'g)^{-1} L_{(g'g)^{-1}}^* \left[h\left(L_g^* h\right)\right] s = \\ &\sum_{g,g'\in\Gamma} dg\, dg'(g'g)^{-1} \left(L_{(g'g)^{-1}}^* h\right) \left(L_{g'^{-1}}^* h\right) s. \end{aligned} \tag{3.23}$$

If $\tau \in Z_e^2$ then

$$\left\langle Z_\tau, \mathrm{Tr}\left(\sum_{g,g'\in\Gamma} dg\, dg'\, h\left(L_g^* h\right) L_{g'g}^*\right)\right\rangle = \tag{3.24}$$

$$\begin{aligned} &\sum_{g,g'\in\Gamma} Z_\tau\left(dg\, dg'(g'g)^{-1}\right) \left(L_{(g'g)^{-1}}^* h\right) \left(L_{g'^{-1}}^* h\right) \\ &\sum_{g,g'\in\Gamma} Z_\tau\left((g'g)^{-1}\, dg\, dg'\right) \left(L_{(g'g)^{-1}}^* h\right) \left(L_{g'^{-1}}^* h\right) \\ &\sum_{g,g'\in\Gamma} \tau\left((g'g)^{-1}, g'^{-1}, e\right) \left(L_{(g'g)^{-1}}^* h\right) \left(L_{g'^{-1}}^* h\right) \\ &\sum_{\gamma,\gamma'\in\Gamma} \tau\left(\gamma,\gamma',e\right) \left(L_\gamma^* h\right) \left(L_{\gamma'}^* h\right). \end{aligned} \tag{3.25}$$

Because of the antisymmetry of τ, this vanishes. A similar argument using antisymmetry applies to all terms in $\left\langle Z_\tau, \mathrm{ch}\left(\nabla^{\mathcal{D}^\omega}\right)\right\rangle$ involving $\sum_{g,g'\in\Gamma} dg\, dg'\, h\left(L_g^* h\right) L_{g'g}^*$. □

Remark 1 : There is a universal $\mathbb{C}\Gamma$-vector bundle $\mathcal{D}^0$ on $B\Gamma$. Working simplicially [17, Chapitre V], one can define a natural partially flat connection $\nabla^{\mathcal{D}^0}$ on $\mathcal{D}^0$. Provided that one relaxes the regularity condition on h to being Lipschitz, one can realize $\nabla^{\mathcal{D}^\omega}$ as $\nu^*\nabla^{\mathcal{D}^0}$, extended from $\mathbb{C}\Gamma$ to $\mathfrak{B}^\omega$.

Remark 2 : Let $C_r^*\Gamma$ denote the reduced group C^*-algebra of Γ. Suppose that there is a Fréchet locally m-convex algebra $\mathfrak{B}^\infty$ such that
1. $\mathfrak{B}^\omega \subset \mathfrak{B}^\infty \subset C_r^*\Gamma$.
2. $\mathfrak{B}^\infty$ is dense in $C_r^*\Gamma$.

3. $\mathfrak{B}^\infty$ is stable under the holomorphic functional calculus in $C_r^*\Gamma$.
We can complete $\mathcal{D}^\omega$ to a $\mathfrak{B}^\infty$-vector bundle $\mathcal{D}^\infty$ on M and to a $C_r^*\Gamma$-vector bundle $\mathcal{D}$ on M. The latter represents an element $[\mathcal{D}] \in K^{top}_{C_r^*\Gamma}(M) \cong KK(\mathbb{C}, C(M) \otimes C_r^*\Gamma)$ and so gives a map $\alpha : K_*(M) \to K_*(C_r^*\Gamma) \cong K_*(\mathfrak{B}^\infty)$. Composing with the Chern character gives $(\mathrm{ch} \circ \alpha)_\mathbb{C} : K_*(M) \otimes \mathbb{C} \to \overline{H}_*(\mathfrak{B}^\infty)$, which coincides with the map coming from $\left[\mathrm{ch}\left(\nabla^{\mathcal{D}^\infty}\right)\right] \in \bigoplus_{p+q\, even} \mathrm{H}^p\left(M; \overline{H}_q(\mathfrak{B}^\infty)\right)$. Suppose that for each $[\tau] \in \mathrm{H}^*(\Gamma;\mathbb{C})$, there is a representative $\tau \in Z^*(\Gamma;\mathbb{C})$ such that the cyclic cocycle $Z_\tau \in ZC^*(\mathbb{C}\Gamma)$ extends to a continuous cyclic cocycle on $\mathfrak{B}^\infty$. Proposition 3 shows that if $\nu^*[\tau] \in \mathrm{H}^*(M;\mathbb{C})$ is nontrivial then Z_τ pairs nontrivially with $\mathrm{Im}(\mathrm{ch}\circ\alpha)_\mathbb{C}$. Taking M to be a sufficiently good approximation to $B\Gamma$, we conclude that the Strong Novikov Conjecture (SNC) holds for Γ, meaning that the assembly map $K_*(B\Gamma) \otimes \mathbb{C} \to K_*(C_r^*\Gamma) \otimes \mathbb{C}$ is injective. The fact that the existence of $\mathfrak{B}^\infty$ implies SNC is well-known [9, III.5], but we wish to emphasize how it comes from the computation of $\mathrm{ch}\left(\nabla^{\mathcal{D}^\infty}\right)$.

If Γ acts properly and cocompactly on a smooth manifold X then one can form the $\mathfrak{B}^\omega$-vector bundle $\mathfrak{B}^\omega \times_\Gamma X$ on the orbifold Γ/X and carry out a similar analysis. The upshot is that if a finitely-generated discrete group Γ satisfies Hypothesis 2 below then the Baum-Connes map [9, II.10.ϵ] is rationally injective.

Remark 3 : In [20] we gave a heat kernel proof of the higher index theorem. This proof can be reinterpreted using partially flat connections. For example, let M be a even-dimensional closed connected spin Riemannian manifold and let E be a Hermitian vector bundle on M with Hermitian connection ∇^E. Then [20, Prop. 12] can be interpreted as saying

$$\lim_{s\to 0}\left\langle Z_\tau, \mathrm{TR}_s\left(e^{-D_s^2}\right)\right\rangle = \int_M \widehat{A}\left(\nabla^{TM}\right) \wedge \mathrm{ch}\left(\nabla^E\right) \wedge \left\langle Z_\tau, \mathrm{ch}\left(\nabla^{\mathcal{D}^\omega}\right)\right\rangle. \tag{3.26}$$

Here TR_s is the supertrace, $s > 0$ is a factor which rescales the metric on M and D_s denotes the (rescaled) Dirac operator on M, coupled to $\mathcal{E}^\omega$ using the connection $\nabla^{\mathcal{E}^\omega}$. One can also prove (3.26) using the methods of Section 6 of the present paper.

4. $\mathfrak{B}$-Hermitian Metrics and Characteristic Classes

In this section we discuss the basic properties of a $\mathfrak{B}$-valued Hermitian metric on a $\mathfrak{B}$-vector bundle. We use such a Hermitian metric to define a characteristic class of a $\mathfrak{B}$-vector bundle with a flat structure. (Related ideas occur in [17, 6.31-6.32].) We show that the explicit partially flat connection described in the previous section, in the context of covering spaces, is self-adjoint. We give an application to the question of the rational injectivity of the algebraic K-theory assembly map.

4.1. **$\mathfrak{B}$-Hermitian Metrics.** Let M, $\mathfrak{B}$ and $\mathfrak{E}$ be as in Section 2. We assume that M is compact, possibly with boundary. Suppose that $\mathfrak{B}$ has an anti-involution, meaning a $\mathbb{C}$-antilinear map $* : \mathfrak{B} \to \mathfrak{B}$ such that $(b_1b_2)^* = b_2^*b_1^*$ and $(b^*)^* = b$. We extend $*$ to $\Omega_*(\mathfrak{B})$ by requiring that $(db)^* = -d(b^*)$. Let $\overline{\mathfrak{E}}^*$ be the vector space of $\mathbb{C}$-antilinear maps $t : \mathfrak{E} \to \mathfrak{B}$ such that $t(be) = t(e)b^*$ for all $b \in \mathfrak{B}$ and $e \in \mathfrak{E}$. It is a left $\mathfrak{B}$-module. If $\mathcal{E}$ is a $\mathfrak{B}$-vector bundle on M then there is an associated $\mathfrak{B}$-vector bundle $\overline{\mathcal{E}}^*$ such that $(\overline{\mathcal{E}}^*)_m = \overline{\mathcal{E}_m}^*$. If $\mathcal{E}$

has a flat structure then so does $\overline{\mathcal{E}}^*$. An element $t \in C^\infty(M;\overline{\mathcal{E}}^*)$ extends to a $\mathbb{C}$-antilinear map $t : \Omega(M,\mathfrak{B};\mathcal{E}) \to \Omega(M,\mathfrak{B})$ such that

$$t(\omega \otimes_{\mathfrak{B}} e) = t(e)\omega^* \tag{4.1}$$

for all $\omega \in \Omega_*(\mathfrak{B})$ and $e \in C^\infty(M;\mathcal{E})$. If $\nabla^{\mathcal{E}}$ is a connection on $\mathcal{E}$ then there is an induced connection $\nabla^{\overline{\mathcal{E}}^*}$ on $\overline{\mathcal{E}}^*$ given by

$$d\left(t(e)\right) = \left(\nabla^{\overline{\mathcal{E}}^*} t\right)(e) - t\left(\nabla^{\mathcal{E}} e\right) \in \Omega_1(\mathfrak{B}) \tag{4.2}$$

for all $t \in C^\infty(M;\overline{\mathcal{E}}^*)$ and $e \in C^\infty(M;\mathcal{E})$. (The funny sign in (4.2) comes from the definition of the the involution on $\Omega_1(\mathfrak{B})$.) If $\nabla^{\mathcal{E}}$ is partially flat then so is $\nabla^{\overline{\mathcal{E}}^*}$.

Definition 6. *1. A Hermitian form on $\mathfrak{E}$ is a map $\langle\cdot,\cdot\rangle : \mathfrak{E} \times \mathfrak{E} \to \mathfrak{B}$ which is $\mathbb{C}$-linear in the first variable, $\mathbb{C}$-antilinear in the second variable and satisfies $\langle b_1 e_1, b_2 e_2\rangle = b_1\langle e_1, e_2\rangle b_2^*$ for all $b_1, b_2 \in \mathfrak{B}$ and $e_1, e_2 \in \mathfrak{E}$.*
2. A Hermitian form $\langle\cdot,\cdot\rangle$ is nondegenerate if it induces an isomorphism $h^{\mathfrak{E}} : \mathfrak{E} \to \overline{\mathfrak{E}}^*$ by $\left(h^{\mathfrak{E}}(e_1)\right)(e_2) = \langle e_1, e_2\rangle$.

We can extend $\langle\cdot,\cdot\rangle$ to a Hermitian form on $\Omega_*(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathfrak{E}$ by requiring that

$$\langle \omega_1 \otimes_{\mathfrak{B}} e_1, \omega_2 \otimes_{\mathfrak{B}} e_2\rangle = \omega_1\langle e_1, e_2\rangle\omega_2^* \in \Omega_*(\mathfrak{B}) \tag{4.3}$$

for all $\omega_1, \omega_2 \in \Omega_*(\mathfrak{B})$ and $e_1, e_2 \in \mathfrak{E}$.

There is a canonical Hermitian form $\langle\cdot,\cdot\rangle^0$ on $\mathfrak{B}^n$ given by $\langle \{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n\rangle^0 = \sum_{i=1}^n x_i y_i^*$.

Definition 7. *A Hermitian metric on $\mathfrak{E}$ is a Hermitian form $\langle\cdot,\cdot\rangle$ on $\mathfrak{E}$ which is positive-definite, meaning that there is an embedding $i : \mathfrak{E} \to \mathfrak{B}^n$ for some n such that $\langle\cdot,\cdot\rangle = i^*\langle\cdot,\cdot\rangle^0$.*

The method of proof of [18, Lemme 2.7] shows that a Hermitian metric is nondegenerate. Since $\mathfrak{E}$ is a finitely-generated projective $\mathfrak{B}$-module, it is clear that it admits some Hermitian metric. The method of proof of [18, Lemme 2.9] gives the following proposition.

Proposition 4. *If $\langle\cdot,\cdot\rangle_0$ and $\langle\cdot,\cdot\rangle_1$ are Hermitian metrics on $\mathfrak{E}$ then there is a smooth 1-parameter family $\{\alpha_t\}_{t\in[0,1]}$ in $\mathrm{Aut}_{\mathfrak{B}}(\mathfrak{E})$ such that $\alpha_0 = \mathrm{Id}_{\mathfrak{E}}$ and $\langle\cdot,\cdot\rangle_0 = \alpha_1^*\langle\cdot,\cdot\rangle_1$.*

Definition 8. *A Hermitian metric on a $\mathfrak{B}$-vector bundle $\mathcal{E}$ is given by a smooth family of Hermitian metrics on the fibers $\{\mathcal{E}_m\}_{m\in M}$.*

Proposition 5. *There is a Hermitian metric on a $\mathfrak{B}$-vector bundle $\mathcal{E}$. Any two such Hermitian metrics are related by an automorphism which is isotopic to the identity.*

Proof. The algebra $C^\infty(M;\mathfrak{B})$ is a Fréchet locally m-convex algebra in a natural way. Furthermore, $C^\infty(M;\mathcal{E})$ is a finitely-generated projective $C^\infty(M;\mathfrak{B})$-module. (The proof is similar to that of the usual case when $\mathfrak{B} = \mathbb{C}$, the essential tool being that $\mathrm{Inv}(M_N(\mathfrak{B}))$ is open in $M_N(\mathfrak{B})$.) A Hermitian metric on the $\mathfrak{B}$-vector bundle $\mathcal{E}$ is the same as a Hermitian metric on the $C^\infty(M;\mathfrak{B})$-module $C^\infty(M;\mathcal{E})$. The result now follows from Proposition 4. □

A Hermitian metric on $\mathcal{E}$ gives a $C^\infty(M;\mathfrak{B})$-linear isomorphism $h^{\mathcal{E}} : \mathcal{E} \to \overline{\mathcal{E}}^*$.

Definition 9. *Given a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, its adjoint connection is*

$$\left(\nabla^{\mathcal{E}}\right)^* = \left(h^{\mathcal{E}}\right)^{-1} \circ \nabla^{\overline{\mathcal{E}}^*} \circ h^{\mathcal{E}}, \tag{4.4}$$

another connection on $\mathcal{E}$.

Explicitly,

$$d\langle e_1, e_2\rangle = \langle \nabla^{\mathcal{E}} e_1, e_2\rangle - \langle e_1, (\nabla^{\mathcal{E}})^* e_2\rangle \in \Omega_1(M, \mathfrak{B}). \tag{4.5}$$

We say that $\nabla^{\mathcal{E}}$ is self-adjoint if $(\nabla^{\mathcal{E}})^* = \nabla^{\mathcal{E}}$.

Suppose that $\mathcal{E}$ has a flat structure. The triple $\left(\mathcal{E}, \overline{\mathcal{E}}^*, h^{\mathcal{E}}\right)$ defines an element of $K_{\mathfrak{B}}^{rel}(M)$. In this case, we write

$$CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right) = CS\left(\nabla^{\mathcal{E}}, \nabla^{\overline{\mathcal{E}}^*}\right) \in \overline{\Omega}^{\prime\prime,odd}(M, \mathfrak{B}). \tag{4.6}$$

Proposition 5 implies that

$$\left[CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right)\right] \in \bigoplus_{\substack{p>q \\ p+q \text{ odd}}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B}))$$

only depends on the flat structure on $\mathcal{E}$. To put it another way, the assignment of $\left(\mathcal{E}, \overline{\mathcal{E}}^*, h^{\mathcal{E}}\right)$ to $\mathcal{E}$ gives an explicit map $K_{\mathfrak{B}}^{alg}(M) \to K_{\mathfrak{B}}^{rel}(M)$. We can then apply CS to obtain an invariant of $K_{\mathfrak{B}}^{alg}(M)$. In total, we have defined a map

$$\mathrm{ch} \oplus CS : K_{\mathfrak{B}}^{alg}(M) \to \left(\bigoplus_p \mathrm{H}^p(M; \overline{Z}_p(\mathfrak{B}))\right) \oplus \left(\bigoplus_{\substack{p<q \\ p+q \text{ even}}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B}))\right) \oplus \left(\bigoplus_{\substack{p>q \\ p+q \text{ odd}}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B}))\right). \tag{4.7}$$

Let $(M, *)$ and $(M', *')$ be smooth connected manifolds with basepoints. Let $\mathcal{E}$ be a $\mathfrak{B}$-vector bundle on M and similarly for $\mathcal{E}'$. Let $\mathcal{T} = M \times \mathcal{E}\big|_*$ denote the trivial $\mathfrak{B}$-vector bundle on M with the same fiber at $*$ as $\mathcal{E}$, and similarly for $\mathcal{T}'$. Then $[\mathcal{E}] - [\mathcal{T}]$ is an element of the reduced group $\widetilde{K}_{\mathfrak{B}}^{top}(M)$, and similarly for $[\mathcal{E}'] - [\mathcal{T}']$. The virtual $\mathfrak{B} \otimes_{\mathbb{C}} \mathfrak{B}'$-vector bundle $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{E}' - \mathcal{T} \otimes_{\mathbb{C}} \mathcal{E}' - \mathcal{E} \otimes_{\mathbb{C}} \mathcal{T}' + \mathcal{T} \otimes_{\mathbb{C}} \mathcal{T}'$ on $M \times M'$ is trivial on $(M \times \{*'\}) \cup (\{*\} \times M')$ and so passes to an element of $\widetilde{K}_{\mathfrak{B} \otimes_{\mathbb{C}} \mathfrak{B}'}^{top}(M \wedge M')$ which represents the product $([\mathcal{E}] - [\mathcal{T}]) \cdot ([\mathcal{E}'] - [\mathcal{T}'])$. If $\mathcal{E}$ and $\mathcal{E}'$ have flat structures then we get the product in $\widetilde{K}^{alg}$ [19, Chapitre II].

Let $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{E}'}$ be connections on $\mathcal{E}$ and $\mathcal{E}'$, respectively. There are induced connections on $\mathcal{T}$ and $\mathcal{T}'$. Let $z \in Z_*(M, *; \mathbb{C})$ and $z' \in Z_*(M', *'; \mathbb{C})$ be relative cycles. Let $zz' \in Z_*(M \wedge M', *; \mathbb{C})$ be the product. Then with an obvious notation, (2.25) implies that

$$\int_{zz'} \mathrm{ch}\left(\nabla^{(\mathcal{E}-\mathcal{T})\cdot(\mathcal{E}'-\mathcal{T}')}\right) = \int_z \mathrm{ch}\left(\nabla^{\mathcal{E}-\mathcal{T}}\right) \cdot \int_{z'} \mathrm{ch}\left(\nabla^{\mathcal{E}'-\mathcal{T}'}\right). \tag{4.8}$$

Suppose that $\mathcal{E}$ and $\mathcal{E}'$ have partially flat connections and Hermitian metrics. Suppose that $\nabla^{\mathcal{E}'}$ is self-adjoint. Then from (2.37),

$$\int_{zz'} CS\left(\nabla^{(\mathcal{E}-\mathcal{T})\cdot(\mathcal{E}'-\mathcal{T}')}, h^{(\mathcal{E}-\mathcal{T})\cdot(\mathcal{E}'-\mathcal{T}')}\right) = \int_z CS\left(\nabla^{\mathcal{E}-\mathcal{T}}, h^{\mathcal{E}-\mathcal{T}}\right) \cdot \int_{z'} \mathrm{ch}\left(\nabla^{\mathcal{E}'-\mathcal{T}'}\right). \tag{4.9}$$

4.2. $\mathfrak{B}$-Hermitian Metrics, Group Algebras and Assembly Maps. We use the notation of Subsection 3.2. Define an involution on $\mathfrak{B}^\omega$ by $*\left(\sum_{g\in\Gamma} c_g g\right) = \sum_{g\in\Gamma} \overline{c_g} g^{-1}$. Considering $\mathfrak{B}^\omega$ as a left-module over itself, it has a Hermitian form given by $\langle b_1, b_2\rangle = b_1 b_2^*$. We can transfer this Hermitian form fiberwise to $\mathcal{D}^\omega$. In what follows, we will freely identify differential forms on M and Γ-invariant differential forms on M'.

Definition 10. *Given $s_1, s_2 \in C^\infty(M; \mathcal{D}^\omega)$, consider them to be elements of $C^\infty(M')$ by (3.8). Then the Hermitian form $\langle s_1, s_2\rangle^\omega \in C^\infty(M; \mathfrak{B}^\omega)$ is given by*

$$\begin{aligned} \langle s_1, s_2\rangle^\omega(x) &= \sum_{g,g'\in\Gamma} g\,(L^*_{gg'} s_1)(x)\,(L^*_{g'}\overline{s_2})(x) \\ &= \sum_{g,g'\in\Gamma} g\, s_1(gg'x)\,\overline{s_2}(g'x). \end{aligned} \tag{4.10}$$

We do not claim that $\langle\cdot,\cdot\rangle^\omega$ is a Hermitian metric, in that it may be degenerate.

