# Some Global Correspondences for General Linear Groups and their Metaplectic Coverings

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**Abstract**

Let $\mathbf{A}$ be the adele ring of a number field containing the $n$th roots of unity, and $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ be an $n$-fold metaplectic covering of $\mathrm{GL}(r, \mathbf{A})$. Under an assumption on $n$, we prove identities between all of the terms in Arthur's invariant trace formulas for $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ and $\mathrm{GL}(r, \mathbf{A})$. We then establish a correspondence between the automorphic representations of these groups.

## Contents

# 1 Introduction

Correspondences of metaplectic covering groups have their origin in the work of Shimura ([Shi73]). Shimura constructed a correspondence between modular forms of half-integral weight and cusp forms of even weight, which preserves L-functions. He suggested that this correspondence be studied further by using the representation-theoretic techniques developed by Jacquet and Langlands ([JL70]). This approach was explored by Flicker ([Fli80]), who gave a complete description of a correspondence between the automorphic representations of the $n$-fold covering of GL(2) and the automorphic representations of GL(2). Flicker's correspondence was proved using the Selberg trace formula and followed Langlands' proof of base change for GL(2). [Lan80]. Trace formula methods were also exploited by Flicker, Kazhdan and

Patterson ([FK86], [KP86]) in the proof of some additional correspondences between the automorphic representations of $n$-fold metaplectic coverings of $\mathrm{GL}(r)$, $r \geq 2$, and automorphic representations of $\mathrm{GL}(r)$. We prove correspondences of automorphic representations for the same groups under some assumptions on $n$, the order of the covering. Our approach is novel in that we use the invariant trace formula of Arthur ([Art88b]) and follow Arthur and Clozel's proof of base change for $\mathrm{GL}(r)$ ([AC89]). We refer the reader to the introductions of [Fli80], [KP84], [KP86] and [FK86] for the ramifications of metaplectic correspondences.

In §§2-4 we recapitulate and expand upon the local results of [FK86]. For a number field $F$ containing the $n$th roots of unity, we define the orbit map,

$$\gamma \mapsto \gamma',$$

which is a map from the general linear group over a local completion $F_v$ to its $n$-fold metaplectic covering $\widetilde{\mathrm{GL}}(r, F_v)$. This map preserves conjugacy classes and is the foundation of all our correspondences. There is a dual map

$$\tilde{f} \mapsto \tilde{f}',$$

from the Hecke space of $\widetilde{\mathrm{GL}}(r, F_v)$ to the Paley-Wiener space of $\mathrm{GL}(r, F_v)$, which matches orbital integrals compared under the orbit map. The local metaplectic correspondence amounts to a map,

$$\tilde{\pi} \mapsto \tilde{\pi}',$$

from certain tempered representations of $\widetilde{\mathrm{GL}}(r, F_v)$ to tempered representations of $\mathrm{GL}(r, F_v)$, which satisfies the character identity,

$$\mathrm{tr}\left(\tilde{\pi}(\tilde{f})\right) = \tilde{f}'(\tilde{\pi}').$$

We make two assumptions in these sections. The first is that the local metaplectic correspondence commutes with parabolic induction as claimed in Proposition 26.2 [FK86]. The second is that the trace Paley-Wiener theorem holds for $\widetilde{\mathrm{GL}}(r, F_v)$ (cf. [BDK86]). Metaplectic coverings of the general linear group over the real numbers are not considered although they are well-understood ([AH97]).

The local terms in Arthur's invariant trace formula depend on the normalization of intertwining operators between induced representations. This is the subject of §5. We show that the intertwining operators for the metaplectic coverings can be suitably normalized by following §2 II [AC89]. The argument depends on the Plancherel formula for $\widetilde{\mathrm{GL}}(r, F_v)$. The proof of this formula has not been written out, but can be seen to follow from the proof for $\mathrm{GL}(r, F_v)$ once the observations at the beginning of §5 are taken into consideration.

The local terms of the geometric side of the invariant trace formula are considered in §§6-8. These terms are distributions parameterized by conjugacy classes. They are grouped according to the orbit map in §6, and the groupings $I_M^{\Sigma}(\gamma)$ are shown to preserve properties of descent and splitting. In §7, we assume that the construction and properties of the local geometric terms passes to metaplectic coverings . From these terms we define the distributions $I_M^{\mathcal{M}}(\gamma)$ that are expected to match $I_M^{\Sigma}(\gamma)$. Descent and splitting properties are proven for $I_M^{\mathcal{M}}(\gamma)$ as well. In order to compare $I_M^{\mathcal{M}}(\gamma)$ and $I_M^{\Sigma}(\gamma)$ using the invariant trace formula, we must show that the distributions which occur in the trace formula of the metaplectic coverings, but are not of the form $I_M^{\mathcal{M}}(\gamma)$, are inconsequential. Ideally, every such distribution can be shown to vanish. In §8 we show that the ideal situation holds only if $n$ is relatively prime to the positive integers less than or equal to $r$. Conversely, we show that if $n$ is relatively prime to an additional integer, which depends on the metaplectic covering, then the ideal situation holds.

The conjectural invariant trace formula for a metaplectic covering of the general linear group is introduced in §9. This formula is conjectural not because of any expected complications in its proof; rather, because its proof is very long. The most formidable hurdle in its proof is quite possibly the trace Paley-Wiener theorem for metaplectic coverings (cf. [BDK86]). Due to the volume of the material, the reader is assumed to be familiar with the basic notions of the invariant trace formula for reductive groups.

The topic of §§10-20 is the comparison of invariant trace formulas. This comparison is made under an assumption on the order of the covering. We first describe the content of these sections and then consider the assumption. Let $\mathbf{A}$ be the adele ring of $F$. The invariant trace formula for $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ is placed into the form

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{F,S}/\mu_n^M} a^{\tilde{M}}(S, \gamma') I_M^{\mathcal{M}}(\gamma, \tilde{f})$$
$$= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{t \geq 0} \int_{\Pi(\tilde{M}, t)} a^{\tilde{M}}(\tilde{\pi}) I_{\tilde{M}}(\tilde{\pi}, \tilde{f}) d\tilde{\pi},$$

and the invariant trace formula for $\mathrm{GL}(r, \mathbf{A})$ is placed into the form

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(F))_{F,S}/\mu_n^M} a^M(S,\gamma) I_M^\Sigma(\gamma,\tilde{f})$$
$$= \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{t\geq 0} \int_{\Pi^\Sigma(M,t)} a^{M,\Sigma}(\tilde{\pi}) I_M^\Sigma(\tilde{\pi},\tilde{f}) d\tilde{\pi}.$$

The function $\tilde{f}$ in these equations is taken to be arbitrary in the global Hecke space of $\widetilde{\mathrm{GL}}(r, \mathbf{A})$. The principal results of §§10-20 are Theorems A and B which match the global terms, $a^{\tilde{M}}(S,\gamma')$ with $a^M(S,\gamma)$ and $a^{\tilde{M}}(\tilde{\pi})$ with $a^{M,\Sigma}(\tilde{\pi})$; and the local distributions, $I_M^{\mathcal{M}}(\gamma)$ with $I_M^\Sigma(\gamma)$ and $I_{\tilde{M}}(\tilde{\pi})$ with $I_M^\Sigma(\tilde{\pi})$. The structure of the proof of Theorems A and B follow II [AC89] very closely. There are however two notable deviations. The first is our use of the local invariant trace formula ([Art89c]) in §§11, 20. We assume that this formula holds for metaplectic coverings. The second is the use of strong approximation in §20. Otherwise, the reader familiar with [AC89] should have no difficulties in relating the ideas of §§10-20 to II [AC89]. To make this relation more transparent, the results of these sections, which have counterparts in II [AC89], have references to these counterparts in parentheses immediately following their own numbering.

Let us now consider assumption on the order of the covering, which is labeled as **(†)** in §10. Under this assumption $n$ is relatively prime to the positive integers less than or equal to $r$ and is also relatively prime to $i(1+2m)-1$, where $1 < i \leq r$ and $0 \leq m \leq n-1$ is a fixed integer. The reader may find it helpful to consider assumption **(†)** with $m = 0$, in which case the assumption is greatly simplified, but the covering groups are still non-trivial. We list below the obstructions that are removed under this assumption. The list is given in increasing order of the author's perception of their difficulty.

Assumption **(†)** excludes even coverings. If $n$ is even the orbit map is defined only on a proper subset of $G(F_v)$ and contains a term defined in terms of K-theory. The former circumstance would require the proof of a vanishing property for the geometric side of the trace formula of $\mathrm{GL}(r, \mathbf{A})$, or restrictions on $\tilde{f}$. The latter would make computations involving the orbit map more complicated.

Proposition 26.2 [FK86] and the Appendix ensure that the local metaplectic correspondence commutes with parabolic induction under the assumption that $n$ is relatively prime to $i(1+2m)-1$, $1 < i \leq r$. It is expected that this assumption can be lifted, but to date there is no proof of this expectation.

Assumption **(†)** implies that $n \geq 3$. In this case the archimedean completions of $F$ are all complex. Since metaplectic coverings of $\mathrm{GL}(r, \mathbf{C})$ are trivial, the representation theory at the archimedean valuations is very straightforward. Jordan canonical form may also be used to simplify the comparison at the archimedean valuations. In order to complete the comparison in the case $n = 2$, one would presumably have to prove identities of differential operators on $\widetilde{\mathrm{GL}}(r, \mathbf{R})$ parallel to those in [Mezb].

The assumption that $n$ is relatively prime to the positive integers less than or equal to $r$ simplifies the comparison of $\gamma$ and $\gamma^n$, and of their centralizers in $G(F_v)$. Many terms in the invariant trace formulas are expressed in terms of these objects (see §§11, 12). Under the above assumption, the discrepancies in these terms may be described purely in terms of $n$th roots of unity.

As already mentioned, the vanishing property for the geometric distributions of $\widetilde{\mathrm{GL}}(r, F_v)$ does not follow for general $n$. This obstacle might be circumvented by making restrictions on $\tilde{f}$ or by showing that the sum of the undesirable distributions vanish.

Last, but certainly not least, the matching of weighted orbital integrals required for Lemma 11.4 and the approximation arguments of §20 is only proved in the case that $n$ is relatively prime to certain integers which are included in **(†)** ([Meza]). Even under the assumption that the local trace formula ([Art89c]) holds for metaplectic coverings, this matching is not immediate.

Theorem B entails some global metaplectic correspondences which are listed in Theorem 21.1. In broad terms, there is a correspondence between unitary automorphic representations of $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ and unitary automorphic representations of Levi subgroups of $\mathrm{GL}(r, \mathbf{A})$. This correspondence preserves a character relation at almost every valuation of $F$. If a representation $\tilde{\pi}$ of $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ corresponds as above to a cuspidal representation $\pi$ of $\mathrm{GL}(r, \mathbf{A})$ then a character relation is preserved at every valuation of $F$ and $\tilde{\pi}$ is the only representation of $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ which corresponds to $\pi$. This implies the multiplicity one and strong multiplicity one properties for $\tilde{\pi}$.

Aside from the elimination of the rather vexing assumption **(†)**, there remains much more work to be done in obtaining a general metaplectic correspondence. The paper is concluded with two conjectures concerning the general correspondence and suggestions for their possible solutions.

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# 2 Preliminaries

We begin by briefly describing metaplectic coverings of the general linear group as in §2 [FK86]. We then fix Haar measures and define some vector spaces which are fundamental to the representation theory of our groups.

Let $n$ be a positive integer and let $F$ be a totally imaginary number field containing the group $\mu_n$ of $n$th roots of unity. If $n \geq 3$ then the condition that $F$ be totally imaginary is superfluous. The completion of $F$ at a valuation $v$ is denoted by $F_v$, and its absolute value, which is determined by Haar measure, is denoted by $|\cdot|_v$. Throughout, $S$ denotes a finite set of valuations of $F$. Let $F_S$ be the ring $\prod_{v\in S} F_v$ and set

$$|\gamma|_S = \prod_{v\in S} |\gamma_v|_v, \ \ \gamma = \prod_{v\in S} \gamma_v \in F_S.$$

We write $\mathbf{A}$ for the adele ring of $F$.

We take $r \geq 2$ to be an integer and write $G$ for the algebraic group $\mathrm{GL}(r)$. Clearly, $\mathrm{GL}(r)$ is defined over $F$. Thus we may form the group of $F$- or $F_v$-valued points of $G$. They are $G(F) = \mathrm{GL}(r, F)$ and $G(F_v) = \mathrm{GL}(r, F_v)$ respectively. $F$ also embeds diagonally into $F_S$ and so we may form the group of $F_S$-valued points of $G$. This is simply $G(F_S) = \prod_{v\in S} \mathrm{GL}(r, F_v)$. The adele group of $\mathrm{GL}(r, F)$ is denoted by $G(\mathbf{A})$.

We now recall the construction of the metaplectic coverings as in §2 [FK86]. For each valuation $v$, there are two-cocycles,

$$\sigma_{mv} : G(F_v) \times G(F_v) \to \mu_n, \ \ 0 \leq m \leq n-1.$$

If $v$ is archimedean, that is $F_v = \mathbf{C}$, then $\sigma_{mv}$ is the trivial cocycle.

Suppose $v$ is nonarchimedean. Then these cocycles are "twists" of Matsumoto's cocycle $\sigma_{0v}$. In this case there exists a compact open subgroup $K_v^0$ and a Borel function,

$$\kappa_v : K_v^0 \to \mu_n,$$

such that

$$\sigma_{mv}(k_1, k_2) = \frac{\kappa_v(k_1k_2)}{\kappa_v(k_1)\kappa_v(k_2)}, \ \ k_1, k_2 \in K_v^0.$$

We extend $\kappa_v$ to a Borel function on $G(F_v)$ such that for any upper-triangular unipotent element $u \in G(F_v)$ we have

$$\kappa_v(\gamma u) = \kappa_v(\gamma), \; \gamma \in G(F_v).$$

This can be achieved, for example, by setting $\kappa_v$ equal to one outside of $K_v^0$ ((2.2) [FK86]).

Define the cocycle $\tau_{mv}$ by

$$\tau_{mv}(\gamma_1, \gamma_2) = \sigma_{mv}(\gamma_1, \gamma_2) \frac{\kappa_v(\gamma_1)\kappa_v(\gamma_2)}{\kappa_v(\gamma_1\gamma_2)}, \; \gamma_1, \gamma_2 \in G(F_v),$$

if $v$ is nonarchimedean, and by $\tau_{mv} = \sigma_{mv}$ otherwise. By definition, $\tau_{mv}$ is cohomologous to $\sigma_{mv}$. It follows from our choice of $\kappa_v$ that for any upper-triangular unipotent elements $u_1, u_2 \in G(F_v)$ we have

$$\tau_{mv}(u_1\gamma_1, \gamma_2 u_2) = \tau_{mv}(\gamma_1, \gamma_2), \; \gamma_1, \gamma_2 \in G(F_v). \tag{1}$$

Furthermore, $\tau_{mv}$ is trivial when restricted to $K_v^0 \times K_v^0$.

The cocycle $\tau_{mv}$ yields a central extension,

$$1 \to \mu_n \to \tilde{G}_m(F_v) \to G(F_v) \to 1,$$

called an $n$-fold metaplectic covering of $G(F_v)$. Note that $\tilde{G}_m(F_v)$ is not identical to the extension investigated in [KP84], [KP86] and [FK86], which is defined from just $\sigma_{mv}$. Nevertheless, $\tilde{G}_m(F_v)$ shares all of the relevant properties of the extension used in these works. We fix $m$ and write $\tilde{G}(F_v)$ in place of $\tilde{G}_m(F_v)$. The $n$-fold metaplectic covering,

$$1 \to \mu_n \xrightarrow{\mathbf{i}} \tilde{G}(F_S) \underset{\mathbf{s}}{\overset{\mathbf{p}}{\rightleftarrows}} G(F_S) \to 1,$$

is defined by way of the two-cocycle $\tau_S = \prod_{v \in S} \tau_{mv}$. It is comforting to keep in mind that, as a set, $\tilde{G}(F_S)$ is equal to $G(F_S) \times \mu_n$. The maps of the above sequence may therefore be expressed more concretely by

$$\mathbf{i}(\zeta) = (1, \zeta), \; \mathbf{p}(\gamma, \zeta) = \gamma, \; \text{and } \mathbf{s}(\gamma) = (\gamma, 1),$$

for all $\gamma \in G(F_S)$ and $\zeta \in \mu_n$. Multiplication is in given by

$$(\gamma_1, \zeta_1)(\gamma_2, \zeta_2) = (\gamma_1\gamma_2, \zeta_1\zeta_2\, \tau_S(\gamma_1, \gamma_2)), \; \gamma_1, \gamma_2 \in G(F_S), \; \zeta_1, \zeta_2 \in \mu_n.$$

Given a subgroup $H$ of $G(F_S)$, we write $\tilde{H}$ for $\mathbf{p}^{-1}(H)$. We say that $\tilde{G}(F_S)$ splits over $H$ if $\tilde{H}$ is isomorphic to $H \times \mu_n$ as a group. The group $\tilde{G}(\mathbf{C})$ splits over $G(\mathbf{C})$.

One can also form the product, $\tau = \prod_v \tau_{mv}$, over all valuations of $F$ to obtain a two-cocycle, which gives rise to an $n$-fold metaplectic covering $\tilde{G}(\mathbf{A})$ of $G(\mathbf{A})$. The previous notions for $\tilde{G}(F_S)$ have obvious parallels for $\tilde{G}(\mathbf{A})$. It is important to realize that $\tilde{G}(\mathbf{A})$ splits over $G(F)$. The splitting homomorphism,

$$\mathbf{s_0} : G(F) \to \tilde{G}(\mathbf{A}),$$

is given by

$$\mathbf{s_0}(\gamma) = \mathbf{s}(\gamma)\mathbf{i}\left(\prod_v \kappa_v(\gamma)^{-1}\right), \ \gamma \in G(F). \tag{2}$$

Suppose $v$ is nonarchimedean. We define the orbit map,

$$G_0(F_v) \xrightarrow{\ \prime\ } \tilde{G}(F_v), \tag{3}$$

as the map given in §4 [FK86]. If $n$ is odd then $G_0(F_v) = G(F_v)$ and the orbit map is given by

$$\gamma' = \mathbf{s}(\gamma)^n = (\gamma, 1)^n, \ \gamma \in G(F_v).$$

If $n$ is even then $G_0(F_v)$ is a dense open subset of $G(F_v)$ and

$$\gamma' = \mathbf{u}(\gamma)\mathbf{s}(\gamma)^n = (1, \pm 1)(\gamma, 1)^n, \ \gamma \in G_0(F_v),$$

where $\mathbf{u}$ is a class function taking values in $\mathbf{i}(\mu_2)$. At the archimedean places, we define the orbit map by

$$\gamma \mapsto \gamma' = \mathbf{s}(\gamma)^n = \mathbf{s}(\gamma^n), \ \gamma \in G(\mathbf{C}).$$

The orbit map is so called, because it preserves conjugacy classes, i.e. orbits under the action of conjugation. The orbit map extends to maps on dense subsets of $G(F_S)$ and $G(\mathbf{A})$ by acting on each factor. We abuse notation by denoting these maps by $'$ and referring to them as "the orbit map" as well.

We shall concern ourselves primarily with the regular semisimple elements and conjugacy classes of $\tilde{G}(F_S)$. We therefore define $G_{\mathrm{reg}}$ to be the set of regular and semisimple elements in $G$. The sets $G_{\mathrm{reg}}(F)$ and $G_{\mathrm{reg}}(F_S)$ are the respective subsets of $F$- and $F_S$-valued points of $G_{\mathrm{reg}}$. Set $\tilde{G}_{\mathrm{reg}}(F_S) = \mathbf{p}^{-1}(G_{\mathrm{reg}}(F_S))$. Taking the orbit map into consideration we define the set,

$$G_{\mathrm{oreg}}(F_S) = \{\gamma \in G(F_S) : \gamma^n \in G_{\mathrm{reg}}(F_S)\}.$$

In the terminology of §1.i [Vig81], the image of $G_{\mathrm{oreg}}(F_S)$ under the orbit map is contained in the set of ordinary regular elements of $\tilde{G}(F_S)$ (Proposition 3 [FK86]). Flicker and Kazhdan use the word *good* in place of *ordinary* (§8 [FK86]).

Let $M_0$ be the diagonal subgroup of $G$.The set of Levi subgroups of $G$ containing the minimal Levi subgroup $M_0$ is denoted by $\mathcal{L}$. We reserve the letter $M$ for an element of $\mathcal{L}$. As usual, the center of $M$ is denoted by $A_M$. The set of Levi subgroups of some element $L \in \mathcal{L}$, which also contain $M \subset L$, is written as $\mathcal{L}^L(M)$. Similarly, the set of parabolic subgroups of $L$ with Levi component $M \subset L$ is written as $\mathcal{P}^L(M)$. Given a parabolic subgroup $P \in \mathcal{P}(M)$, we write $U_P$ for its unipotent radical. Let $P_0$ be the subgroup of upper-triangular matrices in $G$. Its unipotent radical $U_{P_0}$ is the subgroup of upper-triangular unipotent matrices. Clearly, we may adelize or take the $F_S$-valued points of these groups. We therefore write $M(F_S)$, $M(\mathbf{A})$, etc. without further comment.

The following decomposition of $M$ will be useful in inductive arguments made later on. Recall that there exist positive integers, $r_1, \ldots, r_\ell$, and subgroups, $M(1) \cong \mathrm{GL}(r_1), \ldots, M(\ell) \cong \mathrm{GL}(r_\ell)$, of $M$ such that $\sum_{i=1}^{\ell} r_i = r$ and

$$M = M(1) \times \cdots \times M(\ell). \tag{4}$$

For each valuation $v$ we fix a maximal compact subgroup $K_v$ of $G(F_v)$ as follows. If $v$ is nonarchimedean set $K_v = \mathrm{GL}(r, R_v)$, where $R_v$ is the ring of integers of $F_v$. If $F_v = \mathbf{C}$ then $K_v$ is the unitary group $U(r, \mathbf{C})$. Set $K_S = \prod_{v \in S} K_v$.

In order to perform harmonic analysis on these groups without any ambiguity, we endow them with measures. Fix measures on $M(F_S)$ and $U_P(F_S)$, where $P \in \mathcal{P}^L(M)$, $L \in \mathcal{L}(M)$ as in §1 [Art81]. If $H$ is a subgroup of $G(F_S)$ with a fixed Haar measure $dh$, we choose $d\tilde{h}$ to be the unique Haar measure on $\tilde{H}$ satisfying

$$d\tilde{h} \circ \mathbf{p}^{-1} = n \cdot dh.$$

We define quotient measures as follows. Given subgroups $H_1 \subset H_2$ of $G(F_S)$ with a fixed measure on $H_1 \backslash H_2$, we define the measure on $\tilde{H}_1 \backslash \tilde{H}_2$ to be the pull-back of the measure on $H_1\backslash H_2$ via the obvious homeomorphism,

$$\tilde{H}_1\backslash\tilde{H}_2 \to H_1\backslash H_2.$$

We close this section with a description of some vector spaces which are basic to representation theory. Let $X(M)$ be the group of rational characters of $M$. Set $\mathfrak{a}_M = \mathrm{Hom}_{\mathbf{Z}}(X(M), \mathbf{R})$, $\mathfrak{a}_M^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_M, \mathbf{R})$ and $\mathfrak{a}_{M,\mathbf{C}}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_M, \mathbf{C})$. The group $X(M)$ embeds into $\mathfrak{a}_{M,\mathbf{C}}^*$ as a lattice. Fix a Euclidean norm on $\mathfrak{a}_{M_0}$, which is invariant under $W_0^G$, the Weyl group of $(G, A_{M_0})$. There is an obvious embedding of $\mathfrak{a}_M$ into $\mathfrak{a}_{M_0}$ and we endow $\mathfrak{a}_M$ with the Euclidean norm obtained from this embedding by restriction. This norm provides a measure on $\mathfrak{a}_M$.

As usual, we define the homomorphism,

$$H_M : M(F_S) \to \mathfrak{a}_M,$$

by the equation,

$$e^{\langle H_M(\gamma), \xi\rangle} = \prod_{v\in S} |\xi(\gamma_v)|_v,\ \gamma = \prod_{v\in S} \gamma_v \in M(F_S),\ \xi \in X(M).$$

The composition of $H_M$ with $\mathbf{p}$ is evidently a homomorphism of $\tilde{M}(F_S)$ into $\mathfrak{a}_M$. The product formula allows us to generalize the above homomorphisms to apply to $M(\mathbf{A})$ and $\tilde{M}(\mathbf{A})$ as well. These maps provide the spaces $M(\mathbf{A})^1 = \ker(H_M)$ and $\tilde{M}(\mathbf{A})^1 = \ker(H_M \circ \mathbf{p})$ with measures.

Suppose that $\gamma$ belongs to either $M(F_S)$ or $M(\mathbf{A})$. Then

$$H_M(\mathbf{p}(\gamma')) = H_M(\gamma^n) = nH_M(\gamma). \tag{5}$$

This equation motivates the definition of the isomorphism of $\mathfrak{a}_M$ defined by

$$X \mapsto X' = n^{-1}X,\ X \in \mathfrak{a}_M,$$

and the adjoint isomorphism of $\mathfrak{a}_{M,\mathbf{C}}^*$ defined by

$$\lambda \mapsto \lambda' = n\lambda.$$

# 3 The local metaplectic correspondence

The local metaplectic correspondence is a map from certain representations of $\tilde{G}(F_S)$ to representations of $G(F_S)$, which also preserves a specific character relation. It also has dual characterizations in terms of a map of Hecke spaces or Paley-Wiener spaces. In the case of Hecke spaces, the map preserves an identity of orbital integrals. We examine each of these characterizations, giving appropriate definitions along the way.

A representation $\tilde{\pi}$ of $\tilde{M}(F_S)$ or $\tilde{M}(\mathbf{A})$ is said to be genuine if

$$\tilde{\pi}(\gamma, \zeta) = \zeta\tilde{\pi}(\gamma, 1),$$

for all $\zeta \in \mu_n$ and $\gamma$ in $M(F_S)$ or $M(\mathbf{A})$. Let $\Pi(\tilde{M}(F_S))$ be the set of (equivalence classes of) genuine, irreducible and admissible representations of $\tilde{M}(F_S)$. Let $\Pi_{\text{temp}}(\tilde{M}(F_S))$ and $\Pi_{\text{unit}}(\tilde{M}(F_S))$ be the subsets of $\Pi(\tilde{M}(F_S))$ which consist respectively of tempered and unitary representations.

Given $M_1 \in \mathcal{L}^M$, $P \in \mathcal{P}^M(M_1)$ and $\tilde{\pi}_1 \in \Pi(\tilde{M}_1(F_S))$, we denote the unitarily induced representation $\mathrm{Ind}_P^M \tilde{\pi}_1$ by $\tilde{\pi}_1^{\tilde{M}}$. If $\lambda \in \mathfrak{a}^*_{M_1,\mathbf{C}}$ then the representation $\tilde{\pi}_{1,\lambda}$, given by

$$\tilde{\pi}_1(\tilde{\gamma}) = \tilde{\pi}_1(\tilde{\gamma})e^{\lambda(H_{M_1}(\mathbf{p}(\tilde{\gamma})))},\ \tilde{\gamma} \in \tilde{M}_1(F_S),$$

belongs to $\Pi(\tilde{M}_1(F_S))$ as well.

Given $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$, we denote its central character by $\omega_{\tilde{\pi}}$ and its distribution character by $\Theta_{\tilde{\pi}}$. For $\gamma = \prod_{v\in S}\gamma_v \in M(F_S)$, the Weyl discriminant is defined as

$$D^M(\gamma) = \prod_{v\in S} \det(1 - \mathrm{Ad}(\gamma_v))_{\mathfrak{m}/\mathfrak{m}_{\sigma_v}} \in F_S.$$

Here $\sigma_v$ is the semisimple component in the Jordan decomposition of $\gamma_v$, and $\mathfrak{m}$ and $\mathfrak{m}_{\sigma_v}$ are the respective Lie algebras of $M(F_v)$ and the centralizer of $\sigma_v$ in $M(F_v)$.

Suppose $v$ is nonarchimedean, $\tilde{\pi} \in \Pi(\tilde{G}(F_v))$ and $\pi \in \Pi(G(F_v))$. Let $d = \gcd(n, r-1+2rm)$. Following (26.1) [FK86], $\tilde{\pi}$ is said to correspond (or lift) to $\pi$ if

$$\omega_{\tilde{\pi}}(\gamma') = \omega_\pi(\gamma),\ \gamma \in A_G(F_v),$$

and for any $\gamma \in G_{\text{oreg}}(F_v)$,

$$(6) \quad |D^G(\gamma^n)|_v^{1/2}\Theta_{\tilde{\pi}}(\gamma') = \frac{n|d|_v^{1/2}}{d|n|_v^{r/2}} \sum_{\{\gamma_1,\gamma_2:\gamma_1'\gamma_2=\gamma'\}} |D^G(\gamma)|_v^{1/2}\Theta_\pi(\gamma_1)\omega_{\tilde{\pi}}(\gamma_2).$$

The sum on the right is parameterized by $\gamma_1 \in G_\gamma(F_v)/A_G(F_v)$ and $\gamma_2 \in \widetilde{A^{n/d}}(F_v)/A'_G(F_v)$. The group $G_\gamma(F_v)$ is the set of $F_v$-valued points of $G_\gamma$, the centralizer of $\gamma$ in $G$. The set $A'_G(F_v)$ is equal to $\{\gamma_0' : \gamma_0 \in A_G(F_v)\}$.

Theorem 27.3 [FK86] implies that there is a map,

$$\Pi_{\text{temp}}(\tilde{G}(F_v)) \xrightarrow{\prime} \Pi_{\text{temp}}(G(F_v)), \tag{7}$$

such that $\tilde{\pi}$ corresponds to $\tilde{\pi}'$. Proposition 26.2 implies that (7) can be generalized to a map,

$$\Pi_{\text{temp}}(\tilde{M}(F_v)) \xrightarrow{\prime} \Pi_{\text{temp}}(M(F_v)), \tag{8}$$

which preserves a character relation analogous to (6). In addition, map (8) commutes with parabolic induction. These results depend on the method of induction delineated in §26.2 [FK86]. Unfortunately, a counterexample in [Sun97] shows that this method is faulty. Despite this circumstance, we shall assume that the above results are true. We refer the reader to the Appendix for further analysis of this issue.

It is simple to show (§3 [Mezb]) the existence of a map,

$$\Pi_{\text{temp}}(\tilde{M}(\mathbf{C})) \xrightarrow{\prime} \Pi_{\text{temp}}(M(\mathbf{C})), \tag{9}$$

which preserves a character relation analogous to (6), and commutes with parabolic induction.

Combining (8) and (9), we define the local metaplectic correspondence (of tempered representations) by

$$\otimes_{v\in S}\tilde{\pi}_v \overset{\prime}{\mapsto} \otimes_{v\in S}\tilde{\pi}'_v,\ \otimes_{v\in S}\tilde{\pi}_v \in \Pi_{\text{temp}}(\tilde{M}(F_S)). \tag{10}$$

The set of genuine standard representations $\Sigma(\tilde{M}(F_S))$ is defined to be the set of (equivalence classes of) representations of the form $\otimes_{v\in S}\tilde{\pi}^{\tilde{M}}_{v,\lambda_v}$, where $\tilde{\pi}_v \in \Pi_{\text{temp}}(\tilde{M}_v(F_v))$, $\lambda_v \in \mathfrak{a}^*_{M_v}$ is regular and $M_v \in \mathcal{L}^M$, for $v \in S$. We generalize the local metaplectic correspondence to $\Sigma(\tilde{M}(F_S))$ by

$$\otimes_{v\in S}\tilde{\pi}^{\tilde{M}}_{v,\lambda_v} \overset{\prime}{\mapsto} \otimes_{v\in S}(\tilde{\pi}'_{v,\lambda'_v})^M.$$

It follows from equation (5) and some basic properties of distribution characters that this map preserves a character relation analogous to that of the local metaplectic correspondence.

We now extend the local metaplectic correspondence to $\Pi(\tilde{M}(F_S))$. By using the Jacquet modules introduced in §14 [FK86], and following the proof of [Sil78], it can be shown that the Langlands quotient theorem holds for

$\tilde{M}(F_S)$. More precisely, every $\tilde{\pi} \in \Pi(\tilde{M}(F_v))$ may be written uniquely as the quotient of some induced representation $\tilde{\pi}_{1,\lambda}^{\tilde{M}}$, where $\tilde{\pi}_1 \in \Pi_{\text{temp}}(\tilde{M}_1(F_v))$, $\lambda$ is in a fixed positive Weyl chamber of $\mathfrak{a}^*_{\tilde{M}_1}$ and $M_1 \in \mathcal{L}^M$. With the Langlands quotient theorem in place, we may extend map (10) by assigning the Langlands quotient of the representation $\tilde{\pi}_{1,\lambda}^{\tilde{M}}$ to the Langlands quotient of $(\tilde{\pi}'_{1,\lambda'})^M$. This prescribes a map

$$\Pi(\tilde{M}(F_S)) \xrightarrow{\prime} \Pi(M(F_S)). \tag{11}$$

The reader is warned that, unlike the local metaplectic correspondence on $\Pi_{\text{temp}}(\tilde{M}(F_S))$ or $\Sigma(\tilde{M}(F_S))$, this map does not necessarily preserve any character relations.

The dual formulation of the local metaplectic correspondence is defined in terms of Hecke spaces or Paley-Wiener spaces. A function $\tilde{f} : \tilde{M}(F_S) \to \mathbf{C}$ is said to be antigenuine if

$$\tilde{f}(\gamma, \zeta) = \zeta^{-1} \tilde{f}(\gamma, 1), \ \gamma \in M(F_S), \ \zeta \in \mu_n.$$

Let $C_c^\infty(\tilde{M}(F_S))$ be the space of antigenuine, smooth and compactly supported functions on $\tilde{M}(F_S)$. The Hecke space $\mathcal{H}(\tilde{M}(F_S))$ is defined to be the subspace of functions in $C_c^\infty(\tilde{M}(F_S))$ which are $(\tilde{K}_S \cap \tilde{M}(F_S))$-finite under left and right translation.

Given $M_1 \in \mathcal{L}^M$ and $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$, we define the map, $\tilde{f}_{\tilde{M}_1}$ on the set $\Pi_{\text{temp}}(\tilde{M}_1(F_S))$ by

$$\tilde{f}_{\tilde{M}_1}(\tilde{\pi}) = \text{tr}\left(\tilde{\pi}^{\tilde{M}}(\tilde{f})\right) = \text{tr}\left(\int_{\tilde{M}(F_S)} \tilde{\pi}^{\tilde{M}}(x)\tilde{f}(x)dx\right). \tag{12}$$

This definition is justified by the well-known fact that $\tilde{\pi}^{\tilde{M}}(\tilde{f})$ is a trace-class operator. Indeed, $\tilde{\pi}$ is admissible and $\tilde{f}$ has compact support.

The Paley-Wiener theorem holds for $\tilde{M}(\mathbf{C})$ by [Art83] and the fact that $\tilde{M}(\mathbf{C})$ splits over $M(\mathbf{C})$. We assume that the trace Paley-Wiener theorem of [BDK86] generalizes to $\tilde{M}(F_v)$ for nonarchimedean $F_v$. Under this assumption we may take the Paley-Wiener space of $\tilde{M}(F_S)$ to be

$$\mathcal{I}(\tilde{M}(F_S)) = \{\tilde{f}_{\tilde{M}} : \tilde{f} \in \mathcal{H}(\tilde{M}(F_S))\}.$$

We recall some ideas presented on p. 79 [AC89]. Suppose $\theta$ is a continuous linear map from $\mathcal{H}(\tilde{M}(F_S))$ to a topological vector space $\mathcal{V}$. Then $\theta$ is said to be supported on characters if it vanishes on any function $\tilde{f}$ with $\tilde{f}_{\tilde{M}} = 0$. If $\theta$ is supported on characters then there exists a unique continuous map,

$$\hat{\theta} : \mathcal{I}(\tilde{M}(F_S)) \to \mathcal{V},$$

such that

$$\hat{\theta}(\tilde{f}_{\tilde{M}}) = \theta(f),\ \ \tilde{f} \in \mathcal{H}(\tilde{M}(F_S)).$$

The map $\tilde{f} \mapsto \tilde{f}_{\tilde{M}_1}$ defined by (12) is supported on characters and factors through a map $\phi \mapsto \phi_{\tilde{M}_1}$ from $\mathcal{I}(\tilde{M}(F_S))$ to $\mathcal{I}(\tilde{M}_1(F_S))$.

The set of valuations $S$ is said to have the closure property if

$$\mathfrak{a}_{M,S} = \{H_M(\gamma) : \gamma \in M(F_S)\}$$

is a closed subgroup of $\mathfrak{a}_M$. If $S$ contains an archimedean valuation, it has the closure property. If not, $S$ has the closure property if and only if it is comprised entirely of valuations which divide a fixed rational prime. For the remainder of this paper, $S$ is assumed to have the closure property unless otherwise specified. Put $i\mathfrak{a}^*_{M,S} = i\mathfrak{a}^*_M / i\mathrm{Hom}(\mathfrak{a}_{M,S}, \mathbf{Z})$. The group $i\mathfrak{a}^*_{M,S}$ inherits a measure from the Euclidean measure defined previously on $\mathfrak{a}_M$. We identify $\phi \in \mathcal{I}(\tilde{M}(F_S))$ with its Fourier transform,

$$\phi(\tilde{\pi}, X) = \int_{i\mathfrak{a}^*_{M,S}} \phi(\tilde{\pi}_\lambda) e^{-\lambda(X)} d\lambda.$$

Define a map $\tilde{f} \mapsto \tilde{f}'$ from $\mathcal{H}(\tilde{M}(F_S))$ to functions on $\Pi_{\mathrm{temp}}(M(F_S))$ by

$$\tilde{f}'(\pi) = \left\{ \begin{array}{ll} \mathrm{tr}\left(\tilde{\pi}(\tilde{f})\right), & \text{if } \tilde{\pi}' = \pi \\ 0, & \text{otherwise} \end{array} \right\},\ \pi \in \Pi_{\mathrm{temp}}(M(F_S)). \tag{13}$$

It is simple to check that $\tilde{f}'$ satisfies the axioms defining the Paley-Wiener space listed in [BDK86] and [Art83]. Thus $\tilde{f}'$ belongs to $\mathcal{I}(M(F_S))$. We define a map,

$$\mathcal{I}(\tilde{M}(F_S)) \xrightarrow{\ '\ } \mathcal{I}(M(F_S)),$$

which is compatible with (13), by

$$\phi'(\pi, X') = \left\{ \begin{array}{ll} n^{\dim(A_M)} \phi(\tilde{\pi}, X), & \text{if } \tilde{\pi}' = \pi \\ 0, & \text{otherwise} \end{array} \right\},\ \pi \in \Pi_{\mathrm{temp}}(M(F_S)).$$

Once again, it is a simple matter to check that the image of this map lies in $\mathcal{I}(M(F_S))$. To see that this map is compatible with (13), suppose $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$, $\tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}(F_S))$ and $X \in \mathfrak{a}_{M,S}$. Then we have

$$\begin{aligned}
(\tilde{f}_{\tilde{M}})'(\tilde{\pi}', X, ) &= n^{\dim(A_M)} \tilde{f}_{\tilde{M}}(\tilde{\pi}, X) \\
&= n^{\dim(A_M)} \int_{i\mathfrak{a}^*_{M,S}} \mathrm{tr}\left(\tilde{\pi}_\lambda(\tilde{f})\right) e^{-\lambda(X)} d\lambda \\
&= n^{\dim(A_M)} \int_{i\mathfrak{a}^*_{M,S}} \tilde{f}'((\tilde{\pi}_\lambda)') e^{-\lambda(X)} d\lambda \\
&= \int_{i\mathfrak{a}^*_{M,S}} \tilde{f}'(\tilde{\pi}'_{\lambda'}) e^{-\lambda'(X')} d\lambda' \\
&= \tilde{f}'(\tilde{\pi}', X')
\end{aligned}$$

as to be desired.

One would hope that map (13) commutes with the map (12). This is indeed the case, but in order to prove this fact we must describe map (13) purely in terms of Hecke spaces. The trace Paley-Wiener theorem tells us that for each $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$ there exists $f \in \mathcal{H}(M(F_S))$ such that $f_M = \tilde{f}'$. The relationship between the Hecke functions, $\tilde{f}$ and $f$, is expressed in terms of orbital integrals.

Suppose $\tilde{\gamma} \in \tilde{M}(F_S)$ such that $\mathbf{p}(\tilde{\gamma})$ is semisimple. Let $\tilde{M}_{\tilde{\gamma}}(F_S)$ denote the centralizer of $\tilde{\gamma}$ in $\tilde{M}(F_S)$. We define

$$I_{\tilde{M}}^{\tilde{M}}(\tilde{\gamma}, \tilde{h}) = |D^M(\mathbf{p}(\tilde{\gamma}))|_S^{1/2} \int_{\tilde{M}_{\tilde{\gamma}}(F_S)\backslash \tilde{M}(F_S)} \tilde{h}(x^{-1}\tilde{\gamma}x) dx, \ \tilde{h} \in C_c^\infty(\tilde{M}(F_S)).$$

These integrals converge (§6 [FK86], §2 [Art88c]) and are called orbital integrals. Now suppose that $\tilde{\gamma}$ has Jordan decomposition $\tilde{\gamma} = \tilde{\sigma}\mathbf{s}(u)$, where $u = \prod_{v \in S} u_v$ is a unipotent element of $M(F_S)$. A general orbital integral is given by

$$I_{\tilde{M}}^{\tilde{M}}(\tilde{\gamma}, \tilde{h}) = \lim_{a \to 1} I_{\tilde{M}}^{\tilde{M}}(\tilde{\sigma}\mathbf{s}(a^n), \tilde{h}),$$

where $a = \prod_{v \in S} a_v$ and $a_v$ is regular, lying in the center of a parabolic subgroup for which $u_v$ is a Richardson element. It is obvious that $I_{\tilde{M}}^{\tilde{M}}(\tilde{\gamma}, \tilde{h})$ depends only on the conjugacy class of $\tilde{\gamma}$ in $\tilde{M}(F_S)$.

For any valuation $v$ set

$$\Lambda_v^M(\gamma) = |n|_v^{\dim(M_\gamma)/2}, \ \gamma \in M(F_v).$$

and

$$\Lambda^M(\gamma) = \prod_{v\in S} \Lambda_v^M(\gamma_v),\ \gamma = \prod_{v\in S}\gamma_v \in M(F_S).$$

It follows from Proposition 27.3 [FK86], §24 [FK86], §4 [Mezb] and the normalization of our measures that

$$I_M^M(\gamma, f) = \Lambda^M(\gamma) I_{\tilde{M}}^{\tilde{M}}(\gamma', \tilde{f}),\ \gamma \in M(F_S)\cap G_{\mathrm{oreg}}(F_S), \tag{14}$$

where $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$ and $f \in \mathcal{H}(M(F_S))$ are as above. Any two functions, $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$ and $f \in \mathcal{H}(M(F_S))$, satisfying (14) are said to match.

Using this terminology, we can characterize the image of the local metaplectic correspondence on $\Pi_{\mathrm{temp}}(\tilde{M}(F_S))$. An admissible representation $\pi$ of $M(F_S)$ is called metic if there exist matching functions $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$ and $f \in \mathcal{H}(M(F_S))$ such that $\mathrm{tr}\,(\pi_1(f)) \neq 0$ for every subquotient $\pi_1$ of $\pi$. Theorem 27.3 [FK86] and Proposition 4.1 [Mezb] imply that the image of the local metaplectic correspondence on $\Pi_{\mathrm{temp}}(\tilde{M}(F_S))$ is the subset of metic representations in $\Pi_{\mathrm{temp}}(M(F_S))$.

With this characterization in hand, we can prove that map (12) and map (13) commute. Suppose $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$, $\pi \in \Pi_{\mathrm{temp}}(M_1(F_S))$ and $M_1 \in \mathcal{L}^M$. Then, by definition

$$(\tilde{f}_{\tilde{M}_1})'(\pi) = \begin{cases} \tilde{f}_{\tilde{M}}(\tilde{\pi}^{\tilde{M}}), & \text{if } \pi = \tilde{\pi}',\ \tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}_1(F_S)) \\ 0, & \text{otherwise} \end{cases},$$

and

$$(\tilde{f}')_{M_1}(\pi) = \begin{cases} \tilde{f}_{\tilde{M}}(\tilde{\pi}), & \text{if } \pi^M = \tilde{\pi}',\ \tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}(F_S)) \\ 0, & \text{otherwise} \end{cases}$$

Thus, in order to show that $(\tilde{f}_{\tilde{M}_1})' = (\tilde{f}')_{M_1}$ we need only show that $\pi$ is metic if and only if $\pi^M$ is metic. If $\pi$ is metic then $\pi^M$ is metic since the local metaplectic correspondence commutes with parabolic induction. Conversely, if $\pi^M$ is metic then there exist matching $\tilde{f}_1 \in \mathcal{H}(\tilde{M}(F_S))$ and $f_1 \in \mathcal{H}(M(F_S))$ such that $\mathrm{tr}\left(\pi^M(f_1)\right) \neq 0$. Applying a well-known descent property for orbital integrals (§7 [FK86]), there exist matching functions $f_{1,\tilde{P}} \in \mathcal{H}(\tilde{M}_1(F_S))$ and $f_{1,P} \in \mathcal{H}(M_1(F_S))$ for $P \in \mathcal{P}^M(M_1)$ such that

$$\mathrm{tr}\,(\pi(f_{1,P})) = \mathrm{tr}\left(\pi^M(f_1)\right) \neq 0.$$

This means that $\pi$ is metic.

# 4 Comparison of orbital integrals at singular elements

In this section we show how the identity of orbital integrals (14) may be generalized to some singular elements of $G(F_S)$.

**Lemma 4.1** *Suppose $E$ is a field and $u$ is a unipotent element of* $\mathrm{GL}(k,E)$. *Then $u$ is conjugate to $u^n$ in* $\mathrm{GL}(k,E)$.

*Proof.* Set $u_s$ to be the $s \times s$ matrix of the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Jordan canonical form tells us that any unipotent element $u$ of $\mathrm{GL}(k,E)$ is conjugate to

$$\begin{pmatrix} u_{s_1} & & 0 \\ & \ddots & \\ 0 & & u_{s_j} \end{pmatrix},$$

for some positive integers, $s_1, \ldots s_j$, satisfying $\sum_{i=1}^{j} s_i = k$. It therefore suffices to prove the lemma in the case that $j = 1$ and $u = u_k$.

Observe that $u - 1$ is a nilpotent matrix of order $k - 1$. Now

$$u^n - 1 = (1 + (u-1))^n - 1 = \sum_{i=0}^{n} \binom{n}{i} (u-1)^i - 1 = \sum_{i=1}^{n} \binom{n}{i} (u-1)^i.$$

From this it is apparent that $u^n - 1$ is a nilpotent matrix of order $k - 1$. Another application of Jordan canonical form tells us that $u^n$ is conjugate to

$$\begin{pmatrix} u_{b_1} & & 0 \\ & \ddots & \\ 0 & & u_{b_t} \end{pmatrix},$$

where $\sum_{i=1}^{t} b_i = k$. Clearly,

$$\begin{pmatrix} u_{b_1} & & 0 \\ & \ddots & \\ 0 & & u_{b_t} \end{pmatrix} - \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

is a nilpotent matrix of order $\max_{1\leq i\leq t}(b_i - 1)$. This implies that $t = 1$ and $b_1 = k$, whence the lemma. $\square$

Given a nonarchimedean valuation $v$ and a semisimple element $\sigma \in M(F_v)$, let $\mathcal{U}_{M_\sigma}(F_v)$ be the set of unipotent elements in $M_\sigma(F_v)$ and let $(\mathcal{U}_{M_\sigma}(F_v))$ be the corresponding set of $M_\sigma(F_v)$-conjugacy classes. The latter set is the parameter set in the Shalika germ expansion of an orbital integral at $\sigma$. Such germ expansions exist for $\tilde{M}(F_v)$ as well ([Vig81]). We shall use these germ expansions to compare orbital integrals at elements which are not regular. As preparation for this, we show that $(\mathcal{U}_{M_\sigma}(F_v))$ parameterizes the germs in the Shalika germ expansion for $\tilde{M}(F_v)$.

**Lemma 4.2** *Suppose $v$ is nonarchimedean, $\tilde{\gamma} \in \tilde{G}(F_v)$, $\mathbf{p}(\tilde{\gamma})$ has Jordan decomposition $\sigma u$ and $\tilde{\gamma} = \tilde{\sigma}\mathbf{s}(u)$. Suppose further that $\sigma u_0$ belongs to the closure of the $G(F_v)$-conjugacy class of $\mathbf{p}(\tilde{\gamma})$. Then $\tilde{\sigma}\mathbf{s}(u_0)$ belongs to the closure of the $\tilde{G}(F_v)$-conjugacy class of $\tilde{\gamma}$.*

*Proof.* We prove the lemma in a special case and leave the details of the general case to the interested reader. Suppose $r = 2$, $\sigma_1 \in F_v^\times$, $\zeta \in \mu_n$, $\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$, $\tilde{\sigma} = \mathbf{i}(\zeta)\mathbf{s}(\sigma)$, $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $u_0$ is the identity element. Let $x \in F_v^\times$ such that $|x|_v < 1$. Denote the $n$th Hilbert symbol of $F_v$ by $(\cdot,\cdot)_{F_v}$. Using equation (1) we can show that

$$\mathbf{s}\begin{pmatrix} x^{kn} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{s}(u)\mathbf{s}\begin{pmatrix} x^{kn} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \mathbf{s}\begin{pmatrix} 1 & x^{kn} \\ 0 & 1 \end{pmatrix}.$$

Together with Proposition 0.1.5 [KP86], this implies that

$$\begin{aligned} &\lim_{k\to\infty} \mathbf{s}\begin{pmatrix} x^{kn} & 0 \\ 0 & 1 \end{pmatrix}\tilde{\sigma}\mathbf{s}(u)\mathbf{s}\begin{pmatrix} x^{kn} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \mathbf{i}(\zeta)\mathbf{s}(\sigma)\lim_{k\to\infty}(\sigma_1^2, x^{kn})_{F_v}^{(1+2m)}(\sigma_1, x^{kn})_{F_v}^{-1}(\sigma_1, 1)_{F_v}^{-1}\mathbf{s}\begin{pmatrix} 1 & x^{kn} \\ 0 & 1 \end{pmatrix} \\ &= \tilde{\sigma}. \end{aligned}$$

In the general case $G_\sigma(F_v)$ is isomorphic to $\prod_{i=1}^k \mathrm{GL}(m_i, F_i)$ where $F_1, \ldots, F_k$ are extensions of $F_v$ (§1 [KP86]). One can imitate the above computation in $\mathrm{GL}(m_i, F_i)$ for each $1 \leq i \leq k$ to obtain the lemma.$\square$

In the notation of the previous lemma, §1.j [Vig81] tells us that if the closure of the $G(F_v)$-conjugacy class of $\sigma u$ is equal to the disjoint union of the

$G(F_v)$-conjugacy classes of $\sigma u_1, \ldots, \sigma u_k$ then there exist $\zeta_1, \ldots, \zeta_k \in \mu_n$ such that the closure of the $\tilde{G}(F_v)$-conjugacy class of $\tilde{\sigma}\mathbf{s}(u)$ is equal to the disjoint union of the $\tilde{G}(F_v)$-conjugacy classes of $\mathbf{i}(\zeta_1)\mathbf{s}(\sigma)\mathbf{s}(u_1), \ldots, \mathbf{i}(\zeta_k)\mathbf{s}(\sigma)\mathbf{s}(u_k)$. Lemma 4.2 implies that $\zeta_1 = \cdots = \zeta_k$ and $\tilde{\sigma} = \mathbf{i}(\zeta_1)\mathbf{s}(\sigma)$. In particular, the closure of the two conjugacy classes may be identified by way of the map $\mathbf{s}$ acting on the unipotent classes. Bearing in mind decomposition (4), it follows that, $(\mathcal{U}_{M_\sigma}(F_v))$ is an index set for the germs in the Shalika germ expansion for $\tilde{M}(F_v)$ via the map $\mathbf{s}$.

**Lemma 4.3** *Suppose $v$ is nonarchimedean, and $\sigma \in G(F_v)$ is a semisimple element which lies in the domain of the orbit map and satisfies $G_\sigma = G_{\sigma^n}$. Suppose further that $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$ and $f \in \mathcal{H}(G(F_v))$ match. Then*

$$I_G(\sigma, f) = \Lambda^G(\sigma) I_{\tilde{G}}(\sigma', \tilde{f}).$$

*Proof.* According to Corollaire 2.4 [Vig81], §3.3 [Vig81] and the remarks preceding this lemma, we have the following two germ expansions:

$$I_{\tilde{G}}(\gamma', \tilde{f}) = \sum_{u \in (\mathcal{U}_{G_\sigma}(F_v))} I_G(\gamma^n, h_u) I_{\tilde{G}}(\sigma'\mathbf{s}(u), \tilde{f}),$$

$$I_G(\gamma, f) = \sum_{u \in (\mathcal{U}_{G_\sigma}(F_v))} I_G(\gamma, f_u) I_G(\sigma u, f).$$

In these equations, the element $\gamma$ belongs to $G_{\mathrm{oreg}}(F_v)$ and is taken to be in a very small open neighborhood of $\sigma$ depending on $\tilde{f}$ and $f$. The functions $f_u$, $h_u \in C_c^\infty(G(F_v))$ have the following properties:

$$I_G(\sigma^n u_1, h_u) = \left\{ \begin{array}{ll} 1, & \text{if } u_1 = u \\ 0, & \text{otherwise} \end{array} \right\},$$

$$I_G(\sigma u_1, f_u) = \left\{ \begin{array}{ll} 1, & \text{if } u_1 = u \\ 0, & \text{otherwise} \end{array} \right\}, \ u, \ u_1 \in (\mathcal{U}_{G_\sigma}(F_v)).$$

We may assume that

$$f_u(\gamma_1) = \left| \frac{D^G(\sigma^n)}{D^G(\sigma)} \right|_v^{1/2} h_u(\gamma_1^n) = |n|_v^{(r^2 - \dim(G_\sigma))/2} h_u(\gamma_1^n), \ \gamma_1 \in G(F_v)$$

( Application 1 §4.1 [Vig81]). Indeed, for $u_1 \in (\mathcal{U}_{G_\sigma}(F_v))$, we have

$$\begin{aligned} I_G(\sigma u_1, f_u) &= |D^G(\sigma)|_v^{1/2} \int_{G_\sigma(F_v)\backslash G(F_v)} f_u(x^{-1}\sigma u_1 x)dx \\ &= |D^G(\sigma^n)|_v^{1/2} \int_{G_{\sigma^n}(F_v)\backslash G(F_v)} h_u(x^{-1}\sigma^n u_1^n x)dx \\ &= I_G(\sigma^n u_1^n, h_u) \\ &= \begin{cases} 1, & \text{if } u_1^n = u \\ 0, & \text{otherwise} \end{cases}, \end{aligned}$$

and by Lemma 4.1, $u_1^n = u_1$, so $f_u$ satisfies the required property. As a consequence $I_G(\gamma, f_u)$ equals

$$\begin{aligned} & |D^G(\gamma)|_v^{1/2} \int_{G_\gamma(F_v)\backslash G(F_v)} f_u(x^{-1}\gamma x)dx \\ = \;& |D^G(\gamma^n)|_v^{1/2} \left|\frac{D^G(\gamma)}{D^G(\gamma^n)}\right|_v^{1/2} |n|_v^{(r^2-\dim(G_\sigma))/2} \int_{G_{\gamma^n}(F_v)\backslash G(F_v)} h_u(x^{-1}\gamma^n x)dx \\ = \;& |n|_v^{(-r^2+r+r^2-\dim(G_\sigma))/2} |D^G(\gamma^n)|_v^{1/2} \int_{G_{\gamma^n}(F_v)\backslash G(F_v)} h_u(x^{-1}\gamma^n x)dx \\ = \;& |n|_v^{-(\dim(G_\sigma)-r)/2} I_G(\gamma^n, h_u). \end{aligned}$$

The germ expansion for $G$ is thus

$$I_G(\gamma, f) = \sum_{u\in(\mathcal{U}_{G_\sigma}(F_v))} I_G(\gamma^n, h_u)\left(|n|_v^{-(\dim(G_\sigma)-r)/2} I_G(\sigma u, f)\right).$$

After applying Lemma 4.1 to the germ expansion for $\tilde{G}$, we obtain

$$I_{\tilde{G}}(\gamma', \tilde{f}) = \sum_{u\in(\mathcal{U}_{G_\sigma}(F_v))} I_G(\gamma^n, h_u) I_{\tilde{G}}(\sigma'\mathbf{s}(u), \tilde{f}).$$

Let us examine the germ $I_G(\gamma^n, h_u)$ in more detail. According to §2.5 [Vig81],

$$I_G(\gamma^n, h_u) = \left|\frac{D^G(\sigma^n u)D^G(\gamma^n)}{D^{G_\sigma}(\sigma^n u)D^{G_\sigma}(\gamma^n)}\right|_v^{1/2} I_{G_\sigma}(\gamma^n, h_u). \tag{15}$$

Since $\sigma$ is central in $G_\sigma(F_v)$, it follows that $I_{G_\sigma}(\gamma, h_u)$ satisfies a homogeneity property (Application 2 §4.1 [Vig81]), namely

$$I_{G_\sigma}(\gamma^{k^2}, h_u) = |k|_v^{\dim(G_{\sigma u})-r} I_{G_\sigma}(\gamma, h_u), \; k \in \mathbf{Z}.$$

Together with (15), this implies that

$$I_G(\gamma^{nk^2}, h_u) = |k|_v^{((r^2-\dim G_\sigma)/2)+\dim(G_{\sigma u})-r} I_G(\gamma^n, h_u).$$

Hence the germs, $I_G(\gamma^n, h_u)$, for $u \in (\mathcal{U}_{G_\sigma}(F_v))$, may be said to be homogeneous of degree $\dim(G_{\sigma u})$ respectively. It is clear that nonzero germs of different degree are linearly independent. Now suppose $\gamma \in G_{\mathrm{oreg}}(F_v)$ is $F_v$-elliptic in $G_\sigma(F_v)$ and is close to $\sigma$. The difference,

$$\begin{aligned} &\Lambda_v^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}) - I_G(\gamma, f) \\ &= \sum_{u \in (\mathcal{U}_{G_\sigma}(F_v))} \left( \Lambda_v^G(\gamma) I_{\tilde{G}}(\sigma' \mathbf{s}(u), \tilde{f}) - |n|_v^{-(\dim(G_\sigma)-r)/2} I_G(\sigma u, f) \right) I_G(\gamma^n, h_u), \end{aligned}$$

vanishes since $\tilde{f}$ matches $f$. It follows from Proposition 2.1 [Rog83] and Theorem 2.2 [Rog83] that the germ $I_G(\gamma^n, h_1)$ is not zero. It is also the only germ in this linear combination of degree $\dim(G_\sigma)$. Its coefficient must vanish by linear independence. This proves the lemma.□

**Lemma 4.4** *Suppose $\sigma \in G(\mathbf{C})$ is semisimple and $G_\sigma = G_{\sigma^n}$. Suppose further that $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{C}))$ matches $f \in \mathcal{H}(G(\mathbf{C}))$. Then*

$$I_G(\sigma, f) = \Lambda^G(\sigma) I_{\tilde{G}}(\sigma', \tilde{f}).$$

*Proof.* We first consider the case $\sigma = 1$. In this case we must show that

$$f(1) = |n|_{\mathbf{C}}^{r^2/2} \tilde{f}(1).$$

This equation was shown in [Mezb] for a specific choice of $f$ matching $\tilde{f}$. However, it follows from the Weyl integration formula and the Plancherel formula that the value at the identity of any function matching $\tilde{f}$ is equal to $f(1)$. Now suppose that $\sigma \in G(\mathbf{C})$ is any semisimple element satisfying $G_\sigma = G_{\sigma^n}$. Since we will only be considering orbital integrals at semisimple elements, Jordan canonical form allows us to assume that our orbital integrals taken are taken at elements in $M_0(\mathbf{C})$. Clearly, we may assume that $\sigma \in A_M(\mathbf{C})$ for some $M \in \mathcal{L}$, so that $G_\sigma = M$. Fix $P \in \mathcal{P}(M)$ and set $\tilde{f}_P^{\sigma^n}(\gamma)$ equal to

$$\left|\det(1 - \mathrm{Ad}(\gamma\sigma^n))_{|\mathfrak{g}/\mathfrak{m}}\right|_{\mathbf{C}}^{1/2} \int_{U_P(\mathbf{C})} \int_{K_{\mathbf{C}}} \tilde{f}(\mathbf{s}(k)^{-1}\mathbf{s}(u)^{-1}\mathbf{s}(\gamma\sigma^n)\mathbf{s}(u)\mathbf{s}(k))\,du\,dk,$$

where $\gamma \in M(\mathbf{C})$ and $\mathfrak{m}$ is the (real) Lie algebra of $M(\mathbf{C})$. For $\gamma \in M_0(\mathbf{C}) \cap G_{\text{oreg}}(\mathbf{C})$ in a sufficiently small neighborhood $V_\sigma \subset M_0(\mathbf{C})$ of the identity, we have $\gamma\sigma \in M_0(\mathbf{C}) \cap G_{\text{oreg}}(\mathbf{C})$. It then follows from a standard descent formula (§7 [FK86]) that

$$\Lambda^G(\gamma\sigma) I_{\tilde{G}}((\gamma\sigma)', \tilde{f}) = \Lambda^M(\gamma) I_{\tilde{M}}^{\tilde{M}}(\gamma', \tilde{f}_{\tilde{P}}^{\sigma^n}), \ \gamma \in V_\sigma \cap G_{\text{oreg}}(\mathbf{C}).$$

After making similar definitions, we find that

$$I_G(\gamma\sigma, f) = I_M^M(\gamma, f_P^\sigma), \ \gamma \in V_\sigma \cap G_{\text{oreg}}(\mathbf{C}).$$

Combining the last two equations with the matching property, we obtain

$$\Lambda^M(\gamma) I_{\tilde{M}}^{\tilde{M}}(\gamma, \tilde{f}_{\tilde{P}}^{\sigma^n}) = I_M^M(\gamma, f_P^\sigma), \ \gamma \in V_\sigma \cap G_{\text{oreg}}(\mathbf{C}). \tag{16}$$

The functions $\tilde{f}_{\tilde{P}}^{\sigma^n}$ and $f_P^\sigma$ also satisfy the identities,

$$\begin{aligned} \Lambda^M(1)\tilde{f}_{\tilde{P}}^{\sigma^n}(1) &= \Lambda^G(\sigma^n) \left|\det(1 - \mathrm{Ad}(\sigma^n))_{|\mathfrak{g}/\mathfrak{g}_{\sigma^n}}\right|_{\mathbf{C}}^{1/2} \int_{\tilde{G}_{\sigma'}(\mathbf{C})\backslash \tilde{G}(\mathbf{C})} \tilde{f}(x^{-1}\sigma' x)dx \\ &= \Lambda^G(\sigma^n) I_{\tilde{G}}(\sigma', \tilde{f}), \end{aligned}$$

and $f_P^\sigma(1) = I_G(\sigma, f)$. The lemma therefore follows if we show that

$$f_P^\sigma(1) = \Lambda^M(1)\tilde{f}_{\tilde{P}}^{\sigma^n}(1).$$

According to [Mezb], there exists a function $h \in \mathcal{H}(M(\mathbf{C}))$ matching $\tilde{f}_{\tilde{P}}^{\sigma^n} \in \mathcal{H}(\tilde{M}(\mathbf{C}))$. From our proof of the case $\sigma = 1$ it follows that

$$h(1) = \Lambda^M(1)\tilde{f}_{\tilde{P}}^{\sigma^n}(1).$$

Thus, if $h(1) = f_P^\sigma(1)$ the lemma is complete. This last equation follows from the fact that $h(1)$ is determined by the values of $I_M^M(\cdot, h)$ restricted to any neighborhood of the identity in $M_0(\mathbf{C})$. More precisely, there exists a differential operator $\partial(\varpi)$ on $M_0(\mathbf{C})$ such that $h(1) = \partial(\varpi) I_M^M(1, h)$ (Theorem 11.17 [Kna86]). Notice that equation (16) and the matching property imply

$$I_M^M(\gamma, h) = I_M^M(\gamma, f_P^\sigma), \ \gamma \in V_\sigma \cap G_{\text{oreg}}(\mathbf{C}).$$

Since $V_\sigma \cap G_{\text{oreg}}(\mathbf{C})$ contains an open subset of $V_\sigma$, the previous two equations imply that

$$h(1) = \partial(\varpi) I_M^M(1, h) = \partial(\varpi) I_M^M(1, f_P^\sigma) = f_P^\sigma(1). \square$$

The reader might wonder whether one might be able to remove the hypothesis $G_\sigma = G_{\sigma^n}$ from the previous two lemmas. The following example shows that, at least for Lemma 4.4, this hypothesis cannot be removed. Suppose $F_v = \mathbf{C}$, $n = r = 2$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{C}))$ such that $\tilde{f}(1) \neq 0$. It is a simple exercise to show that the function given by,

$$f(\gamma) = \left| \frac{D^G(\gamma^n)}{D^G(\gamma)} \right|_{\mathbf{C}}^{1/2} \tilde{f}(\gamma'), \gamma \in G_{\mathrm{oreg}}(\mathbf{C}),$$

extends to a smooth function on $G(\mathbf{C})$. More precisely, if $\gamma \in G(\mathbf{C})$ has eigenvalues $\gamma_1, \gamma_2 \in \mathbf{C}$ then

$$f(\gamma) = \left| (\gamma_1 + \gamma_2)^2 / \gamma_1 \gamma_2 \right|_{\mathbf{C}}^{1/2} \tilde{f}(\gamma').$$

It is not difficult to show that $f \in \mathcal{H}(G(\mathbf{C}))$ and that $\tilde{f}$ matches $f$. However, for the semisimple element $\sigma = \mathrm{diag}(1, -1) \in G(\mathbf{C})$ the equation of Lemma 4.4 does not hold. Indeed, the equation,

$$f(x^{-1}\sigma x) = 0, \ x \in G_\sigma(\mathbf{C}) \backslash G(\mathbf{C}),$$

implies that $I_G(\sigma, f) = 0$, but $\Lambda^G(\sigma) I_{\tilde{G}}(\sigma', \tilde{f})$ is equal to $n^4 \tilde{f}(1)$, which by assumption is not zero.