Proposition 6. $\nabla^{\mathcal{D}^\omega}$ *is self-adjoint with respect to $\langle\cdot,\cdot\rangle^\omega$, meaning that for all $s_1, s_2 \in C^\infty(M; \mathcal{D}^\omega)$,*

$$d\,\langle s_1, s_2\rangle^\omega = \left\langle \nabla^{\mathcal{D}^\omega} s_1, s_2\right\rangle^\omega - \left\langle s_1, \nabla^{\mathcal{D}^\omega} s_2\right\rangle^\omega \in \Omega_1(M, \mathfrak{B}^\omega). \tag{4.11}$$

Proof. As Γ-invariant differential forms on M', we have

$$d\,\langle s_1, s_2\rangle^\omega = \sum_{g,g'} \left[dg\,(L^*_{gg'} s_1)\, L^*_{g'}\overline{s_2} + g\left(d^{M'} L^*_{gg'} s_1\right) L^*_{g'}\overline{s_2} + g\,(L^*_{gg'} s_1)\, d^{M'} L^*_{g'}\overline{s_2}\right] \tag{4.12}$$

and

$$\begin{aligned} \left\langle \nabla^{\mathcal{D}^\omega,1,0} s_1, s_2\right\rangle^\omega &= \left\langle d^{M'} s_1, s_2\right\rangle^\omega \\ &= \sum_{g,g'} g\left(L^*_{gg'} d^{M'} s_1\right) L^*_{g'}\overline{s_2}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \left\langle \nabla^{\mathcal{D}^\omega,0,1} s_1, s_2\right\rangle^\omega &= \left\langle \sum_\gamma d\gamma\,(hL^*_\gamma s_1), s_2 \right\rangle^\omega \\ &= \sum_{g,g',\gamma} d\gamma\, g\, L^*_{gg'}\,(hL^*_\gamma s_1)\, L^*_{g'}\overline{s_2} \\ &= \sum_{g,g',\gamma} [d(\gamma g) - \gamma\, dg]\; L^*_{gg'} h\,(L^*_{\gamma gg'} s_1)\, L^*_{g'}\overline{s_2} \\ &= \sum_{g,g',\gamma} dg\, L^*_{\gamma^{-1}gg'} h\,(L^*_{gg'} s_1)\, L^*_{g'}\overline{s_2} - \sum_{g,g',\gamma} \gamma\, dg\, L^*_{gg'} h\,(L^*_{\gamma gg'} s_1)\, L^*_{g'}\overline{s_2}. \\ &= \sum_{g,g'} dg\,\;(L^*_{gg'} s_1)\, L^*_{g'}\overline{s_2} - \sum_{g,g',\gamma} \gamma\, dg\, L^*_{gg'} h\,(L^*_{\gamma gg'} s_1)\, L^*_{g'}\overline{s_2}. \end{aligned} \tag{4.14}$$

Switching s_1 and s_2 gives

$$\left\langle \nabla^{\mathcal{D}^\omega,1,0} s_2, s_1\right\rangle^\omega = \sum_{g,g'} g\left(L^*_{gg'} d^{M'} s_2\right) L^*_{g'}\overline{s_1} \tag{4.15}$$

and

$$\left\langle \nabla^{\mathcal{D}^\omega,0,1} s_2, s_1 \right\rangle^\omega = \sum_{g,g',\gamma} d\gamma\, g\, L^*_{gg'} \left(h L^*_\gamma s_2 \right) L^*_{g'} \overline{s_1}. \tag{4.16}$$

Then

$$\begin{aligned} \left\langle s_1, \nabla^{\mathcal{D}^\omega,1,0} s_2 \right\rangle^\omega &= -\sum_{g,g'} g^{-1} \left(L^*_{g'} s_1 \right) L^*_{gg'} d^{M'} \overline{s_2} \\ &= -\sum_{g,g'} g \left(L^*_{g'} s_1 \right) L^*_{g^{-1}g'} d^{M'} \overline{s_2} \\ &= -\sum_{g,g'} g \left(L^*_{gg'} s_1 \right) L^*_{g'} d^{M'} \overline{s_2} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \left\langle s_1, \nabla^{\mathcal{D}^\omega,0,1} s_2 \right\rangle^\omega &= -\sum_{g,g',\gamma} g^{-1}\, d\gamma^{-1}\, L^*_{g'} s_1 \left(L^*_{gg'} h \right) L^*_{\gamma g g'} \overline{s_2} \\ &= -\sum_{g,g',\gamma} \gamma\, dg\, L^*_{g'} s_1 \left(L^*_{\gamma^{-1} g'} h \right) L^*_{g^{-1}\gamma^{-1} g'} \overline{s_2} \\ &= -\sum_{g,g',\gamma} \gamma\, dg\, L^*_{\gamma g g'} s_1 \left(L^*_{gg'} h \right) L^*_{g'} \overline{s_2}. \end{aligned} \tag{4.18}$$

Combining (4.12), (4.13), (4.14), (4.17) and (4.18) gives (4.11). □

We now give examples in which the map (4.7) is nontrivial. Recall the notation of Subsection 3.1.

Hypothesis 2. *There is an involutive Fréchet locally m-convex algebra $\mathfrak{B}^\infty$ such that*
1. $\mathfrak{B}^\omega \subset \mathfrak{B}^\infty \subset C^*_r\Gamma$.
2. $\mathfrak{B}^\infty$ *is dense in* $C^*_r\Gamma$ *and stable under the holomorphic functional calculus in* $C^*_r\Gamma$.
3. For each $\langle x \rangle \in \langle \Gamma \rangle'$ *and* $[\tau] \in \mathrm{H}^*(N_{\langle x \rangle}; \mathbb{C})$, *there is a representative* $\tau \in Z^*_x$ *such that the cyclic cocycle* $Z_\tau \in ZC^*(\mathbb{C}\Gamma)$ *extends to a continuous cyclic cocycle on* $\mathfrak{B}^\infty$.

Hypothesis 2 is known to be satisfied by virtually nilpotent groups [16] and Gromov-hyperbolic groups [30].

We can extend $\mathcal{D}^\omega$ to a $\mathfrak{B}^\infty$-vector bundle $\mathcal{D}^\infty$ on M, $\nabla^{\mathcal{D}^\omega}$ to a connection $\nabla^{\mathcal{D}^\infty}$ on $\mathcal{D}^\infty$ and $\langle \cdot, \cdot \rangle^\omega$ to a Hermitian metric $\langle \cdot, \cdot \rangle^\infty$ on $\mathcal{D}^\infty$.

By [17, Chapitre III], we can consider an element k of $K^{alg}_n(\mathbb{Z}\Gamma)$ to be given by a formal difference of homology n-spheres HS^n equipped with flat bundles $\mathcal{E}^0$ of finitely-generated projective $\mathbb{Z}\Gamma$-modules. For simplicity, we just consider a single HS^n. As in [17, p. 98], we may approximate HS^n by compact manifolds (possibly with boundary), so for simplicity we assume that HS^n is a compact manifold. Let $[HS^n] \in \mathrm{H}^n(HS^n; \mathbb{C})$ denote its "fundamental class". Putting $\mathcal{E}^\infty = \mathfrak{B}^\infty \otimes_{\mathbb{Z}\Gamma} \mathcal{E}^0$, the algebraic K-theory class of $[\mathcal{E}^\infty]$ represents the image of k under the map $K^{alg}_n(\mathbb{Z}\Gamma) \to K^{alg}_n(\mathfrak{B}^\infty)$. Now apply (4.7) to $[\mathcal{E}^\infty]$, pair the result with Z_τ and integrate over $[HS^n]$ to get a number. This procedure gives a map

$$K^{alg}_n(\mathbb{Z}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \to \bigoplus_{\langle x \rangle \in \langle \Gamma' \rangle} \left(\mathrm{H}_n(N_{\langle x \rangle}; \mathbb{C}) \oplus \left(\bigoplus_{k=0}^{\left[\frac{n-1}{2}\right]} \mathrm{H}_{n-1-2k}(N_{\langle x \rangle}; \mathbb{C}) \right) \right) \tag{4.19}$$

which is conjecturally injective.

Example 2 : Take Γ to be the trivial group and $\mathcal{B}^\infty = \mathbb{C}$. In this case, (4.7) becomes

$$ch \oplus CS : K_{\mathbb{C}}^{alg}(M) \to \mathrm{H}^0(M;\mathbb{C}) \oplus \left(\bigoplus_{p \ odd} \mathrm{H}^p(M;\mathbb{C}) \right). \tag{4.20}$$

Applied to a flat complex vector bundle E on M, this represents the rank of E along with its Borel classes [2, Section Ig]. It is known that

$$K_n^{alg}(\mathbb{Z}) \otimes \mathbb{C} = \begin{cases} \mathbb{C} & \text{if } n = 0 \\ \mathbb{C} & \text{if } n \equiv 1 \bmod 4,\ n > 1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.21}$$

with the higher terms being detected by the Borel classes [4]. Thus for all $n \equiv 1 \mod 4$, $n > 1$, there is a homology n-sphere HS^n and a flat bundle $\mathcal{E}^0$ of finitely-generated projective $\mathbb{Z}$-modules on HS^n such that if $\mathcal{E} = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{E}^0$ then $\int_{[HS^n]} CS\,(\mathcal{E}) \neq 0$.

Example 3 : Take Γ to be a finite group and $\mathfrak{B}^\infty = \mathbb{C}\Gamma$. We can write $\mathbb{C}\Gamma = \bigoplus_{\rho_i \in \widehat{\Gamma}} M_{n_i}(\mathbb{C})$, where $n_i = \dim(\rho_i)$. Then (4.7) becomes

$$ch \oplus CS : K_{\mathbb{C}\Gamma}^{alg}(M) \to \bigoplus_{\rho \in \widehat{\Gamma}} \left[\mathrm{H}^0(M;\mathbb{C}) \oplus \left(\bigoplus_{p \ odd} \mathrm{H}^p(M;\mathbb{C}) \right) \right]. \tag{4.22}$$

Consider the case $n = 1$ of (4.19). Take the homology sphere to be a circle S^1. Given $T \in GL_r(\mathbb{Z}\Gamma)$, form a flat $\mathbb{Z}\Gamma$-bundle $\mathcal{E}^0$ on S^1 by gluing the ends of $[0,1] \times (\mathbb{Z}\Gamma)^r$ using T. Then $\mathcal{E}^\infty = \mathbb{C}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathcal{E}^0$ is a flat $\mathbb{C}\Gamma$-vector bundle on S^1 with holonomy T. One computes that

$$\int_{S^1} CS\,([\mathcal{E}^\infty]) = \bigoplus_{\rho \in \widehat{\Gamma}} 2\ \log|\det(\rho(T))|. \tag{4.23}$$

Thus CS detects all of $K_1^{alg}(\mathbb{Z}\Gamma) \otimes \mathbb{C}$ [31].

Example 4 : Suppose that Γ satisfies Hypothesis 1 of the introduction. For an algebra A, let $\mathbf{K}_A$ denote the algebraic K-theory spectrum of A, with $(\mathbf{K}_A)_0 = K_0(A) \times BGL(A)_\delta^+$. For an abelian group G, let $\mathbf{H}_G$ denote the Eilenberg-Maclane spectrum of G. We can think of the class CS as arising from a map

$$\mathbf{K}_{\mathfrak{B}^\infty} \xrightarrow{CS} \prod_{\substack{p>q \\ p+q \ odd}} \Sigma^p(\mathbf{H}_{\overline{H}_q(\mathfrak{B}^\infty)}). \tag{4.24}$$

We recall the assembly map of [19, Chapitre IV], extended from $\mathbb{Z}\Gamma$ to $\mathfrak{B}^\infty$. The inclusion of Γ into $GL(\mathbb{Z}\Gamma)$, as a matrix with one nonzero entry in the upper left corner, induces maps

$$B\Gamma \to BGL(\mathbb{Z}\Gamma)_\delta \to BGL(\mathbb{Z}\Gamma)_\delta^+ \to BGL\,(\mathfrak{B}^\infty)_\delta^+\,, \tag{4.25}$$

which extend to

$$B\Gamma \to K_0(\mathbb{Z}\Gamma) \times BGL(\mathbb{Z}\Gamma)_\delta \to K_0(\mathbb{Z}\Gamma) \times BGL(\mathbb{Z}\Gamma)_\delta^+ \to K_0(\mathfrak{B}^\infty) \times BGL(\mathfrak{B}^\infty)_\delta^+ . \tag{4.26}$$

Smashing with $\mathbf{K_Z}$ gives the assembly map

$$\mathbf{K_Z} \wedge B\Gamma \to \mathbf{K_{Z\Gamma}} \to \mathbf{K}_{\mathfrak{B}^\infty}. \tag{4.27}$$

We can compose with CS to get

$$\mathbf{K_Z} \wedge B\Gamma \to \mathbf{K_{Z\Gamma}} \to \mathbf{K}_{\mathfrak{B}^\infty} \to \prod_{\substack{p>q \\ p+q\ odd}} \Sigma^p(\mathbf{H}_{\overline{H}_q(\mathfrak{B}^\infty)}). \tag{4.28}$$

Taking homotopy groups gives

$$\widetilde{\mathrm{H}}_p(B\Gamma;\mathbf{K_Z}) \to K_p(\mathbb{Z}\Gamma) \to K_p(\mathfrak{B}^\infty) \to \bigoplus_{\substack{q<p \\ q+p\ odd}} \overline{H}_q(\mathfrak{B}^\infty). \tag{4.29}$$

Then tensoring with $\mathbb{C}$, we obtain

$$\bigoplus_{l+m=p} (K_l(\mathbb{Z}) \otimes_{\mathbf{Z}} \mathbb{C}) \otimes_{\mathbf{C}} \widetilde{\mathrm{H}}_m(\Gamma;\mathbb{C}) \to K_p(\mathbb{Z}\Gamma) \otimes_{\mathbf{Z}} \mathbb{C} \to K_p(\mathfrak{B}^\infty) \otimes_{\mathbf{Z}} \mathbb{C} \to \bigoplus_{\substack{q<p \\ q+p\ odd}} \overline{H}_q(\mathfrak{B}^\infty). \tag{4.30}$$

Let HS^l and $\mathcal{E}$ be as in Example 2. Let $[\mathcal{E}]_{\mathbf{C}} \in K_l(\mathbb{Z}) \otimes_{\mathbf{Z}} \mathbb{C}$ be the corresponding K-theory class. Then (4.30) gives a map

$$[\mathcal{E}]_{\mathbf{C}} : \widetilde{\mathrm{H}}_{p-l}(\Gamma;\mathbb{C}) \to K_p(\mathbb{Z}\Gamma) \otimes_{\mathbf{Z}} \mathbb{C} \to K_p(\mathfrak{B}^\infty) \otimes_{\mathbf{Z}} \mathbb{C} \to \bigoplus_{\substack{q<p \\ q+p\ odd}} \overline{H}_q(\mathfrak{B}^\infty). \tag{4.31}$$

Tracing through the definitions, (4.31) can be interpreted concretely as follows. Assume that $B\Gamma$ is a manifold, possibly with boundary. (Otherwise, approximate it by manifolds.) Let $\mathcal{E}'$ be the $\mathfrak{B}^\infty$-vector bundle $\mathcal{D}^\infty$ on $B\Gamma$. The map (4.25) is realized geometrically by $[\mathcal{E}'] - [\mathcal{T}'] \in \widetilde{K}^{alg}_{\mathfrak{B}^\infty}(B\Gamma)$, where $\mathcal{T}'$ is as in the discussion above equation (4.8). Consider the $\mathfrak{B}^\infty$-vector bundle $(\mathcal{E} - \mathcal{T}) \otimes_{\mathbf{C}} (\mathcal{E}' - \mathcal{T}')$ on $HS^l \wedge B\Gamma$. Then

$$\begin{aligned} CS\left([(\mathcal{E} - \mathcal{T}) \otimes_{\mathbf{C}} (\mathcal{E}' - \mathcal{T}')]\right) &\in \bigoplus_{\substack{q<p \\ q+p\ odd}} \widetilde{\mathrm{H}}^p(HS^l \wedge B\Gamma; \overline{H}_q(\mathfrak{B}^\infty)) \\ &= \bigoplus_{\substack{q<p \\ q+p\ odd}} \widetilde{\mathrm{H}}^{p-l}(\Gamma;\mathbb{C}) \otimes \overline{H}_q(\mathfrak{B}^\infty). \end{aligned} \tag{4.32}$$

The map (4.31) comes from pairing $\widetilde{\mathrm{H}}_{p-l}(\Gamma;\mathbb{C})$ with $CS\left([(\mathcal{E} - \mathcal{T}) \otimes_{\mathbf{C}} (\mathcal{E}' - \mathcal{T}')]\right)$.

From Proposition 6, $\nabla^{\mathcal{E}'}$ is self-adjoint. Thus we can apply (4.9), with $z \in Z_l(HS^l, *; \mathbb{C})$ and $z' \in Z_{p-l}(B\Gamma, *'; \mathbb{C})$. If $l \equiv 1 \mod 4$ and $l > 1$ then by Example 2, we can choose HS^l and $\mathcal{E}$ so that $\int_z CS\left(\nabla^{\mathcal{E}-\mathcal{T}}, h^{\mathcal{E}-\mathcal{T}}\right)$ is nonzero. Applying Proposition 3 to $\int_{z'} \mathrm{ch}\left(\nabla^{\mathcal{E}'-\mathcal{T}'}\right)$, we conclude that (4.30) gives an injection

$$\bigoplus_{k=1}^{[\frac{p-1}{4}]} \widetilde{\mathrm{H}}_{p-1-4k}(\Gamma;\mathbb{C}) \to K_p(\mathbb{Z}\Gamma)\otimes_{\mathbf{Z}}\mathbb{C} \to K_p(\mathfrak{B}^\infty)\otimes_{\mathbf{Z}}\mathbb{C} \to \bigoplus_{k=0}^{[\frac{p-1}{2}]} \overline{H}_{p-1-2k}(\mathfrak{B}^\infty) \to \bigoplus_{k=0}^{[\frac{p-1}{2}]} \mathrm{H}_{p-1-2k}(\Gamma;\mathbb{C}). \tag{4.33}$$

Thus

$$\bigoplus_{k=1}^{[\frac{p-1}{4}]} \widetilde{\mathrm{H}}_{p-1-4k}(\Gamma;\mathbb{C}) \to K_p(\mathbb{Z}\Gamma)\otimes_{\mathbf{Z}}\mathbb{C} \tag{4.34}$$

is injective. Including the contribution of the characteristic class ch and taking more care with reduced vs. unreduced homology gives the injectivity of

$$\mathrm{H}_p(\Gamma;\mathbb{C}) \oplus \left(\bigoplus_{k=1}^{[\frac{p-1}{4}]} \mathrm{H}_{p-1-4k}(\Gamma;\mathbb{C})\right) \to K_p(\mathbb{Z}\Gamma)\otimes_{\mathbf{Z}}\mathbb{C}. \tag{4.35}$$

This is not an optimal result, as (4.35) is known to be injective for all groups Γ such that $\dim_{\mathbb{C}} \mathrm{H}_k(\Gamma;\mathbb{C}) < \infty$ for all $k \in \mathbb{N}$, regardless of whether or not they satisfy Hypothesis 1 [3]. The proof of [3] uses more complicated methods.

There is a conjecture that (4.35) is an isomorphism if Γ is torsion-free. This is known to be true when Γ is a discrete cocompact subgroup of a Lie group with a finite number of connected components [14].

5. Noncommutative Superconnections

In this section we first extend the results of the preceding sections from connections to superconnections. For basic information about the superconnection formalism, we refer to the book [1]. We then use superconnections to prove a finite-dimensional analog of our fiber-bundle index theorem. We also construct the associated finite-dimensional analytic torsion form and relate it to various versions of the Reidemeister torsion. The main technical problems of this section involve the large-time behaviour of heat kernels in Fréchet spaces.

5.1. **Partially Flat Superconnections.** Let M, $\mathfrak{B}$ and $\mathcal{E}$ be as in Subsection 4.1. Suppose that $\mathcal{E}$ is $\mathbb{Z}$-graded as a direct sum

$$\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{E}^i \tag{5.1}$$

of $\mathfrak{B}$-vector bundles on M. We use the induced $\mathbb{Z}_2$-grading on $\mathcal{E}$ when defining supertraces. The algebra $\Omega\left(M;\mathrm{Hom}_{\mathfrak{B}}(\mathcal{E},\Omega_*(\mathfrak{B})\otimes_{\mathfrak{B}}\mathcal{E})\right)$ has a trigrading as

$$\Omega\left(M;\mathrm{Hom}_{\mathfrak{B}}(\mathcal{E},\Omega_*(\mathfrak{B})\otimes_{\mathfrak{B}}\mathcal{E})\right) = \bigoplus_{\substack{p,q,r\in\mathbf{Z}\\ p,q\geq 0}} \Omega^{p,q,r}(M,\mathfrak{B},\mathrm{End}(\mathcal{E})), \tag{5.2}$$

where by definition

$$\Omega^{p,q,r}(M,\mathfrak{B},\mathrm{End}(\mathcal{E})) = \Omega^p\left(M;\mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}^\bullet,\Omega_q(\mathfrak{B})\otimes_{\mathfrak{B}}\mathcal{E}^{\bullet+r})\right). \tag{5.3}$$

Define a subalgebra of $\Omega\left(M;\mathrm{Hom}_{\mathfrak{B}}(\mathcal{E},\Omega_*(\mathfrak{B})\otimes_{\mathfrak{B}}\mathcal{E})\right)$ by

$$\Omega'\left(M;\mathrm{Hom}_{\mathfrak{B}}(\mathcal{E},\Omega_*(\mathfrak{B})\otimes_{\mathfrak{B}}\mathcal{E})\right) = \bigoplus_{p+r\leq q}\Omega^{p,q,r}(M,\mathfrak{B},\mathrm{End}(\mathcal{E})). \tag{5.4}$$

Definition 11. *A degree-1 superconnection A' on $\mathcal{E}$ is a sum*

$$A' = \sum_{p\geq 0} A'_p \tag{5.5}$$

where
1. A'_1 is a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ which preserves the $\mathbb{Z}$-grading.
2. For $p\neq 1$, $A'_p\in\bigoplus_{p+q+r=1}\Omega^{p,q,r}(M,\mathfrak{B},\mathrm{End}(\mathcal{E}))$.

We will sometimes omit the phrase "degree-1". For $p\neq 1$, let $A'_{p,q,r}$ denote the component of A'_p in $\Omega^{p,q,r}(M,\mathfrak{B},\mathcal{E})$. As in (2.16), we write $\nabla^{\mathcal{E}} = \nabla^{\mathcal{E},1,0}\oplus\nabla^{\mathcal{E},0,1}$.

The superconnection A' gives a $\mathbb{C}$-linear map

$$A' : C^\infty(M;\mathcal{E}) \to \Omega^*(M,\mathfrak{B};\mathcal{E}) \tag{5.6}$$

which satisfies the Leibniz rule. We extend A' to a $\mathbb{C}$-linear map on $\Omega^*(M,\mathfrak{B};\mathcal{E})$ by requiring that for all $\omega\in\Omega^k(M,\mathfrak{B})$ and $s\in\Omega^l(M,\mathfrak{B};\mathcal{E})$,

$$\nabla^{\mathcal{E}}(\omega s) = (-1)^k\,\omega\wedge\nabla^{\mathcal{E}}s + d\omega\otimes_{C^\infty(M;\mathfrak{B})}s. \tag{5.7}$$

The curvature of A' is

$$(A')^2\in\bigoplus_{p+q+r=2}\Omega^{p,q,r}(M,\mathfrak{B},\mathrm{End}(\mathcal{E})). \tag{5.8}$$

Let $(A')^2_{p,q,r}$ denote the component of $(A')^2$ in $\Omega^{p,q,r}(M,\mathfrak{B},\mathrm{End}(\mathcal{E}))$.

The Chern character of A' is

$$\mathrm{ch}\,(A') = \mathrm{Tr}_s\left(e^{-(A')^2}\right)\in\overline{\Omega}^{even}(M,\mathfrak{B}). \tag{5.9}$$

It is a closed form whose cohomology class $[\mathrm{ch}\,(A')]\in H^{even}_{\mathfrak{B}}(M)$ is independent of the choice of A'.

Definition 12. *The superconnection A' is partially flat if $(A')^2_{p,0,2-p}=0$ for all $p\geq 0$.*

Definition 13. *A superflat structure on $\mathcal{E}$ is given by a degree-1 superconnection*

$$A'^{,flat} : C^\infty(M;\mathcal{E}) \to \Omega^*(M;\mathcal{E}) \tag{5.10}$$

which is $\mathfrak{B}$-linear and whose extension to $\Omega^*(M;\mathcal{E})$ satisfies $\left(A'^{,flat}\right)^2 = 0$.

Note that the map in (5.10) does not involve any $\mathfrak{B}$-differentiation. A partially flat superconnection determines a superflat structure on $\mathcal{E}$ by

$$A'^{,flat} = A'_{0,0,1} + \nabla^{\mathcal{E},1,0} + \sum_{p=2}^{\infty} A'_{p,0,1-p}. \tag{5.11}$$

Conversely, given a superflat structure on $\mathcal{E}$, there is a partially flat superconnection on $\mathcal{E}$ which is compatible with the superflat structure, although generally not a unique one.

Example 5 : If $\mathfrak{B} = \mathbb{C}$ then a partially flat degree-1 superconnection on $\mathcal{E}$ is the same as a flat degree-1 superconnection on $\mathcal{E}$ in the sense of [2, Section IIa].

Example 6 : If $\mathcal{E}$ is concentrated in degree 0 then a partially flat degree-1 superconnection on $\mathcal{E}$ is the same as a partially flat connection on $\mathcal{E}$ in the sense of Subsection 2.2.

Hereafter, we assume that A' is a partially flat degree-1 superconnection.

Proposition 7. *We have* $\mathrm{ch}(A') \in \overline{\Omega}^{\prime,even}(M,\mathfrak{B})$.

Proof. As $(A')^2 \in \Omega'(M; \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}, \Omega_*(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E}))$, the same is true for $e^{-(A')^2}$. As Tr_s vanishes outside of $\bigoplus_{p,q\geq 0} \Omega^{p,q,0}(M, \mathfrak{B}, \mathrm{End}(\mathcal{E}))$, the proposition follows. □

Thus $[\mathrm{ch}(A')] \in H^{\prime,even}_{\mathfrak{B}}(M)$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be smooth $\mathfrak{B}$-vector bundles on M with superflat structures. Suppose that there is a smooth isomorphism $\alpha : \mathcal{E}_1 \to \mathcal{E}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ as topological $\mathfrak{B}$-vector bundles. Choose partially flat superconnections A'_1, A'_2 on $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, which are compatible with the superflat structures. For $u \in [0,1]$, put $A(u) = uA'_1 + (1-u)\,\alpha^* A'_2$. Note that for $u \in (0,1)$, $A(u)$ may not be partially flat on $\mathcal{E}_1$.

Definition 14. *The relative Chern-Simons class* $CS\,(A'_1, A'_2) \in \overline{\Omega}^{\prime\prime,odd}(M,\mathfrak{B})$ *is*

$$CS\,(A'_1, A'_2) = -\int_0^1 \mathrm{Tr}_s\left((\partial_u A(u))\, e^{-A^2(u)} \right) du. \tag{5.12}$$

By construction,

$$dCS\,(A'_1, A'_2) = \mathrm{ch}(A'_1) - \mathrm{ch}(A'_2) \tag{5.13}$$

vanishes in $\overline{\Omega}^{\prime\prime,even}(M,\mathfrak{B})$. Thus there is a class $[CS\,(A'_1, A'_2)] \in H^{\prime\prime,odd}_{\mathfrak{B}}(M)$.

Proposition 8. $[CS\,(A'_1, A'_2)]$ *actually lies in* $\bigoplus_{\substack{p>q \\ p+q\ odd}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B}))$.

Proof. The proof is like that of Proposition 1. We omit the details. □

Proposition 9. *The class* $[CS\,(A'_1, A'_2)]$ *is independent of the choice of partially flat connections* A'_1, A'_2; *hence we denote it by* $\left[CS\left(A_1^{\prime,flat}, A_2^{\prime,flat}\right)\right]$. *It only depends on* α *through its isotopy class. More precisely, let* $\{\alpha(\epsilon)\}_{\epsilon\in\mathbb{R}}$ *be a smooth* 1*-parameter family of* α*'s. Then the variation of* $CS\,(A'_1, A'_2) \in \overline{\Omega}^{\prime\prime,odd}(M,\mathfrak{B})$ *is given by*

$$\partial_\epsilon CS\,(A'_1, A'_2) = d\left(\int_0^1 \mathrm{Tr}_s\left(\alpha^{-1}(\epsilon)\, \frac{d\alpha(\epsilon)}{d\epsilon}\, e^{-A^2(u)} \right) du + \int_0^1 \int_0^1 u(1-u) \right. \tag{5.14}$$
$$\left. \mathrm{Tr}_s\left(\alpha^{-1}(\epsilon) \frac{d\alpha(\epsilon)}{d\epsilon} \left[A'_1 - \alpha^* A'_2,\, e^{-rA^2(u)}\, (A'_1 - \alpha^* A'_2)\, e^{-(1-r)A^2(u)} \right] \right) dr du \right)$$

Proof. We first prove (5.14). Put $\widetilde{M} = \mathbb{R} \times M$. Let $p : \mathbb{R} \times M \to M$ be the projection onto the second factor. For $i \in \{1,2\}$, put $\widetilde{\mathcal{E}}_i = p^* \mathcal{E}_i$, $\widetilde{A}'_i = p^* A'_i$. Then $\widetilde{A}'_i$ is partially flat on $\widetilde{\mathcal{E}}_i$. Define $\widetilde{\alpha} : \widetilde{\mathcal{E}}_1 \to \widetilde{\mathcal{E}}_2$ by saying that $\widetilde{\alpha}\big|_{\{\epsilon\} \times M} = \alpha(\epsilon)$. Let $\widetilde{d} = d\epsilon\, \partial_\epsilon + d$ denote the differential on $\overline{\Omega}^{\prime\prime,*}(\widetilde{M}, \mathfrak{B})$. Write $CS\left(\widetilde{A}'_1, \widetilde{A}'_2\right) \in \overline{\Omega}^{\prime\prime,odd}(\widetilde{M}, \mathfrak{B})$ as

$$CS\left(\widetilde{A}'_1, \widetilde{A}'_2\right) = CS\left(A'_1, A'_2\right)(\epsilon) + d\epsilon \wedge T(\epsilon) \tag{5.15}$$

with $T(\epsilon) \in \overline{\Omega}^{\prime\prime,even}(M, \mathfrak{B})$. Then the equation $\widetilde{d}\, CS\left(\widetilde{A}'_1, \widetilde{A}'_2\right) = 0$ implies that

$$\partial_\epsilon CS\left(A'_1, A'_2\right) = dT(\epsilon). \tag{5.16}$$

It remains to work out $T(\epsilon)$ explicitly.

With an obvious notation, we have

$$\begin{aligned} \widetilde{A}'_1 &= d\epsilon \wedge \partial_\epsilon + A'_1, \\ \widetilde{\alpha}^* \widetilde{A}'_2 &= d\epsilon \wedge \left[\partial_\epsilon + \alpha^{-1}(\epsilon) \frac{d\alpha(\epsilon)}{d\epsilon}\right] + \alpha(\epsilon)^* A'_2. \end{aligned} \tag{5.17}$$

Then

$$\begin{aligned} \widetilde{A}(u) &= d\epsilon \wedge \left[\partial_\epsilon + (1-u)\, \alpha^{-1}(\epsilon)\, \frac{d\alpha(\epsilon)}{d\epsilon}\right] + uA'_1 + (1-u)\, \alpha(\epsilon)^* A'_2 \\ &= d\epsilon \wedge \left[\partial_\epsilon + (1-u)\, \alpha^{-1}(\epsilon)\, \frac{d\alpha(\epsilon)}{d\epsilon}\right] + A(u), \\ \partial_u \widetilde{A}(u) &= -\, d\epsilon \wedge \alpha^{-1}(\epsilon)\, \frac{d\alpha(\epsilon)}{d\epsilon} + A'_1 - \alpha(\epsilon)^* A'_2. \end{aligned} \tag{5.18}$$

As $\alpha(\epsilon)^* A'_2 = \alpha^{-1}(\epsilon) \circ A'_2 \circ \alpha(\epsilon)$, it follows that

$$\partial_\epsilon \left[\alpha(\epsilon)^* A'_2\right] = \left[\alpha(\epsilon)^* A'_2, \alpha^{-1}(\epsilon) \frac{d\alpha(\epsilon)}{d\epsilon}\right]. \tag{5.19}$$

Then one finds

$$\widetilde{A}^2(u) = u(1-u)\, d\epsilon \wedge \left[\alpha^{-1}(\epsilon) \frac{d\alpha(\epsilon)}{d\epsilon}, A'_1 - \alpha(\epsilon)^* A'_2\right] + A^2(u). \tag{5.20}$$

Thus $d\epsilon \wedge T(\epsilon)$ is the $d\epsilon$-term of

$$\begin{aligned} &- \int_0^1 \mathrm{Tr}_s \left(\left(\partial_u \widetilde{A}(u)\right) e^{-\widetilde{A}^2(u)}\right) du = \\ &- \int_0^1 \mathrm{Tr}_s \left(\left(-\, d\epsilon \wedge \alpha^{-1}(\epsilon)\, \frac{d\alpha(\epsilon)}{d\epsilon} + A'_1 - \alpha^* A'_2\right)\right. \\ &\qquad\qquad \left. e^{-\left(u(1-u)\, d\epsilon \wedge \left[\alpha^{-1}(\epsilon) \frac{d\alpha(\epsilon)}{d\epsilon}, A'_1 - \alpha(\epsilon)^* A'_2\right] + A^2(u)\right)}\right) du, \end{aligned} \tag{5.21}$$

giving

$$\begin{aligned} d\epsilon \wedge T(\epsilon) &= d\epsilon \wedge \int_0^1 \mathrm{Tr}_s \left(\alpha^{-1}(\epsilon)\, \frac{d\alpha(\epsilon)}{d\epsilon}\, e^{-A^2(u)}\right) du \\ &+ \int_0^1 \int_0^1 \mathrm{Tr}_s \Big((A_1' - \alpha^* A_2')\, e^{-(1-r)A^2(u)} \\ &\qquad u(1-u)\, d\epsilon \wedge \left[\alpha^{-1}(\epsilon)\frac{d\alpha(\epsilon)}{d\epsilon}, A_1' - \alpha^* A_2'\right] e^{-rA^2(u)}\Big)\, dr du \\ &= d\epsilon \wedge \int_0^1 \mathrm{Tr}_s \left(\alpha^{-1}(\epsilon)\, \frac{d\alpha(\epsilon)}{d\epsilon}\, e^{-A^2(u)}\right) du \\ &+ d\epsilon \wedge \int_0^1 \int_0^1 u(1-u) \mathrm{Tr}_s \Big(\left[\alpha^{-1}(\epsilon)\frac{d\alpha(\epsilon)}{d\epsilon}, A_1' - \alpha^* A_2'\right] e^{-rA^2(u)} \\ &\qquad (A_1' - \alpha^* A_2')\, e^{-(1-r)A^2(u)}\Big)\, dr du \\ &= d\epsilon \wedge \int_0^1 \mathrm{Tr}_s \left(\alpha^{-1}(\epsilon)\, \frac{d\alpha(\epsilon)}{d\epsilon}\, e^{-A^2(u)}\right) du \\ &+ d\epsilon \wedge \int_0^1 \int_0^1 u(1-u) \mathrm{Tr}_s \Big(\alpha^{-1}(\epsilon)\frac{d\alpha(\epsilon)}{d\epsilon} \\ &\qquad \left[A_1' - \alpha^* A_2', e^{-rA^2(u)}\, (A_1' - \alpha^* A_2')\, e^{-(1-r)A^2(u)}\right]\Big)\, dr du \end{aligned} \tag{5.22}$$

Equation (5.14) follows from combining (5.16) and (5.22).

Thus $[CS\,(A_1', A_2')]$ only depends on α through its isotopy class. A similar argument, working on $\mathbb{R} \times M$, shows that $[CS\,(A_1', A_2')]$ is independent of the choice of partially flat connections A_1', A_2'. □

Put

$$v = A_{0,0,1}' \in C^\infty\left(M; \mathrm{Hom}_{\mathfrak{B}}\left(\mathcal{E}^\bullet, \mathcal{E}^{\bullet+1}\right)\right). \tag{5.23}$$

Then the partial flatness condition implies that

$$\begin{aligned} v^2 &= 0, \\ \nabla^{\mathcal{E},1,0}\, v &= 0, \\ \left(\nabla^{\mathcal{E},1,0}\right)^2 + \left[v, A_{2,0,-1}'\right] &= 0. \end{aligned} \tag{5.24}$$

Thus there is a cochain complex of $\mathfrak{B}$-vector bundles

$$(\mathcal{E}, v) : 0 \to \mathcal{E}^0 \xrightarrow{v} \mathcal{E}^1 \xrightarrow{v} \dots \xrightarrow{v} \mathcal{E}^n \to 0. \tag{5.25}$$

Definition 15. *For $m \in M$, let $H(\mathcal{E}, v)_m = \bigoplus_{i=0}^n \mathrm{H}^i(\mathcal{E}, v)_m$ be the cohomology of the complex $(\mathcal{E}, v)_m$ over m.*

We cannot conclude immediately that $H(\mathcal{E}, v)_m$ is a projective module. Put $\overline{\mathcal{E}} = \Lambda \otimes_{\mathfrak{B}} \mathcal{E}$ and $\overline{v}_m = \mathrm{Id}_\Lambda \otimes_{\mathfrak{B}} v_m$.

Hypothesis 3. *For all $m \in M$, the map $\overline{v}_m$ has closed image.*

Remark 4 : If $\mathfrak{B} = \Lambda = \mathbb{C}$ then, as the fiber of $\mathcal{E}$ is finitely-generated, Hypothesis 3 is automatically satisfied.

Proposition 10. *Under Hypothesis 3, $H(\mathcal{E}, v)_m$ is a finitely-generated projective $\mathfrak{B}$-module.*

Proof. The claim is that if we have a cochain complex

$$(\mathfrak{E}, v) : 0 \to \mathfrak{E}^0 \xrightarrow{v} \mathfrak{E}^1 \xrightarrow{v} \dots \xrightarrow{v} \mathfrak{E}^n \to 0. \tag{5.26}$$

of finitely-generated projective $\mathfrak{B}$-modules and if $\mathrm{Im}(\overline{v})$ is closed then $\mathrm{H}^*(\mathfrak{E}, v)$ is a finitely-generated projective $\mathfrak{B}$-module. We first prove a small lemma.

Lemma 1. *Suppose that E is a finitely-generated projective left $\mathfrak{B}$-module. Put $\overline{E} = \Lambda \otimes_{\mathfrak{B}} E$. Given $T \in \mathrm{End}_{\mathfrak{B}}(E)$, put $\overline{T} = \mathrm{Id}_\Lambda \otimes_{\mathfrak{B}} T \in \mathrm{End}_\Lambda(\overline{E})$. If $\overline{T}$ is invertible then T is invertible.*

Proof. Write $E = \mathfrak{B}^N e$ for some projection $e \in M_N(\mathfrak{B})$. Then $\overline{E} = \Lambda^N \overline{e}$, with $\overline{e} \in M_N(\Lambda)$. We can consider T to be an element $T' \in M_N(\mathfrak{B})$ satisfying $eT' = T'e = T'$. Put $S = T' + 1 - e \in M_N(\mathfrak{B})$ and $\overline{S} = \overline{T'} + 1 - \overline{e} \in M_N(\Lambda)$. Then $\overline{S}$ is invertible. Hence S is invertible [5, Proposition A.2.2]. The inverse of T is given by the restriction of S^{-1} to $\mathrm{Im}(e)$. □

Now put $\mathfrak{B}$-Hermitian metrics on $\{\mathfrak{E}^i\}_{i=1}^n$. Let $v^* \in \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}^\bullet, \mathfrak{E}^{\bullet-1})$ be the adjoint to v, defined using these metrics. Put $\Delta = vv^* + v^*v$. Put $\overline{\mathfrak{E}} = \Lambda \otimes_{\mathfrak{B}} \mathfrak{E}$, $\overline{v} = \mathrm{Id}_\Lambda \otimes_{\mathfrak{B}} v$ and $\overline{\Delta} = \overline{v}\,\overline{v}^* + \overline{v}^*\,\overline{v}$. Then $(\overline{\mathfrak{E}}, \overline{v})$ is a cochain complex of finitely-generated projective Hilbert Λ-modules with $\mathrm{Im}(\overline{v})$ closed. We use [34, Theorem 15.3.8], about operators with closed image, throughout. It implies that $\mathrm{Ker}(\overline{v})$ is a finitely-generated projective Hilbert Λ-module with $\mathrm{Im}(\overline{v})$ as a Hilbert Λ-submodule. As usual,

$$\mathrm{Ker}(\overline{v}) \cap \mathrm{Im}(\overline{v})^\perp = \mathrm{Ker}(\overline{v}) \cap \mathrm{Ker}(\overline{v}^*) = \mathrm{Ker}(\overline{\Delta}). \tag{5.27}$$

As $\overline{v}^*$ is conjugate to $\overline{v}$, $\mathrm{Im}(\overline{v}^*)$ is also closed. There is an inclusion map $r : \mathrm{Im}(\overline{\Delta}) \to \mathrm{Im}(\overline{v}) \oplus \mathrm{Im}(\overline{v}^*)$. We claim that r is onto. We have that $\overline{v}$ is an isomorphism between $\mathrm{Ker}(\overline{v})^\perp = \mathrm{Im}(\overline{v}^*)$ and $\mathrm{Im}(\overline{v})$. Thus if $z \in \mathrm{Im}(\overline{v})$ then there is a y such that $z = \overline{v}\,\overline{v}^*(y)$. Similarly, there is an x such that $\overline{v}^*(y) = \overline{v}^*\,\overline{v}(x)$, giving that $z = \overline{\Delta}(\overline{v}(x))$. The same argument applies if $z \in \mathrm{Im}(\overline{v}^*)$. Thus r is an isomorphism.

In particular, $\mathrm{Im}(\overline{\Delta}) \cong \mathrm{Im}(\overline{v}) \oplus \mathrm{Im}(\overline{v}^*)$ is closed, implying that $\overline{\Delta}$ restricts to an isomorphism between $\mathrm{Ker}(\overline{\Delta})^\perp = \mathrm{Im}(\overline{\Delta})$ and $\mathrm{Im}(\overline{\Delta})$. It follows that 0 is isolated in the spectrum $\sigma(\overline{\Delta})$ of $\overline{\Delta}$. By Lemma 1, $\sigma(\Delta) = \sigma(\overline{\Delta})$. Hence we can take a small loop γ around 0 and form the projection operator

$$P^{Ker(\Delta)} = \frac{1}{2\pi i} \int_\gamma \frac{d\lambda}{\lambda - \Delta}. \tag{5.28}$$

It follows that $\mathrm{Ker}(\Delta)$ is a finitely-generated projective $\mathfrak{B}$-module.

If γ' is a contour around $\sigma(\Delta) - \{0\}$ then the Green's operator of Δ is given by

$$G = \frac{1}{2\pi i} \int_{\gamma'} \frac{1}{\lambda} \frac{d\lambda}{\lambda - \Delta}. \tag{5.29}$$

For $x \in \mathfrak{E}$, let $\overline{x} = 1 \otimes_{\mathfrak{B}} x$ denote its image in $\overline{\mathfrak{E}}$. If $x \in \mathrm{Ker}(v) \cap \mathrm{Ker}(v^*)$ then $x \in \mathrm{Ker}(\Delta)$. Conversely, if $x \in \mathrm{Ker}(\Delta)$ then $\overline{x} \in \mathrm{Ker}(\overline{\Delta})$, implying that $\overline{x} \in \mathrm{Ker}(\overline{v}) \cap \mathrm{Ker}(\overline{v})$. Hence $x \in \mathrm{Ker}(v) \cap \mathrm{Ker}(v^*)$, showing that $\mathrm{Ker}(v) \cap \mathrm{Ker}(v^*) = \mathrm{Ker}(\Delta)$.

Finally, consider the map $s : \mathrm{Ker}(v) \cap \mathrm{Ker}(v^*) \to \mathrm{H}(\mathfrak{C}, v)$. We claim that s is an isomorphism. If $x \in \mathrm{Ker}(s)$ then $x = v(y)$ for some y. Then $v^*v(y) = 0$, so $\overline{v}^*\overline{v}(\overline{y}) = 0$, so $\overline{v}(\overline{y}) = 0$, so $x = 0$. Thus s is injective. If $h \in \mathrm{H}(\mathfrak{C}, v)$, find some $y \in \mathrm{Ker}(v)$ in the equivalence class of h. Clearly $y - vGv^*(y) \in \mathrm{Ker}(v)$. By usual arguments, $\overline{y - vGv^*(y)} \in \mathrm{Ker}(\overline{v}^*)$ and hence $y - vGv^*(y) \in \mathrm{Ker}(v^*)$. As $s(y - vGv^*(y)) = h$, s is onto.