This example suggests that we should restrict ourselves to the comparison of orbital integrals at elements in the subset $M_{\mathrm{comp}}(F_S)$ of $M(F_S)$, defined by

$$M_{\mathrm{comp}}(F_S) = \{\gamma = \prod_{v \in S} \gamma_v \in M(F_S) : M_{\sigma_v} = M_{\sigma_v^n}, \ \sigma_v \text{ semisimple part of } \gamma_v\}.$$

The set $M_{\mathrm{comp}}(F_S)$ is a dense open subset of $M(F_S)$.

**Lemma 4.5** *Suppose $\gamma \in G(F_S)$ lies in the domain of the orbit map and has Jordan decomposition $\sigma u$. Suppose further that that $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$ matches $f \in \mathcal{H}(G(F_S))$. Then*

$$I_G(\gamma, f) = \Lambda^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}), \ \gamma \in G_{\mathrm{comp}}(F_S).$$

*Proof.* It suffices to prove the lemma in the case that $S$ consists of a single valuation $v$. The unipotent element $u$ is contained in the Richardson orbit of some parabolic subgroup $P_{\sigma^n} = M_\sigma U_\sigma$ of $G_\sigma$. Let $A_\sigma$ be the center of $M_\sigma$. By equation (1) and definition (1.3) II [AC89], we have

$$I_{\tilde{G}}(\gamma', \tilde{f}) = I_{\tilde{G}}(\sigma' \mathbf{s}(u^n), \tilde{f}) = \lim_{a \to 1} I_{\tilde{G}}(\mathbf{s}(a^n)\sigma', \tilde{f}),$$

where $a$ is regular in $A_\sigma(F_v)$. We also have the parallel definition,

$$I_G(\gamma, f) = I_G(\sigma u, f) = \lim_{a \to 1} I_G(a\sigma, f).$$

The lemma now follows from these two definitions and the matching of $\tilde{f}$ and $f$.$\square$

# 5 Normalization of intertwining operators and the Plancherel formula

Our goal here is to normalize the intertwining operators between induced representations. This is necessary for the definition of the distributions occurring in the invariant trace formula of Arthur ([Art89a] and [Art88a]).

Our method of normalization follows §2 II [AC89]. In particular, we make use of the Plancherel formula ([HC84]). The proof of this formula has not been carried out for $\tilde{G}(F_v)$. We nevertheless assert that the Plancherel formula holds in this case as well. In what follows, we justify this assertion by listing the properties of reductive algebraic groups used in Harish-Chandra's proof of the Plancherel formula and show that they also hold for $\tilde{G}(F_v)$. In this justification we assume that $v$ is nonarchimedean.

Suppose $P \in \mathcal{P}(M)$ such that $P \supset P_0$. Then $\tilde{G}(F_v)$ splits over $U_P(F_v)$ (§2 [FK86]). The splitting homomorphism is $\mathbf{s}$. In other words,

$$\mathbf{s}(U_P(F_v)) = \{(u, 1) : u \in U_P(F_v)\}$$

is a subgroup of $\tilde{G}(F_v)$. Every parabolic subgroup in $\mathcal{P}(M)$ is of the form $P^w = w^{-1}Pw$ for some representative $w$ of the Weyl group $W_0^G$. It is easy to check that $\tilde{G}(F_v)$ splits over $U_{P^w}(F_v) = w^{-1}U_P(F_v)w$ with splitting homomorphism $\mathbf{s}_w$ defined by

$$w^{-1}uw \mapsto \mathbf{s}(w)^{-1}\mathbf{s}(u)\mathbf{s}(w).$$

Clearly $\tilde{P^w}(F_v) = \tilde{M}(F_v)\mathbf{s}_w(U_{P^w}(F_v))$ as $\tilde{M}(F_v)$ is stable under conjugation by $\mathbf{s}(w)$.

We define the Jacquet module of an admissible representation $(\tilde{\pi}, V)$ of finite length with respect to $U_{P^w}(F_v)$ in the following way. Let $V_{U_{P^w}}$ be the linear span of

$$\{\tilde{\pi}(\mathbf{s}_w(w^{-1}uw))v - v : u \in U_P(F_v), v \in V\}.$$

It is a consequence of equation (1) that $\tilde{M}(F_v)$ normalizes $\mathbf{s}_w(U_{P^w}(F_v))$. Thus $\tilde{M}(F_v)$ stabilizes $V_{U_{P^w}}$. We define the Jacquet module of $\tilde{\pi}$ with respect to $U_{P^w}$ to be the representation obtained by twisting the quotient representation $V/V_{U_{P^w}}$ with the modular function $\delta_{P^w}^{-1/2}$. We denote this representation by $\tilde{\pi}_{U_{P^w}}$. This is a mild generalization of §14 [FK86]. It is left to the reader to check that this definition yields the expected properties of Jacquet modules.

Another consequence of the splitting of $\tilde{G}(F_v)$ over $U_{P^w}(F_v)$ is the decomposition,

$$\tilde{G}(F_v) = \tilde{M}(F_v)\mathbf{s}_w(U_{P^w}(F_v))K_v = \tilde{M}(F_v)\mathbf{s}(w)^{-1}\mathbf{s}(U_P(F_v))\mathbf{s}(w)K_v.$$

The associated integration formula follows in the usual fashion.

Suppose, for this paragraph only, that $w$ is the representative of $W_0^G$ such that the opposite parabolic subgroup $\bar{P}$ is equal to $P^w$. Then we obtain the Gelfand-Naimark decomposition, which is a decomposition of an open dense subset of $\tilde{G}(F_v)$ as

$$\mathbf{s}_w(U_{P^w}(F_v))\tilde{M}(F_v)\mathbf{s}(U_P(F_v)).$$

Its associated integration formula is given by

$$\int_{\tilde{G}(F_v)} \tilde{f}(x)dx = \gamma_{\tilde{M}} \int_{\mathbf{s}_w(U_{P^w}(F_v))} \int_{\tilde{M}(F_v)} \int_{\mathbf{s}(U_P(F_v))} \tilde{f}(\bar{u}\tilde{m}u)dud\tilde{m}d\bar{u},$$

where

$$\gamma_{\tilde{M}} = \int_{\mathbf{s}_w(U_{P^w}(F_v))} \delta_P(m_P(\mathbf{p}(\bar{u})))d\bar{u} = \gamma_M \tag{17}$$

$\bar{u} = u_P(\bar{u})m_P(\bar{u})k_P(\bar{u})$, $u_P(\bar{u}) \in U_P(F_v)$, $m_P(\bar{u}) \in M(F_v)$ and $k_P(\bar{u}) \in K_v$. The results of this paragraph do not rely on the assumption that $P \supset P_0$.

The abelian group,

$$\widetilde{A_M^{n/d}}(F_v) = \{\tilde{\gamma} \in \tilde{M}(F_v) : \mathbf{p}(\tilde{\gamma}) = \gamma^{n/d}, \gamma \in A_M(F_v)\},$$

lies in the center of $\tilde{M}(F_v)$ (Proposition 0.1.1 [KP84]) and provides a decomposition of $\mathbf{s}_w(U_{P^w}(F_v))$ by way of the adjoint action. Using this decomposition we may define a subset $\widetilde{A_M^{n/d}}^{-}(F_v)$ of $\widetilde{A_M^{n/d}}(F_v)$ as in §1.4 [Cas93]. This subset may be used to examine the asymptotic behavior of matrix coefficients (cf. §4 [Cas93]). The only other ingredient needed to prove the results concerning the asymptotic behavior of matrix coefficients for $\tilde{G}(F_v)$ is the Iwahori factorization for arbitrarily small compact open subgroups. These Iwahori factorizations exist in $\tilde{G}(F_v)$, because there exist arbitrarily small compact open subgroups of $G(F_v)$ over which $\tilde{G}(F_v)$ splits (§2 [FK86]), and Iwahori factorizations hold in $G(F_v)$ (Proposition 1.4.4 [Cas93]).

The only decomposition which still needs to be addressed for $\tilde{G}(F_v)$ is the Cartan decomposition. This may be recast as

$$\tilde{G}(F_v) = \bigcup_{\gamma} K_v \gamma \widetilde{A_{M_0}^{n/d}}(F_v) K_v,$$

where $\gamma$ runs over a set of representatives of $\widetilde{A_{M_0}}(F_v)/\widetilde{A_{M_0}^{n/d}}(F_v)$. This union is finite and disjoint. It is the finiteness which allows us to restrict our attention to $K_v \widetilde{A_{M_0}^{n/d}}(F_v) K_v$ when proving the convergence of integrals or bounds of certain functions.

We make one further remark concerning bounding functions on $\tilde{G}(F_v)$. If $\tilde{f}$ is a genuine or antigenuine function on $\tilde{G}(F_v)$ then clearly

$$\sup_{\tilde{\gamma} \in \tilde{G}(F_v)} |\tilde{f}(\tilde{\gamma})| = \sup_{\tilde{\gamma} \in \tilde{G}(F_v)} |\tilde{f}(\mathbf{p}(\tilde{\gamma}), 1)| = \sup_{\gamma \in G(F_v)} |\tilde{f}(\mathbf{s}(\gamma))|.$$

Therefore, in cases where one is interested in finding uniform bounds of such functions, the techniques of the non-metaplectic groups may be used.

This concludes the discussion of the properties necessary for the proof of the Plancherel formula. The proof may now be imitated after making some apparent definitions.

The normalization of intertwining operators amounts to the definition of functions,

$$r_{\tilde{Q}|\tilde{P}} : \Pi(\tilde{M}(F_S)) \times \mathfrak{a}^*_{M,\mathbf{C}} \to \mathbf{C}, \; Q, P \in \mathcal{P}(M),$$

which satisfy the conditions of Theorem 2.1 [Art89a]. These functions are called normalizing factors. Such normalizing factors exist for general linear groups ([Sha84], §4 [Art89a]). Since we may take the normalizing factors for $\Pi(\tilde{M}(F_S))$ to be the product of the normalizing factors for $\Pi(\tilde{M}(F_v))$, $v \in S$, it suffices to consider the case that $S$ consists of a single valuation. As mentioned above, we already have normalizing factors of the form $r_{Q|P}$. We define candidates for normalizing factors of metaplectic coverings by setting

$$r_{\tilde{Q}|\tilde{P}}(\tilde{\pi}_\lambda) = \Lambda^M(1)^{1/2}\Lambda^G(1)^{-1/2} r_{Q|P}(\tilde{\pi}'_{\lambda'}),$$

for all $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. As indicated in §2 II [AC89] and §§2, 4 [Art89a], there is only one required property which $r_{\tilde{Q}|\tilde{P}}$ does not obviously satisfy. Let $\Pi_{\mathrm{disc}}(\tilde{M}(F_v))$ be the subset of representations in $\Pi_{\mathrm{temp}}(\tilde{M}(F_v))$ which are square-integrable modulo the center of $\tilde{M}(F_v)$. The property which remains to be verified is

$$r_{\tilde{P}|\tilde{\bar{P}}}(\tilde{\pi}_\lambda) r_{\tilde{\bar{P}}|\tilde{P}}(\tilde{\pi}_\lambda) = J_{\tilde{P}|\tilde{\bar{P}}}(\tilde{\pi}_\lambda) J_{\tilde{\bar{P}}|\tilde{P}}(\tilde{\pi}_\lambda),\ \tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(F_v)),\ \lambda \in \mathfrak{a}^*_{M,\mathbf{C}}.$$

The right-hand side is, by definition, the inverse of the Harish-Chandra $\mu$-function $\mu_{\tilde{M}}$ (not to be confused with the group $\mu_n^M$).

**Lemma 5.1** *Suppose $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(F_v))$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. Then*

$$r_{\tilde{P}|\tilde{\bar{P}}}(\tilde{\pi}_\lambda) r_{\tilde{\bar{P}}|\tilde{P}}(\tilde{\pi}_\lambda) = \mu_{\tilde{M}}(\tilde{\pi}_\lambda)^{-1}.$$

*Proof.* The normalizing factors for $P$, as defined in [Sha84], already satisfy the condition of the lemma. Furthermore, $\mu_{\tilde{M}}$ is meromorphic in $\mathfrak{a}^*_{M,\mathbf{C}}$. Therefore it suffices to show that

$$\mu_{\tilde{M}}(\tilde{\pi}) = \Lambda^M(1)\Lambda^G(1)^{-1}\mu_M(\tilde{\pi}'),\ \tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(F_v)).$$

Suppose $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$ such that

$$\tilde{f}_{\tilde{M}_1}(\tilde{\pi}) = 0,\ \tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}_1(F_v)),\ M_1 \in \mathcal{L},\ M_1 \neq M.$$

It follows from the Plancherel formula for $\tilde{G}(F_v)$ that

$$\tilde{f}(1) = \gamma_{\tilde{M}} \sum_{\Pi_{\mathrm{disc}}(\tilde{M}(F_v))/i\mathfrak{a}^*_{M,\{v\}}} \int_{i\mathfrak{a}^*_{M,\{v\}}} d^{\tilde{M}}(\tilde{\pi}_\lambda)\mu_{\tilde{M}}(\tilde{\pi}_\lambda)\tilde{f}_{\tilde{M}}(\tilde{\pi}_\lambda)d\lambda,$$

where $d^{\tilde{M}}(\tilde{\pi}_\lambda)$ is the formal degree of $\tilde{\pi}_\lambda$. Suppose $f \in \mathcal{H}(G(F_v))$ such that $f_G = \tilde{f}'$ (Proposition 27.3 [FK86]). Then, by Lemma 4.3 and Lemma 4.4, we have

$$\tilde{f}(1) = I_{\tilde{G}}(1, \tilde{f}) = \Lambda^G(1)^{-1} I_G(1, f).$$

Combined with the Plancherel formula and equation (17), this equation implies that $\tilde{f}(1)$ is equal to the product of $\Lambda^G(1)^{-1}$ with

$$\begin{aligned}
&\gamma_M \sum_{\Pi_{\mathrm{disc}}(M(F_v))/i\mathfrak{a}^*_{M,\{v\}}} \int_{i\mathfrak{a}^*_{M,\{v\}}} d^M(\pi_{\lambda'})\mu_M(\pi_{\lambda'})\tilde{f}'_M(\pi_{\lambda'})d\lambda' \\
&= \gamma_{\tilde{M}} \sum_{\Pi_{\mathrm{disc}}(\tilde{M}(F_v))/i\mathfrak{a}^*_{M,\{v\}}} \int_{i\mathfrak{a}^*_{M,\{v\}}} d^M((\tilde{\pi}_\lambda)')\mu_M((\tilde{\pi}_\lambda)')\tilde{f}_{\tilde{M}}(\tilde{\pi}_\lambda)d\lambda' \\
&= n^{\dim(A_M)}\gamma_{\tilde{M}} \sum_{\Pi_{\mathrm{disc}}(\tilde{M}(F_v))/i\mathfrak{a}^*_{M,\{v\}}} \int_{i\mathfrak{a}^*_{M,\{v\}}} d^M((\tilde{\pi}_\lambda)')\mu_M((\tilde{\pi}_\lambda)')\tilde{f}_{\tilde{M}}(\tilde{\pi}_\lambda)d\lambda
\end{aligned}$$

Taking the difference of this equation from the earlier Plancherel expansion of $\tilde{f}(1)$ we find that

$$\int_{i\mathfrak{a}^*_{M,\{v\}}} \left(d^{\tilde{M}}(\tilde{\pi})\mu_{\tilde{M}}(\tilde{\pi}_\lambda) - d^M(\tilde{\pi}')n^{\dim(A_M)}\Lambda^G(1)^{-1}\mu_M(\tilde{\pi}'_{\lambda'})\right)\tilde{f}_{\tilde{M}}(\tilde{\pi}_\lambda)d\lambda$$

vanishes for all $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(F_v))$. An application of the trace Paley-Wiener formula (for metaplectic coverings) to this equation implies that

$$d^{\tilde{M}}(\tilde{\pi})\mu_{\tilde{M}}(\tilde{\pi}) = d^M(\tilde{\pi}')n^{\dim(A_M)}\Lambda^G(1)^{-1}\mu_M(\tilde{\pi}'),\ \tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(F_v)).$$

Notice that $\mu_{\tilde{G}} = \mu_G = 1$ and so

$$d^{\tilde{G}}(\tilde{\pi}_0) = n\Lambda^G(1)^{-1}d^G(\tilde{\pi}'_0),\ \tilde{\pi}_0 \in \Pi_{\mathrm{disc}}(\tilde{G}(F_v)).$$

It follows in general that

$$d^{\tilde{M}}(\tilde{\pi}) = \prod_{i=1}^{\ell} d^{\tilde{M}(i)}(\tilde{\pi}_i) = \prod_{i=1}^{\ell} n\Lambda^{M(i)}(1)^{-1}d^{M(i)}(\tilde{\pi}'_i) = n^{\dim(A_M)}\Lambda^M(1)^{-1}d^M(\tilde{\pi}'),$$

where $M = M(1) \times \cdots \times M(\ell)$ is decomposition (4) and $\tilde{\pi}_1, \ldots, \tilde{\pi}_\ell$ are as in the Appendix. The lemma is now complete.□

# 6 The distribution $I^{\Sigma}_M(\gamma)$

The invariant trace formula for $G$ contains invariant distributions of the form $I_M(\gamma)$ parameterized by $\gamma \in M(F_S)$. These distributions were introduced in

§2 [Art88a]. The reader is assumed to be familiar with their definitions and properties. Our goal is to compare these distributions with the analogous distributions on $\tilde{G}(F_S)$ using the orbit map. The orbit map is not injective. Consequently we must group the distributions $I_M(\gamma)$ in terms of the fibers of the orbit map. Since the orbit map raises elements of $\tilde{G}(F_S)$ to the $n$th power, a good guess would be that we should group the distributions in terms of the $n$th roots of unity $\mu_n$. According to decomposition (4),

$$M(F_S) \cong \mathrm{GL}(r_1, F_S) \times \cdots \times \mathrm{GL}(r_\ell, F_S).$$

The center of $\mathrm{GL}(r_i, F_S)$ is isomorphic to $\mathrm{GL}(1, F_S)$. In consequence,

$$A_M(F_S) \cong \prod_{i=1}^{\ell} \mathrm{GL}(1, F_S).$$

Let $\mu_n^M$ be the finite subgroup of $A_M(F_S)$ corresponding to the subgroup generated by the diagonal embedding of $\prod_{i=1}^{\ell} \mu_n$ into $\prod_{i=1}^{\ell} \mathrm{GL}(1, F_S)$. Suppose $\gamma \in M(F_S)$, $\eta \in \mu_n^M$ and that $(\eta\gamma)'$ is defined. Then by Theorem 4.1 (iii) [KP86] we have

$$(\eta\gamma)' = \mathbf{s}((\eta\gamma)^n)\kappa_v((\eta\gamma)^n)^{-1} = \gamma'. \tag{18}$$

This equation suggests that we ought to group our distributions into $\mu_n^M$-invariant sums. The most obvious method is to take the sum,

$$\sum_{\eta \in \mu_n^M} I_M(\eta\gamma, f), \;\; f \in \mathcal{H}(M(F_S)).$$

This grouping has the shortcoming that if $M = G$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$ then

$$\sum_{\eta \in \mu_n^G} \hat{I}_G(\eta\gamma, \tilde{f}') = n\Lambda^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}) \neq \Lambda^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}), \gamma \in \cap G_{\mathrm{oreg}}(F_S).$$

It would be convenient to correct this shortcoming by replacing the sum over $\mu_n^M$ with the sum over $\mu_n^M/\mu_n^G$. The following two lemmas show that this is indeed possible. We assume the reader is familiar with the spaces $\mathcal{H}_{\mathrm{ac}}(M(F_S))$ and $\mathcal{I}_{\mathrm{ac}}(M(F_S))$ of §11 [Art89a] and their respective subspaces of moderate functions.

**Lemma 6.1** *Suppose $L \in \mathcal{L}(M)$ and $\phi$ is a moderate function in $\mathcal{I}_{\mathrm{ac}}(M(F_S))$ such that $\phi(\pi)$ vanishes for all $\pi \in \Pi_{\mathrm{temp}}(M(F_S))$ whose central character is not trivial on $\mu_n^L$. Then there exists a moderate function $f \in \mathcal{H}_{\mathrm{ac}}(M(F_S))$ such that $f_M = \phi$ and $f$ is invariant under translation by $\mu_n^L$.*

*Proof.* By Lemma 6.1 [Art88a] there exists a moderate function $h \in \mathcal{H}_{\text{ac}}(M(F_S))$ such that $h_M = \phi$. Given $\eta \in \mu_n^L$, set

$$h^\eta(\gamma) = h(\eta\gamma),\ \gamma \in M(F_S).$$

Since $\mu_n^L$ is contained in the center of $M(F_S)$ and the nonzero matrix entries of $\eta \in \mu_n^L$ have absolute value one with respect to any valuation in $S$, the $K_S$-finiteness, support and growth conditions of $h^\eta$ are identical with those of $h$. Thus, the function given by

$$f(\gamma) = |\mu_n^L|^{-1} \sum_{\eta \in \mu_n^L} h^\eta(\gamma),\ \gamma \in M(F_S),$$

is a moderate function in $\mathcal{H}_{\text{ac}}(M(F_S))$. It is simple to see that $f$ satisfies the remaining properties asserted by the lemma.□

**Lemma 6.2** *Suppose $L \in \mathcal{L}(M)$, $\eta \in \mu_n^L$ and $\tilde{f} \in \mathcal{H}(\tilde{L}(F_S))$. Then*

$$\hat{I}_M^L(\eta\gamma, \tilde{f}') = \hat{I}_M^L(\gamma, \tilde{f}'),\ \gamma \in M(F_S).$$

*Proof.* Suppose $\gamma \in M(F_S)$ such that $M_\gamma(F_S) = G_\gamma(F_S)$. By Lemma 6.1 we may choose $f \in \mathcal{H}(L(F_S))$ such that $f_L = \tilde{f}'$ and $f$ is invariant under $\mu_n^L$. If $L = M$ then the lemma follows immediately from the $\mu_n^L$-invariance of $f$ and $\mu_n^L \subset A_L(F_S)$. Assume inductively that the lemma holds if $L$ is replaced by $L_1 \in \mathcal{L}^L(M)$ such that $L_1 \subsetneqq L$. By definition

$$\begin{aligned} \hat{I}_M^L(\eta\gamma, \tilde{f}') &= I_M^L(\eta\gamma, f) \\ &= J_M^L(\eta\gamma, f) - \sum_{\{L_1 \in \mathcal{L}^L(M) : L_1 \subsetneqq L\}} \hat{I}_M^{L_1}(\eta\gamma, \phi_{L_1}^L(f)). \end{aligned}$$

The first term on the right-hand side satisfies

$$J_M^L(\eta\gamma, f) = |D^L(\eta\gamma)|_S^{1/2} \int_{L_{\eta\gamma}(F_S)\backslash L(F_S)} f(x^{-1}\eta\gamma x) v_M^L(x) dx = J_M^L(\gamma, f),$$

as $f$ is $\mu_n^L$-invariant and $\mu_n^L \subset A_L(F_S)$. As for the remaining terms on the right-hand side, we know by Theorem 12.1 [Art89a] that $\phi_{L_1}^L(f)$ is a moderate function in $\mathcal{I}_{\text{ac}}(L_1(F_S))$. Furthermore $\phi_{L_1}^L(f)$ is defined by

$$\phi_{L_1}^L(f)(\pi) = \operatorname{tr}\left(\mathcal{R}_{L_1}^L(\pi)\pi^L(f)\right),\ \pi \in \Pi_{\text{temp}}(L_1(F_S)),$$

for a certain linear operator $\mathcal{R}^L_{L_1}(\pi)$. Obviously,

$$\phi^L_{L_1}(f)(\pi) = \operatorname{tr}\left(\mathcal{R}^L_{L_1}(\pi)\int_{\mu_n^L\backslash L(F_S)} f(x)\sum_{\eta\in\mu_n^L}\pi^L(\eta x)dx\right),\ \pi\in\Pi_{\text{temp}}(L_1(F_S)).$$

It follows that $\phi^L_{L_1}(f)$ satisfies the hypotheses of Lemma 6.1 and so there exists a moderate function $h \in \mathcal{H}_{\text{ac}}(L_1(F_S))$ such that $h_{L_1} = \phi^L_{L_1}(f)$ and $h$ is invariant under $\mu_n^L$. Apparently, $\mu_n^L \subset \mu_n^{L_1}$, so by induction

$$\hat{I}^{L_1}_M(\eta\gamma, \phi^L_{L_1}(f)) = I^{L_1}_M(\eta\gamma, h) = I^{L_1}_M(\gamma, h) = \hat{I}^{L_1}_M(\gamma, \phi^L_{L_1}(f)).$$

In consequence, the lemma holds if $M_\gamma(F_S) = G_\gamma(F_S)$. For arbitrary $\gamma \in M(F_S)$ we have ((2.2) [Art88a])

$$\begin{aligned} I^L_M(\eta\gamma, f) &= \lim_{\{a\to 1: a\in A_{M,\text{reg}}(F_S)\}} \sum_{L_1\in\mathcal{L}^L(M)} r^{L_1}_M(\eta\gamma, a) I^L_{L_1}(a\eta\gamma, f) \\ &= \lim_{\{a\to 1: a\in A_{M,\text{reg}}(F_S)\}} \sum_{L_1\in\mathcal{L}^L(M)} r^{L_1}_M(\eta\gamma, a) I^L_{L_1}(a\gamma, f), \end{aligned}$$

where $A_{M,\text{reg}}(F_S)$ is the subset of elements in $A_M(F_S)$ whose centralizer in $G$ is contained in $M$. The function $r^L_M(\gamma, a)$, defined in §5 [Art88c], is easily seen to be invariant under translation by $A_L(F_S)$ in the first variable. In particular we have $r^L_M(\eta\gamma, a) = r^L_M(\gamma, a)$. The lemma now follows.□

Suppose $\gamma \in M(F_S)$. By Lemma 6.2, it makes sense to define the distribution $I^\Sigma_M(\gamma)$ on $\tilde{G}(F_S)$ by

$$I^\Sigma_M(\gamma, \tilde{f}) = \sum_{\eta\in\mu_n^M/\mu_n^G} \hat{I}_M(\eta\gamma, \tilde{f}'),\ \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

More generally, if $L \in \mathcal{L}(M)$ then

$$I^{L,\Sigma}_M(\gamma, \tilde{f}) = \sum_{\eta\in\mu_n^M/\mu_n^L} \hat{I}^L_M(\eta\gamma, \tilde{f}'),\ \ \tilde{f} \in \mathcal{H}(\tilde{L}(F_S)).$$

Two important properties that $I_M(\gamma)$ satisfies are descent (§8 [Art88a]) and splitting (§9 [Art88a]). We show that $I^\Sigma_M(\gamma)$ satisfies these properties as well.

**Lemma 6.3** *Suppose $M$ and $M_1$ belong to $\mathcal{L}$ and $M_1 \subset M$. Suppose further that $L \in \mathcal{L}(M_1)$ such that $d^G_{M_1}(M, L) \neq 0$. Then the map,*

$$\mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G \to \mu_n^{M_1}/\mu_n^G,$$

*given by $(\eta_1\mu_n^G, \eta_2\mu_n^G) \mapsto \eta_1\eta_2\mu_n^G$, is an isomorphism.*

*Proof.* If $d^G_{M_1}(M, L) \neq 0$ as above, then by definition, $\mathfrak{a}^M_{M_1} \oplus \mathfrak{a}^L_{M_1} \cong \mathfrak{a}^G_{M_1}$ (§7 [Art88a]). The vector spaces $\mathfrak{a}^G_M$ and $\mathfrak{a}^G_L$ may be regarded as the respective orthogonal complements of $\mathfrak{a}^M_{M_1}$ and $\mathfrak{a}^L_{M_1}$ in $\mathfrak{a}^G_{M_1}$. As a consequence we also have $\mathfrak{a}^G_M \oplus \mathfrak{a}^G_L \cong \mathfrak{a}^G_{M_1}$. Consider the homomorphism

$$H_{M_1} : M_1(F_S) \to \mathfrak{a}_{M_1}.$$

It is readily verified that it passes to a homomorphism

$$H_{A_{M_1}} : A_{M_1}(F_S)/A_G(F_S) \to \mathfrak{a}^G_{M_1}$$

such that $H_{A_{M_1}}(A_M(F_S)/A_G(F_S)) \subset \mathfrak{a}^G_M$ and $H_{A_{M_1}}(A_L(F_S)/A_G(F_S)) \subset \mathfrak{a}^G_L$. Accordingly

$$H_{A_{M_1}}((A_M(F_S) \cap A_L(F_S))/A_G(F_S)) \subset \mathfrak{a}^G_M \cap \mathfrak{a}^G_L = 0.$$

In other words, $|\xi(\gamma)| = 1$ for all $\gamma$ belonging to the split torus $A_M(F_S) \cap A_L(F_S)$, and all characters $\xi \in X(M_1)$ which are trivial when restricted to $G$. This implies that $A_L(F_S) \cap A_M(F_S) \subset A_G(F_S)$. As a result, the multiplication map

$$A_M(F_S)/A_G(F_S) \times A_L(F_S)/A_G(F_S) \to A_{M_1}(F_S)/A_G(F_S)$$

is injective. It now follows from the commutative diagram,

$$\begin{array}{ccc} \mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G & \hookrightarrow & A_M(F_S)/A_G(F_S) \times A_L(F_S)/A_G(F_S) \\ \downarrow & & \downarrow \\ \mu_n^{M_1}/\mu_n^G & \hookrightarrow & A_{M_1}(F_S)/A_G(F_S) \end{array}$$

that the map of the lemma is injective. The surjectivity of the map can be seen from the following equalities.

$$|\mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G| = n^{\dim(\mathfrak{a}^G_M)} n^{\dim(\mathfrak{a}^G_L)} = n^{\dim(\mathfrak{a}^G_{M_1})} = |\mu_n^{M_1}/\mu_n^G|. \square$$

**Lemma 6.4** *Let $M$, $M_1$ and $L$ be as in Lemma 6.3. The map*

$$\mu_n^M/\mu_n^G \to \mu_n^{M_1}/\mu_n^L,$$

*given by $\eta\mu_n^G \mapsto \eta\mu_n^L$ for $\eta \in \mu_n^M$, is an isomorphism.*

*Proof.* The proof of this lemma follows from arguments similar to those of Lemma 6.3.□

The following proposition proves a descent property for $I_M^\Sigma(\gamma)$. For this we need the notion of an induced space of orbits. Given $\gamma \in M(F_S)$ define the induced space $\gamma^G$ to be the union of the $G(F_S)$-conjugacy classes which intersect $\gamma U_P(F_S)$ in an open set. Here $P \in \mathcal{P}(M)$ is arbitrary (§6 [Art88c]). Since $\gamma^G$ is contained in a single geometric conjugacy class and $\mathrm{GL}(r)$ is stable, the induced space $\gamma^G$ is a single conjugacy class which we call the induced conjugacy class of $\gamma$.