We have shown that $\mathrm{H}(\mathfrak{C}, v) \cong \mathrm{Ker}(v) \cap \mathrm{Ker}(v^*) = \mathrm{Ker}(\Delta)$ is a finitely-generated projective $\mathfrak{B}$-module. □

Hereafter, we assume that Hypothesis 3 is satisfied.

Proposition 11. *The $\{H(\mathcal{E}, v)_m\}_{m \in M}$ fit together to form a $\mathbb{Z}$-graded $\mathfrak{B}$-vector bundle $H(\mathcal{E}, v)$ on M with a flat structure.*

Proof. By (5.24), v is covariantly-constant with respect to the connection $\nabla^{\mathcal{E},1,0}$. Given $m \in M$, we can use the parallel transport of $\nabla^{\mathcal{E},1,0}$ to extend the result of Proposition 10 uniformly to a neighborhood of m, giving the $\mathfrak{B}$-vector bundle structure on $H(\mathcal{E}, v)$. The flat structure on $H(\mathcal{E}, v)$ comes from [2, Prop. 2.5]. □

There is a Hermitian metric $h^{H(\mathcal{E},v)}$ on $H(\mathcal{E}, v)$ coming from its identification with $\mathrm{Ker}(\Delta) \subset \mathcal{E}$. Letting $P^{Ker(\Delta)}$ be as in the proof of Proposition 10, there is an induced connection

$$\nabla^{H(\mathcal{E},v)} = P^{Ker(\Delta)} \nabla^{\mathcal{E}} \tag{5.30}$$

on $H(\mathcal{E}, v)$.

Proposition 12. *The connection $\nabla^{H(\mathcal{E},v)}$ is partially flat and compatible with the flat structure on $H(\mathcal{E}, v)$. Furthermore, $\left(\nabla^{H(\mathcal{E},v)}\right)^* = P^{Ker(\Delta)} \left(\nabla^{\mathcal{E}}\right)^*$.*

Proof. The proof is similar to that of [2, Prop. 2.6]. We omit the details. □

5.2. **A Finite-Dimensional Index Theorem.** Let $\langle \cdot, \cdot \rangle$ be a Hermitian metric on $\mathcal{E}$ as in Subsection 4.1, which respects the $\mathbb{Z}$-grading on $\mathcal{E}$. As in that subsection, there is a partially flat degree-1 superconnection $\overline{A}'^*$ on $\overline{\mathcal{E}}^*$ and an adjoint partially flat degree-1 superconnection $A'' = (A')^*$ on $\mathcal{E}$ given by

$$A'' = \left(h^{\mathcal{E}}\right)^{-1} \circ \overline{A}'^* \circ h^{\mathcal{E}}. \tag{5.31}$$

Explicitly, define an adjoint operation on $\Omega\left(M; \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}, \Omega_*(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E})\right)$ by requiring that
1. For $\alpha, \alpha' \in \Omega\left(M; \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}, \Omega_*(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E})\right)$, $(\alpha\alpha')^* = \alpha'^*\alpha^*$.
2. If $V \in C^\infty\left(M; \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}, \Omega_*(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E})\right)$ then V^* is the adjoint defined using the Hermitian form (4.3).
3. If $\omega \in \Omega^1(M)$ then its extension by the identity to become an element of $\Omega^1\left(M; \mathrm{End}_{\mathfrak{B}}(\mathcal{E})\right)$ satisfies $\omega^* = -\overline{\omega}$.

Then for A' as in (5.5),

$$A'' = \sum_p A''_p, \tag{5.32}$$

where $A''_1 = \left(\nabla^{\mathcal{E}}\right)^*$ and for $p \neq 1$, $A''_p = \left(A'_p\right)^*$. We write

$$CS\left(A', h^{\mathcal{E}}\right) = CS\left(A', \overline{A}'^*\right) \in \overline{\Omega}^{\prime\prime,odd}(M, \mathfrak{B}). \tag{5.33}$$

Let N be the number operator on $\mathcal{E}$, meaning that N acts on $C^\infty(M;\mathcal{E}^j)$ as multiplication by j. For $t>0$, let $\langle\cdot,\cdot\rangle_t$ be the Hermitian metric on $\mathcal{E}$ such that if $e_1,e_2\in C^\infty(M;\mathcal{E}^j)$ then

$$\langle e_1,e_2\rangle_t = t^j\langle e_1,e_2\rangle \in C^\infty(M;\mathfrak{B}). \tag{5.34}$$

Letting $h_t^{\mathcal{E}}:\mathcal{E}\to\overline{\mathcal{E}}^*$ be the isomorphism induced from $\langle\cdot,\cdot\rangle_t$, we have $h_t^{\mathcal{E}}=h^{\mathcal{E}}t^N$. Letting A_t'' denote the adjoint of A' with respect to $\langle\cdot,\cdot\rangle_t$, we have $A_t''=t^{-N}A''t^N$.

Proposition 13. *For $u\in[0,1]$, put $A(u)=uA'+(1-u)A_t''$. Then*

$$\partial_t CS\left(A',h_t^{\mathcal{E}}\right)=\frac{1}{t}d\left(\int_0^1 \mathrm{Tr}_s\left(Ne^{-A^2(u)}\right)du+\int_0^1\int_0^1 u(1-u)\right. \tag{5.35}$$
$$\left.\mathrm{Tr}_s\left(N\left[A'-A_t'',e^{-rA^2(u)}\left(A'-A_t''\right)e^{-(1-r)A^2(u)}\right]\right)drdu\right).$$

Proof. This follows from Proposition 9. □

To make the equations more symmetric, put $B_t'=t^{N/2}A't^{-N/2}$ and $B_t''=t^{-N/2}A''t^{N/2}$. Then B_t'' is the adjoint of B_t' with respect to $\langle\cdot,\cdot\rangle$. Explicitly,

$$B_t'=\sum_{p\geq 0}t^{(1-p)/2}A_p', \tag{5.36}$$
$$B_t''=\sum_{p\geq 0}t^{(1-p)/2}A_p''.$$

Proposition 14. *For $u\in[0,1]$, put $B_t(u)=uB_t'+(1-u)B_t''$. Define $\mathcal{T}(t)\in\overline{\Omega}^{\prime\prime,even}(M,\mathfrak{B})$ by*

$$\mathcal{T}(t)=-\frac{1}{t}\left(\int_0^1\mathrm{Tr}_s\left(Ne^{-B_t^2(u)}\right)du+\int_0^1\int_0^1 u(1-u)\right. \tag{5.37}$$
$$\left.\mathrm{Tr}_s\left(N\left[B_t'-B_t'',e^{-rB_t^2(u)}\left(B_t'-B_t''\right)e^{-(1-r)B_t^2(u)}\right]\right)drdu\right).$$

Then

$$CS\left(A',h_t^{\mathcal{E}}\right)=CS\left(B_t',h^{\mathcal{E}}\right)=-\int_0^1\mathrm{Tr}_s\left(\left(B_t'-B_t''\right)e^{-B_t^2(u)}\right)du \tag{5.38}$$

and

$$\partial_t CS\left(B_t',h^{\mathcal{E}}\right)=-d\mathcal{T}(t). \tag{5.39}$$

Proof. This follows from (5.12) and (5.35) by conjugating within the supertrace by $t^{N/2}$. □

We now discuss the large-t asymptotics of $\mathrm{ch}(B_t(u))$, $CS\left(B_t',h^{\mathcal{E}}\right)$ and $\mathcal{T}(t)$. We must first specify the notion of convergence. Define i_j and $\|\cdot\|_j$ as after equation (2.1).

Let $\mathfrak{E}$ be a finitely-generated projective left $\mathfrak{B}$-module with a $\mathfrak{B}$-Hermitian metric. Write $\mathfrak{E}=\mathfrak{B}^N e$ for some fixed projection $e\in M_N(\mathfrak{B})$. Put $e_j=i_j(e)\in M_N(B_j)$ and $E_j=B_j^N e_j$. Then E_j inherits a Banach space structure as a closed subspace of B_j^N. Furthermore, $\mathrm{End}_{B_j}(E_j)$ inherits a Banach algebra structure as a closed subalgebra of $\mathrm{End}(E_j)$. Note that $\mathrm{End}_{B_0}(E_0)$ is the same underlying algebra as the C^*-algebra $\mathrm{End}_A(\overline{\mathfrak{E}})$, but may have a different norm.

We can identify $\mathrm{End}_{\mathfrak{B}}(\mathfrak{E})$ with the projective limit of Banach algebras

$$\ldots \longrightarrow \mathrm{End}_{B_{j+1}}(E_{j+1}) \longrightarrow \mathrm{End}_{B_j}(E_j) \longrightarrow \ldots \longrightarrow \mathrm{End}_{B_0}(E_0). \tag{5.40}$$

We again write $\|\cdot\|_j$ for the induced submultiplicative seminorm on $\mathrm{End}_{\mathfrak{B}}(\mathfrak{E})$. Given $T \in \mathrm{End}_{\mathfrak{B}}(\mathfrak{E})$, let T_j be its image in $\mathrm{End}_{B_j}(E_j)$. Then

$$\ldots \supseteq \sigma(T_{j+1}) \supseteq \sigma(T_j) \supseteq \ldots \supseteq \sigma(T_0) \tag{5.41}$$

and $\sigma(T) = \bigcup_{j=0}^{\infty} \sigma(T_j)$. As $\mathrm{End}_{B_0}(E_0)$ is the same underlying algebra as the C^*-algebra $\mathrm{End}_{\Lambda}(\overline{\mathfrak{E}})$, $\sigma(T_0) = \sigma(\overline{T})$. By Lemma 1, $\sigma(T) = \sigma(\overline{T})$. Thus $\sigma(T) = \sigma(T_j) = \sigma(\overline{T})$ for all j.

Using the description of $\Omega_*(\mathfrak{B})$ in [20, Section II], there is a sequence of seminorms $\{\|\cdot\|_j\}_{j=0}^{\infty}$ on each $\Omega_k(\mathfrak{B})$ coming from the norms on B_j. We obtain seminorms $\|\cdot\|_j$ on $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}, \Omega_k(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathfrak{E})$ and $\overline{\Omega}_k(\mathfrak{B})$, with respect to which (2.8) is continuous. Convergence of $\mathrm{ch}\left(B_t(u)\right)$ or $CS\left(B_t', h^{\mathcal{E}}\right)$ will mean convergence in all seminorms $\{\|\cdot\|_j\}_{j=0}^{\infty}$.

Proposition 15. *For all $u \in (0,1)$, as $t \to \infty$,*

$$\mathrm{ch}\left(B_t(u)\right) = \mathrm{ch}\left(\nabla^{H(\mathcal{E},v)}(u)\right) + O(t^{-1/2}) \tag{5.42}$$

uniformly on M. Also,

$$CS\left(B_t', h^{\mathcal{E}}\right) = CS\left(\nabla^{H(\mathcal{E},v)}, h^{H(\mathcal{E},v)}\right) + O(t^{-1/2}) \tag{5.43}$$

uniformly on M.

Proof. We will only prove (5.43), as the proof of (5.42) is similar but easier. We begin with some generalities. Suppose that E is a finitely-generated projective left $\mathfrak{B}$-module, T_1 is an element of $\mathrm{End}_{\mathfrak{B}}(E)$ and T_2 is an element of $\mathrm{End}_{\mathfrak{B}}(E) \otimes \mathfrak{G}$ for some Grassmann algebra $\mathfrak{G}$. We assume that T_2 has positive Grassmann degree. Suppose that $\sigma(T_1) \subset \mathbb{R}$ and that 0 is isolated in $\sigma(T_1)$. Let γ_1 be a small loop around 0 and let γ_2 be a contour around $\sigma(T_1) - \{0\}$. We orient γ_1 and γ_2 counterclockwise. Then we can write

$$P^{Ker(T_1)} = \int_{\gamma_1} \frac{1}{z - T_1} \frac{dz}{2\pi i} \tag{5.44}$$

and

$$P^{Im(T_1)} = \int_{\gamma_2} \frac{1}{z - T_1} \frac{dz}{2\pi i}. \tag{5.45}$$

As $\sigma(T_1 + T_2) = \sigma(T_1)$, we can write

$$e^{-t(T_1+T_2)^2} = \int_{\gamma_1} \frac{e^{-tz^2}}{z - (T_1 + T_2)} \frac{dz}{2\pi i} + \int_{\gamma_2} \frac{e^{-tz^2}}{z - (T_1 + T_2)} \frac{dz}{2\pi i}. \tag{5.46}$$

Using the series

$$\frac{1}{z - (T_1 + T_2)} = \frac{1}{z - T_1} + \frac{1}{z - T_1} T_2 \frac{1}{z - T_1} + \ldots, \tag{5.47}$$

the first contour integral becomes

$$\int_{\gamma_1} \frac{e^{-tz^2}}{z-(T_1+T_2)} \frac{dz}{2\pi i} = P^{Ker(T_1)} - P^{Ker(T_1)} T_2 G - G T_2 P^{Ker(T_1)} + P^{Ker(T_1)} T_2 G T_2 G + G T_2 P^{Ker(T_1)} T_2 G + G T_2 G T_2 P^{Ker(T_1)} - t P^{Ker(T_1)} T_2 P^{Ker(T_1)} T_2 P^{Ker(T_1)} + \ldots \tag{5.48}$$

where G is the Green's operator of T_1.

Writing out B_t' and B_t'' explicitly, we have

$$\begin{aligned} B_t' - B_t'' &= \sqrt{t}\,(v - v^*) + \nabla^{\mathcal{E}} - \left(\nabla^{\mathcal{E}}\right)^* + O(t^{-1/2}), \\ B_t(u) &= \sqrt{t}\,(uv + (1-u)v^*) + u\nabla^{\mathcal{E}} + (1-u)\left(\nabla^{\mathcal{E}}\right)^* + O(t^{-1/2}). \end{aligned} \tag{5.49}$$

We apply equations (5.44) - (5.48) with

$$\begin{aligned} T_1 &= uv + (1-u)v^*, \\ T_2 &= t^{-1/2}B_t(u) - T_1 = t^{-1/2}\left(u\nabla^{\mathcal{E}} + (1-u)\left(\nabla^{\mathcal{E}}\right)^*\right) + O(t^{-1}). \end{aligned} \tag{5.50}$$

For $u \in (0,1)$, $\mathrm{Ker}(T_1) = \mathrm{Ker}(u(1-u)\triangle) = \mathrm{Ker}(\triangle)$. Let us write

$$CS\left(B_t', h^{\mathcal{E}}\right) = CS_1 + CS_2, \tag{5.51}$$

with

$$\begin{aligned} CS_1 &= -\int_0^1 \mathrm{Tr}_s\left((B_t' - B_t'')\int_{\gamma_1} \frac{e^{-tz^2}}{z-(T_1+T_2)}\frac{dz}{2\pi i}\right) du, \\ CS_2 &= -\int_0^1 \mathrm{Tr}_s\left((B_t' - B_t'')\int_{\gamma_2} \frac{e^{-tz^2}}{z-(T_1+T_2)}\frac{dz}{2\pi i}\right) du. \end{aligned} \tag{5.52}$$

Substituting the series (5.48) into CS_1, we see that the leading terms in t come from the terms in (5.48) without any factors of G. Using (5.30) and Proposition 12, one finds

$$\begin{aligned} CS_1 &= -\int_0^1 \mathrm{Tr}_s\Bigg(\left(\sqrt{t}\,(v-v^*) + \nabla^{\mathcal{E}} - \left(\nabla^{\mathcal{E}}\right)^*\right) P^{Ker(\triangle)} \\ &\quad \int_{\gamma_1} \frac{e^{-tz^2}}{z - t^{-1/2}P^{Ker(\triangle)}\left(u\nabla^{\mathcal{E}} + (1-u)\left(\nabla^{\mathcal{E}}\right)^*\right)P^{Ker(\triangle)}} P^{Ker(\triangle)}\frac{dz}{2\pi i}\Bigg) du + O(t^{-1/2}) \\ &= -\int_0^1 \mathrm{Tr}_s\left(\left(\nabla^{H(\mathcal{E},v)} - \left(\nabla^{H(\mathcal{E},v)}\right)^*\right) e^{-\left(u\nabla^{H(\mathcal{E},v)} + (1-u)\left(\nabla^{H(\mathcal{E},v)}\right)^*\right)^2}\right) du + O(t^{-1/2}) \\ &= CS\left(\nabla^{H(\mathcal{E},v)}, h^{H(\mathcal{E},v)}\right) + O(t^{-1/2}). \end{aligned} \tag{5.53}$$

Note that only a finite number of terms of the series (5.48) contribute to the component of CS_1 of a given degree in $\overline{\Omega}^{\prime\prime,odd}(M,\mathfrak{B})$. Thus the derivation of (5.53) is purely algebraic.

It remains to estimate CS_2. Put $X = T_1T_2 + T_2T_1 + T_2^2$ and $e^{-rT_1'^2} = P^{Im(T_1)}e^{-rT_1^2}P^{Im(T_1)}$. We use the heat kernel expansion

$$\int_{\gamma_2} \frac{e^{-tz^2}}{z-(T_1+T_2)} \frac{dz}{2\pi i} = \int_0^\infty e^{-r_0T_1'^2}\delta(r_0-t)\,dr_0 \tag{5.54}$$
$$- \int_0^\infty\int_0^\infty e^{-r_0T_1'^2}Xe^{-r_1T_1'^2}\delta(r_0+r_1-t)\,dr_0dr_1$$
$$+ \int_0^\infty P^{Ker(T_1)}Xe^{-r_0T_1'^2}\theta(r_0-t)\,dr_0$$
$$+ \int_0^\infty e^{-r_0T_1'^2}XP^{Ker(T_1)}\theta(r_0-t)\,dr_0 + \dots$$

Here θ is the Heaviside function : $\theta(r) = \frac{1+sign(r)}{2}$. The series (5.54) is similar to the Duhamel expansion of $e^{-t(T_1^2+X)}$, with each intermediate factor of $e^{-rT_1^2}$ in the Duhamel expansion being replaced by either $e^{-rT_1'^2}$ or $P^{Ker(T_1)}$. A term on the right-hand-side of (5.54) with k X's and l $P^{Ker(T_1)}$'s, $k \geq l$, will have a factor of

$$\begin{array}{ll} (-1)^k\,\delta\left(\sum_{i=0}^k r_i - t\right) & \text{if} \quad l=0, \\ \frac{(-1)^{k+l}}{(l-1)!}\,\theta\left(\sum_{i=0}^{k-l} r_i - t\right)\left(\sum_{i=0}^{k-l} r_i - t\right)^{l-1} & \text{if} \quad l \geq 1. \end{array}$$

In our case, from (5.50),

$$X = t^{-1/2}\left(u^2\nabla^{\mathcal{E}}v + u(1-u)\left(\left(\nabla^{\mathcal{E}}\right)^*v + \nabla^{\mathcal{E}}v^*\right) + (1-u)^2\left(\nabla^{\mathcal{E}}\right)^*v^*\right) + O(t^{-1}). \tag{5.55}$$

Put

$$e^{-r\Delta'} = P^{Im(\Delta)}e^{-r\Delta}P^{Im(\Delta)}. \tag{5.56}$$

Using (5.54) gives

$$CS_2 = -\int_0^1 \mathrm{Tr}_s\left[\left(\nabla^{\mathcal{E}} - \left(\nabla^{\mathcal{E}}\right)^*\right)e^{-tu(1-u)\Delta'}\right]du \tag{5.57}$$
$$+\int_0^1\int_0^\infty\int_0^\infty \mathrm{Tr}_s\Big[(v-v^*)e^{-r_0u(1-u)\Delta'}\left(u^2\nabla^{\mathcal{E}}v + u(1-u)\left(\left(\nabla^{\mathcal{E}}\right)^*v + \nabla^{\mathcal{E}}v^*\right)\right.$$
$$\left.+(1-u)^2\left(\nabla^{\mathcal{E}}\right)^*v^*\right)e^{-r_1u(1-u)\Delta'}\Big]\,\delta(r_0+r_1-t)\,dr_0dr_1du + \dots$$

We now use the crucial fact [11, Theorem 1.22] that if $\{\alpha_r\}_{r>0}$ is a 1-parameter semigroup in a Banach algebra then the number

$$a = \lim_{r\to\infty} r^{-1}\log \parallel \alpha_r \parallel \tag{5.58}$$

exists. Furthermore, for all $r > 0$, the spectral radius of α_r is given by

$$\mathrm{SpRad}(\alpha_r) = e^{ar}. \tag{5.59}$$

Let $\lambda_0 > 0$ be the infimum of the nonzero spectrum of Δ. Then by the spectral mapping theorem, $\mathrm{SpRad}(e^{-r\Delta'}) = e^{-r\lambda_0}$. Thus for any $j \geq 0$, there is a constant $C_j > 0$ such that for all $r > 1$,

$$\| e^{-r\Delta'} \|_j \leq C_j \, e^{-r\lambda_0/2}. \tag{5.60}$$

Consider the $\overline{\Omega}^{\prime\prime,1}(M, \mathfrak{B})$-component of CS_2, which is given explicitly in (5.57). First, we have

$$\| \int_0^1 \mathrm{Tr}_s \left[\left(\nabla^{\mathcal{E}} - (\nabla^{\mathcal{E}})^* \right) e^{-tu(1-u)\Delta'} \right] du \, \|_j \leq \text{const.} \int_0^1 e^{-tu(1-u)\lambda_0/2} du = O(t^{-1}). \tag{5.61}$$

Next, consider the second term in (5.57). Recall that $CS\left(B'_t, h^{\mathcal{E}}\right)$ lives in the quotient space $\overline{\Omega}^{\prime\prime,odd}(M, \mathfrak{B})$ defined in (2.10). It follows from (5.24) that $\nabla^{\mathcal{E}} v$ will not contribute to the $\overline{\Omega}^{\prime\prime,1}(M, \mathfrak{B})$-component of CS_2, and similarly for $\left(\nabla^{\mathcal{E}}\right)^* v^*$. Hence the second term in (5.57) is bounded above in the $\| \cdot \|_j$-seminorm by

$$\text{const.} \int_0^1 \int_0^\infty \int_0^\infty u(1-u) e^{-tu(1-u)\lambda_0/2} \delta(r_0 + r_1 - t) \, dr_0 dr_1 du = O(t^{-1}). \tag{5.62}$$

Thus we have shown that for each j, the $\| \cdot \|_j$-seminorm of the $\overline{\Omega}^{\prime\prime,1}(M, \mathfrak{B})$-component of CS_2 is $O(t^{-1})$. One can carry out a similar analysis for all of the terms in CS_2. The point is that for large t, the $e^{-tu(1-u)\lambda_0/2}$ factor ensures that in the u-integral, only the behavior near $u = 0$ and $u = 1$ is important. Consider, for example, what happens when u is close to 1. When $u = 1$, $B_t(1)$ is partially flat and so X lies in $\Omega'\left(M; \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}, \Omega_*(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{E})\right)$, as defined in (5.4). Consider the value at $u = 1$ of a given term of CS_2. The contribution from $B'_t - B''_t$ lies in $\bigoplus_{p+q\pm r=1} \Omega^{p,q,r}(M, \mathfrak{B}, \mathrm{End}(\mathcal{E}))$ and the contribution from the X's lies in $\bigoplus_{p'+r' \leq q'} \Omega^{p',q',r'}(M, \mathfrak{B}, \mathrm{End}(\mathcal{E}))$. For the supertrace to be nonzero, we must have $r + r' = 0$. The explicit factor of t appearing is

$$t^{\frac{1}{2}} t^{-\frac{p+p'+q+q'}{2}} t^{\frac{p'+q'+r'}{2}} = t^{\frac{1-p-q-r}{2}}. \tag{5.63}$$

Suppose first that $p + q + r = 1$. The integral near $u = 1$ of $e^{-tu(1-u)\lambda_0/2}$ yields an additional factor of t^{-1}, giving a total estimate of $O(t^{-1})$. Suppose now that $p + q - r = 1$. The explicit factor of t is t^{-r}. Along with the integral of $e^{-tu(1-u)\lambda_0/2}$, we obtain a total estimate of $O(t^{-1-r})$. For this to be more significant than $O(t^{-1})$, we must have $r < 0$. As $r = p + q - 1$ and $p, q \geq 0$, the only possibility is $p = q = 0$ and $r = -1$. Then $r' = 1$ and so $p' + 1 \leq q'$. The supertrace of such a term lies in $\overline{\Omega}^{p',q'}(M, \mathfrak{B}) \subset \overline{\Omega}^{\prime,odd}(M, \mathfrak{B})$ and so vanishes in $\overline{\Omega}^{\prime\prime,odd}(M, \mathfrak{B})$.

One can carry out a similar analysis near $u = 0$. Finally, the convergence is clearly uniform on M. □

Corollary 1. *We have*

$$[\mathrm{ch}\,(A')] = \left[\mathrm{ch}\left(\nabla^{H(\mathcal{E},v)}\right)\right] \;\; in \;\; H^{\prime,even}_{\mathfrak{B}}(M) \tag{5.64}$$

and

$$\left[CS\left(A', h^{\mathcal{E}}\right)\right] = \left[CS\left(\nabla^{H(\mathcal{E},v)}, h^{H(\mathcal{E},v)}\right)\right] \;\; in \bigoplus_{\substack{p>q \\ p+q \; odd}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B})). \tag{5.65}$$

Proof. For all $u \in (0,1)$, $[\mathrm{ch}\,(A(u))] = [\mathrm{ch}\,(A')]$ and $\left[\mathrm{ch}\left(\nabla^{H(\mathcal{E},v)}(u)\right)\right] = \left[\mathrm{ch}\left(\nabla^{H(\mathcal{E},v)}\right)\right]$. As $A' = B'_1$, the corollary follows from (5.39) and Proposition 15. □

Remark 5 : If $\mathfrak{B} = \mathbb{C}$ then (5.65) is equivalent to [2, Theorem 2.14].

Proposition 16. *As $t \to \infty$,*

$$\mathcal{T}(t) = O(t^{-3/2}) \tag{5.66}$$

uniformly on M.

Proof. Let ϵ be a new variable which commutes with the other variables and satisfies $\epsilon^2 = 0$. Then

$$\mathcal{T}(t) = -\frac{1}{t}\left(\int_0^1 \mathrm{Tr}_s\left(Ne^{-B_t^2(u)}\right)du + \partial_\epsilon \int_0^1 u(1-u)\right. \tag{5.67}$$
$$\left.\mathrm{Tr}_s\left(N\left[B_t' - B_t'', e^{-B_t^2(u)+\epsilon(B_t'-B_t'')}\right]\right)du\right).$$

The implication is that we can again write $\mathcal{T}(t)$ as $\mathcal{T}_1(t) + \mathcal{T}_2(t)$, where $\mathcal{T}_i(t)$ is a contour integral around γ_i. Using the method of proof of Proposition 15, one finds

$$\mathcal{T}_1(t) = -\frac{1}{t}\left(\int_0^1 \mathrm{Tr}_s\left(Ne^{-\nabla^{H(\mathcal{E},v)^2}(u)}\right)du + \int_0^1\int_0^1 u(1-u)\mathrm{Tr}_s\left(N\left[\nabla^{H(\mathcal{E},v)} - \left(\nabla^{H(\mathcal{E},v)}\right)^*,\right.\right.\right. \tag{5.68}$$
$$\left.\left.\left. e^{-r\nabla^{H(\mathcal{E},v)^2}(u)}\left(\nabla^{H(\mathcal{E},v)} - \left(\nabla^{H(\mathcal{E},v)}\right)^*\right)e^{-(1-r)\nabla^{H(\mathcal{E},v)^2}(u)}\right]\right)drdu\right) + O(t^{-3/2})$$
$$= -\frac{1}{t}\int_0^1 \mathrm{Tr}_s\left(Ne^{-\nabla^{H(\mathcal{E},v)^2}(u)}\right)du + O(t^{-3/2}).$$

As $\mathrm{Tr}_s\left(Ne^{-\nabla^{H(\mathcal{E},v)^2}(u)}\right) \in \overline{\Omega}^{\prime,even}(M,\mathfrak{B})$, it follows that $\mathcal{T}_1(t)$ is $O(t^{-3/2})$ in $\overline{\Omega}^{\prime\prime,even}(M,\mathfrak{B})$.

Next, consider $\mathcal{T}_2(t)$. Counting powers of t as in the proof of Proposition 15, one finds that $\mathcal{T}_2(t)$ is $O(t^{-3/2})$ except for possible terms which decay like t^{-1} and lie in $\bigoplus_p \overline{\Omega}^{p,p}(M,\mathfrak{B})$ mod $\overline{\Omega}^{\prime,even}(M,\mathfrak{B})$. Let us write such a term as $t^{-1}\mathcal{T}_2^{p,p}$, with $\mathcal{T}_2^{p,p} \in \overline{\Omega}^{p,p}(M,\mathfrak{B})$. From (5.39), we see that $t^{-1}d^{1,0}\mathcal{T}_2^{p,p}$ comes from the t-derivative of the $\overline{\Omega}^{p+1,p}(M,\mathfrak{B})$-component of $CS\left(B_t', h^{\mathcal{E}}\right)$. However, as $CS\left(B_t', h^{\mathcal{E}}\right)$ has no $\log(t)$-term in its asymptotics, it follows that $d^{1,0}\mathcal{T}_2^{p,p} = 0$. Thus $\mathcal{T}_2^{p,p}$ lies in $Z^p\left(M;\overline{\Omega}_p(\mathfrak{B})\right)$. As we quotient by this subspace in defining $\overline{\Omega}^{\prime\prime,even}(M,\mathfrak{B})$, equation (5.66) follows.

Again, the convergence is clearly uniform on M. □

Remark 6 : There may seem to be a contradiction between Proposition 16 and [2, Theorem 2.13], in which a nonzero $O(t^{-1})$ term for $\mathcal{T}(t)$ was found. However, in the present paper we quotient by $Z^k\left(M;\overline{\Omega}_k(\mathfrak{B})\right)$ in defining $\overline{\Omega}^{2k}\left(M;\overline{\Omega}_p(\mathfrak{B})\right)$. When $\mathfrak{B} = \mathbb{C}$, as in [2], this quotienting removes the $O(t^{-1})$ term of [2, Theorem 2.13] and so there is no contradiction.

5.3. **The Analytic Torsion Form.** In this subsection we consider the special case when $\mathcal{E}$ has not only a partially flat degree-1 superconnection, but has a partially flat connection in the sense of Definition 3. Let

$$(\mathcal{E},v): 0 \to \mathcal{E}^0 \xrightarrow{v} \mathcal{E}^1 \xrightarrow{v} \ldots \xrightarrow{v} \mathcal{E}^n \to 0 \tag{5.69}$$

be a cochain complex of $\mathfrak{B}$-vector bundles on M. Let

$$\nabla^{\mathcal{E}} = \bigoplus_{i=0}^{n} \nabla^{\mathcal{E}^i} \tag{5.70}$$

be a partially-flat connection on $\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{E}^i$. Suppose that $\nabla^{\mathcal{E},1,0} v = 0$. Put

$$A' = v + \nabla^{\mathcal{E}}. \tag{5.71}$$

Then A' is a partially flat degree-1 superconnection. In the notation of Subsection 5.2,

$$\begin{aligned} B'_t &= \sqrt{t}\, v + \nabla^{\mathcal{E}}, \\ B''_t &= \sqrt{t}\, v^* + \left(\nabla^{\mathcal{E}}\right)^*. \end{aligned} \tag{5.72}$$

Proposition 17. *As* $t \to 0$,

$$\mathrm{ch}\left(B_t(u)\right) = \mathrm{ch}\left(\nabla^{\mathcal{E}}(u)\right) + O(t), \tag{5.73}$$

$$CS\left(B'_t, h^{\mathcal{E}}\right) = CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right) + O(t) \tag{5.74}$$

and

$$\mathcal{T}(t) = O(1). \tag{5.75}$$

Proof. Equations (5.73) and (5.74) are evident. From (5.37), the t^{-1}-term of $\mathcal{T}$ is

$$\begin{aligned} &-\frac{1}{t}\left(\int_0^1 \mathrm{Tr}_s\left(Ne^{-\nabla^{\mathcal{E}^2}(u)}\right) du + \int_0^1\int_0^1 u(1-u)\mathrm{Tr}_s\left(N\left[\nabla^{\mathcal{E}} - \left(\nabla^{\mathcal{E}}\right)^*,\right.\right.\right. \\ &\qquad \left.\left.\left. e^{-r\nabla^{\mathcal{E}^2}(u)}\left(\nabla^{\mathcal{E}} - \left(\nabla^{\mathcal{E}}\right)^*\right) e^{-(1-r)\nabla^{\mathcal{E}^2}(u)}\right]\right) dr du\right) \\ &= -\frac{1}{t}\int_0^1 \mathrm{Tr}_s\left(Ne^{-\nabla^{\mathcal{E}^2}(u)}\right) du. \end{aligned} \tag{5.76}$$

As $\mathrm{Tr}_s\left(Ne^{-\nabla^{\mathcal{E}^2}(u)}\right) \in \overline{\Omega}^{\prime,even}(M, \mathfrak{B})$, this vanishes in $\overline{\Omega}^{\prime\prime,even}(M, \mathfrak{B})$. It is easy to check that there is no $O(t^{-1/2})$-term. □

Corollary 2. *We have*

$$\left[\mathrm{ch}\left(\nabla^{\mathcal{E}}\right)\right] = \left[\mathrm{ch}\left(\nabla^{H(\mathcal{E},v)}\right)\right] \ \text{in} \ H^{\prime,even}_{\mathfrak{B}}(M) \tag{5.77}$$

and

$$\left[CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right)\right] = \left[CS\left(\nabla^{H(\mathcal{E},v)}, h^{H(\mathcal{E},v)}\right)\right] \ \text{in} \bigoplus_{\substack{p>q \\ p+q \text{ odd}}} \mathrm{H}^p(M; \overline{H}_q(\mathfrak{B})). \tag{5.78}$$

Proof. For all $u \in (0,1)$, $\left[\mathrm{ch}\left(\nabla^{\mathcal{E}}(u)\right)\right] = \left[\mathrm{ch}\left(\nabla^{\mathcal{E}}\right)\right]$ and $\left[\mathrm{ch}\left(\nabla^{H(\mathcal{E},v)}(u)\right)\right] = \left[\mathrm{ch}\left(\nabla^{H(\mathcal{E},v)}\right)\right]$. The corollary now follows from Corollary 1 and Proposition 17. □

Remark 7 : If $\mathfrak{B} = \mathbb{C}$ then Corollary 2 is equivalent to [2, Theorem 2.19].

Definition 16. *The analytic torsion form* $\mathcal{T} \in \overline{\Omega}^{\prime\prime,even}(M, \mathfrak{B})$ *is given by*

$$\mathcal{T} = \int_0^\infty \mathcal{T}(t)\, dt. \tag{5.79}$$

By Propositions 16 and 17, the integral in (5.79) makes sense.

Proposition 18. *We have*

$$d\mathcal{T} = CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right) - CS\left(\nabla^{H(\mathcal{E},v)}, h^{H(\mathcal{E},v)}\right) \ \text{in} \ \overline{\Omega}^{\prime\prime,odd}(M, \mathfrak{B}). \tag{5.80}$$

Proof. This follows from (5.39) and Propositions 15 and 17. □

Let us look more closely at $\mathcal{T}_{[0]}$, the component of $\mathcal{T}$ in $\overline{\Omega}^{\prime\prime,0}(M,\mathfrak{B})$. Assume for simplicity that M is connected. Then

$$\overline{\Omega}^{\prime\prime,0}(M,\mathfrak{B}) = (C^\infty(M)/\mathbb{C}) \otimes \left(\mathfrak{B}/\overline{[\mathfrak{B},\mathfrak{B}]}\right). \tag{5.81}$$

From (5.37) and (5.79),

$$\mathcal{T}_{[0]} \equiv -\int_0^\infty \left(\int_0^1 \mathrm{Tr}_s\left(Ne^{-tu(1-u)\Delta}\right) du + t\int_0^1\int_0^1 u(1-u)\right. \tag{5.82}$$

$$\left.\mathrm{Tr}_s\left(N\left[v-v^*, e^{-ru(1-u)\Delta}\left(v-v^*\right)e^{-(1-r)u(1-u)\Delta}\right]\right) dr du\right)\frac{dt}{t}$$

$$= -\int_0^\infty\int_0^1 \mathrm{Tr}_s\left(N\left(1-2tu(1-u)\Delta\right)e^{-tu(1-u)\Delta}\right) du\frac{dt}{t}$$

To give a specific lifting of $\mathcal{T}_{[0]}$ to $C^\infty(M)\otimes\left(\mathfrak{B}/\overline{[\mathfrak{B},\mathfrak{B}]}\right)$, we use the fact that $\mathrm{Tr}_s\left(N|_{\mathcal{E}}\right)$ and $\mathrm{Tr}_s\left(N|_{H(E,v)}\right)$ are constant on M. Put

$$g(t) = -\int_0^1 (1-2u(1-u)t)e^{-tu(1-u)}du. \tag{5.83}$$

The asymptotics of g are given by

$$g(0) = -1 \text{ and } \lim_{t\to\infty} g(t) = 0. \tag{5.84}$$

Then we can define the lifting to be

$$\mathcal{T}_{[0]} = \int_0^\infty \left[\mathrm{Tr}_s\left(Ng\left(t\Delta\right)\right) - \left(\mathrm{Tr}_s\left(N|_{\mathcal{E}}\right) - \mathrm{Tr}_s\left(N|_{H(E,v)}\right)\right)g(t)\right. \tag{5.85}$$

$$\left.+\mathrm{Tr}_s\left(N|_{H(E,v)}\right)\right]\frac{dt}{t}.$$

Let Δ' be the restriction of Δ to $\mathrm{Im}(\Delta)$. Then

$$\mathcal{T}_{[0]} = \int_0^\infty \left[\mathrm{Tr}_s\left(Ng\left(t\Delta'\right)\right) - \mathrm{Tr}_s\left(N|_{Im(\Delta)}\right)g(t)\right]\frac{dt}{t}. \tag{5.86}$$

It follows from (5.84) that for $\lambda > 0$,

$$\int_0^\infty [g(\lambda t) - g(t)]\frac{dt}{t} = \log(\lambda). \tag{5.87}$$

Thus by the holomorphic functional calculus,

$$\mathcal{T}_{[0]} = \mathrm{Tr}_s\left(N\log(\Delta')\right) \in C^\infty(M)\otimes\left(\mathfrak{B}/\overline{[\mathfrak{B},\mathfrak{B}]}\right). \tag{5.88}$$

Example 7 : If $\mathfrak{B} = \mathbb{C}$ then $\mathcal{T}_{[0]}$ is the usual Reidemeister torsion of the cochain complex $(\mathcal{E}, v)$ [32], considered to be a function on M.

Example 8 : Suppose that Γ is a finite group and $\mathfrak{B} = \mathbb{C}\Gamma$. Then $\mathcal{T}_{[0]}$ is equivalent to the equivariant Reidemeister torsion of [25].

Example 9 : Suppose that Γ is a discrete group and $\mathfrak{B} = C_r^*\Gamma$. Let τ be the trace on $\mathfrak{B}$ given by $\tau(\sum_{\gamma\in\Gamma} c_\gamma \gamma) = c_e$. Then $\tau\left(\mathcal{T}_{[0]}\right)$ is the L^2-Reidemeister torsion of [8], in the case when there is a gap in the spectrum. (One can define the L^2-Reidemeister torsion for a cochain complex of modules over the group von Neumann algebra, not just the group C^*-algebra.)

6. Fiber Bundles

In this section we extend the results of Section 5 to the fiber bundle setting. The translation is that the algebra of endomorphisms of a finitely-generated projective $\mathfrak{B}$-module gets replaced by an algebra of $\mathfrak{B}$-pseudodifferential operators. In the case $\mathfrak{B} = \mathbb{C}$, we recover the fiber-bundle results of [2]. We emphasize the necessary modifications to [2] and refer to [2] for some computations.

6.1. **$\mathfrak{B}$-Pseudodifferential Calculus.** Let Z^n be a smooth closed manifold. Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be smooth $\mathfrak{B}$-vector bundles on Z, with fibers isomorphic to $\mathfrak{E}^1$ and $\mathfrak{E}^2$, respectively. In the case when $\mathfrak{B}$ is a C^*-algebra, an algebra $\Psi^\infty_{\mathfrak{B}}(Z;\mathcal{E}^1,\mathcal{E}^2)$ of classical $\mathfrak{B}$-pseudodifferential operators was defined in [29, §3]. We extend this notion to the Fréchet locally m-convex algebra $\mathfrak{B}$ as follows. First, define seminorms $\{\|\cdot\|\}_{j=0}^\infty$ on $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}^1,\mathfrak{E}^2)$ similarly to the discussion after Proposition 14. Let $U \cong \mathbb{R}^n$ be a coordinate patch of Z equipped with isomorphisms $\mathcal{E}^1\big|_U \cong U\times\mathfrak{E}^1$ and $\mathcal{E}^2\big|_U \cong U\times\mathfrak{E}^2$. We define an algebra $\Psi^\infty_{\mathfrak{B}}(U;\mathcal{E}^1,\mathcal{E}^2)$ of classical $\mathfrak{B}$-pseudodifferential operators on U by requiring that the symbol $\sigma(z,\xi) \in \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}^1,\mathfrak{E}^2)$ of an order-m operator $T\in\Psi^m_{\mathfrak{B}}(Z;\mathcal{E}^1,\mathcal{E}^2)$ be compactly supported in z and satisfy

$$\| \partial_{z^\alpha}\partial_{\xi^\beta}\sigma(z,\xi)\|_j \leq C_{\alpha,\beta,j}(1+|\xi|)^{m-|\beta|} \tag{6.1}$$

for all multi-indices α and β. Then we define $\Psi^\infty_{\mathfrak{B}}(Z;\mathcal{E}^1,\mathcal{E}^2)$ using a partition of unity as in [29, §3].

Using the representation of $\mathfrak{B}$ as the projective limit of the sequence (2.1) of Banach algebras $\{B_j\}_{j=0}^\infty$, with B_0 a C^*-algebra, we can say that $\Psi^\infty_{\mathfrak{B}}(Z;\mathcal{E}^1,\mathcal{E}^2)$ is the projective limit of the sequence of pseudodifferential operator algebras

$$\ldots \longrightarrow \Psi^\infty_{B_{j+1}}(Z;E^1_{j+1},E^2_{j+1}) \longrightarrow \Psi^\infty_{B_j}(Z;E^1_j,E^2_j) \longrightarrow \ldots \longrightarrow \Psi^\infty_{B_0}(Z;E^1_0,E^2_0). \tag{6.2}$$

Let $\mathcal{E}$ be a $\mathfrak{B}$-vector bundle on Z. Given $T\in\Psi^\infty_{\mathfrak{B}}(Z;\mathcal{E},\mathcal{E})$, let $i_j(T)$ be its image in $\Psi^\infty_{B_j}(Z;E_j,E_j)$.

Proposition 19. *If $i_0(T)$ is invertible in $\Psi^\infty_{B_0}(Z;E_0,E_0)$ then T is invertible in $\Psi^\infty_{\mathfrak{B}}(Z;\mathcal{E},\mathcal{E})$.*

Proof. It is enough to show that each $i_j(T)$ is invertible in $\Psi^\infty_{B_j}(Z;E_j,E_j)$, as then T^{-1} will be the inverse limit of $\{(i_j(T))^{-1}\}_{j=0}^\infty$. So suppose that B is a Banach algebra which is dense in a C^*-algebra $\overline{B}$ and stable under the holomorphic functional calculus in $\overline{B}$. Let E be a B-vector bundle on Z. Let $\overline{E} = \overline{B}\otimes_B E$ be the corresponding $\overline{B}$-vector bundle on Z. Given $T\in\Psi^m_B(Z;E,E)$, let $\overline{T}$ be its image in $\Psi^m_{\overline{B}}(Z;\overline{E},\overline{E})$. We will show that if $\overline{T}$ is invertible in $\Psi^\infty_{\overline{B}}(Z;\overline{E},\overline{E})$ then T is invertible in $\Psi^\infty_B(Z;E,E)$.