**Proposition 6.1** *Suppose $M$ and $M_1$ belong to $\mathcal{L}$ with $M_1 \subset M$. Moreover suppose $\gamma \in M_1(F_S)$. Then*

$$I_M^\Sigma(\gamma^M, \tilde{f}) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L)\hat{I}_{M_1}^{L,\Sigma}(\gamma, \tilde{f}_{\tilde{L}}), \;\; \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

*Proof.* Suppose $\eta \in \mu_n^M$. Since $\eta$ lies in the center of $M(F_S)$,

$$\eta\gamma^M = \{\eta\gamma_1 : \gamma_1 \in \gamma^M\}$$

is a conjugacy class in $M(F_S)$. Suppose $P \in \mathcal{P}^M(M_1)$. Clearly, left multiplication by $\eta$ is a homeomorphism between $\gamma U_P(F_S)$ and $\eta\gamma U_P(F_S)$. It follows that $\eta\gamma^M$ is a conjugacy class of $M(F_S)$ which intersects $\eta\gamma U_P(F_S)$ in an open set. In other words $\eta\gamma^M = (\eta\gamma)^M$. The descent property for $I_M(\gamma)$ (Theorem 8.1 [Art88a]) together with Lemma 6.4 imply that

$$\begin{aligned}
I_M^\Sigma(\gamma^M, \tilde{f}) &= \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M(\eta\gamma^M, \tilde{f}') \\
&= \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M((\eta\gamma)^M, \tilde{f}') \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_{M_1}^L(\eta\gamma, \tilde{f}'_L) \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \sum_{\eta \in \mu_n^{M_1}/\mu_n^L} \hat{I}_{M_1}^L(\eta\gamma, \tilde{f}'_L) \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L)\hat{I}_{M_1}^{L,\Sigma}(\gamma, \tilde{f}_{\tilde{L}}).\square
\end{aligned}$$

**Corollary 6.1** *Suppose $M$ and $M_1$ belong to $\mathcal{L}$ with $M_1 \subset M$. Moreover suppose $\gamma \in M_1(F_S)$ and $M_{1,\gamma} = M_\gamma$. Then*

$$I_M^\Sigma(\gamma, \tilde{f}) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \hat{I}_{M_1}^{L,\Sigma}(\gamma, \tilde{f}_{\tilde{L}}), \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

**Proposition 6.2** *Suppose $S$ is the disjoint union of nonempty sets $S_1$ and $S_2$, and that $\tilde{f} = \tilde{f}_1 \tilde{f}_2 \in \mathcal{H}(\tilde{G}(F_S))$, $\gamma = \gamma_1\gamma_2 \in M(F_S)$ are corresponding decompositions. Then*

$$I_M^\Sigma(\gamma, \tilde{f}) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \hat{I}_M^{L_1,\Sigma}(\gamma_1, \tilde{f}_{1,\tilde{L_1}}) \hat{I}_M^{L_2,\Sigma}(\gamma_2, \tilde{f}_{2,\tilde{L_2}}).$$

*Proof.* We begin by applying the splitting property to the summands of $I_M^\Sigma(\gamma)$.

$$\begin{aligned} I_M^\Sigma(\gamma, \tilde{f}) &= \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M(\eta\gamma, \tilde{f}') \\ &= \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M^{L_1}(\eta\gamma_1, \tilde{f}'_{1,L_1}) \hat{I}_M^{L_2}(\eta\gamma_2, \tilde{f}'_{2,L_2}). \end{aligned}$$

Suppose $d_M^G(L_1, L_2)$ is not zero. By using arguments similar to those of the proof of Lemma 6.3, it may be established that

$$\mu_n^M/\mu_n^G \cong \mu_n^M/\mu_n^{L_1} \times \mu_n^M/\mu_n^{L_2},$$

$$\mu_n^M/\mu_n^G \cong \mu_n^{L_1}/\mu_n^G \times \mu_n^{L_2}/\mu_n^G,$$

and $\mu_n^{L_1} \cap \mu_n^{L_2} = \mu_n^G$. From $\mu_n^{L_1} \cap \mu_n^{L_2} = \mu_n^G$ it follows that the homomorphism

$$\mu_n^{L_1}/\mu_n^G \to \mu_n^M/\mu_n^{L_2},$$

given by $\eta\mu_n^G \mapsto \eta\mu_n^{L_2}$, is injective. This homomorphism is also surjective as

$$|\mu_n^{L_1}/\mu_n^G| = n^{\dim(\mathfrak{a}_M^{L_1}) - \dim(\mathfrak{a}_M^G)} = n^{\dim(\mathfrak{a}_M^{L_2})} = |\mu_n^M/\mu_n^{L_2}|.$$

It may be deduced in the same manner that $\mu_n^{L_2}/\mu_n^G \cong \mu_n^M/\mu_n^{L_1}$. Thus the previous sum is equal to

$$\sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta_1 \in \mu_n^M/\mu_n^{L_1}} \sum_{\eta_2 \in \mu_n^M/\mu_n^{L_2}} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1, \tilde{f}'_{1,L_1}) \hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2, \tilde{f}'_{2,L_2})$$

$$
\begin{aligned}
&= \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) \sum_{\eta_1\in\mu_n^{L_2}/\mu_n^G}\ \sum_{\eta_2\in\mu_n^{L_1}/\mu_n^G} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1,\tilde{f}'_{1,L_1})\hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2,\tilde{f}'_{2,L_2}) \\
&= \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) \sum_{\eta_1\in\mu_n^{L_2}/\mu_n^G} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1,\tilde{f}'_{1,L_1}) \sum_{\eta_2\in\mu_n^{L_1}/\mu_n^G} \hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2,\tilde{f}'_{2,L_2}) \\
&= \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) \sum_{\eta_1\in\mu_n^{M}/\mu_n^{L_1}} \hat{I}_M^{L_1}(\eta_1\gamma_1,\tilde{f}'_{1,L_1}) \sum_{\eta_2\in\mu_n^{M}/\mu_n^{L_2}} \hat{I}_M^{L_2}(\eta_2\gamma_2,\tilde{f}'_{2,L_2}) \\
&= \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2)\hat{I}_M^{L_1,\Sigma}(\gamma_1,\tilde{f}_{1,\tilde{L_1}})\hat{I}_{M,\Sigma}^{L_2}(\gamma_2,\tilde{f}_{2,\tilde{L_2}}).\square
\end{aligned}
$$

# 7 The distribution $I_M^{\mathcal{M}}(\gamma)$

From here on we assume that the invariant trace formula of [Art88b] is valid for metaplectic coverings of the general linear group. In particular, we are assuming that there exist invariant distributions $I_{\tilde{M}}(\tilde{\gamma})$ parameterized by $\tilde{\gamma}\in\tilde{M}(F_S)$, which are defined in the same fashion as $I_M(\gamma)$ of the previous section, and also satisfy splitting and descent properties. We would like to compare these distributions to $I_M^{\Sigma}(\gamma)$. To this end, we define

$$I_M^{\mathcal{M}}(\gamma,\tilde{f}) = \Lambda^M(\gamma)I_{\tilde{M}}(\gamma',\tilde{f}),\ \tilde{f}\in\mathcal{H}(\tilde{G}(F_S)),$$

for all $\gamma\in M(F_S)$ in the domain of the orbit map such that $M_\gamma(F_S) = G_\gamma(F_S)$. This definition is very similar to the definition of $I_M^{\mathcal{E}}(\gamma)$ of (3.8) II [AC89]. Indeed $I_M^{\mathcal{M}}(\gamma)$ shares most of the properties of $I_M^{\mathcal{E}}(\gamma)$ listed in §3 II [AC89]. We list them below and invite the reader to verify that the proofs of that section apply to $I_M^{\mathcal{M}}(\gamma)$. From now on we shall tacitly assume that $\gamma$ belongs to the domain of the orbit map whenever $\gamma'$ appears.

A mild paraphrase of Lemma 3.1 and Corollary 3.2 II [AC89] implies that

$$(19)\qquad \lim_{\{a\to 1: a\in A_{M,\mathrm{reg}}(F_S)\}} \sum_{L\in\mathcal{L}(M)} r_M^L(\gamma^n,a)I_L^{\mathcal{M}}(a\gamma,\tilde{f}),\ \gamma\in M(F_S)$$

exists and is equal to

$$(20)\qquad \Lambda^M(\gamma)\sum_{L\in\mathcal{L}(M)} c_M^L(\gamma^n,n)I_{\tilde{L}}((\gamma^L)',\tilde{f}).$$

The functions $c_M^L(\gamma,n)$ are derived from the $(G,M)$ family (§6 [Art81]),

$$c_P(\nu,\gamma,n) = \prod_{v\in S}\prod_\beta |n|_v^{-\rho(\beta,u_v)\nu(\beta^\vee)/2},\ P\in\mathcal{P}(M),$$

where $\gamma = \prod_{v\in S} \gamma_v$ has Jordan decomposition $\prod_{v\in S} \sigma_v u_v$, and the remaining terms stem from the $(G, M)$ family $r_P(\nu, \gamma, a)$ defined in §5 [Art88c]. We define $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ to be equal to (19) for general $\gamma \in M(F_S)$. It is clear from the proof of Corollary 3.4 II [AC89] that if $\gamma \in M(F)$ is embedded diagonally into $M(F_S)$ and $S$ contains $\{v : |n|_v \neq 1\}$ then

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = I_{\tilde{M}}(\gamma', \tilde{f}),\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)). \tag{21}$$

The descent property,

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = \sum_{L\in\mathcal{L}(M_1)} d_{M_1}^G(M, L) \hat{I}_{M_1}^{L,\mathcal{M}}(\gamma, \tilde{f}_{\tilde{L}}), \tag{22}$$

for $\gamma \in M_1(F_S)$ with $M_1 \in \mathcal{L}^M$ and $M_{1,\gamma} = M_\gamma$; and the splitting property

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1, L_2) \hat{I}_M^{L_1,\mathcal{M}}(\gamma_1, \tilde{f}_{1,\tilde{L_1}}) \hat{I}_M^{L_2,\mathcal{M}}(\gamma_2, \tilde{f}_{2,\tilde{L_2}}), \tag{23}$$

for $\gamma = \gamma_1\gamma_2 \in M(F_S)$ and $\tilde{f} = \tilde{f}_1\tilde{f}_2$ as in Proposition 6.2 also hold. This is immediate for $\gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S)$ and can be established for general $\gamma \in M(F_S)$ by using Corollary 3.2 [Art88a] (see also the end of the proof of Theorem 8.1 [Art88a]).

# 8 Geometric vanishing properties

As alluded to earlier, we wish to compare $I_M^{\mathcal{M}}(\gamma)$ with $I_M^{\Sigma}(\gamma)$. At least in the case $M = G$, our wish is partially fulfilled as

$$I_G^{\Sigma}(\gamma, \tilde{f}) = \hat{I}_G(\gamma, \tilde{f}') = \Lambda^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}) = I_G^{\mathcal{M}}(\gamma, \tilde{f})$$

for $\gamma \in G_{\mathrm{oreg}}(F_S)$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. One might wonder whether, in our definition of $I_G^{\mathcal{M}}(\gamma)$, we are neglecting to take into account the distributions $I_{\tilde{G}}(\tilde{\gamma})$ for which $\tilde{\gamma} \in \tilde{G}_{\mathrm{reg}}(F_S)$ does not lie in the image of the orbit map. In this regard, Proposition 3 [FK86] tells us that if $\tilde{\gamma} \in \tilde{G}_{\mathrm{reg}}(F_S)$ then $I_{\tilde{G}}(\tilde{\gamma})$ is zero unless $\mathbf{p}(\tilde{\gamma}) = \gamma_0^{n/r}\gamma^n$ for some $\gamma_0 \in A_G(F_S)$ and $\gamma \in G_{\mathrm{oreg}}(F_S)$. This is a local vanishing property for $I_{\tilde{G}}(\tilde{\gamma})$. We shall extend this local vanishing property to the case $M \neq G$ and arbitrary $\tilde{\gamma} \in \tilde{M}(F_S)$. However, our real interest lies in proving a global vanishing property, for which the local vanishing property is just a prelude.

**Lemma 8.1** *Suppose $v$ is nonarchimedean, $\tilde{\sigma}$ is an element of $\tilde{M}(F_v)$ such that $\sigma = \mathbf{p}(\tilde{\sigma})$ is semisimple, and $I_{\tilde{M}}(\tilde{\sigma}, \tilde{f}) \neq 0$ for some $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$. Then there exist elements $\gamma_0 \in A_G(F_v)$ and $\gamma \in M(F_v)$ such that $\sigma = \gamma_0^{n/d}\gamma^n$.*

*Proof.* We will first prove the lemma for $M = G$. After inverting the germ expansion for orbital integrals at $\tilde{\sigma}$ (Corollaire 2.4, §3.3 [Vig81]), we find that $I_{\tilde{G}}(\tilde{\sigma}, \tilde{f})$ is a finite linear combination of nonzero orbital integrals of $\tilde{f}$ at elements in $\tilde{G}_{\tilde{\sigma}}(F_v) \cap \tilde{G}_{\mathrm{reg}}(F_v)$ close to $\tilde{\sigma}$. Let $\tilde{\delta}$ be one of these elements. By Proposition 3 [FK86], $\mathbf{p}(\tilde{\delta})$ is equal to $\gamma_0^{n/d}\delta^n$ for some $\gamma_0 \in A_G(F_v)$ and $\delta \in G_{\mathrm{oreg}}(F_v)$. By a standard argument (§1 [KP86]) there exist extensions $E_1, \ldots, E_k$ of $F_v$ such that $G_\delta(F_v)$ is isomorphic to $\prod_{i=1}^k E_i^\times$. Let $\sigma_i$ and $\delta_i$, $1 \leq i \leq k$, be the respective $i$th components of $\sigma$ and $\delta$ with respect to this isomorphism and regard $\gamma_0$ as an element of $F_v^\times \subset E_i^\times$. Applying Hensel's lemma to the polynomial $X^n - \gamma_0^{-n/d}\sigma_i \in E_i[X]$, we obtain an element $\gamma_i \in E_i$ such that $|\gamma_i - \delta_i|_{E_i} < 1$ and $\gamma_0^{n/d}\gamma_i^n = \sigma_i$, $1 \leq i \leq k$. It is now clear that the element $\gamma \in G_\delta(F_v)$ which maps to $(\gamma_1, \ldots, \gamma_k)$ under the above isomorphism and $\gamma_0$ are the desired elements of the lemma in the case $M = G$. For the general case, note that we are assuming that the metaplectic version of (2.3) [Art88a] holds. Namely, if $\tilde{M}_{\tilde{\sigma}}(F_v) = \tilde{G}_{\tilde{\sigma}}(F_v)$ then there exists $\tilde{h} \in C_c^\infty(\tilde{M}(F_v))$ such that for all $\tilde{\gamma} \in \tilde{M}_{\tilde{\sigma}}(F_v)$ close to $\tilde{\sigma}$

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = I_{\tilde{M}}^{\tilde{M}}(\tilde{\gamma}, \tilde{h}).$$

Since the $K_v$-finiteness of $\tilde{f}$ did not play role in the proof of the case $M = G$, that case together with the substitution $\tilde{\gamma} = \tilde{\sigma}$ in this equation imply the lemma whenever $\tilde{M}_{\tilde{\sigma}}(F_v) = \tilde{G}_{\tilde{\sigma}}(F_v)$. The lemma now follows for general $\tilde{\sigma}$ from the definition ((2.2*) [Art88a]),

$$I_{\tilde{M}}(\tilde{\sigma}, \tilde{f}) = \lim_{\{a \to 1 : a \in A_{M,\mathrm{reg}}(F_v)\}} \sum_{L \in \mathcal{L}(M)} r_M^L(\mathbf{p}(\tilde{\sigma}), a^n) I_{\tilde{L}}(\mathbf{s}(a^n)\tilde{\sigma}, \tilde{f}),$$

and the fact that

$$\tilde{L}_{\mathbf{s}(a^n)\tilde{\sigma}}(F_v) = \tilde{M}_{\mathbf{s}(a^n)\tilde{\sigma}}(F_v) = \tilde{G}_{\mathbf{s}(a^n)\tilde{\sigma}}(F_v)$$

for $a \in A_{M,\mathrm{reg}}(F_v)$ close to the identity.□

It is unfortunate that Lemma 8.1 can not be extended to general elements in $\tilde{M}(F_v)$ without some additional hypotheses. To see this, suppose $F_v$ is nonarchimedean, $n = r = 2$, $m = 0$, $x, y \in F_v^\times$, $x \notin F_v^{\times 2}$ and

$\tilde{\gamma} = \mathbf{s}\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$. Using Proposition 0.1.5 [KP84] and equation (1), one can compute $\tilde{G}_{\tilde{\gamma}}(F_v)$ and show that it is equal to

$$\mathbf{p}^{-1}\left(G_{\mathbf{p}(\tilde{\gamma})}(F_v)\right) = \mathbf{p}^{-1}\left(\left\{\begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix} : z_1 \in F_v^{\times},\ z_2 \in F_v\right\}\right).$$

It follows from §1.i [Vig81] that $I_{\tilde{G}}(\tilde{\gamma}, \tilde{f}) \neq 0$ for some $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$. On the other hand, it is simple to show that if $\mathbf{p}(\tilde{\gamma}) = \gamma^2$ for some $\gamma \in G(F_v)$ then $\gamma$ must lie in $P_0(F_v)$ and in turn $x$ must lie in $F_v^{\times 2}$. Our assumption on $x$ therefore excludes this possibility.

The pathology in this example stems from the fact that $r$ divides $n$. In fact, one can construct similar examples for any $n$ and $r$ for which $\gcd(i,n) \neq 1$, for some $1 < i \leq r$. Thus, $n$ must be relatively prime to the positive integers less than or equal to $r$ if Lemma 8.1 is to be generalized. The following lemma converts this condition on $n$ into properties which are compatible with the structure of $G(F_v)$.

**Lemma 8.2** *Suppose that $v$ is nonarchimedean and that $E/F_v$ is a finite extension such that $n$ is relatively prime to $[E : F_v]$. If $x \in E^{\times}$ such that $N_{E/F_v}(x) \in F_v^{\times n}$ then $x \in E^{\times n}$.*

*Proof.* We first show that $E^{\times}/E^{\times n} \cong F_v^{\times}/F_v^{\times n}$ and that we may take coset representatives of $E^{\times}/E^{\times n}$ to lie in $F_v^{\times}$. The homomorphism, $F_v^{\times}/F_v^{\times n} \to E^{\times}/E^{\times n}$, given by

$$zF_v^{\times n} \mapsto zE^{\times n},\ z \in F_v^{\times},$$

is injective. Indeed, suppose $z$ does not belong to $F^{\times n}$, but does belong to $E^{\times n}$. By Theorem 10 (b) VIII §6 [Lan84], $[F(z^{1/n}) : F_v]$ divides $n$. Moreover $z^{1/n} \in E$, so

$$[E : F_v] = [E : F_v(z^{1/n})][F_v(z^{1/n}) : F_v].$$

This contradicts $\gcd(n, [E : F_v]) = 1$. The surjectivity of this map follows at once from the fact that (Lemma 0.3.2 [KP84])

$$|E^{\times}/E^{\times n}| = |F_v^{\times}/F_v^{\times n}| = n^2/|n|_v. \tag{24}$$

The enunciation of the lemma amounts to showing the injectivity of the homomorphism given by

$$xE^{\times n} \mapsto (N_{E/F_v}(x))F_v^{\times n} = x^{[E:F_v]}F_v^{\times n},\ x \in F_v^{\times}.$$

If $x^{[E:F_v]}F_v^{\times n} = F_v^{\times n}$ and $x \notin F_v^{\times n}$, then $[E : F_v]$ must divide $n^2/|n|_v$ by (24), thereby contradicting $\gcd([E : F_v], n) = 1$. Thus this map is injective. $\square$

**Lemma 8.3** *Suppose $v$ is nonarchimedean and $n$ is relatively prime to $r - 1 + 2rm$ and all positive integers less than or equal to $r$. Moreover suppose that $\tilde{\gamma} \in \tilde{M}(F_v)$ such that $I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) \neq 0$ for some $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$. Then there exists $\gamma \in M(F_v)$ such that $\mathbf{p}(\tilde{\gamma}) = \gamma^n$.*

*Proof.* Let $\sigma u$ be the Jordan decomposition of $\mathbf{p}(\tilde{\gamma})$. Suppose $M = G$. In this case, by the arguments of §1 [KP86], we may take $\sigma$ to be a block diagonal matrix of the form

$$\begin{pmatrix} \sigma_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \sigma_1 & & & & \\ & & & \ddots & & & \\ & & & & \sigma_k & & \\ & & & & & \ddots & \\ 0 & & & & & & \sigma_k \end{pmatrix},$$

where $\sigma_i \in \mathrm{GL}(m_i, F_v)$, $1 \leq i \leq k$ are pairwise distinct, generate respective extensions $F_i/F_v$ of degree $m_i$ and each appear $b_i$ times. We proceed by showing that $\sigma_i$ is an $n$th power in $\mathrm{GL}(m_i, F_v)$. Our hypothesis on $\tilde{\gamma}$ is equivalent to $\tilde{G}_{\tilde{\gamma}}(F_v) = \mathbf{p}^{-1}(G_{\mathbf{p}(\tilde{\gamma})})(F_v)$ (cf. 1.i [Vig81]). The set $\mathbf{p}^{-1}(G_{\mathbf{p}(\tilde{\gamma})})(F_v)$ contains the diagonal matrices of the form

$$\tilde{\delta} = \mathbf{s} \begin{pmatrix} \delta_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \delta_1 & & & & \\ & & & \ddots & & & \\ & & & & \delta_k & & \\ & & & & & \ddots & \\ 0 & & & & & & \delta_k \end{pmatrix},$$

where $\delta_i \in F_v^\times$ appears $m_i b_i$ times, $1 \leq i \leq k$. In particular, $\tilde{\delta}$ commutes with $\mathbf{s}(\sigma)$. Let $(\cdot, \cdot)_{F_v}$ and $(\cdot, \cdot)_{F_i}$ be the $n$th Hilbert symbol of $F_v$ and $F_i$ respectively, $1 \leq i \leq k$. Suppose that $\delta_1 = \cdots = \delta_k$. Then Proposition 0.1.5

[KP84], the fact that $\tilde{\delta}$ commutes with $\mathbf{s}(\sigma)$ and the properties of the Hilbert symbol imply that

$$\begin{aligned} 1 &= (\det(\sigma), \delta_1^r)_{F_v}^{1+2m} \prod_{i=1}^k (\sigma_i, \delta_1^{b_i})_{F_i}^{-1} \\ &= (\det(\sigma)^{r(1+2m)}, \delta_1)_{F_v} \left( \prod_{i=1}^k N_{F_i/F_v}(\sigma_i)^{b_i}, \delta_1 \right)_{F_v}^{-1} \\ &= (\det(\sigma)^{r-1+2rm}, \delta_1)_{F_v}. \end{aligned}$$

As $\delta_1 \in F_v^\times$ is arbitrary, $\det(\sigma)^{r-1+2rm}$ must lie in $F_v^{\times n}$. It follows from $\gcd(n, r-1+2rm) = 1$ and (24) that $\det(\sigma) \in F_v^{\times n}$. Now fix $1 \leq j \leq k$ and let $\tilde{\delta}$ be such that $\delta_i = 1$ for $1 \leq i \leq k$ and $i \neq j$. Then, repeating the above procedure and noting that $n$ is relatively prime to $b_j$, we find that

$$\begin{aligned} 1 &= \Big(\det(\sigma), \det(\mathbf{p}(\tilde{\delta}))\Big)_{F_v}^{1+2m} (\sigma_j, \delta_j^{b_j})_{F_j}^{-1} \\ &= (N_{F_j/F_v}(\sigma_j), \delta_j)_{F_v}^{-b_j} \\ &= (N_{F_j/F_v}(\sigma_j), \delta_j)_{F_v}. \end{aligned}$$

Consequently $N_{F_j/F_v}(\sigma_j) \in F_v^{\times n}$. By Lemma 8.2 and our hypotheses on $n$ we have $\sigma_j \in F_j^{\times n}$. Therefore there exists $\sigma_{0j} \in \mathrm{GL}(m_j, F_v)$ such that $\sigma_{0j}^n = \sigma_j$, $1 \leq j \leq k$. Set $\sigma_0$ to be the obvious block diagonal matrix composed of the blocks $\sigma_{01}, \ldots, \sigma_{0k}$ such that $\sigma_0^n = \sigma$. It is evident from the inclusion $G_{\sigma_0} \subset G_\sigma$ that we have

$$G_{\sigma_0}(F_v) = G_\sigma(F_v) \cong \prod_{i=1}^k \mathrm{GL}(b_i, F_i).$$

Now set $u_0 = \exp\left(\frac{1}{n}\log(u)\right)$. By construction, $u_0^n = u$ and $u_0 \in G_\sigma(F_v) = G_{\sigma_0}(F_v)$. The lemma now follows in the case $M = G$ by setting $\gamma = \sigma_0 u_0$. The lemma follows in the general case by using the arguments near the end of the proof of Lemma 8.1.□
The assumption placed on $n$ in Lemma 8.3 will remain to be a restriction in the comparison of the trace formulas later on.

We can use the local vanishing properties to prove global vanishing properties by way of some local-global results on $n$th roots in $F$ and $G(F)$.

**Lemma 8.4** *Suppose $x \in F^\times$ such that $x \in F_v^{\times n}$ for almost all valuations $v$. Then $x \in F^{\times n}$.*

*Proof.* We will prove this lemma by contradiction. To this end, suppose that $\sigma$ is a non-trivial element in the Galois group of the abelian extension $F(x^{1/n})$. (Observe that $F(x^{1/n})$ is well-defined since $\mu_n \subset F$.) It is immediate from the hypothesis of the lemma that the density of the valuations for which $\sigma$ is the Frobenius automorphism is zero. However, the Tchebotarev density theorem (VIII, §4 Theorem 10 [Lan70]) tells us that this density is $1/[F(x^{1/n}) : F]$.$\square$

**Lemma 8.5** *Suppose that $\gamma \in M(F)$ and that for almost every valuation $v$ of $F$, there exists $\gamma_v \in M(F_v)$ such that $\gamma_v^n = \gamma$. Then there exists $\gamma_0 \in M(F)$ such that $\gamma_0^n = \gamma$.*

*Proof.* By decomposition (4) of $M$ it suffices to prove the case $M = G$. Let $\gamma = \sigma u$ be the Jordan decomposition of $\gamma$. By §1 [KP86], we may take the semisimple element $\sigma$ to be a block diagonal matrix of the form

$$\begin{pmatrix} \sigma_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \sigma_1 & & & & \\ & & & \ddots & & & \\ & & & & \sigma_k & & \\ & & & & & \ddots & \\ 0 & & & & & & \sigma_k \end{pmatrix},$$

where $\sigma_i \in \mathrm{GL}(m_i, F)$, $1 \leq i \leq k$, are pairwise distinct, generate respective extensions $F_i/F$ of degree $m_i$ and each appear $b_i$ times. Clearly, $G_\sigma(F_v)$ may be identified with a subgroup of $\prod_{i=1}^k \mathrm{GL}(m_i b_i, F_v)$. Since $\gamma_v \in G_\sigma(F_v)$, we may assume that $k = 1$ without any loss of generality. Let $\sigma_v u_v$ be the Jordan decomposition of $\gamma_v$. We obviously have $\sigma_v^n = \sigma$ and $u_v^n = u$. The latter equation implies that $u_v = u_0 = \exp\left(\frac{1}{n}\log(u)\right) \in G(F)$ so we need only produce an appropriate semisimple factor. We first consider the case $u \neq 1$. Using Jordan canonical form in $G_\sigma(F) \cong \mathrm{GL}(b_1, F_1)$ we may assume that $\gamma$ is of the form

$$\begin{pmatrix} \sigma_1 & 1 & 0 & 0 \\ 0 & \sigma_1 & * & 0 \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \sigma_1 \end{pmatrix} \in G(F),$$

where the 1 in the $(1,2)$ entry represents the $m_1 \times m_1$ identity matrix. Since $\sigma_v$ commutes with $u_v = u_0$, it also commutes with $u$. This implies that $\sigma_v$ is of the form the form

$$\begin{pmatrix} \sigma_{v0} & * & * \\ 0 & \sigma_{v0} & * \\ 0 & * & * \end{pmatrix} \in G(F_v),$$

for some $\sigma_{v0} \in \mathrm{GL}(m_1, F_v)$. Consequently, $\sigma_{v0}^n = \sigma_1$ for almost every $v$. If $F_1 = F$ then $m_1 = 1$ and we may apply Lemma 8.4 to $\sigma_1$ to conclude that there exists $\sigma_0 \in F^\times$ such that the $n$th power of $\gamma_0 = \sigma_0 u_0$ is equal to $\gamma$. Otherwise, suppose $w_1, \ldots, w_t$ are the valuations of $F_1$ which divide $v$. Then $\sigma_1$ is $\mathrm{GL}(m_1, F_v)$-conjugate to

$$\begin{pmatrix} \sigma_{11} & & 0 \\ & \ddots & \\ 0 & & \sigma_{1t} \end{pmatrix},$$

where $\sigma_{1i}$ generates the completion $F_{1,w_i}$ over $F_v$, $1 \le i \le t$. This implies that the centralizer of $\sigma_1$ in $\mathrm{GL}(m_1, F_v)$ is isomorphic to $\prod_{i=1}^t F_{1,w_i}^\times$. In accordance with this last isomorphism, we decompose $\sigma_{v0}$ as $(\sigma_{v1}, \ldots, \sigma_{vt})$, where $\sigma_{vi} \in F_{1,w_i}^\times$, $1 \le i \le t$. Observe that the map $\sigma_1 \mapsto \sigma_{1i}$ corresponds to the embedding $F_1 \hookrightarrow F_{1,w_i}$, $1 \le i \le t$. Therefore $\sigma_1$, regarded as an element of $F_1$, has an $n$th root in almost every completion of $F_1$. Applying Lemma 8.4 (with $F$ replaced by $F_1$) to $\sigma_1$ proves the case $u \ne 1$. Now suppose $u = 1$ and $F_1 = F$. Then

$$\sigma_1^r = \det(\sigma) = \det(\sigma_v^n) \in F_v^{\times n},$$

for almost every $v$. Set $c = n/\gcd(n,r)$. Since $F_v$ contains the $\gcd(n,r)$th roots of unity, we have $\sigma_1^{r/\gcd(n,r)} \in F_v^{\times c}$. Furthermore, since $r/\gcd(n,r)$ is relatively prime to $c^2/|c|_v$, the order of $F_v^\times/F_v^{\times c}$ (cf. equation (24)), we have $\sigma_1 \in F_v^{\times c}$. Lemma 8.4 (with $n$ replaced by $c$) implies that $\sigma_1 \in F^{\times c}$. The lemma now follows in this case by setting $\gamma_0$ to be the block diagonal matrix in $G(F)$ with blocks of the form

$$\begin{pmatrix} 0 & 0 & 0 & \sigma_1^{1/c} \\ 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}(\gcd(n,r), F).$$

The remaining case that $u = 1$ and $F_1 \neq F$, is proven similarly, but with the added complication of using the embeddings $F_1 \hookrightarrow F_{1,w_i}$, $1 \leq i \leq t$, mentioned earlier. We leave the details to the interested reader.□

**Corollary 8.1 (Kazhdan-Patterson)** *Suppose $\sigma \in G(F)$ is semisimple and for almost every valuation $v$ there exist elements $\delta_v \in A_G(F_v)$ and $\sigma_v \in G(F_v)$ such that $\sigma = \delta_v^{n/d}\sigma_v^n$. Then there exist $\delta_0 \in A_G(F)$ and $\sigma_0 \in G(F)$ such that $\sigma = \delta_0^{n/d}\sigma_0^n$.*

*Proof.* Our proof follows the sketch on p. 226 [KP86]. Let $x = \det(\sigma)$, $y_v = \det(\sigma_v)$ and $z_v \in F_v^\times$ be the scalar corresponding to $\delta_v$ for almost every $v$. Clearly $x = z_v^{rn/d} y_v^n \in F_v^{n/d}$. Applying Lemma 8.4 (with $n/d$ in place of $n$) and recalling that $F_v \supset \mu_n \supset \mu_{n/d}$, we find that there exists $x_0 \in F^\times$ such that $x_0 = z_v^r y_v^d$ for almost every $v$. It is simple to verify that $r$ and $d$ are relatively prime. In turn, $r$ and $d^2$, the order of $F_v^\times / F_v^{\times d}$ (cf. equation (24)) whenever $|d|_v = 1$, are relatively prime. Combining this fact with the equality $x_0 F_v^{\times d} = z_v^r F_v^{\times d}$, we conclude that $z_v = x_0^t w_v^d$ for some $t \in \mathbf{Z}$ and $w_v \in F_v^\times$ for almost every $v$. We may therefore assume that $\delta_v = \delta_0 \in A_G(F)$. The application of Lemma 8.5 to $\delta_0^{-1}\sigma$ completes the proof.□

The following two lemmas are global vanishing properties (cf. Proposition 8.1 [Art88b]) which follow from the local vanishing properties, Lemma 8.1 and Lemma 8.3.