Write E as the image under a projection $e\in C^\infty(Z;M_N(B))$ of a trivial B-vector bundle $Z\times B^N$. Let $E' = \mathrm{Im}(1-e)$ be the complementary B-vector bundle. Choose

$T' \in \Psi^m_B(Z; E, E)$ such that $\overline{T'}$ is invertible in $\Psi^\infty_{\overline{B}}(Z; \overline{E'}, \overline{E'})$. If we can show that $T \oplus T'$ is invertible in $\Psi^\infty_B(Z; B^N, B^N)$ then the inverse of T will be given by the restriction of $(T \oplus T')^{-1}$ to $\mathrm{Im}(e)$. So we may as well assume that E is a trivial B-vector bundle with fiber B^N.

Note that as $\overline{T}$ is invertible, T is elliptic. By the usual parametrix construction, we can find $U \in \Psi^{-m}_B(Z; E, E)$ such that $TU = I - K$ with $K \in \Psi^{-\infty}_B(Z; E, E)$, and similarly for UT. Perturbing U a bit if necessary, we can assume that $\overline{U}$ is invertible. Then $I - \overline{K}$ is invertible, with inverse $\overline{U}^{-1}\overline{T}^{-1}$. If we can show that $I - K$ is invertible then $T^{-1} = U(I - K)^{-1}$.

Thus we are reduced to showing that if $K \in \Psi^{-\infty}_B(Z; E, E)$ and $I - \overline{K}$ is invertible then $I - K$ is invertible. Fix a Riemannian metric on Z. Let $\{e_i\}_{i=1}^\infty$ be the orthonormal basis of smooth functions on Z given by the eigenfunctions of the Laplacian. For $M \geq 1$, let $p_M \in \Psi^{-\infty}_{\mathbb{C}}(Z; \mathbb{C}, \mathbb{C})$ be the obvious projection operator from $L^2(Z)$ to $\left(\bigoplus_{i=1}^M e_i\right)$ and let $P_M \in \Psi^{-\infty}_B(Z; E, E)$ be its extension by the identity on B^N. Consider the operator $D_M = I - (I - P_M)K(I - P_M)$. We claim that if M is large enough then D_M is invertible. To see this, write the Schwartz kernel of $(I - P_M)K(I - P_M)$ as

$$\begin{aligned}\left[(I - P_M)K(I - P_M)\right](z, z') = {} & K(z, z') - \int_Z P_M(z, w)K(w, z')dw - \int_Z K(z, w')P_M(w', z')dw \\ & + \int_Z \int_Z P_M(z, w)K(w, w')P_M(w', z')dwdw'.\end{aligned} \tag{6.3}$$

The sequence of Schwartz kernels $\{P_M(w, w')\}_{M=1}^\infty$ forms an approximate identity. By assumption, $K(z, z')$ is a smooth function from $Z \times Z$ to $M_N(B)$. It follows that for any $\epsilon > 0$, there is an $M \geq 1$ such that for all $z, z' \in Z$, in the Banach norm,

$$\left| \left[(I - P_M)K(I - P_M)\right](z, z') \right| \leq \epsilon. \tag{6.4}$$

Taking ϵ small enough, the sum of convolutions

$$D_M^{-1} = \sum_{k=0}^\infty ((I - P_M)K(I - P_M))^k \tag{6.5}$$

converges in the algebra $I + \Psi^{-\infty}_B(Z; E, E)$.

With respect to the decomposition $I = P_M + (I - P_M)$, write

$$I - K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{6.6}$$

We have shown that δ is invertible. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha - \beta\delta^{-1}\gamma & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta^{-1}\gamma & 1 \end{pmatrix}. \tag{6.7}$$

As $I - \overline{K}$ is invertible, it follows that $\overline{\alpha - \beta\delta^{-1}\gamma}$ is invertible in $M_{MN}(\overline{B})$. Then $\alpha - \beta\delta^{-1}\gamma$ is invertible in $M_{MN}(B)$ [5, Proposition A.2.2]. Hence

$$(I - K)^{-1} = \begin{pmatrix} 1 & 0 \\ -\delta^{-1}\gamma & 1 \end{pmatrix} \begin{pmatrix} (\alpha - \beta\delta^{-1}\gamma)^{-1} & 0 \\ 0 & \delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\beta\delta^{-1} \\ 0 & 1 \end{pmatrix} \tag{6.8}$$

is well-defined in $I + \Psi^{-\infty}_B(Z; E, E)$. □

Note that $\Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$ is an algebra in its own right (without unit). Given $T \in \Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$, let $\sigma_{\Psi^{-\infty}}(T)$ denote its spectrum in $\Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$ and let $\sigma_{\Psi^{\infty}}(T)$ denote its spectrum in $\Psi_{\mathfrak{B}}^{\infty}(Z;\mathcal{E},\mathcal{E})$.

Lemma 2. $\sigma_{\Psi^{-\infty}}(T) = \sigma_{\Psi^{\infty}}(T)$.

Proof. As $\Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$ has no unit, $0 \in \sigma_{\Psi^{-\infty}}(T)$. As T is not invertible in $\Psi_{\mathfrak{B}}^{\infty}(Z;\mathcal{E},\mathcal{E})$, $0 \in \sigma_{\Psi^{\infty}}(T)$. If $\lambda \neq 0$ then by definition, $\lambda \notin \sigma_{\Psi^{-\infty}}(T)$ if and only if we can solve the equation $TU - \lambda U - \lambda^{-1}T = UT - \lambda U - \lambda^{-1}T = 0$ for some $U \in \Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$. Thus if $\lambda \notin \sigma_{\Psi^{-\infty}}(T)$ then in $\Psi_{\mathfrak{B}}^{\infty}(Z;\mathcal{E},\mathcal{E})$, we have $(T-\lambda)(U-\lambda^{-1}) = (U-\lambda^{-1})(T-\lambda) = I$ and hence $\lambda \notin \sigma_{\Psi^{\infty}}(T)$. Conversely, if $\lambda \notin \sigma_{\Psi^{\infty}}(T)$ then we can solve $(T-\lambda)(U-\lambda^{-1}) = (U-\lambda^{-1})(T-\lambda) = I$ for some $U \in \Psi_{\mathfrak{B}}^{\infty}(Z;\mathcal{E},\mathcal{E})$. By the pseudodifferential operator calculus, $U = \lambda^{-1}T(T-\lambda)^{-1} = \lambda^{-1}(T-\lambda)^{-1}T \in \Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$ and so $\lambda \notin \sigma_{\Psi^{-\infty}}(T)$. □

Fix a Riemannian metric on Z. Given a $\mathfrak{B}$-vector bundle $\mathcal{E}$ on Z, write $\mathcal{E} = \mathfrak{B}^N e$ for some N and some projection $e \in C^\infty(Z; M_N(\mathfrak{B}))$. Then

$$\mathrm{Hom}_{\mathfrak{B}}\left(\mathcal{E}_{z_2}, \mathcal{E}_{z_1}\right) \cong \{k \in M_N(\mathfrak{B}) : k = e(z_1)ke(z_2)\}. \tag{6.9}$$

Consider the algebra $\mathfrak{A}$ of integral operators whose kernels $K(z_1,z_2) \in \mathrm{Hom}_{\mathfrak{B}}\left(\mathcal{E}_{z_2}, \mathcal{E}_{z_1}\right)$ are continuous in z_1 and z_2, with multiplication

$$(KK')(z_1,z_2) = \int_Z K(z_1,z)K'(z,z_2)\, d\mathrm{vol}(z). \tag{6.10}$$

Let A_j be the analogous algebra with continuous kernels $K(z_1,z_2) \in \mathrm{Hom}_{B_j}\left((E_j)_{z_2}, (E_j)_{z_1}\right)$. Give $\mathrm{Hom}_{B_j}\left((E_j)_{z_2}, (E_j)_{z_1}\right)$ the Banach space norm $|\cdot|_j$ induced from $\mathrm{Hom}\left(B_j^N, B_j^N\right)$. Define a norm $|\cdot|_j$ on A_j by

$$|K|_j = (\mathrm{vol}(Z))^{-1} \max_{z_1,z_2 \in Z} |K(z_1,z_2)|_j. \tag{6.11}$$

Then one can check that A_j is a Banach algebra (without unit). Furthermore, $\mathfrak{A}$ is the projective limit of $\{A_j\}_{j\geq 0}$ and so is a Fréchet locally m-convex algebra with seminorms $\{\|\cdot\|_j\}_{j\geq 0}$ coming from $\{|\cdot|_j\}_{j\geq 0}$. Any $T \in \Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$ gives an element of $\mathfrak{A}$ through its Schwartz kernel. Let $\sigma_{\mathfrak{A}}(T)$ be its spectrum in $\mathfrak{A}$.

Lemma 3. $\sigma_{\mathfrak{A}}(T) = \sigma_{\Psi^{-\infty}}(T)$.

Proof. As $\mathfrak{A}$ and $\Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$ have no unit, $0 \in \sigma_{\mathfrak{A}}(T)$ and $0 \in \sigma_{\Psi^{-\infty}}(T)$. For $\lambda \neq 0$, suppose that $\lambda \notin \sigma_{\Psi^{-\infty}}(T)$. Then we can solve $TU - \lambda U - \lambda^{-1}T = UT - \lambda U - \lambda^{-1}T = 0$ for some $U \in \Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$. As U defines an element of $\mathfrak{A}$, we have $\lambda \notin \sigma_{\mathfrak{A}}(T)$. Now suppose that $\lambda \notin \sigma_{\mathfrak{A}}(T)$. Then we can solve $TU - \lambda U - \lambda^{-1}T = UT - \lambda U - \lambda^{-1}T = 0$ for some $U \in \mathfrak{A}$. As $U = \lambda^{-1}T(U-\lambda^{-1}) = \lambda^{-1}(U-\lambda^{-1})T$, it follows that U has a smooth kernel and so defines an element of $\Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$. Thus $\lambda \notin \sigma_{\Psi^{-\infty}}(T)$. □

Define $\mathrm{TR} : \mathfrak{A} \to \mathfrak{B}/\overline{[\mathfrak{B},\mathfrak{B}]}$ as in (2.39).

Corollary 3. *Suppose that $\{\alpha_r\}_{r>0}$ is a 1-parameter semigroup in $\Psi_{\mathfrak{B}}^{-\infty}(Z;\mathcal{E},\mathcal{E})$ whose spectral radius in $\Psi_{\mathfrak{B}}^{\infty}(Z;\mathcal{E},\mathcal{E})$ is given by $\mathrm{SpRad}(\alpha_r) = e^{ar}$ for some $a < 0$. Then for all $j \geq 0$, as $r \to \infty$, in $\mathfrak{A}$ we have $\| \alpha_r \|_j = o\left(e^{ar/2}\right)$. In particular, $\mathrm{TR}(\alpha_r) = o\left(e^{ar/2}\right)$.*

Proof. By Lemmas 2 and 3, the spectral radius of α_r in $\mathfrak{A}$ is e^{ar}. Then its spectral radius in the Banach algebra A_j is less than or equal to e^{ar}. By (5.58) and (5.59), $\| \alpha_r \|_j = o\left(e^{ar/2}\right)$. As TR is continuous on $\mathfrak{A}$, the corollary follows. □

6.2. **Induced Superconnections.** Let $Z \to M \xrightarrow{\pi} B$ be a smooth fiber bundle with total space M, compact base B and connected closed fibers $\{Z_b\}_{b\in B}$ of dimension n. We use the notation of [2, Section 3] when discussing the topology or geometry of such a fiber bundle. Let TZ be the vertical tangent bundle of the fiber bundle, an $\mathbb{R}^n$-bundle on M, and let $o(TZ)$ be its orientation bundle, a flat $\mathbb{R}$-bundle on M. Let T^HM be a horizontal distribution for the fiber bundle. Let $\mathcal{E}$ be a $\mathfrak{B}$-vector bundle on M. There is an induced $\mathbb{Z}$-graded $\mathfrak{B}$-vector bundle $\mathcal{W}$ on B whose fiber over $b \in B$ consists of the smooth $\mathcal{E}$-valued differential forms on Z_b, i.e. $\mathcal{W}_b = \Omega^*\left(Z_b; \mathcal{E}\big|_{Z_b}\right)$. If $n > 0$ then $\mathcal{W}_b$ is infinitely-generated. Using the horizontal distribution, there is an isomorphism

$$\Omega(B, \mathfrak{B}; \mathcal{W}) \cong \Omega(M, \mathfrak{B}; \mathcal{E}). \tag{6.12}$$

We equip $\mathcal{E}$ with a partially flat connection $\nabla^{\mathcal{E}}$ as in (2.21). Using (6.12), this induces a partially flat degree-1 superconnection A' on $\mathcal{W}$. The connection component $\nabla^{\mathcal{W}}$ of A' has two pieces :

$$\begin{aligned} \nabla^{\mathcal{W},1,0} &: \Omega^{p,q}(M, \mathfrak{B}; \mathcal{E}) \to \Omega^{p+1,q}(M, \mathfrak{B}; \mathcal{E}), \\ \nabla^{\mathcal{W},0,1} &: \Omega^{p,q}(M, \mathfrak{B}; \mathcal{E}) \to \Omega^{p,q+1}(M, \mathfrak{B}; \mathcal{E}). \end{aligned} \tag{6.13}$$

As in [2, Proposition 3.4], $\nabla^{\mathcal{W},1,0}$ is given by Lie differentiation with respect to a horizontal vector field on M. On the other hand, $\nabla^{\mathcal{W},0,1}$ comes from the action of $\partial^{\mathcal{E}}$ as in (2.19). The other nonzero components of A' are $A'_{0,0,1} = d^Z$ and $A'_{2,0,-1} = i_T$. The degree-1 superconnection $A'^{,flat}$ defining the superflat structure on $\mathcal{W}$ is essentially the same as the flat degree-1 superconnection of [2, Section 3b]. The main difference between [2, Section 3] and the present paper is that we take into account $\nabla^{\mathcal{W},0,1}$, so that A' is not completely flat.

Let g^{TZ} be a family of vertical Riemannian metrics on the fiber bundle. Let $*$ be the corresponding fiberwise Hodge duality operator, extended linearly from $C^\infty(M; \Lambda(T^*Z))$ to $C^\infty(M; \Lambda(T^*Z) \otimes \mathcal{E}) \cong C^\infty(B; \mathcal{W})$. Let $h^{\mathcal{E}}$ be a Hermitian metric on $\mathcal{E}$. Let $\left(\nabla^{\mathcal{E}}\right)^*$ be the adjoint connection to $\nabla^{\mathcal{E}}$, with respect to $h^{\mathcal{E}}$. There is a self-adjoint connection on $\mathcal{E}$ given by

$$\nabla^{\mathcal{E},sa} = \frac{1}{2}\left(\nabla^{\mathcal{E}} + (\nabla^{\mathcal{E}})^*\right). \tag{6.14}$$

Put

$$\psi = \left(\nabla^{\mathcal{E},1,0}\right)^* - \nabla^{\mathcal{E},1,0} \in \Omega^1(M; \mathrm{End}_{\mathfrak{B}}(\mathcal{E})). \tag{6.15}$$

Let us assume that $\left(\partial^{\mathcal{E}}\right)^* = \partial^{\mathcal{E}}$; this can always be achieved by replacing $\partial^{\mathcal{E}}$ by $\frac{1}{2}\left(\partial^{\mathcal{E}} + \left(\partial^{\mathcal{E}}\right)^*\right)$ if necessary. Then

$$\nabla^{\mathcal{E},sa} = \nabla^{\mathcal{E}} + \frac{\psi}{2}. \tag{6.16}$$

For notational convenience, put

$$\nabla^{\mathcal{E},flat,sa} = \frac{1}{2}\left(\nabla^{\mathcal{E},flat} + \left(\nabla^{\mathcal{E},flat}\right)^*\right). \tag{6.17}$$

Then

$$\nabla^{\mathcal{E},flat,sa} = \nabla^{\mathcal{E},flat} + \frac{\psi}{2}, \tag{6.18}$$

$$\nabla^{\mathcal{E},sa} = \nabla^{\mathcal{E},flat,sa} + \partial^{\mathcal{E}} \tag{6.19}$$

and

$$\left(\nabla^{\mathcal{E},flat,sa}\right)^2 = -\frac{\psi^2}{4}. \tag{6.20}$$

Furthermore, one can check that $\nabla^{TM\otimes\mathcal{E},flat,sa}\psi$ vanishes in $\Omega^2\left(M; \mathrm{End}_{\mathfrak{B}}(\mathcal{E})\right)$.

There is a Hermitian metric $h^{\mathcal{W}}$ on $\mathcal{W}$ such that for $s, s' \in \mathcal{W}_b$,

$$\langle s, s'\rangle_{h^{\mathcal{W}}} = \int_{Z_b} \langle s \wedge *s'\rangle_{h^{\mathcal{E}}} \in \mathfrak{B}. \tag{6.21}$$

Let $(A')^*$ be the adjoint superconnection to A', with respect to $h^{\mathcal{W}}$. There is a self-adjoint connection $\nabla^{\mathcal{W},sa}$ on $\mathcal{W}$ given by

$$\nabla^{\mathcal{W},sa} = \frac{1}{2}\left(\nabla^{\mathcal{W}} + \left(\nabla^{\mathcal{W}}\right)^*\right). \tag{6.22}$$

Let $\{e_j\}_{j=1}^n$ be a local orthonormal basis for TZ, with dual basis $\{\tau^j\}_{j=1}^n$. Let E^j denote exterior multiplication by τ^j and let I^j denote interior multiplication by e_j. Put

$$\begin{aligned} c^j &= E^j - I^j, \\ \widehat{c}^j &= E^j + I^j. \end{aligned} \tag{6.23}$$

Then

$$\begin{aligned} c^j c^k + c^k c^j &= -2\delta^{jk}, \\ \widehat{c}^j \widehat{c}^k + \widehat{c}^k \widehat{c}^j &= 2\delta^{jk}, \\ c^j \widehat{c}^k + \widehat{c}^k c^j &= 0. \end{aligned} \tag{6.24}$$

Thus c and $\widehat{c}$ generate two graded-commuting Clifford algebras.

Let ∇^{TZ} be Bismut's connection on TZ [1, p. 322], with curvature R^{TZ}. Let $e\left(TZ, \nabla^{TZ}\right) \in \Omega^n(M; o(TZ))$ be the corresponding Euler form. Define $\mathcal{R} \in \Omega^2\left(M; \mathrm{End}(\Lambda(T^*Z) \otimes \mathcal{E})\right)$ by

$$\mathcal{R} = \frac{1}{4}\left(\langle e_j, R^{TZ} e_k\rangle_{g^{TZ}}\, \widehat{c}^j \widehat{c}^k \otimes I_{\mathcal{E}}\right) - \frac{1}{4}\left(I_{\Lambda(T^*Z)} \otimes \psi^2\right). \tag{6.25}$$

Let $R \in C^\infty(M)$ be the scalar curvature of the fibers. Let τ^α be a local basis of T^*B and let E^α denote exterior multiplication by τ^α. Define the superconnection $B_t(u)$ on $\mathcal{W}$ as in Proposition 14.

Proposition 20. *We have*

$$\begin{aligned} B_t(u) = {} & \frac{\sqrt{t}}{2} c^j \nabla^{TZ\otimes\mathcal{E},sa}_{e_j} - \left(\frac{1}{2} - u\right)\sqrt{t}\, \widehat{c}^j \nabla^{TZ\otimes\mathcal{E},sa}_{e_j} - \frac{\sqrt{t}}{4} \widehat{c}^j \psi_j + \left(\frac{1}{2} - u\right)\frac{\sqrt{t}}{2} c^j \psi_j \\ & + \nabla^{\mathcal{W},sa} + \left(\frac{1}{2} - u\right) E^\alpha \psi_\alpha + \left(\frac{1}{2} - u\right)\omega_{\alpha jk} E^\alpha c^j \widehat{c}^k \\ & + \left(\frac{1}{2} - u\right)\frac{1}{\sqrt{t}}\omega_{\alpha\beta j} E^\alpha E^\beta \widehat{c}^j + \frac{1}{2\sqrt{t}}\omega_{\alpha\beta j} E^\alpha E^\beta c^j, \end{aligned} \tag{6.26}$$

where

$$\nabla^{\mathcal{W},sa,1,0} = E^{\alpha}\left(\nabla^{TZ\otimes\mathcal{E},sa}_{e_\alpha} + \frac{1}{2}k_\alpha\right), \tag{6.27}$$
$$\nabla^{\mathcal{W},sa,0,1} = \partial^{\mathcal{E}}.$$

Proof. This follows from a computation using [2, Prop. 3.5, 3.7]. We omit the details. ☐

Let z be an odd Grassmann variable which anticommutes with all of the Grassmann variables previously introduced. Put

$$\mathcal{D}_j = \nabla^{TZ\otimes\mathcal{E},sa}_{e_j} - \frac{1}{2\sqrt{t}}\,\omega_{\alpha jk}E^{\alpha}c^{k} - \frac{1}{4t}\,\omega_{\alpha\beta j}E^{\alpha}E^{\beta} - \frac{1}{2\sqrt{t}}\,z\widehat{c}^{j}, \tag{6.28}$$
$$\mathcal{D}^2 = \mathcal{D}_j\mathcal{D}_j - \mathcal{D}_{\nabla^{TZ}_{e_j}e_j}.$$

Proposition 21. *The following Lichnerowicz-type formula holds :*

$$\begin{aligned} B^2_{4t}(u) + 2u(1-u)z\left(B'_{4t} - B''_{4t}\right) = 4u(1-u)\Bigg[&t\left(-\mathcal{D}^2 + \frac{R}{4}\right) \\ &+ \frac{t}{2}c^ic^j\mathcal{R}(e_i,e_j) + \sqrt{t}c^iE^{\alpha}\mathcal{R}(e_i,e_\alpha) + \frac{1}{2}E^{\alpha}E^{\beta}\mathcal{R}(e_\alpha,e_\beta) \\ &+ t\left(\frac{1}{4}\psi_j^2 + \frac{1}{8}\widehat{c}^j\widehat{c}^k[\psi_j,\psi_k] - \frac{1}{2}c^j\widehat{c}^k\left(\nabla^{TZ\otimes\mathcal{E},sa}_{e_j}\psi_k\right)\right) \\ &-\frac{\sqrt{t}}{2}E^{\alpha}\widehat{c}^j\left(\nabla^{TZ\otimes\mathcal{E},sa}_{e_\alpha}\psi_j\right) - \frac{z\sqrt{t}}{2}c^j\psi_j - \frac{z}{2}E^{\alpha}\psi_\alpha\Bigg] \\ &+ \sqrt{t}\,c^j\left(\nabla^{\mathcal{E},sa}_{e_j}\partial^{\mathcal{E}}\right) - 2\left(\frac{1}{2}-u\right)\sqrt{t}\,\widehat{c}^j\left(\nabla^{\mathcal{E},sa}_{e_j}\partial^{\mathcal{E}}\right) \\ &+ E^{\alpha}\left(\nabla^{\mathcal{E},sa}_{e_\alpha}\partial^{\mathcal{E}}\right) + \left(\frac{1}{2}-u\right)E^{\alpha}[\psi_\alpha,\partial^{\mathcal{E}}] - \frac{\sqrt{t}}{2}\,\widehat{c}^j[\psi_j,\partial^{\mathcal{E}}] \\ &+ \sqrt{t}\left(\frac{1}{2}-u\right)c^j[\psi_j,\partial^{\mathcal{E}}] + \left(\partial^{\mathcal{E}}\right)^2. \end{aligned} \tag{6.29}$$

Proof. Let us write $B_{4t}(u) = uB'^{,flat}_{4t} + (1-u)B''^{,flat}_{4t} + \partial^{\mathcal{E}}$. Then

$$\begin{aligned} B^2_{4t}(u) =& u(1-u)\left(B'^{,flat}_{4t}B''^{,flat}_{4t} + B''^{,flat}_{4t}B'^{,flat}_{4t}\right) \\ &+ u\left[B'^{,flat}_{4t},\partial^{\mathcal{E}}\right] + (1-u)\left[B''^{,flat}_{4t},\partial^{\mathcal{E}}\right] + \left(\partial^{\mathcal{E}}\right)^2. \end{aligned} \tag{6.30}$$

A formula for $\frac{1}{4}\left(B'^{,flat}_{4t}B''^{,flat}_{4t} + B''^{,flat}_{4t}B'^{,flat}_{4t}\right) + \frac{1}{2}z\left(B'^{,flat}_{4t} - B''^{,flat}_{4t}\right)$ was given in [2, Theorem 3.11]. The rest of (6.29) can be derived using Proposition 20. ☐

6.3. **Small Time Limits.** For $t > 0$ and $u \in (0,1)$, the restriction of $B^2_t(u)$ to a fiber Z_b is an element of $\Psi^2_{\mathfrak{B}}\left(Z_b; \Lambda(T^*Z_b)\otimes\mathcal{E}\big|_{Z_b}, \Lambda(T^*_bB)\otimes\Omega_*(\mathfrak{B})\otimes_{\mathfrak{B}}\left(\Lambda(T^*Z_b)\otimes\mathcal{E}\big|_{Z_b}\right)\right)$ with principal symbol $\sigma(z,\xi) = u(1-u)t|\xi|^2$. It follows that on Z_b,

$$e^{-B^2_t(u)} \in \Psi^{-\infty}_{\mathfrak{B}}\left(Z_b; \Lambda(T^*Z_b)\otimes\mathcal{E}\big|_{Z_b}, \Lambda(T^*_bB)\otimes\Omega_*(\mathfrak{B})\otimes_{\mathfrak{B}}\left(\Lambda(T^*Z_b)\otimes\mathcal{E}\big|_{Z_b}\right)\right). \tag{6.31}$$

Hence $e^{-B_t^2(u)}$ has a smooth kernel $e^{-B_t^2(u)}(z,z')$ and using the notion of TR_s from Subsection 2.3, we can define $\mathrm{TR}_s\left(e^{-B_t^2(u)}\right) \in \overline{\Omega}^{even}(B,\mathfrak{B})$.

Put $\nabla^{\mathcal{E}}(u) = u\nabla^{\mathcal{E}} + (1-u)\left(\nabla^{\mathcal{E}}\right)^*$.

Proposition 22. *For all* $u \in (0,1)$, *as* $t \to 0$,

$$\mathrm{TR}_s\left(e^{-B_t^2(u)}\right) = \begin{cases} \int_Z e\left(TZ, \nabla^{TZ}\right) \wedge \mathrm{ch}\left(\nabla^{\mathcal{E}}(u)\right) + O(t) & \text{if } n \text{ is even,} \\ O(\sqrt{t}) & \text{if } n \text{ is odd} \end{cases} \tag{6.32}$$

uniformly on B.

Proof. Consider a rescaling in which $\partial_j \to \epsilon^{-1/2}\partial_j$, $c^j \to \epsilon^{-1/2}E^j - \epsilon^{1/2}I^j$, $E^\alpha \to \epsilon^{-1/2}E^\alpha$, $\widehat{c}^j \to \widehat{c}^j$ and $\partial^{\mathcal{E}} \to \epsilon^{-1/2}\partial^{\mathcal{E}}$. One finds from (6.29) that as $\epsilon \to 0$, in adapted coordinates the rescaling of $\epsilon B_4^2(u)$ approaches

$$\begin{aligned} &-4u(1-u)\left(\partial_j - \frac{1}{4}R_{jk}^{TZ}x^k\right)^2 + 4u(1-u)\mathcal{R} \\ &+ E^j\left(\nabla_{e_j}^{\mathcal{E},sa}\partial^{\mathcal{E}}\right) + E^\alpha\left(\nabla_{e_\alpha}^{\mathcal{E},sa}\partial^{\mathcal{E}}\right) \\ &+ \left(\frac{1}{2}-u\right)E^j[\psi_j,\partial^{\mathcal{E}}] + \left(\frac{1}{2}-u\right)E^\alpha[\psi_\alpha,\partial^{\mathcal{E}}] + \left(\partial^{\mathcal{E}}\right)^2. \end{aligned} \tag{6.33}$$

Using local index methods as in the proof of [2, Theorem 3.15], one finds

$$\begin{aligned} \lim_{t\to 0}\mathrm{TR}_s\left(e^{-B_t^2(u)}\right) &= \int_Z (4u(1-u))^{-n/2}\,\mathrm{Pf}\left[\frac{4u(1-u)R^{TZ}}{2\pi}\right] \wedge \\ &\qquad \mathrm{Tr}_s e^{-\left[\left(\nabla^{\mathcal{E},flat,sa}\partial^{\mathcal{E}}\right) + \left(\frac{1}{2}-u\right)[\psi,\partial^{\mathcal{E}}] + \left(\partial^{\mathcal{E}}\right)^2 - u(1-u)\psi^2\right]} \\ &= \int_Z e\left(TZ,\nabla^{TZ}\right) \wedge \mathrm{Tr}_s e^{-\left[\left(\nabla^{\mathcal{E},flat,sa}\partial^{\mathcal{E}}\right) + \left(\frac{1}{2}-u\right)[\psi,\partial^{\mathcal{E}}] + \left(\partial^{\mathcal{E}}\right)^2 - u(1-u)\psi^2\right]}. \end{aligned} \tag{6.34}$$

On the other hand,

$$\nabla^{\mathcal{E}}(u) = \nabla^{\mathcal{E},flat,sa} + \left(\frac{1}{2}-u\right)\psi + \partial^{\mathcal{E}} \tag{6.35}$$

and so

$$\left(\nabla^{\mathcal{E}}(u)\right)^2 = \left(\nabla^{\mathcal{E},flat,sa}\partial^{\mathcal{E}}\right) + \left(\frac{1}{2}-u\right)[\psi,\partial^{\mathcal{E}}] + \left(\partial^{\mathcal{E}}\right)^2 - u(1-u)\psi^2. \tag{6.36}$$

This gives the $t \to 0$ limit of (6.32).

We have error estimates as in [2, Theorem 3.16], from which the proposition follows. □

Define $CS\left(B_t', h^{\mathcal{W}}\right) \in \overline{\Omega}^{\prime\prime,odd}(B,\mathfrak{B})$ and $\mathcal{T}(t) \in \overline{\Omega}^{\prime\prime,even}(B,\mathfrak{B})$ as in Proposition 14.

Proposition 23. *As* $t \to 0$,

$$CS\left(B_t', h^{\mathcal{W}}\right) = \begin{cases} \int_Z e\left(TZ,\nabla^{TZ}\right) \wedge CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right) + O(t) & \text{if } n \text{ is even,} \\ O(\sqrt{t}) & \text{if } n \text{ is odd} \end{cases} \tag{6.37}$$

uniformly on B.

Proof. Given $\alpha_1, \alpha_2 \in \overline{\Omega}^{\prime\prime,*}(B, \mathfrak{B})$, let us write

$$\partial_z (\alpha_1 + z\alpha_2) = \alpha_2. \tag{6.38}$$

Then

$$CS\left(B'_t, h^{\mathcal{W}}\right) = \frac{1}{2}\,\partial_z \int_0^1 \frac{1}{u(1-u)}\,\mathrm{Tr}_s e^{-\left[B_t^2(u)+2u(1-u)z(B'_t-B''_t)\right]} du. \tag{6.39}$$

Let us do a rescaling as in the proof of Proposition 22, with $z \rightarrow \epsilon^{-1/2} z$ in addition. One finds from (6.29) that as $\epsilon \rightarrow 0$, in adapted coordinates the rescaling of $\epsilon\left(B_4^2(u) + 2u(1-u)z(B'_4 - B''_4)\right)$ approaches

$$\begin{aligned} &- 4u(1-u)\left(\partial_j - \frac{1}{4} R^{TZ}_{jk} x^k\right)^2 + 4u(1-u)\mathcal{R} - 2u(1-u)z\psi \\ &+ E^j\left(\nabla^{\mathcal{E},sa}_{e_j} \partial^{\mathcal{E}}\right) + E^\alpha\left(\nabla^{\mathcal{E},sa}_{e_\alpha} \partial^{\mathcal{E}}\right) \\ &+ \left(\frac{1}{2} - u\right) E^j[\psi_j, \partial^{\mathcal{E}}] + \left(\frac{1}{2} - u\right) E^\alpha[\psi_\alpha, \partial^{\mathcal{E}}] + \left(\partial^{\mathcal{E}}\right)^2. \end{aligned} \tag{6.40}$$

Proceeding as in the proof of [2, Theorem 3.16], one obtains

$$\begin{aligned} \lim_{t\rightarrow 0} CS\left(B'_t, h^{\mathcal{W}}\right) &= \frac{1}{2}\,\partial_z \int_0^1 \frac{1}{u(1-u)} \int_Z e\left(TZ, \nabla^{TZ}\right) \wedge \mathrm{Tr}_s e^{-\left[\left(\nabla^{\mathcal{E}}(u)\right)^2 - 2u(1-u)z\psi\right]} du \\ &= \int_0^1 \int_Z e\left(TZ, \nabla^{TZ}\right) \wedge \mathrm{Tr}_s\left[\psi\, e^{-\left(\nabla^{\mathcal{E}}(u)\right)^2}\right] du \\ &= \int_Z e\left(TZ, \nabla^{TZ}\right) \wedge CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right). \end{aligned} \tag{6.41}$$

Although we are integrating over u, there is no problem with the $t \rightarrow 0$ limit as the effective time parameter is $u(1-u)t$, which only improves the convergence. □

Proposition 24. *As* $t \rightarrow 0$,

$$\mathcal{T}(t) = \begin{cases} O(1) & \text{if } n \text{ is even,} \\ O(t^{-1/2}) & \text{if } n \text{ is odd} \end{cases} \tag{6.42}$$

uniformly on B.

Proof. Using the method of proof of [2, Theorem 3.21], one finds

$$\mathcal{T}(t) = \begin{cases} -\frac{n}{2}\frac{1}{t}\int_0^1 \mathrm{Tr}_s\left(e^{-B_t^2(u)}\right) du + O(1) & \text{if } n \text{ is even,} \\ O(t^{-1/2}) & \text{if } n \text{ is odd.} \end{cases} \tag{6.43}$$

By Proposition 22, if n is even then

$$\lim_{t\rightarrow 0} \int_0^1 \mathrm{Tr}_s\left(e^{-B_t^2(u)}\right) du = \int_Z e\left(TZ, \nabla^{TZ}\right) \wedge \int_0^1 \mathrm{ch}\left(\nabla^{\mathcal{E}}(u)\right) du \in \overline{\Omega}^{\prime,even}(B, \mathfrak{B}). \tag{6.44}$$

(Again, as the effective time parameter is $u(1-u)t$, there is no problem in switching the $t \rightarrow 0$ limit and the u-integration.) As we quotient by $\overline{\Omega}^{\prime,even}(B, \mathfrak{B})$ in defining $\overline{\Omega}^{\prime\prime,even}(B, \mathfrak{B})$, the proposition follows. □

6.4. **Index Theorems.** We continue with the setup of Subsection 6.2. For each $b \in B$, let $H\left(Z_b; \mathcal{E}\big|_{Z_b}\right)$ denote the cohomology of the complex $(\mathcal{W}_b, d^Z)$. Put $\triangle_b = d^Z (d^Z)^* + (d^Z)^* d^Z \in \Psi^2_{\mathfrak{B}}\left(Z_b; \Lambda(T^*Z_b) \otimes \mathcal{E}\big|_{Z_b}, \Lambda(T^*Z_b) \otimes \mathcal{E}\big|_{Z_b}\right)$. Put $\overline{\mathcal{E}} = \Lambda \otimes_{\mathfrak{B}} \mathcal{E}$ and $\overline{\mathcal{W}}_b = \Omega\left(Z_b; \overline{\mathcal{E}}\big|_{Z_b}\right)$. Let $\overline{\triangle}_b \in \Psi^2_{\Lambda}\left(Z_b; \Lambda(T^*Z_b) \otimes \overline{\mathcal{E}}\big|_{Z_b}, \Lambda(T^*Z_b) \otimes \overline{\mathcal{E}}\big|_{Z_b}\right)$ be the corresponding Laplacian in the Λ-pseudodifferential operator calculus.

Hypothesis 4. *For each $b \in B$, the operator $\overline{d^Z} \in \mathrm{End}_{\Lambda}\left(\overline{\mathcal{W}}_b\right)$ has closed image.*

Proposition 25. *Hypothesis 4 is satisfied if and only if 0 is isolated in $\sigma(\overline{\triangle}_b)$.*

Proof. This follows from standard arguments. We omit the details. □

Hereafter, we assume that Hypothesis 4 is satisfied.

Proposition 26. *For each $b \in B$, $\mathrm{H}\left(Z_b; \mathcal{E}\big|_{Z_b}\right)$ is a finitely-generated projective $\mathcal{B}$-module. The $\{\mathrm{H}\left(Z_b; \mathcal{E}\big|_{Z_b}\right)\}_{b\in B}$ fit together to form a $\mathbb{Z}$-graded $\mathfrak{B}$-vector bundle $H\left(Z; \mathcal{E}\big|_Z\right)$ on B with a flat structure.*

Proof. The proof is similar to the proofs of Propositions 10 and 11, with Proposition 19 replacing Lemma 1. □

There is an induced Hermitian metric $h^{H(Z;\mathcal{E}|_Z)}$ on $H\left(Z; \mathcal{E}\big|_Z\right)$ and an induced partially flat connection $\nabla^{H(Z;\mathcal{E}|_Z)}$ as in (5.30).

Proposition 27. *For all $u \in (0,1)$, as $t \to \infty$,*

$$\mathrm{ch}\,(B_t(u)) = \mathrm{ch}\left(\nabla^{H(Z;\mathcal{E}|_Z)}(u)\right) + O(t^{-1/2}) \tag{6.45}$$

uniformly on B. Also,

$$CS\,(B'_t, h^{\mathcal{W}}) = CS\left(\nabla^{H(Z;\mathcal{E}|_Z)}, h^{H(Z;\mathcal{E}|_Z)}\right) + O(t^{-1/2}) \tag{6.46}$$

uniformly on B.

Proof. Let $\lambda_0 > 0$ be the infimum of the nonzero spectrum of $\overline{\triangle}$. For $r > 0$, put $\alpha_r = P^{Im(\triangle)} e^{-r\triangle} P^{Im(\triangle)}$. By Proposition 19 and Corollary 3, for each $j \geq 0$ there is a constant $C_j > 0$ such that for all $r > 1$,

$$\parallel \alpha_r \parallel_j \leq C_j\, e^{-r\lambda_0/2}. \tag{6.47}$$

The proof of the proposition is now formally the same as that of Proposition 15, with (6.47) replacing (5.60). □

Proposition 28. *We have*

$$\left[\mathrm{ch}\left(\nabla^{H(Z;\mathcal{E}|_Z)}\right)\right] = \int_Z e(TZ) \wedge \left[\mathrm{ch}\left(\nabla^{\mathcal{E}}\right)\right] \text{ in } H^{l,even}_{\mathfrak{B}}(B) \tag{6.48}$$

and

$$\left[CS\left(\nabla^{H(Z;\mathcal{E}|_Z)}, h^{H(Z;\mathcal{E}|_Z)}\right)\right] = \int_Z e(TZ) \wedge \left[CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right)\right] \text{ in } \bigoplus_{\substack{p>q \\ p+q \text{ odd}}} \mathrm{H}^p(B; \overline{H}_q(\mathfrak{B})). \tag{6.49}$$

Proof. As in the finite-dimensional setting, one can verify that $[\mathrm{ch}\,(B_t(u))]$ and $\left[CS\left(B_t', h^{\mathcal{W}}\right)\right]$ are independent of t. For all $u \in (0,1)$, $\left[\mathrm{ch}\left(\nabla^{H(Z;\mathcal{E}|_Z)}(u)\right)\right] = \left[\mathrm{ch}\left(\nabla^{H(Z;\mathcal{E}|_Z)}\right)\right]$ and $\left[\mathrm{ch}\left(\nabla^{\mathcal{E}}(u)\right)\right] = \left[\mathrm{ch}\left(\nabla^{\mathcal{E}}\right)\right]$. The proposition now follows from Propositions 22, 23 and 27. □

Remark 8 : Proposition 28 is also a consequence of the topological index theorem of [12]. Namely, Proposition 26 ensures that we can apply (0-3) of their paper, as given in (1.4) of the present paper. Proposition 28 follows from (1.4) by applying ch and CS.

6.5. **The Analytic Torsion Form II.** We continue with the assumptions of Subsection 6.4. Let N be the number operator on $\mathcal{W}$. For $t > 0$, define $\mathcal{T}(t)$ as in (5.37).

Proposition 29. *As* $t \to \infty$,

$$\mathcal{T}(t) = O(t^{-3/2}) \tag{6.50}$$

uniformly on B.

Proof. The proof is formally the same as that of Proposition 16. We omit the details. □

Again, we have

$$\partial_t CS\left(B_t', h^{\mathcal{W}}\right) = -\, d\mathcal{T}(t). \tag{6.51}$$

Definition 17. *The analytic torsion form* $\mathcal{T} \in \overline{\Omega}^{\prime\prime,even}(M, \mathfrak{B})$ *is given by*

$$\mathcal{T} = \int_0^\infty \mathcal{T}(t)\, dt. \tag{6.52}$$

By Propositions 24 and 29, the integral in (6.52) makes sense.

Proposition 30. *We have*

$$d\mathcal{T} = \int_Z e\left(TZ, \nabla^{TZ}\right) \wedge CS\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right) - CS\left(\nabla^{H(Z;\mathcal{E}|_Z)}, h^{H(Z;\mathcal{E}|_Z)}\right) \text{ in } \overline{\Omega}^{\prime\prime,odd}(B, \mathfrak{B}). \tag{6.53}$$

Proof. This follows from Proposition 23, Proposition 27 and (6.51). □

Corollary 4. *If* Z *is odd-dimensional and* $H(Z; \mathcal{E}|_Z) = 0$ *then* $\mathcal{T}$ *is closed and so represents a class* $[\mathcal{T}] \in H^{\prime\prime,even}(B, \mathfrak{B})$.

Proof. If Z is odd-dimensional then $e\left(TZ, \nabla^{TZ}\right) = 0$. The corollary now follows from Proposition 30. □

Proposition 31. *If* Z *is odd-dimensional and* $H(Z; \mathcal{E}|_Z) = 0$ *then* $[\mathcal{T}] \in H^{\prime\prime,even}(B, \mathfrak{B})$ *is independent of* g^{TZ}, $T^H M$, $h^{\mathcal{E}}$ *and* $\partial^{\mathcal{E}}$. *Thus it only depends on the (smooth) topological fiber bundle* $Z \to M \to B$ *and the flat structure on* $\mathcal{E}$.

Proof. Put $\mathcal{F} = \{g^{TZ}, T^H M, h^{\mathcal{E}}, \partial^{\mathcal{E}}\}$ and let $\mathcal{F}'$ be another choice of such data. We can find a smooth 1-parameter family $\{\mathcal{F}(\epsilon)\}_{\epsilon\in\mathbb{R}}$ such that $\mathcal{F}(0) = \mathcal{F}$ and $\mathcal{F}(1) = \mathcal{F}'$. Put $\widetilde{Z} = Z$, $\widetilde{M} = \mathbb{R} \times M$ and $\widetilde{B} = \mathbb{R} \times B$. Let $p : \widetilde{M} \to M$ be projection onto the second factor and put $\widetilde{\mathcal{E}} = p^*\mathcal{E}$. Then the family $\{\mathcal{F}(\epsilon)\}_{\epsilon\in\mathbb{R}}$ provides the data $g^{T\widetilde{Z}}$, $T^H\widetilde{M}$, $h^{\widetilde{\mathcal{E}}}$ and $\partial^{\widetilde{\mathcal{E}}}$ on the fiber bundle $\widetilde{Z} \to \widetilde{M} \xrightarrow{\widetilde{\pi}} \widetilde{B}$. Let $\widetilde{d} = d\epsilon\, \partial_\epsilon + d$ denote the differential on $\overline{\Omega}^{\prime\prime,*}(\widetilde{B}, \mathfrak{B})$. By the preceding constructions, there is an analytic torsion form on $\widetilde{B}$ which we can write as $\widetilde{\mathcal{T}} = \mathcal{T}(\epsilon) + d\epsilon \wedge \mathcal{T}'(\epsilon)$, satisfying

$$\widetilde{d}\widetilde{\mathcal{T}} = \int_{\widetilde{Z}} e\left(T\widetilde{Z}, \nabla^{T\widetilde{Z}}\right) \wedge CS\left(\nabla^{\widetilde{\mathcal{E}}}, h^{\widetilde{\mathcal{E}}}\right) - CS\left(\nabla^{H(\widetilde{Z};\widetilde{\mathcal{E}}|_{\widetilde{Z}})}, h^{H(\widetilde{Z};\widetilde{\mathcal{E}}|_{\widetilde{Z}})}\right). \tag{6.54}$$

By our assumptions, the right-hand-side of (6.54) vanishes. Thus $\partial_\epsilon \mathcal{T}(\epsilon) = d\mathcal{T}'(\epsilon)$, from which the proposition follows. □

Proposition 32. *Suppose that*
1. *Z is even-dimensional*
2. *TZ is oriented*
3. *$\nabla^{\mathcal{E}}$ is self-adjoint with respect to $h^{\mathcal{E}}$.*

Then $\mathcal{T} = 0$.