**Proposition 8.1** *Suppose $\tilde{\sigma} \in \tilde{G}(F)$ is embedded diagonally into $\tilde{G}(F_S)$, for some large finite set of valuations $S$. Suppose further that $\mathbf{p}(\tilde{\sigma})$ is semisimple. Then*

$$I_{\tilde{G}}(\tilde{\sigma}, \tilde{f}) = 0, \;\; \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

*unless $\mathbf{p}(\tilde{\sigma}) = \gamma_0^{n/d}\gamma^n$ for some $\gamma_0 \in A_G(F)$ and $\gamma \in G(F)$.*

*Proof.* Suppose $\mathbf{p}(\tilde{\sigma})$ is not of the form $\gamma_0^{n/d}\gamma^n$ for any $\gamma_0 \in A_G(F)$ and $\gamma \in G(F)$. Then by Corollary 8.1 we may assume that there exists a nonarchimedean valuation $v_1 \in S$ such that $\mathbf{p}(\tilde{\gamma})$ is not of the form $\delta_{v_1}^{n/d}\sigma_{v_1}^n$ for any $\delta_{v_1} \in A_G(F_{v_1})$ and $\sigma_{v_1} \in G(F_{v_1})$. The proposition now follows from Lemma 8.1.□

**Proposition 8.2** *Suppose $\tilde{\gamma} \in \tilde{M}(F)$ is embedded diagonally into $\tilde{M}(F_S)$, for some large finite set of valuations $S$. Suppose further that the assumption of Lemma 8.3 on $n$ holds. Then*

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = 0, \;\; \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

*unless* $\mathbf{p}(\tilde{\gamma})$ *is an nth power in* $M(F)$.

*Proof.* Suppose $\mathbf{p}(\tilde{\gamma})$ is not an $n$th power in $M(F)$. Then by Lemma 8.5 we may assume that there exists a nonarchimedean valuation $v_1 \in S$ such that $\mathbf{p}(\tilde{\gamma})$ is not an $n$th power in $M(F_{v_1})$. Set $S_1 = \{v_1\}$, $S_2 = S - \{v_1\}$ and let $\tilde{f} = \tilde{f}_1 \tilde{f}_2$ be the corresponding decomposition of $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. It follows from Lemma 8.1 that

$$\hat{I}^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, \tilde{f}_{1,\tilde{L}}) = 0, \;\; L \in \mathcal{L}(M).$$

The proposition follows by combining this equation with the splitting property (Proposition 9.1 [Art88a]),

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d^G_M(L_1, L_2) \hat{I}^{\tilde{L_1}}_{\tilde{M}}(\tilde{\gamma}, \tilde{f}_{1,\tilde{L_1}}) \hat{I}^{\tilde{L_2}}_{\tilde{M}}(\mathbf{s}(\mathbf{p}(\tilde{\gamma})), \tilde{f}_{2,\tilde{L_2}}). \square$$

# 9 The invariant trace formula

We set forth the invariant trace formula of a function $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ as the equality of

$$\sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\tilde{\gamma} \in (\mathbf{s_0}(M(F)))_{\tilde{M},S}} a^{\tilde{M}}(S, \tilde{\gamma}) I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) \tag{25}$$

with

$$\sum_t \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(\tilde{M},t)} a^{\tilde{M}}(\tilde{\pi}) I_{\tilde{M}}(\tilde{\pi}, \tilde{f}) d\tilde{\pi}. \tag{26}$$

It will be convenient to denote (25) as $I(\tilde{f})$. This trace formula is extrapolated from the trace formula given in [Art88b] for reductive algebraic groups. Since non-trivial metaplectic coverings of algebraic groups are not algebraic, one ought to verify the results of Arthur for metaplectic groups in order to rigorously assert the existence of a trace formula as we have done above. There are, lamentably, too many results to be checked to be included here. Those results which have been checked (§5 for example) follow in a straightforward manner. There is no reason to doubt that the other results do not follow in the same way. We therefore assume that the invariant trace formula is correct as stated.

Expansion (26) is known as the spectral side of the trace formula, as its terms depend on representations of $\tilde{M}(\mathbf{A})$. It will be further elaborated upon in §14. Expansion (25) is known as the geometric side of the trace formula, as its terms depend (in our case) on conjugacy classes in $\tilde{M}(F_S)$. Both sides of the trace formula contain terms which are local, *i.e.* determined by $\tilde{M}(F_S)$, and global, *i.e.* determined by the subgroup $\mathbf{s_0}(M(F))$ of $\tilde{M}(\mathbf{A})$.

The local terms of the geometric side have already been introduced in §§6-7. In order to introduce some of the global terms of the geometric side we perform a familiar calculation. Recall that the map $\mathbf{s_0}$ given by (2) is a splitting homomorphism for $\tilde{G}(\mathbf{A})$ over $G(F)$. Let $L^2(\mathbf{s_0}(G(F))\backslash\tilde{G}(\mathbf{A}))$ be the space of genuine square-integrable functions on $\tilde{G}(\mathbf{A})$, which are left-invariant under $\mathbf{s_0}(G(F))$. This space admits a theory of automorphic representations through the decomposition of the (right-)regular representation $R$ of $\tilde{G}(\mathbf{A})$. The metaplectic version of the geometric side of the trace formula originates from the following calculation. Let $\varphi \in L^2(\mathbf{s_0}(G(F))\backslash\tilde{G}(\mathbf{A}))$. We then have

$$\begin{aligned} (R(\tilde{f})\varphi)(y) &= \int_{\tilde{G}(\mathbf{A})} \tilde{f}(x)\varphi(yx)dx \\ &= \int_{\mathbf{s_0}(G(F))\backslash\tilde{G}(\mathbf{A})} \sum_{\gamma\in\mathbf{s_0}(G(F))} \tilde{f}(y^{-1}\gamma x)\varphi(\gamma x)dx \\ &= \int_{\mathbf{s_0}(G(F))\backslash\tilde{G}(\mathbf{A})} \left( \sum_{\gamma\in\mathbf{s_0}(G(F))} \tilde{f}(y^{-1}\gamma x)\right) \varphi(x)dx. \end{aligned}$$

Roughly speaking, the trace of the operator $R(\tilde{f})$ is obtained by integrating the integral kernel,

$$(x,y) \mapsto \sum_{\gamma\in\mathbf{s_0}(G(F))} \tilde{f}(y^{-1}\gamma x),\ \ x,y \in \mathbf{s_0}(G(F))\backslash\tilde{G}(\mathbf{A}),$$

over the diagonal.

Suppose $S$ contains $\{v : |n|_v \neq 1\}$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. For nonarchimedean valuations $v$ define $\tilde{f}_v^0$ by

$$\tilde{f}_v^0(\gamma,\zeta) = \left\{ \begin{array}{ll} \zeta^{-1}, & \text{if } \gamma \in K_v \\ 0, & \text{otherwise} \end{array} \right\},\ \ \gamma \in G(F_v),\ \zeta \in \mu_n. \tag{27}$$

We embed $\tilde{f}$ into $\mathcal{H}(\tilde{G}(\mathbf{A}))$ by taking its product with $\prod_{v\notin S} \tilde{f}_v^0$. If $S$ satisfies some additional properties, which are given in §3 [Art88b], then $I(\tilde{f})$ equals

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\tilde{\gamma}\in(\mathbf{s_0}(M(F)))_{\tilde{M},S}} a^{\tilde{M}}(S,\tilde{\gamma})I_{\tilde{M}}(\tilde{\gamma},\tilde{f}).$$

The set $(\mathbf{s_0}(M(F)))_{\tilde{M},S}$ denotes the set of $(\tilde{M},S)$-equivalence classes of elements in $\mathbf{s_0}(M(F))$ (§8 [Art86]). In the present case $(\mathbf{s_0}(M(F)))_{\tilde{M},S}$ is simply the set of $\tilde{M}(F_S)$-conjugacy classes of elements in $\mathbf{s_0}(M(F))$. Since conjugacy classes play such an prominent role we shall, for the sake of convenience, often abuse notation by identifying an element $\gamma \in M(F_S)$ with its conjugacy class.

The coefficient $a^{\tilde{M}}(S,\tilde{\gamma})$ requires more explanation. Let $\sigma u$ be the Jordan decomposition of $\mathbf{p}(\tilde{\gamma}) \in M(F)$. Set $i^M(S,\sigma) = 1$ if $\sigma$ is $F$-elliptic in $M(F)$, and the $M(F_v)$-orbit of $\sigma$ meets $K_v \cap M(F_v)$ for every valuation $v \notin S$. Otherwise set $i^M(S,\sigma) = 0$. It follows from the nature of the conjugacy classes of $M(F_S)$ and (3.2) [Art88b] that

$$a^{\tilde{M}}(S,\gamma) = i^M(S,\sigma) a^{\tilde{M}_{\mathbf{s_0}(\sigma)}}(S,u). \tag{28}$$

For a description of $a^{M_\sigma}(S,u)$ see §7 [Art86]. The term $a^{\tilde{M}_{\mathbf{s_0}(\sigma)}}(S,u)$ is defined analogously.

# 10 Statement of Theorem A

Henceforth, we assume that

**(†)** $n$ is relatively prime to $i(1+2m)-1$ and $i$ for all $1 < i \leq r$.

We shall cite each use of this assumption in §§10-22 with **(†)**. Assumption **(†)** is more severe than the assumption needed for Lemma 8.3 and Proposition 8.2. The additional restrictions of this assumption are required for the arguments of the Appendix, which ensure that parabolic induction commutes with the local metaplectic correspondence.

We begin by comparing the geometric sides of the trace formulas. Suppose that $S$ is as in §9 and consider the summand of (25) indexed by $M = G$, namely

$$\sum_{\tilde{\gamma} \in (\mathbf{s_0}(G(F)))_{\tilde{G},S}} a^{\tilde{G}}(S,\tilde{\gamma}) I_{\tilde{G}}(\tilde{\gamma},\tilde{f}).$$

If we are to have any hope in comparing this term with its counterpart,

$$\sum_{\gamma \in (G(F))_{G,S}} a^{G}(S,\gamma) I_{\tilde{G}}(\gamma,\tilde{f}),$$

by using the orbit map, then we must eliminate those $\tilde{\gamma}$ from the former sum such that $\tilde{\gamma} \neq \gamma'$ for all $\gamma \in G(F)$. The global vanishing properties, Proposition 8.1 and Proposition 8.2, were proven with this in mind. However, there is also the additional concern that the orbit map is not injective. The following lemma determines the extent to which it is not injective on the elliptic set.

**Lemma 10.1** *Suppose $\gamma_1$ and $\gamma_2$ are $F$-elliptic in $M(F)$. Then $\gamma_1^n$ is $M(F_v)$-conjugate to $\gamma_2^n$ if and only if there exists $\eta \in \mu_n^M$ such that $\gamma_1$ is $M(F_v)$-conjugate to $\eta\gamma_2$ .*

*Proof.* The "if" direction is trivial to prove. Suppose therefore that $\gamma_1^n$ is $M(F_v)$-conjugate to $\gamma_2^n$. Using decomposition (4) it suffices to prove the lemma in the case that $M = G$ (cf. (4)) and $\gamma_1^n$ is actually equal to $\gamma_2^n$. In this case, there exist extensions $E_1$, $E_2$ of $F$ such that the elliptic torus of $G(F)$ containing $\gamma_i$ is isomorphic to $E_i^\times$, $1 = 1, 2$. Let $E = E_1 \cap E_2$. Regarding $\gamma_1$ and $\gamma_2$ as $n$th roots of the elements $\gamma_1^n, \gamma_2^n \in E$, it is obvious that $E(\gamma_1)$ is isomorphic to $E(\gamma_2)$ over $E$. In particular these fields are isomorphic over $F$ and so any two embeddings of these fields into $G(F)$ must be conjugate over $G(F)$. Thus a $G(F)$-conjugate of $\gamma_2$ lies in the torus isomorphic to $E(\gamma_1)$. Since this element may also be regarded as a $n$th root of $\gamma_1$ it must be of the form $\eta\gamma_1$ for some $\eta \in \mu_n^G \cong \mu_n$.$\square$

**Proposition 10.1 (5.1)** *Suppose $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$. Then $I(\tilde{f})$ is equal to*

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^{\tilde{M}}(S, \gamma') I_{\tilde{M}}^{\mathcal{M}}(\gamma, \tilde{f})$$

*for a sufficiently large finite set of valuations $S$.*

*Proof.* According to Proposition 8.2 (which contains **(†)** in its hypotheses), the distribution $I_{\tilde{M}}(\mathbf{s_0}(\gamma))$, $\gamma \in M(F)$, vanishes unless $\gamma$ is an $n$th power in $M(F)$. From Lemma 10.1, we see that the map,

$$(M(F))_{M,S}/\mu_n^M \xrightarrow{\prime} (\mathbf{s_0}(M(F)))_{\tilde{M},S},$$

given by (equation (18), equation (2))

$$\gamma\mu_n^M \mapsto \gamma' = \mathbf{s}(\gamma)^n \mathbf{i}\left(\prod_v \kappa_v(\gamma)^{-1}\right)^n = \mathbf{s_0}(\gamma)^n = \mathbf{s_0}(\gamma^n),$$

is injective when restricted to the conjugacy classes of $F$-elliptic elements in $M(F)$. The proposition now follows from equation (21) and the fact that $a^{\tilde{M}}(S, \mathbf{s_0}(\gamma))$ vanishes unless $\gamma$ is $F$-elliptic in $M(F)$ (equation (28)).$\square$

The trace formula for $G(\mathbf{A})$, which we expect to match $I(\tilde{f})$, is

$$\begin{aligned} I(\tilde{f}') &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S,\gamma)\hat{I}_M(\gamma, \tilde{f}') \\ &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} \sum_{\eta \in \mu_n^M} a^M(S,\eta\gamma)\hat{I}_M(\eta\gamma, \tilde{f}'). \end{aligned}$$

Equation (28) specializes to

$$a^M(S,\gamma) = i^M(S,\sigma)a^{M_\sigma}(S,u), \tag{29}$$

for the Jordan decomposition $\sigma u$ of $\gamma \in M(F)$. Since every element of $\mu_n^M \subset A_M(F)$ is $F$-elliptic in $M(F)$ and also lies in $K_v \cap M(F_v)$ for all valuations $v$, it is immediate that $i^M(S,\eta\sigma) = i^M(S,\sigma)$ for all $\eta \in \mu_n^M$ and semisimple $\sigma \in M(F)$. This implies that $a^M(S,\eta\gamma) = a^M(S,\gamma)$ in the previous sum. Explicitly, we have

$$\begin{aligned} I(\tilde{f}') &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S,\gamma) \sum_{\eta \in \mu_n^M} \hat{I}_M(\eta\gamma, \tilde{f}') \\ &= n \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S,\gamma) I_M^\Sigma(\gamma, \tilde{f}). \end{aligned}$$

We shall shall carry the coefficient $n$ over to the spectral side of $I(\tilde{f}')$ in §§13-14 and set $I^\Sigma(\tilde{f}) = n^{-1}I(\tilde{f}')$. The present forms of the two trace formulas and Theorem A II [AC89] motivate the first main theorem of the comparison.

**Theorem A** *(i) Suppose that $S$ is a finite set of valuations with the closure property. Then*

$$I_M^\Sigma(\gamma, \tilde{f}) = I_M^{\mathcal{M}}(\gamma, \tilde{f}), \ \gamma \in M_{\mathrm{comp}}(F_S), \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

*(ii) Suppose $\gamma \in M(F)$. Then*

$$a^{\tilde{M}}(S, \gamma') = a^M(S,\gamma)$$

*for any suitably large finite set $S$.*

The proof of Theorem A will be completed in §20. Observe that by the splitting properties ((23), Proposition 6.2), Theorem A (i) holds if and only if it holds in the case that $S$ consists of a single valuation.

The proof of Theorem A consists of several induction arguments. Suppose $\sigma$ is a semisimple element of $G(F)$. We make the induction hypothesis that Theorem A holds if $G$ is replaced by $G_\sigma$ and $\dim(G_\sigma) < \dim(G)$. Every Levi subgroup in $\mathcal{L}$ is of the form $G_\sigma$. Furthermore, it follows from Proposition 3 §2 II [Lan70] (Krasner's Lemma) and §1 [KP86] that for any nonarchimedean valuation $v$ and semisimple element $\sigma_1 \in G(F_v)$ there exists a semisimple element $\sigma \in G(F)$ such that $G_{\sigma_1}(F_v) = G_\sigma(F_v)$. This fact allows us to apply the induction hypothesis in the local nonarchimedean context. Let us consider some immediate implications of our induction hypothesis. By combining the induction hypothesis with the descent properties, we obtain the following lemma.

**Lemma 10.2** *Suppose $M_1$ and $M$ are in $\mathcal{L}$ such that $M_1 \subsetneqq M$. Suppose further that $\gamma \in M_1(F_S) \cap M_{\mathrm{comp}}(F_S)$ satisfies $M_{1,\gamma} = M_\gamma$. Then*

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = I_M^{\Sigma}(\gamma, \tilde{f}), \;\; \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

*In particular, this equation holds for $\gamma \in M_1(F_S) \cap G_{\mathrm{oreg}}(F_S)$.*

*Proof.* It follows from Jordan canonical form and exercises 33-34 Chapter 5 [AW92] that $M_{1,\gamma} = M_\gamma$ if and only if $M_{1,\sigma} = M_\sigma$. We therefore have

$$M_{1,\sigma} \subset M_{1,\sigma^n} \subset M_{\sigma^n} = M_\sigma = M_{1,\sigma},$$

and so $M_{1,\sigma} = M_{1,\sigma^n}$. That is, $\gamma \in M_{1,\mathrm{comp}}(F_S)$. By the descent properties ((22), Corollary 6.1), we have

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \left( \hat{I}_{M_1}^{L,\mathcal{M}}(\gamma, \tilde{f}_{\tilde{L}}) - \hat{I}_{M_1}^{L,\Sigma}(\gamma, \tilde{f}_{\tilde{L}}) \right).$$

The induction hypothesis and the fact that $d_{M_1}^G(M, G) = 0$ (§7 [Art88a]) imply that the right-hand side vanishes.□

Applying our induction hypothesis to the geometric sides of the trace formulas and also to expansions (28) and (29), we obtain the following lemma whose proof can be gleaned from the proof of Lemma 5.2 II [AC89].

**Lemma 10.3 (5.2)** *The distribution,*

$$\tilde{f} \mapsto I(\tilde{f}) - I^{\Sigma}(\tilde{f}'), \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

*is the sum of*

$$\sum_{M \in \mathcal{L}, M \neq G} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S,\gamma) \left( I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) \right)$$

*and*

$$\sum_{\delta \in A_G(F) \backslash \mu_n^G} \sum_{u \in (\mathcal{U}_G(F))_{G,S}} \left( a^{\tilde{G}}(S,u) - a^G(S,u) \right) I_G^{\mathcal{M}}(\delta u, \tilde{f}).$$

# 11 Comparison of $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ and $I_M^{\Sigma}(\gamma, \tilde{f})$

Our aim here is to show that $I_M^{\Sigma}(\gamma, \tilde{f})$ is equal to $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ under various circumstances. We begin by showing that Theorem A (i) holds if it holds at certain regular elements. In order to do this, we need to be able to compare the functions $r_M(\gamma, a)$, which appear in the definitions of our invariant distributions. The following two lemmas afford such a comparison.

**Lemma 11.1** *Suppose that $\gamma \in M(F_v)$ is not $G(F_v)$-conjugate to any element $\gamma_0 \in L(F_v)$, such that $L \in \mathcal{L}$, $L \subsetneq M$ and $L_{\gamma_0} = M_{\gamma_0}$. Suppose further that $\gamma = \sigma u$ is the Jordan decomposition of $\gamma$ in $M(F_v)$. Then there exists $\eta \in \mu_n^M$ such that $G_{\eta\gamma} = G_{\gamma^n}$ and $G_{\eta\sigma} = G_{\sigma^n}$. Moreover, $M_\sigma = M_{\sigma^n}$.*

*Proof.* We shall show that $\dim(G_{\eta\gamma}) = \dim(G_{\gamma^n})$ for some $\eta \in \mu_n^M$. The first assertion of the lemma then follows from the fact that centralizers in $G$ are closed and connected. The remaining assertions shall follow easily from the proof of the first. Recall decomposition (4),

$$M = M(1) \times \cdots \times M(\ell) \cong \mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_\ell).$$

Identify $\gamma \in M(F_v)$ with $(\gamma_1, \ldots, \gamma_\ell)$, where $\gamma_i \in \mathrm{GL}(r_i, F_v)$, $1 \leq i \leq \ell$. As explained at the beginning of §1 [KP86], there exist positive integers, $k_i$, $r_{ij}$, and commuting elements, $\sigma_{ij}, u_{ij} \in \mathrm{GL}(r_{ij}, F_v)$, for $1 \leq i \leq \ell$, $1 \leq j \leq k_i$, such that the elements $\sigma_{i1}, \ldots, \sigma_{ik_i}$ are semisimple, pairwise distinct and generate field extensions, $E_{i1}, \ldots, E_{ik_i}$, respectively; the elements $u_{i1}, \ldots, u_{ik_i}$

are unipotent and upper-triangular; and $\gamma_i$ is $\mathrm{GL}(r_i, F_v)$-conjugate to the block diagonal matrix

$$(30) \qquad \begin{pmatrix} \sigma_{i1}u_{i1} & & 0 \\ & \ddots & \\ 0 & & \sigma_{ik_i}u_{ik_i} \end{pmatrix} \in \mathrm{GL}(r_i, F_v) \cong M(i)(F_v).$$

This matrix determines a unique Levi subgroup,

$$L_i \cong \mathrm{GL}(r_{i1}) \times \cdots \times \mathrm{GL}(r_{ik_i}),$$

of $M(i) \cong \mathrm{GL}(r_i)$. In turn, we obtain a unique Levi subgroup $L \in \mathcal{L}$ such that $L \cong L_1 \times \cdots \times L_\ell$ and $L \subset M$. Suppose $\gamma_0$ is the element of $L(F_v)$ determined by (30). Then $L_{\gamma_0} = M_{\gamma_0}$. To see this, observe that for $1 \leq j_1 < j_2 \leq k_i$, the eigenvalues of $\sigma_{ij_1}$ are pairwise distinct from the eigenvalues of $\sigma_{ij_2}$. This implies that $L_{i,\gamma_i} = M(i)_{\gamma_i}$ (exercise 33-34 Chapter 5 [AW92]). This last equality clearly implies $L_{\gamma_0} = M_{\gamma_0}$. By the hypothesis of the lemma we must have $L = M$. This means that $k_1 = \cdots = k_\ell = 1$ and that $\gamma_i$ is $\mathrm{GL}(r_i, F_v)$-conjugate to $\sigma_i u_i$, where $\sigma_i = \sigma_{i1}$ and $u_i = u_{i1}$. An immediate consequence is that $\gamma_i^n$ is $\mathrm{GL}(r_i, F_v)$-conjugate to $\sigma_i^n u_i^n$. The element $\sigma_i^n$ generates a field $F_i$ which lies between $E_i = E_{i1}$ and $F_v$. Assumption **(†)** implies that $E_i = F_i$. Indeed, after identifying $\sigma_i$ with an element of $E_i$, we have $E_i = F_i(\sigma_i) = F_i((\sigma_i^n)^{1/n})$. According to Theorem 10 (b) VIII §6 [Lan84], the index $[E_i : F_i]$ divides $n$. At the same time we have $[E_i : F_i] \leq r_i \leq r$ and so **(†)** implies that $[E_i : F_i] = 1$. It is immediate that the degree of the minimal polynomial of $\sigma_i$, namely $[E_i : F_v]$, is the same as that of $\sigma_i^n$. Furthermore, by applying Jordan canonical form (in $\mathrm{GL}(r_i/[E_i : F_v], E_i)$) and Lemma 4.1 to $\sigma_i^n u_i^n$, we find that it is $\mathrm{GL}(r_i, F_v)$-conjugate to $\sigma_i^n u_i$. Suppose that

$$(31) \qquad \sigma_{j_1}^n \neq \sigma_{j_2}^n, \text{ whenever } \sigma_{j_1} \neq \sigma_{j_2},\ 1 \leq j_1 < j_2 \leq \ell.$$

Then the previous two observations imply that there is a degree-preserving bijection between the elementary divisors of $\gamma$ and $\gamma^n$. Since the dimensions of $G_\gamma$ and $G_{\gamma^n}$ are determined by the degrees of the elementary divisors of $\gamma$ and $\gamma^n$ respectively (Theorem 5.15 [AW92]), we have $\dim(G_\gamma) = \dim(G_{\gamma^n})$. Now suppose that $\sigma_{j_1}^n = \sigma_{j_2}^n$, but $\sigma_{j_1} \neq \sigma_{j_2}$ for some $1 \leq j_1 < j_2 \leq \ell$. By regarding $\sigma_{j_1}$ and $\sigma_{j_2}$ as elements of $E_{j_1} (= E_{j_2})$, we find that $\sigma_{j_1} = \zeta \sigma_{j_2}$ for some $\zeta \in \mu_n$. This implies the existence of an element $\eta \in \mu_n^M$, such that if $\gamma$ is replaced by $\eta\gamma$ in the earlier argument, then (31) holds. This proves the first assertion of the lemma. The second assertion is seen to follow easily from the above argument by taking $u=1$. The final assertion of the lemma is a consequence of $E_i = F_i$.$\square$

We already mentioned in the proof of Lemma 6.2 that $r_M(\gamma, a)$ is invariant under translation by $A_G(F_S)$ in the first variable. This justifies the appearance of the quotient in the index set of the sum in the following lemma.

**Lemma 11.2** *Suppose $\gamma \in M(F_v)$ satisfies the hypotheses of Lemma 11.1 and $a \in A_{M,\mathrm{reg}}(F_v)$. Then*

$$\sum_{\eta \in \mu_n^M/\mu_n^G} r_M(\eta\gamma, a) = r_M(\gamma^n, a).$$

*Proof.* By Lemma 11.1, we may assume that $\gamma$ satisfies $G_\gamma = G_{\gamma^n}$, $G_\sigma = G_{\sigma^n}$ and $M_\sigma = M_{\sigma^n}$. Suppose $\eta \in \mu_n^M$. We would like to use Lemma 8.2 [Art88c] to show that

$$r_M(\eta\gamma, a) = \begin{cases} r_M^{G_\eta}(\gamma, a), & \text{if } \mathfrak{a}_{G_\eta} = \mathfrak{a}_G \\ 0, & \text{otherwise} \end{cases}. \tag{32}$$

In order to be able to invoke Lemma 8.2 [Art88c] we must verify the three conditions listed on p. 262 [Art88c]. To satisfy the first condition, we must have $\eta \in G(F)$. This is trivial. The second condition is satisfied if $\mathfrak{a}_{M_\eta} = \mathfrak{a}_M$. This is obvious as $M_\eta = M$. Finally, we must verify that $G_{\eta\gamma}(F_v)$ is contained in $G_\gamma(F_v)$. This is simple to verify since,

$$G_{\eta\gamma}(F_v) \subset G_{(\eta\gamma)^n}(F_v) = G_{\gamma^n}(F_v) = G_\gamma(F_v).$$

We now have equation (32). A moments thought reveals that $\mathfrak{a}_{G_\eta} = \mathfrak{a}_G$ if and only if $\eta \in \mu_n^G$. Thus,

$$\sum_{\eta \in \mu_n^M/\mu_n^G} r_M(\eta\gamma, a) = r_M(\gamma, a).$$

Now $r_M(\gamma, a)$ is defined in the usual fashion (cf. §6 [Art81]) from the $(G, M)$ family

$$r_P(\nu, \gamma, a) = r_P(\nu, \sigma u, a) = \prod_\beta \left|a^\beta - a^{-\beta}\right|^{\rho(\beta,u)\nu(\beta^\vee)/2}, \ P \in \mathcal{P}(M).$$

The product on the right is taken over the roots of $(P \cap G_\sigma, A_{M_\sigma})$, and $\rho(\beta, u)$ is a real number (defined in §3 [Art88c]) which depends only on the conjugacy class of $u$ in $M_\sigma$. Since $u^n$ is conjugate to $u$ in $M_\sigma$ (Lemma 4.1) and

$$(P \cap G_\sigma, A_{M_\sigma}) = (P \cap G_{\sigma^n}, A_{M_{\sigma^n}}),$$

we have in turn that

$$r_P(\nu, \gamma^n, a) = r_P(\nu, \sigma^n u^n, a) = r_P(\nu, \sigma u, a) = r_P(\nu, \gamma, a)$$

and

$$r_M(\gamma^n, a) = r_M(\gamma, a).$$

This completes the lemma.$\square$

**Lemma 11.3 (3.6)** *Suppose that $\tilde{f}$ is a function in $\mathcal{H}(\tilde{G}(F_S))$ such that*

$$I_M^{\Sigma}(\gamma, \tilde{f}) = I_M^{\mathcal{M}}(\gamma, \tilde{f})$$

*for every element $\gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S)$. Then the same formula holds for any element $\gamma \in M_{\mathrm{comp}}(F_S)$.*

*Proof.* By the splitting properties ((23), Proposition 6.2), it suffices to prove the lemma in the case that $S$ consists of a single valuation $v$. By Lemma 10.2, it suffices to prove the lemma under the assumption that $\gamma \in M_{\mathrm{comp}}(F_v)$ satisfies the hypotheses of Lemma 11.1. By following the argument of Lemma 3.6 II [AC89], we may conclude that

$$I_M^{\Sigma}(\gamma, \tilde{f}) = I_M^{\mathcal{M}}(\gamma, \tilde{f}),$$

if the semisimple component $\sigma$ of $\gamma$ satisfies $G_\sigma = M_\sigma$. Now suppose that $a \in A_{M,\mathrm{reg}}(F_v)$ is close to the identity so that

$$I_M^{\Sigma}(a\gamma, \tilde{f}) = I_M^{\mathcal{M}}(a\gamma, \tilde{f})$$

holds. According to the definitions of our invariant distributions and Lemma 11.2, we have

$$\begin{aligned} I_M^{\Sigma}(\gamma, \tilde{f}) &= \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} \sum_{\eta \in \mu_n^M / \mu_n^L} r_M^L(\eta\gamma, a) I_L^{\Sigma}(a\eta\gamma, \tilde{f}) \\ &= \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_M^L(\gamma^n, a) I_L^{\mathcal{M}}(a\gamma, \tilde{f}) \\ &= I_M^{\mathcal{M}}(\gamma, \tilde{f}). \square \end{aligned}$$

Lemma 11.3 tells us that in order to prove Theorem A (i) it is enough to consider the case that $\gamma$ belongs to $M(F_S) \cap G_{\mathrm{oreg}}(F_S)$. This is an important observation which we shall, often implicitly, make use of in the rest of this paper.

We now consider a very different comparison. At almost every place $v$, a function in $\mathcal{H}(\tilde{G}(\mathbf{A}))$ is of the form $\tilde{f}_v^0$ (cf. (27)). It is therefore valuable to compare $I_M^{\mathcal{M}}(\gamma, \tilde{f}_v^0)$ with $I_M^{\Sigma}(\gamma, \tilde{f}_v^0)$.

**Lemma 11.4** *Suppose $v$ is a valuation such that $|n|_v = 1$. Then*

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}_v^0) = I_M^{\Sigma}(\gamma, \tilde{f}_v^0), \ \ \gamma \in M_{\mathrm{comp}}(F_v).$$

*Proof.* By Lemma 11.3, it suffices to prove the lemma for $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$. Obviously, $|n|_v = 1$ implies

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}_v^0) = I_{\tilde{M}}(\gamma', \tilde{f}_v^0).$$

By Lemma 2.1 [Art88a] the term on the right is equal to the weighted orbital integral $J_{\tilde{M}}(\gamma', \tilde{f}_v^0)$ (cf. §2 [Art88c]). Let $f_v^0 \in \mathcal{H}(G(F_v))$ be the characteristic function of $K_v$. It may easily be verified that $f_{v,G}^0 = (\tilde{f}_v^0)'$. Under assumption **(†)** we may apply the Theorem of [Meza] to conclude that

$$J_{\tilde{M}}(\gamma', \tilde{f}_v^0) = \sum_{\eta \in \mu_n^M / \mu_n^G} J_M(\eta\gamma, f_v^0).$$

Again, by Lemma 2.1 [Art88a], the sum on the right is equal to

$$\sum_{\eta \in \mu_n^M / \mu_n^G} I_M(\eta\gamma, f_v^0) = I_M^{\Sigma}(\gamma, \tilde{f}_v^0). \square$$

We now turn to comparisons which are more obviously connected to Theorem A. Through the comparison of trace formulas in §20, we will prove Theorem A (i) in the special case that $S$ contains $\{v : |n|_v \neq 1\}$. The purpose of the next theorem and its corollary is to show that the general case of Theorem A (i) follows from this special case.

**Theorem 11.1 (6.1)** *In the special case that $S$ contains $\{v : |n|_v \neq 1\}$, we suppose that*

$$I_L^{\mathcal{M}}(\gamma, \tilde{f}) = I_L^{\Sigma}(\gamma, \tilde{f}),$$

*for any $\gamma \in L_{\mathrm{comp}}(F_S)$ and $L \in \mathcal{L}(M)$. Then there are unique constants*

$$\varepsilon_L(S) = \varepsilon_L^G(S), \ \ L \in \mathcal{L}(M),$$

*such that*

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = \sum_{L \in \mathcal{L}(M)} \hat{I}_M^{L,\Sigma}(\gamma, \varepsilon_L(S)\tilde{f}_{\tilde{L}}),\ \gamma \in M(F_S).$$

*The constants have the descent property*

$$\varepsilon_M(S) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L)\varepsilon_{M_1}^L(S),\ M_1 \subset M,$$

*and the splitting property*

$$\varepsilon_M(S) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2)\varepsilon_M^{L_1}(S_1)\varepsilon_M^{L_2}(S_2),\ S = S_1 \cup S_2.$$

*Proof.* This theorem follows from the proof of Theorem 6.1 II [AC89], with $I_M^{\mathcal{E}}$ replaced by $I_M^{\mathcal{M}}$ and $I_M$ replaced by $I_M^{\Sigma}$.$\square$

**Corollary 11.1 (6.4)** *In the special case that $S$ contains $\{v : |n|_v \neq 1\}$, we suppose that*

$$I_L^{\mathcal{M}}(\gamma, \tilde{f}) = I_L^{\Sigma}(\gamma, \tilde{f}),$$

*for any $\gamma \in L_{\mathrm{comp}}(F_S)$ and $L \in \mathcal{L}(M)$. Then*

$$\varepsilon_M(S) = \begin{cases} 1, & M = G \\ 0, & M \neq G \end{cases}$$

*for any finite set $S$ of valuations with the closure property.*

*Proof.* Fix $M \subsetneqq G$. By the induction hypothesis following Theorem A we have that $\varepsilon_M^L(S) = 0$, if $M \subsetneqq L \subsetneqq G$. It follows from the the descent property of Theorem 11.1 that $\varepsilon_M(S) = 0$ unless $M = M_0$. It follows from the splitting property of Theorem 11.1 that

$$\varepsilon_{M_0}(S) = \sum_{v \in S} \varepsilon_{M_0}(v).$$

The corollary therefore follows if we show that $\varepsilon_{M_0}(v) = 0$. We assume that the invariant local trace formula of Arthur (Proposition 8.1 [Art89c]) holds for $\tilde{G}(F_v)$ and shall compare local trace formulas to show that $\varepsilon_{M_0}(v) = 0$. Our assumption is that the geometric side of the local trace formula ((8.5) [Art89c]),

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{(\tilde{M}(F_v)_{\mathrm{ell}})} I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}_1 \times \tilde{f}_2) d\tilde{\gamma},$$

is equal to the spectral side ((8.6) [Art89c]),

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}$$
$$\times \sum_{\tilde\pi\in\Pi_{\mathrm{disc}}(\tilde M)} a^{\tilde M}_{\mathrm{disc}}(\tilde\pi)\int_{i\mathfrak{a}_M^*} r_{\tilde M}(\tilde\pi_\lambda,\tilde P)\mathrm{tr}\left(\mathrm{Ind}_{\tilde P}^{\tilde G}(\tilde\pi_\lambda,\tilde f_1\times\tilde f_2)\right)d\lambda.$$

We shall make a few remarks concerning the notations. The set $(\tilde M(F_v)_{\mathrm{ell}})$ is the set of $\tilde M(F_v)$-conjugacy classes of elements $\tilde\gamma$ in $M(F_v)$ such that $\mathbf{p}(\tilde\gamma)$ is $F_v$-elliptic in $M(F_v)$. It is a measure space by virtue of an identification with a set of anisotropic tori. The invariant distribution $I_{\tilde M}(\tilde\gamma,\tilde f_1\times\tilde f_2)$ is defined analogously to the distribution $I_{\tilde M}(\tilde\gamma,\tilde f)$ of §7, the only differences being that $\tilde f_1$ and $\tilde f_2$ are Hecke functions on the same group and that $\tilde f_1$ is genuine, not antigenuine. The set $\Pi_{\mathrm{disc}}(\tilde M)$ is, in our case (Proposition 27 [FK86]), a subset of (equivalence classes of) representations of the form

$$\tilde\pi=\tilde\pi_1^\vee\otimes\tilde\pi_1,\ \ \tilde\pi_1\in\Pi_{\mathrm{temp}}(\tilde M(F_v)),$$

where $\tilde\pi_1^\vee$ denotes the contragredient of $\tilde\pi_1$. The term $r_{\tilde M}(\tilde\pi_\lambda,\tilde P)$ is obtained from the $(G,M)$ family of normalizing factors given in §5. The reader is referred to [Art89c] for a description of the remaining notations. Applying the local vanishing property (Lemma 8.1), making the change of variable from $\gamma$ to $\gamma'$, and making some obvious definitions, we find that the geometric side is equal to

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{(M(F_v)_{\mathrm{ell}})/\mu_n^M} |n|_v^r I_{\tilde M}(\gamma',\tilde f_1\times\tilde f_2)d\gamma$$
$$= \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{(M(F_v)_{\mathrm{ell}})/\mu_n^M} I_M^{\mathcal{M}}(\gamma,\tilde f_1\times\tilde f_2)d\gamma.$$

The geometric side of the local trace formula for $G(F_v)$ at $\tilde f_1'\times\tilde f_2'$ is equal to

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{(M(F_v)_{\mathrm{ell}})} \hat I_M(\gamma,\tilde f_1'\times\tilde f_2')d\gamma$$
$$= n\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{(M(F_v)_{\mathrm{ell}})/\mu_n^M} I_M^{\Sigma}(\gamma,\tilde f_1\times\tilde f_2)d\gamma.$$

The comparison of the spectral side of the local trace formula of $G(F_v)$ follows in three steps. First, from a simple computation (cf. §16) and the definition of our normalizing factors (§5), it follows that

$$r_{\tilde{M}}(\tilde{\pi}_\lambda, \tilde{P}) = n^{\dim(A_M/A_G)} r_M(\tilde{\pi}'_{\lambda'}, P), \ \ P \in \mathcal{P}(M).$$

Second, the local metaplectic correspondence for tempered representations (8) implies that

$$\operatorname{tr}\left(\operatorname{Ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\pi}_\lambda, \tilde{f}_1 \times \tilde{f}_2)\right) = (\tilde{f}'_1 \times \tilde{f}'_2)\left(\operatorname{Ind}_P^G(\tilde{\pi}'_{\lambda'})\right).$$

Finally, we leave it to the interested reader to show that once the terms in the definition of $a^{\tilde{M}}_{\text{disc}}(\tilde{\pi})$ and $a^M_{\text{disc}}(\tilde{\pi}')$ are unraveled, they are easily seen to be equal. Taking the forgoing into consideration and noting that $d\lambda'$ equals $n^{\dim(A_M)} d\lambda$, we conclude that the spectral side of the local trace formula for $G(F_v)$ is $n$ times the spectral side of the local trace formula for $\tilde{G}(F_v)$. The coefficient $n$ cancels the $n$ appearing in front of the geometric side of the trace formula for $G(F_v)$. Thus, taking the difference of our modified local trace formulas yields

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}$$
$$\times \int_{(M(F_v)_{\text{ell}})/\mu_n^M} I_M^{\mathcal{M}}(\gamma, \tilde{f}_1 \times \tilde{f}_2) - I_M^{\Sigma}(\gamma, \tilde{f}_1 \times \tilde{f}_2) d\gamma = 0.$$

An application the splitting properties of $I_M^{\mathcal{M}}(\gamma)$ and $I_M^{\Sigma}(\gamma)$ (which follow as in (23) and Proposition 6.2) and the induction hypothesis to the integrand reduces this equation to

$$r^{-1}(-1)^{r-1} \int_{M_0(F_v)/\mu_n^M} 2|n|_v^r \varepsilon_{M_0}(v) \hat{I}_{\tilde{M}_0}^{\tilde{M}_0}(\gamma', \tilde{f}_{1,\tilde{M}_0}) \hat{I}_{\tilde{M}_0}^{\tilde{M}_0}(\gamma', \tilde{f}_{2,\tilde{M}_0}) d\gamma = 0,$$

which in turn implies

$$\int_{M_0(F_v)/\mu_n^M} \varepsilon_{M_0}(v) I_{\tilde{G}}(\gamma', \tilde{f}_1) I_{\tilde{G}}(\gamma', \tilde{f}_2) d\gamma = 0. \tag{33}$$

Choosing appropriate non-negative functions $\tilde{f}_1$, $\tilde{f}_2$ in this equation implies that $\varepsilon_{M_0}(v) = 0$ and the corollary is complete.□

## 12 Comparison of germs

We shall establish germ expansions for $I_M^{\mathcal{M}}(\gamma)$ and $I_M^{\Sigma}(\gamma)$ and then compare them. Before writing the germ expansion for $I_{\tilde{M}}(\tilde{\gamma})$ we extend the notion of $(M,\sigma)$-equivalence (cf. §2 [Art88c]) to metaplectic coverings. Suppose that $\tilde{\sigma}$ is a semisimple element of $\tilde{M}(F_S)$, and $\phi_1$, $\phi_2$ are functions defined on an open subset $\Sigma$ of $\tilde{M}_{\tilde{\sigma}}(F_S)$, whose closure contains and $\tilde{M}_{\tilde{\sigma}}$-invariant neighborhood of $\tilde{\sigma}$. We say $\phi_1$ is $(\tilde{M},\tilde{\sigma})$-equivalent to $\phi_2$ and write

$$\phi_1(\tilde{\gamma}) \overset{(\tilde{M},\tilde{\sigma})}{\sim} \phi_2(\tilde{\gamma}),\ \tilde{\gamma} \in \Sigma,$$

if there exists $\tilde{h} \in C_c^{\infty}(\tilde{M}(F_S))$ and a neighborhood $U$ of $\tilde{\sigma}$ in $\tilde{M}(F_S)$ such that

$$\phi_1(\tilde{\gamma}) - \phi_2(\tilde{\gamma}) = I_{\tilde{M}}^{\tilde{M}}(\tilde{\gamma},\tilde{h}),\ \tilde{\gamma} \in \Sigma \cap U.$$

For the remainder of this section we assume that $v$ is nonarchimedean and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$. Suppose that $\tilde{\gamma} \in \tilde{M}(F_v)$ such that $\mathbf{p}(\tilde{\gamma})$ has Jordan decomposition $\sigma u$. Choose $\tilde{\sigma}$ such that $\mathbf{p}(\tilde{\sigma}) = \sigma$ and $\tilde{\gamma} = \tilde{\sigma} u$. The germ expansion of $I_{\tilde{M}}(\tilde{\gamma},\tilde{f})$ is the extension of (2.5) [Art88a] to metaplectic coverings and reads as

$$I_{\tilde{M}}(\tilde{\gamma},\tilde{f}) \overset{(\tilde{M},\tilde{\sigma})}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\delta}\in\tilde{\sigma}(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma},\tilde{\delta}) I_{\tilde{L}}(\tilde{\delta},\tilde{f}). \tag{34}$$

As in the germ expansions of invariant orbital integrals (§3), the germs $g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma},\tilde{\delta})$ are parameterized by classes in $(\mathcal{U}_{L_\sigma}(F_v))$ (by way of the orbit map $\mathbf{s}$). Therefore in comparing germ expansions at $\gamma = \sigma u \in M(F_v)$ and $\gamma'$ it would be convenient to have a bijection between $(\mathcal{U}_{L_\sigma}(F_v))$ and $(\mathcal{U}_{L_{\sigma^n}}(F_v))$. The following lemma gives us such a bijection provided $\sigma$ is $F_v$-elliptic in $L$.

**Lemma 12.1** *Suppose $L \in \mathcal{L}(M)$ and $\sigma$ is a semisimple element of $M(F_v)$ such that $\sigma$ is $F_v$-elliptic in $M$ and $\sigma^n$ is $F_v$-elliptic in $L$. Then there exists an element $\eta$ in $\mu_n^M$ such that $L_{\eta\sigma}(F_v) = L_{\sigma^n}(F_v)$. In particular $\eta\sigma$ is $F_v$-elliptic in $L$.*

*Proof.* For the sake of convenience, we suppose that $L = G$. Recall decomposition (4),

$$M(F_v) = M(1)(F_v) \times \cdots \times M(\ell)(F_v) \cong \mathrm{GL}(r_1,F_v) \times \cdots \times \mathrm{GL}(r_\ell,F_v).$$

We identify $M(F_v)$ with this direct product of general linear groups. By assumption, $\sigma$ is $F_v$-elliptic in $M(F_v)$. It is therefore $M(F_v)$-conjugate to

$$\begin{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} \sigma_\ell & & 0 \\ & \ddots & \\ 0 & & \sigma_\ell \end{pmatrix} \end{pmatrix},$$

where $\sigma_i \in \mathrm{GL}(m_i, F_v)$ generates an extension $F_i/F_v$ of degree $m_i$, $m_i$ divides $r_i$, and $\sigma_i$ appears $r_i/m_i$ times (cf. §1 [KP86]). Since $\sigma^n$ is $F_v$-elliptic in $G(F_v)$, it is $M(F_v)$-conjugate to

$$\begin{pmatrix} \sigma_0 & & 0 \\ & \ddots & \\ 0 & & \sigma_0 \end{pmatrix},$$

where $\sigma_0 \in \mathrm{GL}(m_0, F_v)$ generates a field extension $F_0$ of degree $m_0$, $m_0$ divides $r$, and $\sigma_0$ appears $r/m_0$ times. The elements $\sigma_i \in \mathrm{GL}(m_i, F_v)$ may also be regarded as elements of the fields $F_i$, $0 \leq i \leq \ell$. From this perspective we have

$$F_i = F_v(\sigma_i) \supset F_v(\sigma_i^n) = F_v(\sigma_0) = F_0, \ 1 \leq i \leq \ell.$$

Obviously $[F_i : F_0] \leq r$ and, by Theorem 10 (b) VIII §6 [Lan84], $[F_i : F_0]$ divides $n$, $1 \leq i \leq \ell$. As we are assuming **(†)**, we must have $F_0 = F_1 = \cdots = F_\ell$. As a result, $m_1 = \cdots = m_\ell = m_0$ and $\sigma_1^n = \cdots = \sigma_\ell^n = \sigma_0$. Since $F_0$ contains $\mu_n$, it follows that there exist $\zeta_1, \ldots, \zeta_\ell \in \mu_n$ such that $\zeta_i \sigma_1 = \sigma_i$. The elements $\zeta_1, \ldots, \zeta_\ell$ determine an element $\eta \in \mu_n^M$ such that

$$\eta\sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix}.$$

The lemma now follows from the fact that

$$G_{\eta\sigma}(F_v) \cong \mathrm{GL}(r/m_0, F_0) \cong G_{\sigma^n}(F_v). \square$$

In the hypothesis of Lemma 12.1 we take $\sigma$ to be $F_v$-elliptic in $M(F_v)$. We shall see in a moment that this restriction does not pose a problem for the comparison of germs because the germs of (34) vanish for non-elliptic $\sigma$. We therefore assume for the rest of this section that $\sigma$ is a semisimple element of $M(F_v)$ which is $F_v$-elliptic in $M(F_v)$.

Let us consider the ellipticity of $\sigma^n$. There exists a unique maximal Levi subgroup $L_0 \in \mathcal{L}(M)$ such that $\sigma^n$ is $F_v$-elliptic in $L_0$. Indeed, if $\sigma^n$ is $F_v$-elliptic in $L_1, L_2 \in \mathcal{L}(M)$, then the centralizer in $G$ of the split torus $A_{L_1} \cap A_{L_2}$ is a Levi subgroup $L \in \mathcal{L}(M)$. It is then simple to verify that

$$A_L(F_v) = A_{L_1}(F_v) \cap A_{L_2}(F_v) = A_{L_{1,\sigma^n}}(F_v) \cap A_{L_{2,\sigma^n}}(F_v) = A_{L_{\sigma^n}}(F_v),$$

which is equivalent to $\sigma^n$ being $F_v$-elliptic in $L$. By Lemma 12.1 we know that there exists $\eta_0 \in \mu_n^M$ such that $L_{0,\eta_0\sigma}(F_v) = L_{0,\sigma^n}(F_v)$. We shall assume that $\eta_0$ is the identity for the rest of this section. As a result, $\sigma$ is $F_v$-elliptic in $L \in \mathcal{L}(M)$ if and only if $\sigma^n$ is $F_v$-elliptic in $L$.

**Lemma 12.2** *Suppose that $L \in \mathcal{L}(M)$, $\sigma$ is $F_v$-elliptic in $L$, and $\eta \in \mu_n^M$. Then $\eta\sigma$ is $F_v$-elliptic in $L$ if and only if $\eta \in \mu_n^L$.*

*Proof.* If $\eta \in \mu_n^L$ then $L_{\eta\sigma}(F_v) = L_\sigma(F_v)$, so $\eta\sigma$ is clearly seen to be $F_v$-elliptic in $L$. Conversely, suppose $\eta\sigma$ is $F_v$-elliptic in $L$ and, for the sake of simplicity, that $L = G$. Since $\sigma$ is $F_v$-elliptic in $G$, we may assume by rational canonical form that

$$\sigma = \begin{pmatrix} \sigma_0 & & 0 \\ & \ddots & \\ 0 & & \sigma_0 \end{pmatrix},$$

where $\sigma_0 \in \mathrm{GL}(m_0, F_v)$ generates a field extension $F_0$ of degree $m_0$ and $\sigma_0$ appears $r/m_0$ times. Consequently,

$$\eta\sigma = \begin{pmatrix} \eta_1\sigma_0 & & 0 \\ & \ddots & \\ 0 & & \eta_{r/m_0}\sigma_0 \end{pmatrix},$$

where the scalar matrices, $\eta_1, \ldots, \eta_{r/m_0}$, are projections of $\eta$ into the $\mathrm{GL}(m_0, F_v)$-blocks. In order for $\eta\sigma$ to be $F_v$-elliptic in $G$, we must have $\eta_i\sigma_0 = \eta_j\sigma_0$ for $1 \leq i, j \leq r/m_0$. This implies that $\eta \in \mu_n^G$.$\square$

We are now ready to give germ expansions for the local geometric terms of the trace formula.

**Lemma 12.3** *Suppose $\gamma \in M_\sigma(F_v) \cap G_{\text{oreg}}(F_v)$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$. Then there exist functions $\gamma \mapsto g_M^L(\gamma, \delta)$ such that*

$$I_M^\Sigma(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma, \delta) I_L^\Sigma(\delta, \tilde{f}).$$

*Proof.* Suppose $f \in \mathcal{H}(G(F_v))$. If $\eta \in \mu_n^M$ then $\eta\sigma$ is a semisimple element and by (2.5) [Art88a], we have

$$I_M(\gamma, f) \overset{(M,\eta\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in (\eta\sigma\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\gamma, \delta) I_L(\delta, f), \ \gamma \in M_{\eta\sigma}(F_v) \cap G_{\text{oreg}}(F_v).$$

As $\eta$ lies in the center of $M(F_v)$, it is easy to see that $(M, \eta\sigma)$-equivalence of a function in $\gamma$ is the same as $(M,\sigma)$-equivalence of a function in $\eta\gamma$. The above expansion may thus be written as

$$I_M(\eta\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in (\eta\sigma\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\eta\gamma, \delta) I_L(\delta, f), \ \gamma \in M_\sigma(F_v) \cap G_{\text{oreg}}(F_v).$$

Consequently

$$\begin{aligned} I_M^\Sigma(\gamma, \tilde{f}) &= \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M(\eta\gamma, \tilde{f}') \\ &\overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\eta \in \mu_n^M/\mu_n^G} \sum_{\delta \in \eta\sigma(\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\eta\gamma, \delta) \hat{I}_L(\delta, \tilde{f}'). \end{aligned}$$

Now if $\gamma = \sigma\gamma_1$ and $\delta = \eta\sigma u$, where $u \in \mathcal{U}_{L_{\eta\sigma}}(F_v)$, then by Lemma 9.2 [Art88a] we have

$$g_M^L(\eta\gamma, \delta) = \begin{cases} g_{M_\sigma}^{L_{\eta\sigma}}(\gamma_1, u), & \text{if } \eta\sigma \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases}.$$

If $\sigma^n$ is not $F_v$-elliptic in $L$ then neither is $\eta\sigma$ for any $\eta \in \mu_n^M/\mu_n^G$ and $g_M^L(\eta\gamma, \delta) = 0$. On the other hand, if $\sigma^n$ is $F_v$-elliptic in $L$ then, by our assumption on $\sigma$, Lemma 12.2 and Lemma 9.2 [Art88a],

$$\begin{aligned} g_M^L(\eta\gamma, \delta) &= \begin{cases} g_{M_\sigma}^{L_{\eta\sigma}}(\gamma_1, u), & \text{if } \eta \in \mu_n^L/\mu_n^G \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} g_{M_\sigma}^{L_\sigma}(\gamma_1, u), & \text{if } \eta \in \mu_n^L/\mu_n^G \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} g_M^L(\gamma, \delta), & \text{if } \eta \in \mu_n^L/\mu_n^G \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Combining these last two observations, we find that

$$\begin{aligned} I_M^{\Sigma}(\gamma, \tilde{f}) &\overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma,\delta) \sum_{\eta\in\mu_n^L/\mu_n^G} \hat{I}_L(\eta\delta, \tilde{f}') \\ &= \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma,\delta) I_L^{\Sigma}(\delta, \tilde{f}).\square \end{aligned}$$

The next lemma shows that there is a similar germ expansion for the distribution $I_M^{\mathcal{M}}(\gamma, f)$. We shall abuse notation slightly by identifying the index set $(\mathcal{U}_{L_\sigma}(F_v))$ below with $(\mathbf{s}(\mathcal{U}_{L_\sigma}(F_v)))$.

**Lemma 12.4** *Suppose $\gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v)$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$. Then there exist functions $\gamma \mapsto g_{\tilde{M}}^{\tilde{L}}(\mathbf{s}(\gamma), \mathbf{s}(\delta))$ such that*

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\mathbf{s}(\gamma), \mathbf{s}(\delta)) I_L^{\mathcal{M}}(\delta, \tilde{f}).$$

*Proof.* Expansion (2.5) [Art88a] translates into the metaplectic context as

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) \overset{(\tilde{M},\sigma')}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\delta}\in\sigma'(\mathcal{U}_{L_{\sigma^n}}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \tilde{\delta}) I_{\tilde{L}}(\tilde{\delta}, \tilde{f})$$

for $\tilde{\gamma} \in \tilde{M}_{\sigma'}(F_v) \cap \tilde{G}_{\mathrm{reg}}(F_v)$. Lemma 9.2 [Art88c] translates as

$$g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \tilde{\delta}) = \begin{cases} g_{\widetilde{M_{\sigma^n}}}^{\widetilde{L_{\sigma^n}}}(\gamma_1, \mathbf{s}(u)), & \text{if } \sigma^n \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases},$$

where $\tilde{\gamma} = \sigma'\gamma_1$, $\tilde{\delta} = \sigma'\mathbf{s}(u)$ and $u \in \mathcal{U}_{L_{\sigma^n}}(F_v)$. If $\sigma^n$ is $F_v$-elliptic in $L$, it follows from our assumptions on $\sigma$ that $L_\sigma = L_{\sigma^n}$. Consequently, $\widetilde{L_\sigma} = \widetilde{L_{\sigma^n}}$ and the previous equation becomes

$$g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \tilde{\delta}) = \begin{cases} g_{\widetilde{M_{\sigma}}}^{\widetilde{L_{\sigma}}}(\gamma_1, \mathbf{s}(u)), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases}. \tag{35}$$

By taking these facts into consideration, we obtain the expansion,

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) \overset{(\tilde{M},\sigma')}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\delta}\in\sigma'(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \tilde{\delta}) I_{\tilde{L}}(\tilde{\delta}, \tilde{f}),\ \tilde{\gamma} \in \tilde{M}_{\sigma'}(F_v) \cap \tilde{G}_{\mathrm{reg}}(F_v).$$

The local vanishing property, Proposition 8.3, tells us that $I_{\tilde{L}}(\tilde{\delta}, \tilde{f})$ vanishes unless $\mathbf{p}(\tilde{\delta}) = \delta^n$ for some $\delta \in G(F_v)$. The set $\sigma(\mathcal{U}_{L_\sigma}(F_v))$ maps bijectively onto the set $\sigma'(\mathcal{U}_{L_\sigma}(F_v))$ under the orbit map. This can be deduced from Lemma 4.1 and the identity of cocycles (1). Hence,

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) \overset{(\tilde{M},\sigma')}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \delta') I_{\tilde{L}}(\delta', \tilde{f}), \; \tilde{\gamma} \in \tilde{M}_{\sigma'}(F_v) \cap \tilde{G}_{\mathrm{reg}}(F_v).$$

Once again, by the local vanishing property, $I_{\tilde{M}}(\tilde{\gamma}, \tilde{f})$ vanishes unless $\mathbf{p}(\tilde{\gamma}) = \gamma^n$ for some $\gamma \in G(F_v) \cap G_{\mathrm{oreg}}(F_v)$. We claim that for each $L \in \mathcal{L}(M)$ and $\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))$, the function $g_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \delta')$ has the same vanishing property. We may assume inductively that this is true for $L \neq G$. Fix $\delta_1 \in \sigma(\mathcal{U}_{L_\sigma}(F_v))$. There exists a function $\tilde{f}_1 \in \mathcal{H}(\tilde{G}(F_v))$ such that

$$I_{\tilde{G}}(\tilde{\delta}, \tilde{f}_1) = \left\{ \begin{array}{ll} 1, & \text{if } \tilde{\delta} = \delta_1' \\ 0, & \text{otherwise} \end{array} \right\}, \; \tilde{\delta} \in \sigma'(\mathcal{U}_{G_\sigma}(F_v))$$

(cf. §3.3 [Vig81]). In particular,

$$I_{\tilde{G}}(\delta', \tilde{f}_1) = \left\{ \begin{array}{ll} 1, & \text{if } \delta' = \delta_1' \\ 0, & \text{otherwise} \end{array} \right\}, \; \delta \in \sigma(\mathcal{U}_{G_\sigma}(F_v)).$$

It is easily shown using Lemma 4.1 in this instance that $\delta' = \delta_1'$ if and only if $\delta = \delta_1$. Thus

$$I_{\tilde{G}}(\delta', \tilde{f}_1) = \left\{ \begin{array}{ll} 1, & \delta = \delta_1 \\ 0, & \text{otherwise} \end{array} \right\}, \; \delta \in \sigma(\mathcal{U}_{G_\sigma}(F_v)).$$

If we substitute $\tilde{f}_1$ into our last germ expansion, the desired vanishing property for $g_{\tilde{M}}^{\tilde{G}}(\tilde{\gamma}, \delta')$ follows. Our germ expansion now has the form

$$I_{\tilde{M}}(\gamma', \tilde{f}) \overset{(\tilde{M},\sigma')}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\gamma', \delta') I_{\tilde{L}}(\delta', \tilde{f}), \; \gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

As noted in §3, the orbital integral of any function in $C_c^\infty(\tilde{M}(F_v))$ is equal to the orbital integral of a function in $C_c^\infty(M(F_v))$. Therefore $(\tilde{M}, \sigma')$-equivalence may be taken to be $(M, \sigma)$-equivalence and we may write

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} |n|_v^{r/2} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\gamma', \delta') I_{\tilde{L}}(\delta', \tilde{f}), \; \gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

We now follow an argument on pp. 121-122 [AC89]. Suppose $\sigma$ is $F_v$-elliptic in $L$. Then by equation (35) we have

$$g_{\tilde{M}}^{\tilde{L}}(\gamma', \delta') = g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\gamma_1', u') = g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1)^n, \mathbf{s}(u)^n),$$

where $\gamma = \sigma\gamma_1$ and $\delta = \sigma u$. The metaplectic version of the homogeneity property of germs (Proposition 10.2 [Art88c]) implies that $g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1)^n, \mathbf{s}(u)^n)$ is equal to the product of $|n|_v^{(\dim(L_{\sigma u})-r)/2}$ with

$$\sum_{L_1 \in \mathcal{L}^L(M)} \sum_{u_1 \in (\mathcal{U}_{L_{1,\sigma}}(F_v))} g_{\widetilde{M_\sigma}}^{\widetilde{L_{1,\sigma}}}(\mathbf{s}(\gamma_1), \mathbf{s}(u_1)) c_{L_{1,\sigma}}^{L_\sigma}(u_1, n)[u_1^{L_\sigma} : u],$$

where $c_{L_{1,\sigma}}^{L_\sigma}(u_1, n)$ is defined as in (20) and

$$[u_1^{L_\sigma} : u] = \begin{cases} 1, & \text{if } u = u_1^{L_\sigma} \\ 0, & \text{otherwise} \end{cases}.$$

Suppose that $u = u_1^{L_\sigma}$ and set $\delta_1 = \sigma u_1$. Then

$$|n|_v^{r/2}|n|^{(\dim(L_{\sigma u})-r)/2} = |n|^{\dim(L_{1,\delta_1})/2} = \Lambda^{L_1}(\delta_1).$$

Furthermore, by Lemma 3.5 II [AC89], we have

$$c_{L_{1,\sigma^n}}^{L_{\sigma^n}}(u_1, n) = c_{L_{1,\sigma^n}}^{L_{\sigma^n}}(u_1^n, n) = c_{L_1}^{L}(\sigma_1^n u, n) = c_{L_1}^{L}(\delta_1^n, n).$$

Again, by equation (35) we have

$$g_{\widetilde{M_\sigma}}^{\widetilde{L_{1,\sigma}}}(\mathbf{s}(\gamma_1), \mathbf{s}(u_1)) = g_{\tilde{M}}^{\tilde{L_1}}(\mathbf{s}(\gamma), \mathbf{s}(\delta_1)).$$

Putting these facts together, we see that

$$|n|_v^{r/2} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_{\tilde{M}}^{\tilde{L}}(\gamma', \delta') I_{\tilde{L}}(\delta', \tilde{f})$$

is $(M, \sigma)$-equivalent to

$$\sum_{L_1 \in \mathcal{L}^L(M)} \sum_{\delta_1 \in \sigma(\mathcal{U}_{L_{1,\sigma}}(F_v))} \Lambda^{L_1}(\delta_1) g_{\tilde{M}}^{\tilde{L_1}}(\mathbf{s}(\gamma), \mathbf{s}(\delta_1)) c_{L_1}^{L}(\delta_1^n, n) I_{\tilde{L}}((\delta_1^L)', \tilde{f}),$$

if $\sigma$ is $F_v$-elliptic in $L$. If $\sigma$ is not $F_v$-elliptic in $L$ then equation (35) and Lemma 3.5 II [AC89] imply that both of these expressions vanish, and are thus still equal. In consequence, $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ is $(M, \sigma)$-equivalent to the sum of the latter expression over $L \in \mathcal{L}(M)$. Interchanging the sums over $L$ and $L_1$ and applying (20) then completes the lemma.$\square$

In what is left of this section we compare the germ expansions of Lemma 12.3 and Lemma 12.4.