Proof. This follows from an argument using Hodge duality, as in [2, Theorem 3.26]. We omit the details. □

Let us look more closedly at $\mathcal{T}_{[0]}$, the component of $\mathcal{T}$ in $\overline{\Omega}^{\prime\prime,0}(B, \mathfrak{B})$. Assume for simplicity that B is connected. Then

$$\overline{\Omega}^{\prime\prime,0}(B, \mathfrak{B}) = (C^\infty(B)/\mathbb{C}) \otimes \left(\mathfrak{B}/\overline{[\mathfrak{B}, \mathfrak{B}]}\right). \tag{6.55}$$

As in (5.82),

$$\mathcal{T}_{[0]} \equiv -\int_0^\infty \int_0^1 \mathrm{Tr}_s \left(N\left(1 - 2tu(1-u)\triangle\right) e^{-tu(1-u)\triangle}\right) du \frac{dt}{t}. \tag{6.56}$$

Define g as in (5.83). Then a specific lifting of $\mathcal{T}_{[0]}$ to $C^\infty(B) \otimes \left(\mathfrak{B}/\overline{[\mathfrak{B}, \mathfrak{B}]}\right)$ is given by

$$\mathcal{T}_{[0]} = \int_0^\infty \left[\mathrm{Tr}_s\left(N g\left(t\triangle\right)\right) - \left(\frac{n}{2}\chi(Z)\,\mathrm{Tr}_s\left(I\big|_{\mathcal{E}}\right) - \mathrm{Tr}_s\left(N\big|_{H(Z;\mathcal{E}|_Z)}\right)\right) g(t) + \mathrm{Tr}_s\left(N\big|_{H(Z;\mathcal{E}|_Z)}\right)\right] \frac{dt}{t}. \tag{6.57}$$

Example 10 : If $\mathfrak{B} = \mathbb{C}$ then as in [2, Theorem 3.29], $\mathcal{T}_{[0]}$ is the usual Ray-Singer analytic torsion [32], considered to be a function on B.

Example 11 : Suppose that Γ is a finite group and $\mathfrak{B} = \mathbb{C}\Gamma$. Then $\mathcal{T}_{[0]}$ is equivalent to the equivariant analytic torsion of [25].

Example 12 : Suppose that Γ is a discrete group and $\mathfrak{B} = C_r^*\Gamma$. Let τ be the trace on $\mathfrak{B}$ given by $\tau(\sum_{\gamma\in\Gamma} c_\gamma\gamma) = c_e$. Then $\tau\left(\mathcal{T}_{[0]}\right)$ is the L^2-analytic torsion of [22, 27]. (In the cited papers, the L^2-torsion is defined using the group von Neumann algebra and without the assumption of a gap in the spectrum of Δ, but with the assumption of positive Novikov-Shubin invariants.)

7. Diffeomorphism Groups

Let Z be a connected closed manifold and let $\mathrm{Diff}(Z)$ be its diffeomorphism group, endowed with the natural smooth topology [28]. For $i > 1$, let $\alpha : (S^i, *) \to (\mathrm{Diff}(Z), \mathrm{Id})$ be a smooth map. We want to find invariants of $[\alpha] \in \pi_i(\mathrm{Diff}(Z))$. Put $M_1 = M_2 = D^{i+1} \times Z$. Glue M_1 and M_2 along their common boundary $S^i \times Z$ by identifying $(\theta, z) \in \partial M_1$ with $(\theta, (\alpha(\theta))(z)) \in \partial M_2$. Let $M = M_1 \cup_{S^i \times Z} M_2$ be the resulting manifold. Then M is the total space of a fiber bundle with base $B = S^{i+1}$ and fiber Z. Any (smooth) topological invariant of such fiber bundles gives an invariant of $[\alpha]$.

As mentioned in the introduction, we are interested in the case when Z is a $K(\Gamma, 1)$-manifold. Then $\pi_1(M) = \Gamma$. Suppose that Γ satisfies Hypothesis 1 of the introduction. There is a $\mathfrak{B}$-vector bundle $\mathcal{E} = \mathfrak{B} \times_\Gamma \widetilde{M}$ on M. Choosing $h \in C_0^\infty(\widetilde{M})$ satisfying (3.11), Proposition 2 gives a partially flat connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$. Let us add vertical Riemannian metrics g^{TZ} and a horizontal distribution $T^H M$ to the fiber bundle.

We would like to use the formalism of Section 6 to define the analytic torsion form. By Proposition 6, $\nabla^{\mathcal{E}}$ is self-adjoint and so Proposition 32 implies that the torsion form vanishes if $\dim(Z)$ is even. (As the analysis is effectively done on the universal cover $\widetilde{Z}$, the orientation assumption on TZ is irrelevant.) So assume that $\dim(Z)$ is odd. Let $\tau \in Z^q(\Gamma; \mathbb{C})$ be a group cocycle and let $Z_\tau \in ZC^q(\mathfrak{B})$ be the cyclic cocycle coming from (3.4) (with $x = e$). We want to use Proposition 31 to define the analytic torsion class

$$[\mathcal{T}] \in H^{\prime\prime,even}_{\mathfrak{B}}(B) = \bigoplus_{\substack{p+q \text{ even} \\ p>q}} \mathrm{H}^p(B; \overline{H}_q(\mathfrak{B})),$$

take its integral over B to get

$$\int_B [\mathcal{T}] \in \bigoplus_{\substack{q \equiv i+1 \mod 2 \\ q < i+1}} \overline{H}_q(\mathfrak{B})$$

and pair the result with Z_τ.

In order to satisfy the hypotheses of Proposition 31, we need to know that $H(Z; \mathcal{E}|_Z) = 0$ and that Hypothesis 4 is satisfied. Equivalently, we need to know that the p-form Laplacian on $\widetilde{Z}_b$ is invertible for all $0 \leq p \leq \dim(Z)$. This is a topological condition on Z, but it seems likely that it is never satisfied [23]. To understand the nature of the problem, let us look in detail at the case when Γ is a free abelian group.

7.1. Free Abelian Fundamental Groups.

Suppose that $\Gamma = \mathbb{Z}^n$. Then $Z = T^n$. Let $B\Gamma$ and $\widehat{\Gamma}$ denote the classifying space of Γ and the Pontryagin dual of Γ, respectively. They are again n-tori, but it will be convenient to distinguish them from Z.

Under Fourier transform, $C_r^*\Gamma \cong C(\widehat{\Gamma})$. Take $\mathfrak{B} = C^\infty(\widehat{\Gamma})$. Instead of using the universal GDA of $\mathfrak{B}$, we will simplify things and use the GDA of smooth differential forms on $\widehat{\Gamma}$. This allows us to use ordinary "commutative" analysis. All of the relevant steps of the paper go through with this replacement. We now summarize the statements.

First, there is a natural Hermitian line bundle H with Hermitian connection ∇^H on $B\Gamma \times \widehat{\Gamma}$ [21, Section 3.1.1]. (The third line of that section should read $\mathrm{H}_1(M;\mathbb{Z})_{mod\,Tor} \subset \mathrm{H}_1(M;\mathbb{R})$.) For all $\theta \in \widehat{\Gamma}$, the restriction of ∇^H to $B\Gamma \times \{\theta\}$ is the flat connection on $B\Gamma$ with holonomy specified by θ.

We assume that we are given a fiber bundle $Z \to M \to B$ as above, endowed with a vertical Riemannian metric and a horizontal distribution. Consider the fiber bundle $Z \to M \times \widehat{\Gamma} \to B \times \widehat{\Gamma}$. It inherits a vertical Riemannian metric and a horizontal distribution. Let $f : M \to B\Gamma$ be a classifying map for the universal cover $\widetilde{M} \to M$. Put $E_0 = (f \times \mathrm{Id})^* H$, a Hermitian line bundle on $M \times \widehat{\Gamma}$. The pulled-back connection ∇^{E_0} is partially flat.

Let $\triangle$ be the vertical Laplacian of the fiber bundle $Z \to M \times \widehat{\Gamma} \to B \times \widehat{\Gamma}$, acting on $\Omega\left(Z; E_0\big|_Z\right)$. Then $\triangle$ is invertible except on the fibers over $B \times \{1\} \subset B \times \widehat{\Gamma}$. This lack of invertibility on $B \times \{1\}$ is responsible for the fact that Hypothesis 4 is not satisfied. The effect is that the analytic torsion form may be singular on $B \times \widehat{\Gamma}$, with singularity along $B \times \{1\}$.

In order to get around this problem, one approach is to just remove the singular subspace from consideration. Let $U \subset \widehat{\Gamma}$ be a small neighborhood of $1 \in \widehat{\Gamma}$. Consider the restriction of the fiber bundle to $B \times (\widehat{\Gamma} - U)$. Then the vertical Laplacian is invertible and we can define the analytic torsion class

$$[\mathcal{T}] \in \bigoplus_{\substack{p+q\ even \\ p>q}} \mathrm{H}^p(B;\mathbb{C}) \otimes \mathrm{H}^q(\widehat{\Gamma} - U;\mathbb{C}).$$

Now $\mathrm{H}^q(\widehat{\Gamma} - U;\mathbb{C}) \cong \mathrm{H}_q(\Gamma;\mathbb{C})$ if $0 \le q < n$, and $\mathrm{H}^n(\widehat{\Gamma} - U;\mathbb{C}) = 0$. Thus there is a (smooth) topological invariant of the fiber bundle given by

$$\int_B [\mathcal{T}] \in \bigoplus_{\substack{q \equiv i+1 \mod 2 \\ q < \min(i+1,n)}} \mathrm{H}_q(\Gamma;\mathbb{C}). \tag{7.1}$$

In fact, an argument involving complex conjugation shows that the component of $\int_B [\mathcal{T}]$ in $\mathrm{H}_q(\Gamma;\mathbb{C})$ vanishes unless $q \equiv i+1 \mod 4$.

Comparing (7.1) with (1.1) (in the case $\Gamma = \mathbb{Z}^n$) shows that $\int_B [\mathcal{T}]$ potentially detects all of $\pi_*(\mathrm{Diff}(T^n)) \otimes_{\mathbb{Z}} \mathbb{C}$ in the stable range. By removing U from $\widehat{\Gamma}$, we have lost the component of $\int_B [\mathcal{T}]$ in $\mathrm{H}_n(\Gamma;\mathbb{C})$ if $i+1 > n$, but this lies outside of the stable range, anyway.

Remark 9 : Although Hypothesis 4 is not satisfied for the bundle $Z \to M \to B$, we have seen that it is nevertheless possible to extract most of the information in $[\mathcal{T}]$, due to the fact that $\triangle$ is noninvertible only on a high-codimension subset of $B \times \widehat{\Gamma}$. Although $[\mathcal{T}]$ is possibly singular on $B \times \{1\}$, it may be that its singularity is sufficiently mild to still define the component of $\int_B [\mathcal{T}]$ in $\mathrm{H}_n(\Gamma;\mathbb{C})$. We have not looked at this point in detail.

In summary, when $\Gamma = \mathbb{Z}^n$ then a certain part of the analytic torsion form is well-defined directly. We do not know what the situation is for the analytic torsion form in the case of general Γ. In the next subsection we will make use of the homotopic triviality of the fiber bundle in order to define a relative analytic torsion class for general Γ.

7.2. **General Fundamental Groups.** Let $Z \to M \xrightarrow{\pi} B$ and $\mathcal{E}$ be as described at the beginning of Section 7. We again assume that Γ satisfies Hypothesis 1, that Z is a $K(\Gamma,1)$-manifold and that $\dim(Z)$ is odd. Let $M' = Z \times B$ be the product bundle over B. Let $\mathcal{E}'$ be the corresponding $\mathfrak{B}$-vector bundle on M'. From homotopy theory, we know that M and M' are fiber-homotopy equivalent by some smooth map $f : M \to M'$. Furthermore, f is unique up to homotopy. It induces an isomorphism between the local systems $\mathcal{E}$ and $\mathcal{E}'$. We will show that the problems with invertibility of the Laplacian cancel out when we consider M relative to M'.

Consider the restriction of f to a single fiber Z_b. It acts by pullback on differential forms. However, this need not be a bounded, or even closable, operator. To get around this problem, we use the trick of [15], which involves modifying f to make it a submersion.

Let $i : Z \to \mathbb{R}^N$ be an embedding of Z in Euclidean space. For $\epsilon > 0$ sufficiently small, let U be an ϵ-tubular neighborhood of $i(Z)$, with projection $P : U \to Z$. Let $p_1 : Z \times B \to Z$ be projection on the first factor. Let B^N denote the unit ball in $\mathbb{R}^N$. Consider the fiber bundle $\mathrm{B}^N \times Z \to \mathrm{B}^N \times M \to B$. Define $F : \mathrm{B}^N \times M \to M'$ by

$$F(\vec{x}, m) = \left(P\left(\frac{\epsilon}{2}\,\vec{x} + (i \circ p_1 \circ f)(m)\right), \pi(m)\right). \tag{7.2}$$

Then F is a fiber-homotopy equivalence which is a fiberwise submersion. Choose $\nu \in \Omega^N(\mathrm{B}^N)$ with support near $0 \in \mathrm{B}^N$ and total integral 1. Define $\mathcal{W}$ as in Subsection 6.2 and let $\widetilde{\mathcal{W}}$ be the analogous object for the fiber bundle M'. Define a cochain map $T : \widetilde{\mathcal{W}} \to \mathcal{W}$ by

$$T(s') = \int_{\mathrm{B}^N} \nu \wedge F^* s', \tag{7.3}$$

where F^* acts fiberwise. Then T is bounded.

Put $\widehat{\mathcal{W}} = \mathcal{W} \oplus \widetilde{\mathcal{W}}$, with the $\mathbb{Z}$-grading $\widehat{\mathcal{W}^i} = \mathcal{W}^i \oplus (\widetilde{\mathcal{W}'})^{i+1}$. Let A' be the superconnection on $\mathcal{W}$ defined in Subsection 6.2 and let $\widetilde{A}'$ be the analogous superconnection on $\widetilde{\mathcal{W}}$. For $r \in \mathbb{R}$, define a superconnection $\widehat{A}'_r$ on $\widehat{\mathcal{W}}$ by

$$\widehat{A}'_r = \begin{pmatrix} A' & rT(-1)^N \\ 0 & \widetilde{A}' \end{pmatrix}. \tag{7.4}$$

The cochain part $\widehat{A}'_{r,0,0,1}$ of $\widehat{A}'_r$ is

$$\widehat{d}^Z_r = \begin{pmatrix} d^Z & rT(-1)^N \\ 0 & \widetilde{d}^Z \end{pmatrix}. \tag{7.5}$$

Proposition 33. *The superconnection $\widehat{A}'_r$ is partially flat on $\widehat{\mathcal{W}}$.*

Proof. It is enough to show that $A'^{,flat}T(-1)^N + T(-1)^N \widetilde{A}'^{,flat} = 0$. Now $A'^{,flat}$ acts on $\Omega(B;\mathcal{W}) = \Omega(M;\mathcal{E})$ by exterior differentiation d^M, and similarly for $\widetilde{A}'^{,flat}$. Taking into account that T is an odd variable, in ungraded language we must show that $d^M T = T d^{M'}$. As T acts on $\Omega(M';\mathcal{E}')$ by

$$T(\omega') = \int_{\mathrm{B}^N} \nu \wedge F^* \omega', \tag{7.6}$$

the proposition follows. □

As T is a cochain homotopy equivalence, if $r \neq 0$ then $H(\widehat{\mathcal{W}}, \widehat{d}_r^Z) = 0$, while if $r = 0$ then $H^*(\widehat{\mathcal{W}}, \widehat{d}_r^Z) = H^*(Z; \mathcal{E}_Z) \oplus H^{*+1}(Z; \mathcal{E}_Z)$.

We now want to define an analytic torsion form $\mathcal{T}$ using the superconnection $\widehat{A}'_r$. If t is large, we want r to be nonzero, in order to get the gap in the spectrum necessary for large-t convergence of the integral for $\mathcal{T}$. If t is small, we want r to be zero, in order to use the small-time estimates separately on M and M'.

Choose a $\phi \in C_0^\infty([0,\infty))$ which is identically one near $t = 0$. Put $h(t) = \sqrt{t}\,(1 - \phi(t))$ and

$$\widehat{B}'_t = t^{N/2} \widehat{A}'_{h(t)} t^{-N/2}. \tag{7.7}$$

The cochain part $\widehat{B}'_{t,0,0,1}$ of $\widehat{B}'_t$ is

$$\widehat{B}'_{t,0,0,1} = \sqrt{t} \begin{pmatrix} d^Z & (1-\phi(t))T(-1)^N \\ 0 & \widetilde{d}^Z \end{pmatrix}. \tag{7.8}$$

Let $\widehat{B}''_t$ be the adjoint of $\widehat{B}'_t$. For $u \in (0,1)$, put $\widehat{B}_t(u) = u\widehat{B}'_t + (1-u)\widehat{B}''_t$.

Definition 18. *For $t > 0$, define $CS(t) \in \overline{\Omega}^{\prime\prime,odd}(B, \mathfrak{B})$ and $\mathcal{T}(t) \in \overline{\Omega}^{\prime\prime,even}(B, \mathfrak{B})$ by*

$$CS(t) = -\int_0^1 \mathrm{Tr}_s \left(\left(\widehat{B}'_t - \widehat{B}''_t \right) e^{-\widehat{B}_t^2(u)} \right) du \tag{7.9}$$

and

$$\begin{aligned} \mathcal{T}(t) = & -\frac{1}{t} \left(\int_0^1 \mathrm{Tr}_s \left(N e^{-\widehat{B}_t^2(u)} \right) du + \int_0^1 \int_0^1 u(1-u) \right. \\ & \left. \mathrm{Tr}_s \left(N \left[\widehat{B}'_t - \widehat{B}''_t, e^{-r\widehat{B}_t^2(u)} \left(\widehat{B}'_t - \widehat{B}''_t \right) e^{-(1-r)\widehat{B}_t^2(u)} \right] \right) dr du \right) \\ & + h'(t) \int_0^1 \int_0^1 \mathrm{Tr}_s \left(\begin{pmatrix} 0 & uT(-1)^N \\ (1-u)(-1)^N T^* & 0 \end{pmatrix} \right. \\ & \left. e^{-r\widehat{B}_t^2(u)} \left(\widehat{B}'_t - \widehat{B}''_t \right) e^{-(1-r)\widehat{B}_t^2(u)} \right) dr du. \end{aligned} \tag{7.10}$$

Proposition 34. *We have $\partial_t CS(t) = -d\mathcal{T}(t)$.*

Proof. There is a partially flat superconnection on $\mathbb{R}^+ \times B$ given by $d\epsilon\,\partial_\epsilon + \widehat{A}'_{h(\epsilon)}$. One can then proceed as in the proofs of Propositions 9 and 14. We omit the details. □

Definition 19. *Define $\mathcal{T} \in \overline{\Omega}^{\prime\prime,even}(B, \mathfrak{B})$ by*

$$\mathcal{T} = \int_0^\infty \mathcal{T}(t) dt. \tag{7.11}$$

The integrand in (7.11) is integrable. For small t, this follows from the fact that $1 - \phi$ vanishes identically near $t = 0$, so one effectively has the difference of the torsion integrands of M and M'. The large-t convergence comes from the fact that T is a cochain homotopy equivalence, which implies that the Laplacian

$$\begin{pmatrix} d^Z & T(-1)^N \\ 0 & \widetilde{d}^Z \end{pmatrix}^* \begin{pmatrix} d^Z & T(-1)^N \\ 0 & \widetilde{d}^Z \end{pmatrix} + \begin{pmatrix} d^Z & T(-1)^N \\ 0 & \widetilde{d}^Z \end{pmatrix} \begin{pmatrix} d^Z & T(-1)^N \\ 0 & \widetilde{d}^Z \end{pmatrix}^*$$

is invertible [24, Lemma 2.5].

Proposition 35. *The form $\mathcal{T}$ is closed. Its class $[\mathcal{T}] \in \overline{H}^{\prime\prime,even}(B, \mathfrak{B})$ only depends on $[\alpha] \in \pi_i(\mathrm{Diff}(Z))$.*

Proof. This follows as in Corollary 4 and Proposition 31. The only point to check is that any two choices of the auxiliary data are connected by a smooth 1-parameter family. This is obvious except, perhaps, for the choice of embedding $i : Z \to \mathbb{R}^N$. If $i' : Z \to \mathbb{R}^{N'}$ is another choice, we can find some N'' and isometric embeddings $I : \mathbb{R}^N \to \mathbb{R}^{N''}$, $I' : \mathbb{R}^{N'} \to \mathbb{R}^{N''}$ such that $I \circ i$ and $I' \circ i'$ are connected by a smooth 1-parameter family of embeddings. □

So we have an invariant

$$\int_B [\mathcal{T}] \in \bigoplus_{\substack{q \equiv i+1 \mod 2 \\ q < i+1}} \overline{H}_q(\mathfrak{B}). \tag{7.12}$$

Because of the underlying real structures of the vector bundles involved, one can show that the $\overline{H}_q(\mathfrak{B})$-component of $\int_B [\mathcal{T}]$ vanishes unless $q \equiv i+1 \mod 4$. By Hypothesis 1, for each $[\tau] \in \mathrm{H}^q(\Gamma; \mathbb{C})$, there is a representative $\tau \in Z^q(\Gamma; \mathbb{C})$ such that $Z_\tau \in HC^q(\mathbb{C}\Gamma)$ extends to a continuous cyclic cocycle on $\mathfrak{B}$. Then the pairing $\langle Z_\tau, \int_B [\mathcal{T}] \rangle \in \mathbb{C}$ is a numerical invariant of $[\alpha] \in \pi_i(\mathrm{Diff}(Z))$.

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