**Lemma 12.5 (7.1)** *Suppose Theorem A (i) holds for $G$. Then for each $u \in \mathcal{U}_G(F_v)$ we have*

$$g_M^G(\gamma, u) \overset{(M,1)}{\sim} g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(u)),\ \ \gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

*Proof.* We may assume by induction that

$$g_M^L(\gamma, u) \overset{(M,1)}{\sim} g_{\tilde{M}}^{\tilde{L}}(\mathbf{s}(\gamma), \mathbf{s}(u)),\ \ \gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v),$$

for all $L \in \mathcal{L}(M)$ such that $L \neq G$. We may equate the germ expansions of Lemmas 12.3 and 12.4 since we are assuming Theorem A to hold. Together with the induction assumption, this yields

$$\sum_{u \in (\mathcal{U}_G(F_v))} \left( g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(u)) - g_M^G(\gamma, u) \right) \Lambda^G(u) I_{\tilde{G}}(u', \tilde{f}) \overset{(M,1)}{\sim} 0,$$

for $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$. As in Lemma 12.4, for a a fixed element $u_1 \in (\mathcal{U}_G(F_v))$, we may choose $\tilde{f}_1 \in \mathcal{H}(\tilde{G}(F_v))$ such that

$$I_{\tilde{G}}(u, \tilde{f}_1) = \left\{ \begin{array}{ll} 1, & u = u_1 \\ 0, & \text{otherwise} \end{array} \right\},\ u \in (\mathcal{U}_G(F_v)).$$

The lemma now follows by replacing $\tilde{f}$ with $\tilde{f}_1$ in the last $(M,1)$-equivalence.
$\square$

**Lemma 12.6 (7.2)** *Suppose $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$. Then*

$$g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), 1) \overset{(M,1)}{\sim} g_M^G(\gamma, 1).$$

*Proof.* Fix a supercuspidal representation $\tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{G}(F_v))$ and let $\tilde{f}$ be a matrix coefficient of $\tilde{\pi}$. Although $\tilde{f}$ only has compact support modulo the center of $\tilde{G}(F_v)$, the distributions $I_M(\gamma)$ and $I_M^{\Sigma}(\gamma)$ are still defined at $\tilde{f}$. Indeed, these distributions are defined on the space $\mathcal{H}_{\mathrm{ac}}(\tilde{G}(F_v))$ of "almost compact" Hecke functions (cf. §11 [Art89a]). Furthermore, it is clear from §27.3 [FK86] that there exists a matrix coefficient $f$ of $\tilde{\pi}'$ such that

$$\Lambda^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}) = I_G(\gamma, f),\ \gamma \in G_{\mathrm{oreg}}(F_v).$$

The local trace formula ((9.3) [Art89c]) implies the equations,

$$n I_{\tilde{M}}(\gamma', \tilde{f}) = (-1)^{\dim(A_M)} |D^G(\gamma^n)|_v^{1/2} \Theta_{\tilde{\pi}}(\gamma'),$$
$$I_M(\gamma, f) = (-1)^{\dim(A_M)} |D^G(\gamma)|_v^{1/2} \Theta_{\tilde{\pi}'}(\gamma),$$

for $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$ which are $F_v$-elliptic in $M$ (the coefficient $n$ in the former equation arises from an invariant sum over $\mathbf{i}(\mu_n)$). It is immediate from these equations that we also have

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = n^{-1} |n|_v^{r/2} (-1)^{\dim(A_M)} |D^G(\gamma^n)|_v^{1/2} \Theta_{\tilde{\pi}}(\gamma'),$$
$$I_M^{\Sigma}(\gamma, \tilde{f}) = (-1)^{\dim(A_M)} \sum_{\eta \in \mu_n^M / \mu_n^G} |D^G(\eta\gamma)|_v^{1/2} \Theta_{\tilde{\pi}'}(\eta\gamma).$$

According to character relation (6) the right-hand sides of these equations are equal. If $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$ is not $F_v$-elliptic in $M$ then

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = 0 = I_M^{\Sigma}(\gamma, \tilde{f}),$$

by [Art87]. Applying Lemma 11.3 we obtain

$$I_M^{\mathcal{M}}(1, \tilde{f}) = I_M^{\Sigma}(1, \tilde{f}).$$

Suppose $u \in M(F_v)$ is unipotent and $u \neq 1$. Then $u$ can be represented by an induced unipotent conjugacy class $u_1^M$, where $u_1 \in (\mathcal{U}_{M_1}(F_v))$, $M_1 \in \mathcal{L}$ and $M_1 \subsetneq M$. Equation (20) and the descent formula of Corollary 8.2 [Art88a] (extended to metaplectic coverings) imply that

$$\begin{aligned} I_M^{\mathcal{M}}(u, \tilde{f}) &= \Lambda^M(u) \sum_{L \in \mathcal{L}(M)} c_M^L(u^n, n) I_{\tilde{L}}((u_1^L)', \tilde{f}) \\ &= \Lambda^M(u) \sum_{L \in \mathcal{L}(M)} c_M^L(u^n, n) \sum_{L_1 \in \mathcal{L}(M_1)} d_{M_1}^G(L, L_1) \hat{I}_{\tilde{M}_1}^{\tilde{L}_1}(u_1', \tilde{f}_{\tilde{L}_1}). \end{aligned}$$

Since $\tilde{f}$ is a supercusp form on $\tilde{G}$ we have $\tilde{f}_{\tilde{L_1}} = 0$ for any $L_1 \in \mathcal{L}$ such that $L_1 \subsetneqq G$. It follows that the right-hand side of this equation vanishes. Using a similar, but simpler, argument we see that $I_M^{\Sigma}(u, \tilde{f})$ also vanishes. Collecting our results, we find that many of the terms in the respective germ expansions in Lemma 12.3 and Lemma 12.4 of $I_M^{\Sigma}(\gamma, \tilde{f})$ and $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ about the identity disappear. Assuming inductively that the lemma holds for $G$ replaced by $L \in \mathcal{L}$ with $L \subsetneqq G$, we find that the difference of the remaining terms is

$$\begin{aligned} & g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), 1) I_G^{\mathcal{M}}(1, \tilde{f}) - g_M^G(\gamma, 1) I_G^{\Sigma}(\gamma, \tilde{f}) \\ = & \quad \left( g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), 1) - g_M^G(\gamma, 1) \right) |n|_v^{r^2/2} \tilde{f}(1) \\ \overset{(M,1)}{\sim} & \quad 0. \end{aligned}$$

The lemma now follows from the fact that $\tilde{f}(1) \neq 0$.□

We now define a subspace $\mathcal{H}(\tilde{G}(F_S))^0$ of $\mathcal{H}(\tilde{G}(F_S))$ upon which our distributions shall be easier to compare. Set $\mathcal{H}(\tilde{G}(F_S))^0$ to be the subspace of $\mathcal{H}(\tilde{G}(F_S))$ generated by functions,

$$\tilde{f} = \prod_{v \in S} \tilde{f}_v, \ \tilde{f}_v \in \mathcal{H}(G(F_v)),$$

such that for each nonarchimedean valuation $v \in S$ the orbital integral of $\tilde{f}_v$ vanishes at any element of the form

$$(\delta_v u_v, \zeta), \ \delta \in A_G(F_v), \ u_v \in \mathcal{U}_G(F_v), \ u_v \neq 1, \ \zeta \in \mu_n.$$

**Proposition 12.1 (7.3)** *Suppose $\sigma$ is as above and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))^0$. Then*

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in (\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma, \delta) I_L^{\mathcal{M}}(\delta, \tilde{f}), \ \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

*Proof.* Suppose $\delta = \sigma u$, where $u \in \mathcal{U}_{L_\sigma}(F_v)$, and $\gamma = \sigma\gamma_1$, where $\gamma_1 \in M_\sigma(F_v)$. Then according to Lemma 9.2 [Art88c] and the proof of Lemma 12.4,

$$g_M^L(\gamma, \delta) = \begin{cases} g_{M_\sigma}^{L_\sigma}(\gamma_1, u), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases}$$

and

$$g_{\tilde{M}}^{\tilde{L}}(\mathbf{s}(\gamma), \mathbf{s}(\delta)) = \begin{cases} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1), \mathbf{s}(u)), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases}.$$

By the germ expansion of Lemma 12.4, it suffices to show that

$$g_{M_\sigma}^{L_\sigma}(\gamma_1, u) \overset{(M_\sigma,1)}{\sim} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1), \mathbf{s}(u)),$$

if $\sigma$ is $F_v$-elliptic in $L$, and $L \in \mathcal{L}(M)$. Suppose that $\sigma$ is $F_v$-elliptic in $L$ and that $L_\sigma \neq G$. Then the induction assumption following Theorem A holds for $L_\sigma$ and we may apply Lemma 12.5, with $L_\sigma$ in place of $G$, to obtain

$$g_{M_\sigma}^{L_\sigma}(\gamma_1, u) \overset{(M_\sigma,1)}{\sim} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1), \mathbf{s}(u)).$$

By Lemma 2.1 [Art88c], it follows that $(M_\sigma, 1)$-equivalence of these germs as functions of $\gamma_1$ is that same as $(M, \sigma)$-equivalence as functions of $\gamma$. That is,

$$g_M^L(\gamma, \delta) = g_{M_\sigma}^{L_\sigma}(\gamma_1, u) = g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\mathbf{s}(\gamma_1), \mathbf{s}(u)) = g_{\tilde{M}}^{\tilde{L}}(\mathbf{s}(\gamma), \mathbf{s}(\delta)).$$

The remaining possibility is that $\sigma \in A_G(F_v)$ and $L = G$. Suppose this is the case. Then, by our assumptions on $\sigma$ and Lemma 4.5,

$$\sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(\delta)) I_G^{\mathcal{M}}(\delta, \tilde{f}) = \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(\delta)) \Lambda^G(\delta) I_{\tilde{G}}(\delta, \tilde{f}).$$

Since $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))^0$, we see that $I_{\tilde{G}}(\delta, \tilde{f})$ vanishes in the above sum unless $\delta = \sigma$. Since $\sigma$ is central,

$$g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(\sigma)) = g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma_1), 1) = g_M^G(\gamma_1, 1) = g_M^G(\gamma, \sigma)$$

by Lemma 9.2 [Art88c] and Lemma 12.6. Hence,

$$\begin{aligned} \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(\delta)) I_G^{\mathcal{M}}(\delta, \tilde{f}) &= g_{\tilde{M}}^{\tilde{G}}(\mathbf{s}(\gamma), \mathbf{s}(\sigma)) I_G^{\mathcal{M}}(\sigma, \tilde{f}) \\ &\overset{(M,\sigma)}{\sim} g_M^G(\gamma, \sigma) I_G^{\mathcal{M}}(\sigma, \tilde{f}) \\ &= \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_M^G(\gamma, \delta) I_G^{\mathcal{M}}(\delta, \tilde{f}) \end{aligned}$$

and the lemma is complete.□

# 13 The distributions $I_{\tilde{M}}(\tilde{\pi}, X)$ and $I^{\Sigma}_{M}(\tilde{\pi}, X)$

Leaving the geometric side of the trace formula behind, we examine the spectral side. The spectral side contains invariant distributions,

$$I_{\tilde{M}}(\tilde{\pi}, X, \tilde{f}),\ \tilde{\pi} \in \Pi(\tilde{M}(F_S)),\ X \in \mathfrak{a}_{M,S},\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

which are introduced in §3 [Art88a] in the case of the trivial covering. We assume that the reader is to some degree familiar with this introduction and describe some of it below in the context of non-trivial coverings.

Given $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$, we define

$$I^{G,\Sigma}_{M}(\tilde{\rho}, X, \tilde{f}) = n^{-\dim(A_G)} \hat{I}^{G}_{M}(\tilde{\rho}', X', \tilde{f}') = n^{-1} \hat{I}_{M}(\tilde{\rho}', X', \tilde{f}'),$$

for all $X \in \mathfrak{a}_{M,S}$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. The coefficient $n^{-1}$ appears in this definition in order absorb the same coefficient appearing in $I^{\Sigma}(\tilde{f}) = n^{-1} I(\tilde{f}')$. If $L \in \mathcal{L}(M)$ and $\lambda \in \mathfrak{a}^{*}_{M,S}$ is in general position, then the induced representation $\tilde{\rho}^{\tilde{L}}_{\lambda}$ belongs to $\Sigma(\tilde{L}(F_S))$. When $\tilde{\rho}^{\tilde{L}}_{\lambda}$ appears as an argument of $I_{\tilde{L}}(\cdot)$ or $I^{\Sigma}_{L}(\cdot)$, we will often suppress the superscript $\tilde{L}$. For $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$, we define $I^{\Sigma}_{M}(\tilde{\pi}, X, \tilde{f})$ as

$$\sum_{P} \omega_P \sum_{L} \sum_{\tilde{\rho}} \int_{\varepsilon_P + i\mathfrak{a}^{*}_{M,S}/i\mathfrak{a}^{*}_{L,S}} r^{\tilde{L}}_{\tilde{M}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) I^{\Sigma}_{L}(\tilde{\rho}_\lambda, h_L(X), \tilde{f}) e^{-\lambda(X)} d\lambda, \tag{36}$$

where $P$, $L$ and $\tilde{\rho}$ are summed over $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\Sigma(\tilde{M}(F_S))$ respectively. For the definitions of $h_L(X)$ and $\omega_P$ see §6 [Art89a] and §3 [Art88a]. The definition of $r^{\tilde{L}}_{\tilde{M}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda)$ follows §6 [Art89a].

As on p. 127 [AC89], we identify a representation $\tilde{\pi}$ in $\Pi(\tilde{M}(F_S)^1)$ with the orbit $\{\tilde{\pi}_\lambda : \lambda \in \mathfrak{a}^{*}_{M,\mathbf{C}}\}$ in $\Pi(\tilde{M}(F_S))$, if $\tilde{\pi}$ is not unitary. If $\tilde{\pi}$ belongs to $\Pi_{\mathrm{unit}}(\tilde{M}(F_S)^1)$, then we identify it with the orbit $\{\tilde{\pi}_\lambda : \lambda \in i\mathfrak{a}^{*}_{M}\}$ in $\Pi_{\mathrm{unit}}(\tilde{M}(F_S))$. We make similar identifications for representations in $\Pi(\tilde{M}(\mathbf{A})^1)$ and $\Pi_{\mathrm{unit}}(\tilde{M}(\mathbf{A})^1)$. If $\tilde{\pi} \in \Pi_{\mathrm{unit}}(\tilde{M}(F_S)^1)$, set

$$I_{\tilde{M}}(\pi, \tilde{f}) = I_{\tilde{M}}(\tilde{\pi}_\lambda, 0, \tilde{f}),$$

and

$$I^{\Sigma}_{M}(\tilde{\pi}, \tilde{f}) = I^{\Sigma}_{M}(\tilde{\pi}_\lambda, 0, \tilde{f}),$$

for any $\lambda \in i\mathfrak{a}^{*}_{M}$. It may be verified that these definitions are well-defined. Both of these definitions are independent of $S$, if $S$ is large, and therefore may be extended to representations in $\Pi_{\mathrm{unit}}(\tilde{M}(\mathbf{A})^1)$. In complete analogy with the expressions of the geometric side of the trace formula, we shall compare $I_{\tilde{M}}(\tilde{\pi}, \tilde{f})$ with the term $I_M^{\Sigma}(\tilde{\pi}, \tilde{f})$ occurring in the spectral sides of the trace formulas.

We may draw parallels between the local geometric and the local spectral terms of the trace formulas. In order to compare the local geometric terms of the trace formulas for $G$ and $\tilde{G}$, we use the orbit map. One might surmise that the analogous transfer map for the local spectral terms might be given by the local metaplectic correspondence (11). Unfortunately, this map does not intrinsically relate the characters of the representations to each other. In what follows, we define certain constants which relate representations in $\Pi(\tilde{M}(F_S))$ to representations in $\Pi(M(F_S))$ in a fashion that is compatible with their characters.

By the Langlands quotient theorem and §5 [Art89a] there exist constants $\Delta(\tilde{\pi}, \tilde{\rho})$ and $\Gamma(\tilde{\rho}, \tilde{\pi})$ for arbitrary $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$ and $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$, such that

$$\mathrm{tr}(\tilde{\rho}) = \sum_{\tilde{\pi} \in \Pi(\tilde{M}(F_S))} \Gamma(\tilde{\rho}, \tilde{\pi}) \mathrm{tr}(\tilde{\pi}),$$

and

$$\mathrm{tr}(\tilde{\pi}) = \sum_{\tilde{\rho} \in \Sigma(\tilde{M}(F_S))} \Delta(\tilde{\pi}, \tilde{\rho}) \mathrm{tr}(\tilde{\rho}).$$

For $\pi \in \Pi(M(F_S))$ and $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$ define

$$\Delta(\pi, \tilde{\rho}) = \Delta(\pi, \tilde{\rho}').$$

Given $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$, we set

$$\delta(\pi, \tilde{\pi}) = \sum_{\tilde{\rho} \in \Sigma(\tilde{M}(F_S))} \Delta(\pi, \tilde{\rho}) \Gamma(\tilde{\rho}, \tilde{\pi}).$$

As can be seen from the next proposition, the map of virtual characters,

$$\Theta_\pi \mapsto \sum_{\tilde{\pi} \in \Pi(\tilde{M}(F_S))} \delta(\pi, \tilde{\pi}) \Theta_{\tilde{\pi}}, \tag{37}$$

is the transfer map which allows us to compare characters of representations.

**Proposition 13.1 (8.2)** *For any $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$ and matching $f \in \mathcal{H}(M(F_S))$ we have*

$$\mathrm{tr}\,(\pi(f)) = \sum_{\tilde{\pi} \in \Pi(\tilde{M}(F_S))} \delta(\pi, \tilde{\pi}) \mathrm{tr}(\tilde{\pi}(\tilde{f})), \;\; \pi \in \Pi(M(F_S)).$$

*Proof.* According to the character relations satisfied by the local metaplectic correspondence

$$\begin{aligned}
\mathrm{tr}\,(\pi(f)) &= \sum_{\rho \in \Sigma(M(F_S))} \Delta(\pi, \rho) \mathrm{tr}\,(\rho(f)) \\
&= \sum_{\tilde{\rho} \in \Sigma(\tilde{M}(F_S))} \Delta(\pi, \tilde{\rho}') \mathrm{tr}(\tilde{\rho}(\tilde{f})) \\
&= \sum_{\tilde{\rho} \in \Sigma(\tilde{M}(F_S))} \sum_{\tilde{\pi} \in \Pi(\tilde{M}(F_S))} \Delta(\pi, \tilde{\rho}') \Gamma(\tilde{\rho}, \tilde{\pi}) \mathrm{tr}(\tilde{\pi}(\tilde{f})) \\
&= \sum_{\tilde{\pi} \in \Pi(\tilde{M}(F_S))} \delta(\pi, \tilde{\pi}) \mathrm{tr}(\tilde{\pi}(\tilde{f})). \square
\end{aligned}$$

**Corollary 13.1 (8.3)** *Suppose $S$ consists of a single valuation $v$, for which $|n|_v = 1$, and that $\tilde{\pi} \in \Pi(\tilde{M}(F_v))$ is an unramified representation. Then for any $\pi \in \Pi(M(F_v))$*

$$\delta(\pi, \tilde{\pi}) = \begin{cases} 1, & \text{if } \pi = \tilde{\pi}' \\ 0, & \text{otherwise} \end{cases}.$$

*Proof.* Take $\tilde{f}$ to be an arbitrary function in $\mathcal{H}(M(F_v))$ which is bi-invariant under $\mathbf{s}\,(K_v \cap M(F_v))$. Theorem 16 [FK86] tells us that $\mathrm{tr}\,(\tilde{\pi}'(f)) = \mathrm{tr}(\tilde{\pi}(\tilde{f}))$ for any $f \in \mathcal{H}(M(F_v))$ matching $\tilde{f}$. The corollary now follows from Proposition 13.1 and the linear independence of characters. $\square$

This corollary allows us to define an map

$$\delta(\pi, \tilde{\pi}) = \prod_v \delta(\pi_v, \tilde{\pi}_v)$$

for adelic representations $\tilde{\pi} = \otimes_v \tilde{\pi}_v \in \Pi(\tilde{M}(\mathbf{A}))$ and $\pi = \otimes_v \pi_v \in \Pi(M(\mathbf{A}))$. All of the above formulas remain valid in the adelic context. Given $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A})^1)$ and $\pi \in \Pi(M(\mathbf{A})^1)$, set

$$\delta(\pi, \tilde{\pi}) = \sum_{\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}} \delta(\pi_{\lambda_1}, \tilde{\pi}_\lambda)$$

for arbitrary $\lambda_1 \in \mathfrak{a}^*_{M,\mathbf{C}}$. This definition may be verified to be well-defined.

# 14 Statement of Theorem B

Now that we have a spectral transfer map (37), we can begin to compare the spectral sides of the trace formulas. The spectral side of the (conjectural) trace formula for $\tilde{G}$ is of the form

$$I(\tilde{f}) = \sum_{t \geq 0} I_t(\tilde{f}),$$

where

$$I_t(\tilde{f}) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(\tilde{M},t)} a^{\tilde{M}}(\tilde{\pi}) I_{\tilde{M}}(\tilde{\pi}, \tilde{f}) d\tilde{\pi}. \tag{38}$$

A description of the terms occurring in this formula may be extrapolated from §4 [Art88b]. The definition of $\Pi(\tilde{M}, t)$ is given here (cf. §4 [Art88b]), as we shall often consider it. Let $M_1 \in \mathcal{L}$ and $t$ be a positive real number. We are obliged to first define two other sets, $\Pi(\tilde{M}(\mathbf{A})^1, t)$ and $\Pi_{\mathrm{disc}}(\tilde{M}_1, t)$, before we define $\Pi(\tilde{M}, t)$. Given a representation $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A})^1)$, let $\nu_{\tilde{\pi}}$ be the infinitesimal character of the archimedean factor of $\tilde{\pi}$. The set $\Pi(\tilde{M}(\mathbf{A})^1, t)$ is defined to be the set of (equivalence classes) of representations $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A})^1)$ such that $\|\mathrm{Im}(\nu_{\tilde{\pi}})\| = t$. We write $\Pi_{\mathrm{disc}}(\tilde{M}_1, t)$ for the subset of $\Pi_{\mathrm{unit}}(\tilde{M}(\mathbf{A})^1, t)$ consisting of irreducible constituents of induced representations

$$\tilde{\sigma}_\lambda^{\tilde{M}_1}, \ L \in \mathcal{L}^{M_1}, \ \tilde{\sigma} \in \Pi_{\mathrm{unit}}(\tilde{L}(\mathbf{A})^1, t), \ \lambda \in i\mathfrak{a}_L^*/i\mathfrak{a}_{M_1}^*,$$

in which $\tilde{\sigma}_\lambda$ satisfies the following two conditions:

1. $a_{\mathrm{disc}}^{\tilde{L}}(\tilde{\sigma}) \neq 0$ (see (4.4) [Art88b]).
2. There is an element $s \in W^{M_1}(\mathfrak{a}_L)_{\mathrm{reg}}$ such that $s\tilde{\sigma}_\lambda = \tilde{\sigma}_\lambda$.

Then $\Pi(\tilde{M}, t)$ is defined as the disjoint union over $M_1 \in \mathcal{L}^M$ of the sets

$$\Pi_{\tilde{M}_1}(\tilde{M}, t) = \{\tilde{\pi} = \tilde{\pi}_{1,\lambda} : \tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}_1, t), \ \lambda \in i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*\}.$$

The measure in (38) is given by

$$d\tilde{\pi} = d\tilde{\pi}_{1,\lambda} = |W_0^{M_1}||W_0^M|^{-1} d\lambda.$$

Let us again consider §10, where we examined the geometric sides of the trace formulas. In Proposition 10.1, the geometric side of the trace formula for $\tilde{G}(\mathbf{A})$ was expressed in a manner that was compatible with the orbit map. We shall follow suit by expressing the spectral side of the trace formula for $\tilde{G}(\mathbf{A})$ in a manner that is compatible with the spectral transfer map (37). This will be carried out in §17. For the the time being, we set up the appropriate grouping of representations for the global datum $a^{\tilde{M}}(\tilde{\pi})$. In other words, we define the global datum $a^{M,\Sigma}(\tilde{\pi})$, which ought equal the global datum $a^{\tilde{M}}(\tilde{\pi})$, occurring in the trace formula for $\tilde{G}(\mathbf{A})$. This is similar in spirit to the grouping of the local geometric terms according to $\mu_n^M$ in §6. We first define $a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1)$ for $M_1 \in \mathcal{L}^M$. Set

$$a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1) = \sum_{\pi \in \Pi(M_1(\mathbf{A})^1)} a_{\text{disc}}^{M_1}(\pi)\delta(\pi, \tilde{\pi}_1), \ \tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1).$$

This sum may be shown to be finite using the arguments of Lemma 9.1 II [AC89]. Given $\tilde{\pi} = \tilde{\pi}_{1,\lambda}$, where $\lambda \in \mathfrak{a}^*_{M_1,\mathbf{C}}/\mathfrak{a}^*_{M,\mathbf{C}}$, we set

$$a^{M,\Sigma}(\tilde{\pi}) = a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1) r_{\tilde{M}_1}^{\tilde{M}}(\tilde{\pi}_{1,\lambda}).$$

The number $r_{\tilde{M}_1}^{\tilde{M}}(\pi_{1,\lambda})$ is not defined for arbitrary $\tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1)$, and so the definition of $a^{M,\Sigma}(\tilde{\pi})$ is not valid as it now stands. The obstacle stems from the fact that $r_{\tilde{M}_1}^{\tilde{M}}(\tilde{\pi}_{1,\lambda})$ is derived from the adelic version of the normalizing factors of intertwining operators (§5, §16). As such, it is defined as an infinite product over the valuations of $F$ and might not converge. One expects such normalizing factors to converge and have analytic continuation for automorphic representations. This is borne out by the theory of Eisenstein series (§4 [Art89a]). In order to rectify the above definition, we make the following induction hypothesis. We assume that for any $M_1 \in \mathcal{L}$ with $M_1 \neq G$, that

$$a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1) = a_{\text{disc}}^{M_1}(\tilde{\pi}_1), \ \tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1).$$

In this case $a_{\text{disc}}^{M_1,\Sigma}(\tilde{\pi}_1)$ vanishes unless $\tilde{\pi}_1$ belongs to $\Pi_{\text{disc}}(\tilde{M}_1, t)$. If $M_1 \in \mathcal{L}^M$, $\tilde{\pi}_1 \in \Pi_{\text{disc}}(\tilde{M}_1, t)$ and $\lambda \in i\mathfrak{a}^*_{M_1,\mathbf{C}}/i\mathfrak{a}^*_{M,\mathbf{C}}$ then $r_{\tilde{M}_1}^{\tilde{M}}(\tilde{\pi}_{1,\lambda})$ is defined and the above definition of $a^{M,\Sigma}(\tilde{\pi}_{1,\lambda})$ makes sense.

The global datum $a^{M,\Sigma}(\tilde{\pi}_1)$, suggests the definitions of sets of representations along the made along the same lines as the definitions of $\Pi_{\text{disc}}(\tilde{M}_1, t)$, $\Pi_{\tilde{M}_1}(\tilde{M}, t)$ and $\Pi(\tilde{M}, t)$ above. We define the sets of (equivalence classes of) representations $\Pi^{\Sigma}_{\text{disc}}(\tilde{M}_1, t)$, $\Pi^{\Sigma}_{\tilde{M}_1}(\tilde{M}, t)$ and $\Pi^{\Sigma}(\tilde{M}, t)$ as above, except that $a_{\text{disc}}^{\tilde{M}}$ is replaced with $a_{\text{disc}}^{M,\Sigma}$.

We are now in the position to state the spectral analogue of Theorem A.

**Theorem B** (i) Suppose that $S$ is a finite set of valuations with the closure property. Then

$$I_M^\Sigma(\tilde{\pi}, \tilde{f}) = I_{\tilde{M}}(\tilde{\pi}, \tilde{f}),\ \tilde{\pi} \in \Pi_{\mathrm{unit}}(\tilde{M}(\mathbf{A})^1),\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)).$$

(ii) For any given

$$\tilde{\pi} = \tilde{\pi}_{1,\lambda},\ \tilde{\pi}_1 \in \Pi(\tilde{M_1}(\mathbf{A})^1),\ \lambda \in \mathfrak{a}^*_{M_1,\mathbf{C}}/\mathfrak{a}^*_{M,\mathbf{C}},$$

we have

$$a^{M,\Sigma}(\tilde{\pi}) = a^{\tilde{M}}(\tilde{\pi}).$$

The proof of this theorem will be completed at the end of §20.

# 15 Comparison of $I_{\tilde{M}}(\tilde{\pi}, X, \tilde{f})$ and $\hat{I}_M^\Sigma(\tilde{\pi}, X, \tilde{f})$

The purpose of this section is to show that Theorem A (i) implies Theorem B (i). We achieve this using the maps $\theta_{\tilde{M}}^{\tilde{L}}$ and ${}^c\theta_{\tilde{M}}^{\tilde{L}}$ defined as in §4 [Art88a]. These maps are defined on $\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{L}(F_S))$ and take values in $\tilde{\mathcal{I}}_{\mathrm{ac}}(\tilde{M}(F_S))$ for every pair of Levi subgroups $M \subset L$ in $\mathcal{L}$. The spaces $\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{L}(F_S))$ and $\tilde{\mathcal{I}}_{\mathrm{ac}}(\tilde{M}(F_S))$ contain $\mathcal{H}(\tilde{L}(F_S))$ and $\mathcal{I}(\tilde{M}(F_S))$ respectively and are defined as in §11 [Art89a]. The map (13) extends in an obvious manner from a map on $\mathcal{H}(\tilde{L}(F_S))$ to a map on $\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{L}(F_S))$. The above maps satisfy the following properties:

$$\sum_{L \in \mathcal{L}(M)} \hat{\theta}_{\tilde{M}}^{\tilde{L}}({}^c\theta_{\tilde{L}}(\tilde{f})) = \sum_{L \in \mathcal{L}(M)} {}^c\hat{\theta}_{\tilde{M}}^{\tilde{L}}(\theta_{\tilde{L}}(\tilde{f})) = 0,$$

(39)
$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = \sum_{L \in \mathcal{L}(M)} {}^c\hat{I}_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \theta_{\tilde{L}}(\tilde{f})),$$

and

(40)
$${}^cI_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = \sum_{L \in \mathcal{L}(M)} \hat{I}_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma},\ {}^c\theta_{\tilde{L}}(\tilde{f})),$$

for $\tilde{\gamma} \in \tilde{M}(F_S)$ and $\tilde{f} \in \tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{G}(F_S))$. For the definition of ${}^cI_{\tilde{M}}(\tilde{\gamma})$ see §4 [Art88a]. Set

$${}^cI_M^{\mathcal{M}}(\gamma, \tilde{f}) = \Lambda^M(\gamma)\, {}^cI_{\tilde{M}}(\gamma', \tilde{f}),\ \gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S).$$

The analogue of property (40) for ${}^cI_M^{\mathcal{M}}(\gamma)$ is then seen to be

$$
\begin{aligned}
{}^cI_M^{\mathcal{M}}(\gamma,\tilde f) &= \sum_{L\in\mathcal{L}(M)} \Lambda^M(\gamma)\hat I_{\tilde M}^{\tilde L}(\gamma',\, {}^c\theta_{\tilde L}(\tilde f)) \\
&= \sum_{L\in\mathcal{L}(M)} \hat I_M^{L,\mathcal{M}}(\gamma,\, {}^c\theta_{\tilde L}(\tilde f)),\ \gamma\in M(F_S)\cap G_{\mathrm{oreg}}(F_S).
\end{aligned}
\tag{41}
$$

After a similar computation we may conclude that the analogue of property (39) is

$$
I_M^{\mathcal{M}}(\gamma,\tilde f) = \sum_{L\in\mathcal{L}(M)} {}^c\hat I_M^{L,\mathcal{M}}(\gamma,\theta_{\tilde L}(\tilde f)),\ \gamma\in M(F_S)\cap G_{\mathrm{oreg}}(F_S).
$$

Properties (39) and (40) may also be adapted to the distributions of the form $I_M^{\Sigma}(\gamma)$. We mimic the arguments of §6 to arrive at the equation

$$
{}^c\hat I_M(\eta\gamma,\tilde f') = {}^c\hat I_M(\gamma,\tilde f'),\ \tilde f\in\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde G(F_S)),\, \gamma\in M(F_S),\, \eta\in\mu_n^G.
$$

Thus, imitating the definition of $I_M^{\Sigma}(\gamma)$, we set

$$
{}^cI_M^{\Sigma}(\gamma,\tilde f) = \sum_{\eta\in\mu_n^M/\mu_n^G} {}^c\hat I_M(\eta\gamma,\tilde f'),\ \gamma\in M(F_S),\, \tilde f\in\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde G(F_S)).
$$

The analogue of property (40) for ${}^cI_M^{\Sigma}(\gamma)$ is then seen to be

$$
\begin{aligned}
{}^cI_M^{\Sigma}(\gamma,\tilde f) &= \sum_{\eta\in\mu_n^M/\mu_n^G}\sum_{L\in\mathcal{L}(M)} \hat I_M^L(\eta\gamma,\, {}^c\hat\theta_L(\tilde f')) \\
&= \sum_{L\in\mathcal{L}(M)}\sum_{\eta\in\mu_n^M/\mu_n^L}\sum_{\eta_1\in\mu_n^L/\mu_n^G} \hat I_M^L(\eta_1\eta\gamma,\, {}^c\hat\theta_L(\tilde f')) \\
&= \sum_{L\in\mathcal{L}(M)}\sum_{\eta\in\mu_n^M/\mu_n^L} \hat I_M^L(\eta\gamma,\, \sum_{\eta_1\in\mu_n^L/\mu_n^G} {}^{\eta_1 c}\hat\theta_L(\tilde f')).
\end{aligned}
$$

Here, ${}^{\eta_1 c}\hat\theta_M(\tilde f')$ is defined as $({}^{\eta_1}h)_M$ where $h\in\tilde{\mathcal{H}}_{\mathrm{ac}}(M(F_S))$ such that $h_M = {}^c\hat\theta_M(\tilde f')$ (Lemma 4.1 [Art88a]), and ${}^{\eta_1}h$ is the left-translate of $h$ by $\eta_1\in\mu_n^L$.

**Theorem 15.1 (10.2)** *Fix an element $M\in\mathcal{L}$ and assume that*

$$
I_L^{\mathcal{M}}(\gamma,\tilde f) = I_L^{\Sigma}(\gamma,\tilde f),\ \tilde f\in\mathcal{H}(\tilde G(F_S)),
$$

*for each* $L \in \mathcal{L}(M)$ *and* $\gamma \in L_{\mathrm{comp}}(F_S)$. *Then for any* $\tilde{f} \in \mathcal{H}_{\mathrm{ac}}(\tilde{G}(F_S))$ *and* $X \in \mathfrak{a}_{M,S}$, *we have*

$$^{c}\theta_{\tilde{M}}(\tilde{f})' = \sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}'),$$

$$\theta_{\tilde{M}}(\tilde{f})' = \sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta}\hat{\theta}_M(\tilde{f}'),$$

$$I_{\tilde{M}}(\tilde{\rho}, X, \tilde{f}) = I_M^{\Sigma}(\tilde{\rho}, X, \tilde{f}),\ \tilde{\rho} \in \Sigma(\tilde{M}(F_S)),$$

*and*

$$I_{\tilde{M}}(\tilde{\pi}, X, \tilde{f}) = I_M^{\Sigma}(\tilde{\pi}, X, \tilde{f}),\ \tilde{\pi} \in \Pi(\tilde{M}(F_S)).$$

*Proof.* We shall not prove this theorem in its entirety as its proof is almost identical to the proof of Theorem 10.2 II [AC89]. We shall however, provide a proof of the first and third assertions. We begin with the first assertion. We assume inductively that the theorem holds if $G$ is replaced by $L_1 \in \mathcal{L}$ such that $L_1 \subsetneqq G$. We also make the inductive assumption that the theorem holds if $M$ is replaced by $L_1 \in \mathcal{L}(M)$ such that $L_1 \supsetneqq M$. We may assume that $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$ as the restriction of any function in $\tilde{\mathcal{H}}_{\mathrm{ac}}(\tilde{G}(F_S))$ to

$$\tilde{G}(F_S) = \{\tilde{\gamma} \in \tilde{G}(F_S) : H_G(\mathbf{p}(\tilde{\gamma})) = Z\},\ Z \in \mathfrak{a}_{G,S},$$

lies in $\mathcal{H}(\tilde{G}(F_S))$. Suppose $\gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S)$ and consider the expression

$$\begin{aligned} &{}^{c}I_M^{\mathcal{M}}(\gamma, \tilde{f}) - {}^{c}I_M^{\Sigma}(\gamma, \tilde{f}) \\ &= \sum_{L \in \mathcal{L}(M)} \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^{c}\theta_{\tilde{L}}(\tilde{f})) - \sum_{\eta \in \mu_n^M/\mu_n^L} \hat{I}_M^L(\eta\gamma, \sum_{\eta_1 \in \mu_n^L/\mu_n^G} {}^{\eta_1 c}\hat{\theta}_L(\tilde{f}')). \end{aligned}$$

Under the assumption of the theorem and the above induction hypotheses this difference reduces to

$$\begin{aligned} &\hat{I}_M^{M,\mathcal{M}}(\gamma, \theta_{\tilde{M}}(\tilde{f})) - \hat{I}_M^M(\gamma, \sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}')) \\ &+ \sum_{\{L \in \mathcal{L}(M): L \neq M\}} \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^{c}\theta_{\tilde{L}}(\tilde{f})) - \hat{I}_M^{\Sigma}(\gamma, {}^{c}\theta_{\tilde{L}}(\tilde{f})) \\ &= \hat{I}_M^M(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}')). \end{aligned}$$

By Lemma 4.4 [Art88a], ${}^{c}I_M^{\mathcal{M}}(\cdot, \tilde{f}) - {}^{c}I_M^{\Sigma}(\cdot, \tilde{f})$ is a function of compact support in the space $M(F_S)$-conjugacy classes of $M(F_S)$. Given $X \in \mathfrak{a}_{M,S}$ and $\gamma \in$

$M(F_S) \cap G_{\text{oreg}}(F_S)$ such that $H_M(\gamma) = X$, the expression $\hat{I}_M^M(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}'))$ is the orbital integral of a function defined on

$$M(F_S)^X = \{\gamma_1 \in M(F_S) : H_M(\gamma_1) = X\}.$$

The tempered characters of this function are

$$\theta_{\tilde{M}}(\tilde{f})'(\pi, X) - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X), \ \pi \in \Pi_{\text{temp}}(M(F_S)).$$

Since $\hat{I}_M^M(\gamma, \theta_{\tilde{M}}(\tilde{f})') - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}'))$ has compact support in $\gamma$, and $H_M$ maps $M(F_S) \cap G_{\text{oreg}}(F_S)$ onto $\mathfrak{a}_{M,S}$, this difference has compact support in $X \in \mathfrak{a}_{M,S}$. We shall combine this fact with the classical Paley-Wiener theorem in order to deform contours of integration later in the proof. We proceed by showing that $\sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X)$ vanishes if $\pi \in \Pi_{\text{temp}}(M(F_S))$ is not metic, that is if $\pi$ is not in the image of (10). It follows easily from the definitions that for $\pi \in \Pi_{\text{temp}}(M(F_S))$,

$$\sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X) = \begin{cases} n^{\dim(A_M/A_G)}\, {}^{c}\hat{\theta}_M(\tilde{f}', \pi, X), & \text{if } \pi \text{ is } \mu_n^M\text{-invariant} \\ 0, & \text{otherwise} \end{cases}.$$

If $\pi$ is metic then it is $\mu_n^M$-invariant. Suppose therefore that $\pi$ is not metic but is $\mu_n^M$-invariant. Define

$${}^{c}\hat{I}_M(\rho, X, \tilde{f}') = \sum_{L \in \mathcal{L}(M)} \hat{I}_M^L(\rho, X, {}^{c}\hat{\theta}_L(\tilde{f}')), \ \rho \in \Sigma(M(F_S)), \ X \in \mathfrak{a}_{M,S}.$$

According to Lemma 5.2 [Art88a] and Proposition 5.4 [Art88a] there exist constants $\omega_P$ and a meromorphic function,

$${}^{c}\hat{\theta}_M(\tilde{f}', \rho_\lambda) = \int_{\mathfrak{a}_{M,S}} {}^{c}\hat{\theta}_M(\tilde{f}', \rho_\lambda, X) e^{\lambda(X)} dX, \ \lambda \in \mathfrak{a}_{M,\mathbf{C}}^*,$$

such that

$$(42) \quad {}^{c}\hat{I}_M(\rho, X, \tilde{f}') = \lim_{\beta} \sum_{P \in \mathcal{P}(M)} \omega_P \int_{\varepsilon_P + i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda)\, {}^{c}\hat{\theta}_M(\tilde{f}', \rho_\lambda) e^{-\lambda(X)} d\lambda.$$

Here $\beta$ is a test function in $C_c^\infty(\mathfrak{a}_{M,S})$ which approaches Dirac measure at the origin and $X \in \mathfrak{a}_{M,S}$ is any point at which the left-hand side is smooth.

We take $\rho = \pi_\nu$, where $\nu \in \mathfrak{a}_M^*$ is in general position. We can then deform the contour of integration in the above integral so that

$$\begin{aligned} {}^c\hat{I}_M(\pi_\nu, X, \tilde{f}') &= \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \int_{i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda)\, {}^c\hat{\theta}_M(\tilde{f}', \pi_{\nu+\lambda}) e^{-\lambda(X)} d\lambda \\ &= e^{\nu(X)} \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \int_{\nu + i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda - \nu)\, {}^c\hat{\theta}_M(\tilde{f}', \pi_\lambda) e^{-\lambda(X)} d\lambda. \end{aligned}$$

Notice that the function,

$$\lambda \mapsto \hat{\beta}(\lambda - \nu),$$

is the Fourier transform of the function,

$$X \mapsto e^{-\nu(X)} \beta(X),$$

which approaches Dirac measure at the origin if $\beta$ does. We may therefore replace $\hat{\beta}(\lambda - \nu)$ with $\hat{\beta}(\lambda)$ in the previous equation to obtain

$$e^{-\nu(X)}\, {}^c\hat{I}_M(\pi_\nu, X, \tilde{f}') = \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \int_{\nu + i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda)\, {}^c\hat{\theta}_M(\tilde{f}', \pi_\lambda) e^{-\lambda(X)} d\lambda.$$

By definition

(43) $$\lambda \mapsto {}^c\hat{\theta}_M(\tilde{f}, \pi_\lambda)$$

the Fourier transform of the function

$$X \mapsto {}^c\hat{\theta}_M(\tilde{f}', \pi, X).$$

By our assumptions on $\pi$ we have

$$X \mapsto \theta_{\tilde{M}}(\tilde{f})'(\pi, X) - \sum_{\eta \in \mu_n^M / \mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X) = n^{\dim(A_M/A_G)}\, {}^c\hat{\theta}_M(\tilde{f}', \pi, X).$$

This function has compact support in $\mathfrak{a}_{M,S}$. It follows that (43) is the Fourier transform of a function of compact support and is therefore entire. We may consequently deform the contour of integration in the previous integral to $i\mathfrak{a}_{M,S}^*$. This implies that $e^{-\nu(X)}\, {}^c\hat{I}(\pi_\nu, X, \tilde{f}')$ is independent of $\nu \in \mathfrak{a}_M^*$, for $\nu$ in general position, and by a comment on p. 143 [AC89] it follows that $e^{-\nu(X)}\, {}^c\hat{I}(\pi_\nu, X, \tilde{f}')$ is independent of $\nu \in \mathfrak{a}_M^*$ without restriction. According to Lemma 4.5 [Art88a] we have

$$\sum_{P \in \mathcal{P}(M)} \omega_P(X) e^{-\nu_P(X)}\, {}^c\hat{I}_M(\pi_{\nu_P}, X, \tilde{f}') = 0.$$

Here, $\omega_P(X) = \mathrm{vol}(\mathfrak{a}_P \cap B_X)\mathrm{vol}(B_X)^{-1}$, where $B_X$ is a small ball in $\mathfrak{a}_M$ centered at $X$ and $\nu_P$ is a point in the chamber $(\mathfrak{a}_P^*)^+$ which is far from the walls. It is obvious from this equation and the independence in $\nu$ just mentioned that ${}^c\hat{I}_M(\pi_\nu, X, \tilde{f}') = 0$. Setting $\nu = 0$ and applying Lemma 4.7 [Art88a] we obtain

$$ {}^c\hat{\theta}_M(\tilde{f}', \pi, X) = {}^c\hat{I}_M(\pi, X, \tilde{f}') = 0. $$

We have just proved that $\sum_{\eta \in \mu_n^M/\mu_n^G} {}^{\eta c}\hat{\theta}_M(\tilde{f}', \pi, X)$ vanishes if $\pi$ is not metic. Now suppose that $\tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}(F_S))$. Define

$$
\begin{aligned}
{}^cI_M^\Sigma(\tilde{\rho}, X, \tilde{f}) &= n^{-\dim(A_G)}\, {}^c\hat{I}_M(\tilde{\rho}', X', \tilde{f}') \\
&= \sum_{L \in \mathcal{L}(M)} n^{-\dim(A_G)} \hat{I}_M^L(\tilde{\rho}', X', {}^c\hat{\theta}_L(\tilde{f}')) \\
&= \sum_{L \in \mathcal{L}(M)} n^{-\dim(A_L)} \hat{I}_M^L(\tilde{\rho}', X', \sum_{\eta \in \mu_n^L/\mu_n^G} {}^{\eta c}\hat{\theta}_L(\tilde{f}')),
\end{aligned}
$$

for $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$ and $X \in \mathfrak{a}_{M,S}$. Equation (42) allows us to write ${}^c\hat{I}_M^\Sigma(\tilde{\rho}, X, \tilde{f})$ as

$$
\begin{aligned}
& n^{-\dim(A_G)} \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \int_{\varepsilon_P + i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda')\, {}^c\hat{\theta}_M(\tilde{f}', (\tilde{\rho}_\lambda)') e^{-\lambda'(X')} d\lambda' \\
&= \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \int_{\varepsilon_P + i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda) n^{\dim(A_M/A_G)}\, {}^c\hat{\theta}_M(\tilde{f}', (\tilde{\rho}_\lambda)') e^{-\lambda(X)} d\lambda.
\end{aligned}
$$

Defining

$$ {}^c\hat{I}_{\tilde{M}}(\tilde{\rho}, X, \tilde{f}) = \sum_{L \in \mathcal{L}(M)} \hat{I}_{\tilde{M}}^{\tilde{L}}(\tilde{\rho}, X, {}^c\theta_{\tilde{L}}(\tilde{f})), \;\; \tilde{\rho} \in \Sigma(\tilde{M}(F_S)), \;\; X \in \mathfrak{a}_{M,S}, $$

we follow the earlier argument to show that ${}^cI_{\tilde{M}}(\tilde{\pi}_\nu, X, \tilde{f}) - {}^cI_M^\Sigma(\tilde{\pi}_\nu, X, \tilde{f})$ is equal to

$$
\begin{aligned}
& \lim_\beta \sum_{P \in \mathcal{P}(M)} \omega_P \\
& \times \int_{i\mathfrak{a}_{M,S}^*} \hat{\beta}(\lambda) \left( {}^c\theta_{\tilde{M}}(\tilde{f})'((\tilde{\pi}_{\nu+\lambda})') - n^{\dim(A_M/A_G)}\, {}^c\hat{\theta}_M(\tilde{f}', (\tilde{\pi}_{\nu+\lambda})') \right) e^{-\lambda(X)} d\lambda,
\end{aligned}
$$

and that this expression vanishes. It now follows as before that

$$^{c}\theta_{\tilde{M}}(\tilde{f},\tilde{\pi},X) - n^{-\dim(A_G)}\hat{\theta}_M(\tilde{f}',\tilde{\pi}',X') = 0.$$

This equation may be rewritten as

$$n^{-\dim(A_M)}\, {}^{c}\theta_{\tilde{M}}(\tilde{f})'(\tilde{\pi}',X') - n^{-\dim(A_G)}\, {}^{c}\hat{\theta}_M(\tilde{f}',\tilde{\pi}',X') = 0,$$

and the first assertion follows. The first assertion implies that

$$^{c}\theta_{\tilde{M}}(\tilde{f})'((\tilde{\pi}_\lambda)') = n^{\dim(A_M/A_G)}\, {}^{c}\hat{\theta}_M(\tilde{f}',(\tilde{\pi}_\lambda)'),\ \tilde{\pi}\in\Pi_{\mathrm{temp}}(\tilde{M}(F_S)),\ \lambda\in\mathfrak{a}^*_{M,\mathbf{C}}.$$

By analytic continuation, this formula holds if $\tilde{\pi}$ is replaced by a standard representation. Recalling (42), this implies that

$$^{c}I_{\tilde{M}}(\tilde{\rho},X,\tilde{f}) = {}^{c}I^{\Sigma}_M(\tilde{\rho},X,\tilde{f}),\ \tilde{\rho}\in\Sigma(\tilde{M}(F_S)).$$

By the induction assumption and the first assertion we have

$$\begin{aligned} 0 &= {}^{c}I_{\tilde{M}}(\tilde{\rho},X,\tilde{f}) - {}^{c}I^{\Sigma}_M(\tilde{\rho},X,\tilde{f}) \\ &= \sum_{L\in\mathcal{L}(M)} \hat{I}^{\tilde{L}}_{\tilde{M}}(\tilde{\rho},X,{}^{c}\theta_{\tilde{L}}(\tilde{f})) - \hat{I}^{L,\Sigma}_M(\tilde{\rho},X,{}^{c}\theta_{\tilde{L}}(\tilde{f})) \\ &= I_{\tilde{M}}(\tilde{\rho},X,\tilde{f}) - I^{\Sigma}_M(\tilde{\rho},X,\tilde{f}). \end{aligned}$$

This is the third assertion.□

With Theorem 15.1 in place, the proof of of Theorem B (i) follows *mutatis mutandis* from the argument on p. 145 [AC89]. We include it here for the sake of continuity. We wish to show that

$$I^{\Sigma}_M(\tilde{\pi},0,\tilde{f}) = I_{\tilde{M}}(\tilde{\pi},0,\tilde{f}),\ \tilde{\pi}\in\Pi_{\mathrm{unit}}(\tilde{M}(F_S)).$$

The distribution on the left is defined by equation (36), while the distribution on the right is defined by

$$\sum_P \omega_P \sum_L \sum_{\tilde{\rho}} \int_{\varepsilon_P + i\mathfrak{a}^*_{\tilde{M},S}/i\mathfrak{a}^*_{\tilde{L},S}} r^{\tilde{L}}_{\tilde{M}}(\tilde{\pi}_\lambda,\tilde{\rho}_\lambda) I_{\tilde{L}}(\tilde{\rho}_\lambda, h_{\tilde{L}}(X),\tilde{f}) e^{-\lambda(X)} d\lambda,$$

where $P$, $L$ and $\tilde{\rho}$ are summed over $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\Sigma(\tilde{M}(F_S))$ respectively. The number $r^{\tilde{L}}_{\tilde{M}}(\tilde{\pi}_\lambda,\tilde{\rho}_\lambda)$ vanishes if $\Delta(\tilde{\pi},\tilde{\rho})$ vanishes ((6.4) [Art89a]). Comparing the expansions of these two distributions, it is clear that it suffices to prove

(44) $$I^{\Sigma}_L(\tilde{\rho}_\lambda, h_L(X),\tilde{f}) = I_{\tilde{L}}(\tilde{\rho}_\lambda, h_L(X),\tilde{f}),$$

for all $L \in \mathcal{L}(M)$, $X \in \mathfrak{a}_M$, $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ with small real part, and $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$ such that $\Delta(\tilde{\pi}, \tilde{\rho}) \neq 0$. By using the splitting property for these distributions (Proposition 9.4 [Art88a]) it suffices to prove (44) for $S = \{v\}$. Suppose then that $\tilde{\rho} \in \Sigma(\tilde{M}(F_v))$ and $\Delta(\tilde{\pi}, \tilde{\rho}) \neq 0$. Then the central character of $\tilde{\rho}$ must be unitary. From the definition of standard representations, it follows that $\tilde{\rho}$ is either tempered or induced from a proper parabolic subgroup of $M$. Suppose first that $\rho$ is tempered. Then by the proof of Lemma 3.1 [Art88a], we have

$$\begin{aligned} I^{\Sigma}_L(\tilde{\rho}_\lambda, h_L(X), \tilde{f}) &= \begin{cases} 0, & L \neq G \\ \int_{i\mathfrak{a}^*_{G,v}} \operatorname{tr}\left(\tilde{\rho}_{\lambda+\nu}(\tilde{f})\right) e^{-\nu(h_G(X))} d\nu, & L = G \end{cases} \\ &= I_{\tilde{L}}(\tilde{\rho}_\lambda, h_L(X), \tilde{f}). \end{aligned}$$

Now suppose that $\tilde{\rho} = \tilde{\rho}_1^{\tilde{M}}$, where $\tilde{\rho}_1 \in \Sigma(\tilde{M}_1(F_v))$ and $M_1$ is a proper Levi subgroup of $M$. We apply the descent property (Corollary 8.5 [Art88a]) to (44) and find that it suffices to show

$$\hat{I}^{L_1,\Sigma}_{M_1}(\tilde{\rho}_{1,\lambda}, X_1, \tilde{f}_{\tilde{L}_1}) = \hat{I}^{\tilde{L}_1}_{\tilde{M}_1}(\tilde{\rho}_{1,\lambda}, X_1, \tilde{f}_{\tilde{L}_1}), \tag{45}$$

for $X_1 \in \mathfrak{a}_{M_1}$ and $L_1 \in \mathcal{L}(M_1)$ with $L_1 \neq G$. The induction hypothesis of following Theorem A allows us to apply Theorem 15.1, with $G$ replaced by $L_1$, in order to obtain (45). The proof is now complete. $\square$

# 16 More on normalizing factors

This section is devoted to the construction of a few additional $(G, M)$ families which we shall require for the comparison of the spectral sides of the trace formulas. We shall return to the actual comparison of the trace formulas in the following section.

Take $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ and $\pi = \otimes_v \pi_v$ to be representations in $\Pi(\tilde{M}(\mathbf{A}))$ and $\Pi(M(\mathbf{A}))$ respectively, and suppose that

$$\delta(\pi, \tilde{\pi}) = \prod_v \delta(\pi_v, \tilde{\pi}_v)$$

does not vanish (§13). Define

$$\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_v, \tilde{\pi}_v) = r_{P_1|P_2}(\pi_v)^{-1} r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\pi}_v), \;\; P_1, P_2 \in \mathcal{P}(M).$$

If $|n|_v = 1$ and both $\tilde{\pi}_v$ and $\pi_v$ are unramified representations then $\tilde{\pi}'_v = \pi_v$ by Corollary 13.1. The normalization of §5 then implies that

$$\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_v, \tilde{\pi}_v) = r_{P_1|P_2}(\pi_v)^{-1} r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\pi}_v) = r_{P_1|P_2}(\pi_v)^{-1} r_{P_1|P_2}(\pi_v) = 1.$$

This fact allows us to define the infinite product

$$\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_{\lambda'}, \tilde{\pi}_\lambda) = \prod_v \tilde{r}_{P_1\tilde{|}P_2}(\pi_{v,\lambda'}, \tilde{\pi}_{v,\lambda}),\ P_1, P_2 \in \mathcal{P}(M),\ \lambda \in \mathfrak{a}^*_{M,\mathbf{C}}.$$

Given $P_1 \in \mathcal{P}(M)$ we define a $(G, M)$ family by

$$\tilde{r}_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\pi}_\lambda, P_1) = \tilde{r}_{\tilde{P}|\tilde{P}_1}(\pi_{\lambda'}, \tilde{\pi}_\lambda)^{-1} \tilde{r}_{\tilde{P}|\tilde{P}_1}(\pi_{\lambda'+\nu'}, \tilde{\pi}_{\lambda+\nu}),$$

where $P \in \mathcal{P}(M)$ and $\nu \in i\mathfrak{a}^*_M$.

**Lemma 16.1 (11.3)** *(a) Take $\tilde{\pi}$ and $\pi$ as above. Then for each $L \in \mathcal{L}(M)$, $\tilde{r}^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_\lambda)$ is independent of $P_1$ and is also a rational function of the variables $\{\lambda(\alpha^\vee), q_v^{-\lambda(\alpha^\vee)}\}_{v \in S}$, where $\alpha$ runs over the roots of $(G, A_M)$, $q_v$ is the order of the residue field of $F_v$ and $S$ is a finite set of valuations outside of which $\tilde{\pi}$ and $\pi$ are unramified,.*
*(b) Suppose in addition that $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}, t)$ and $\pi \in \Pi_{\mathrm{disc}}(M, t)$. Then*

$$\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_{\lambda'}, \tilde{\pi}_\lambda) = r_{P_1|P_2}(\pi_{\lambda'})^{-1} r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\pi}_\lambda),\ P_1, P_2 \in \mathcal{P}(M).$$

*In particular, for each $L \in \mathcal{L}(M)$, the function $\tilde{r}^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_\lambda)$ is regular for $\lambda \in i\mathfrak{a}^*_M$. Moreover,*

$$r^{\tilde{L}}_{\tilde{M}}(\tilde{\pi}_\lambda) = \sum_{L_1 \in \mathcal{L}^L(M)} n^{\dim(A_M/A_{L_1})} r^{L_1}_M(\pi_{\lambda'}) \tilde{r}^{\tilde{L}}_{\tilde{L}_1}(\pi_{\lambda'}, \tilde{\pi}_\lambda).$$

*Proof.* Part (a) of the lemma follows from the computations on p.149 [AC89]. Under the hypotheses of part (b), $r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\pi}_\lambda)$ and $r_{P_1|P_2}(\pi_{\lambda'})$ are regular functions in $\lambda \in i\mathfrak{a}^*_M$ (§6 [Art82]). Thus, if one unravels the definition of $\tilde{r}_{\tilde{P}_1|\tilde{P}_2}(\pi_{\lambda'}, \tilde{\pi}_\lambda)$, one obtains the first equality and the the regularity on $i\mathfrak{a}^*_M$. The last equality follows from an application of Lemma 6.5 [Art81] to

$$r_{\tilde{P}}(\nu, \tilde{\pi}_\lambda, \tilde{P}_1) = \tilde{r}_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\pi}_\lambda, \tilde{P}_1) r_P(\nu', \pi_{\lambda'}, P_1). \Box$$

We may define further $(G, M)$ families along the same lines as $\tilde{r}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_\lambda)$. If we replace $\tilde{\pi}$ in the above discussion with some $\tilde{\rho} \in \Sigma(\tilde{M}(\mathbf{A}))$ such that $\Delta(\pi, \tilde{\rho}) \neq 0$, we obtain the $(G, M)$ family

$$\tilde{r}_{\tilde{P}}(\nu, \pi_\lambda, \tilde{\rho}_\lambda, \tilde{P}_1) = \tilde{r}_{\tilde{P}|\tilde{P}_1}(\pi_{\lambda'}, \tilde{\rho}_\lambda)^{-1} \tilde{r}_{\tilde{P}|\tilde{P}_1}(\pi_{\lambda'+\nu'}, \tilde{\rho}_{\lambda+\nu}).$$

We define yet another $(G,M)$ family for representations $\pi_1, \pi_2 \in \Pi(M(\mathbf{A}))$ such that $\delta(\pi_i, \tilde{\pi}) \neq 0$ for $i = 1, 2$. Set

$$\tilde{r}_{\tilde{P}}(\nu, \pi_{1,\lambda'}, \pi_{2,\lambda'}, \tilde{P}_1) = \tilde{r}_{\tilde{P}}(\nu, \pi_{1,\lambda'}, \tilde{\pi}, \tilde{P}_1)\tilde{r}_{\tilde{P}}(\nu, \pi_{2,\lambda'}, \tilde{\pi}_\lambda, \tilde{P}_1)^{-1}.$$

This $(G,M)$ family is independent of $\tilde{\pi}$. Lemma 6.5 [Art81] applied to this last $(G,M)$ family yields

$$\tilde{r}_{\tilde{M}}^{\tilde{L}}(\pi_{1,\lambda'}, \tilde{\pi}_\lambda) = \sum_{L_1 \in \mathcal{L}^L(M)} \tilde{r}_{\tilde{M}}^{\tilde{L_1}}(\pi_{1,\lambda'}, \pi_{2,\lambda'})\tilde{r}_{\tilde{L_1}}^{\tilde{L}}(\pi_{2,\lambda'}, \tilde{\pi}_\lambda). \tag{46}$$

For arbitrary $\pi \in \Pi(M(\mathbf{A}))$, $\tilde{\rho} = \otimes_v \tilde{\rho}_v \in \Sigma(\tilde{M}(\mathbf{A}))$ and $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A}))$. Set

$$r_{\tilde{P}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) = \Delta(\tilde{\pi}, \tilde{\rho})\tilde{r}_{\tilde{P}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda),$$
$$r_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\pi}_\lambda) = \delta(\pi, \tilde{\pi})\tilde{r}_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\pi}_\lambda),$$

and

$$r_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\rho}_\lambda) = \Delta(\pi, \tilde{\rho})\tilde{r}_{\tilde{P}}(\nu, \pi_{\lambda'}, \tilde{\rho}_\lambda).$$

In the following lemma we compare the final $(G,M)$ family in this list to another, keeping in mind the normalization in §5.

**Lemma 16.2 (11.4)** *For each $L \in \mathcal{L}(M)$ we have*

$$r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) = n^{\dim(A_M/A_L)} r_M^L(\pi_{\lambda'}, \tilde{\rho}'_{\lambda'}).$$

*Proof.* According to Lemma 6.2 [Art81], $r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\rho}_\lambda)$ is equal to

$$\Delta(\pi, \tilde{\rho}) \lim_{\nu \to 0} \sum_{P \in \mathcal{P}^L(M)} \theta_P^L(\nu)^{-1} r_{P|P_1}^L(\pi_{\lambda'}) r_{P|P_1}^L(\pi_{\lambda'+\nu'})^{-1} r_{\tilde{P}|\tilde{P}_1}^{\tilde{L}}(\tilde{\rho}_\lambda)^{-1} r_{\tilde{P}|\tilde{P}_1}^{\tilde{L}}(\tilde{\rho}_{\lambda+\nu}),$$

where

$$\theta_P^L(\nu) = \mathrm{vol}\left(\mathfrak{a}_M^L / \mathbf{Z}(\Delta_P^L)^\vee\right)^{-1} \prod_{\alpha \in \Delta_P^L} \nu(\alpha^\vee),$$

and $\Delta_P^L$ are the simple roots of $(P, A_M)$. (We apologize to the reader for the similarity in notation of $\theta_P^L$ with the maps of §15.) Making the change of variables from $\nu$ to $\nu'$ in the limit and noting that $\theta_P^L(n^{-1}\nu) = n^{-\dim(A_M/A_L)}\theta_P^L$, this expression can be rewritten as the product of $\Delta(\pi, \tilde{\rho}')n^{\dim(A_M/A_L)}$ with

$$\lim_{\nu \to 0} \sum_{P \in \mathcal{P}^L(M)} \theta_P^L(\nu)^{-1} r_{P|P_1}^L(\pi_{\lambda'}) r_{P|P_1}^L(\pi_{\lambda'+\nu})^{-1} r_{P|P_1}^L(\tilde{\rho}'_{\lambda'})^{-1} r_{P|P_1}^L(\tilde{\rho}'_{\lambda'+\nu}).$$

This is by definition equal to $n^{\dim(A_M/A_L)} r_M^L(\pi_{\lambda'}, \tilde{\rho}'_{\lambda'})$.□

The other $(G,M)$ families defined in this section satisfy versions of Lemma 16.2 as well. These versions are proven similarly.

# 17 A formula for $I_t^\Sigma$

The object of this section is to express the spectral side

$$\sum_{t\geq 0} I_t^\Sigma(\tilde{f}) = \sum_{t\geq 0} n^{-1} I_t(\tilde{f}')$$

in a manner that is compatible with the spectral expansion (37). This amounts to expressing the spectral side for $G(\mathbf{A})$ in terms of the global datum $a^{M,\Sigma}$ and the set of representations $\Pi^\Sigma(M,t)$.

The integrals in $I_t^\Sigma(\tilde{f})$ are taken over spaces of representations which, on the face of it, are not metic. We need the following spectral vanishing property to ensure that these non-metic representations do not appear in $I^\Sigma(\tilde{f})$.

**Proposition 17.1** *Suppose $\rho \in \Sigma(M(F_S))$. Then*

$$\hat{I}_M(\rho, X, \tilde{f}') = 0,\ \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

*unless $\rho$ is metic.*

*Proof.* Recall that $\rho$ is metic if there exists $\tilde{f}_1 \in \mathcal{H}(\tilde{M}(F_S))$ such that $\tilde{f}_1'(\rho) \neq 0$. Therefore, the proposition follows in the case $M = G$. We assume inductively that the proposition holds if $G$ is replaced by $L \in \mathcal{L}$ with $L \subsetneqq G$. By the splitting property, Proposition 9.4 [Art88a] (and the remark immediately following it), it suffices to prove the proposition for $S = \{v\}$. Suppose that $\rho$ is not metic. We may write $\rho = \pi_\nu^M$, where $M_1 \in \mathcal{L}^M$, $\pi \in \Pi_{\mathrm{temp}}(M_1(F_v))$ and $\nu \in \mathfrak{a}_{M_1}^*$. Since the local metaplectic correspondence commutes with induction it is clear that $\pi_\nu$ is not metic. According to Corollary 8.5 [Art88a], the Fourier transform,

$$\int_{i\mathfrak{a}_{M_1,v}^*/i\mathfrak{a}_{M,v}^*} \hat{I}_M(\rho_\lambda, X, \tilde{f}') e^{-\lambda(X_1)} d\lambda,$$

(where the projection $h_M(X_1)$ of $X_1 \in \mathfrak{a}_{M_1,v}$ onto $\mathfrak{a}_{M,v}$ is $X \in \mathfrak{a}_{M,v}$,) is equal to

$$\sum_{L\in\mathcal{L}(M_1)} d_{M_1}^G(M,L) \hat{I}_{M_1}^L(\pi_\nu, X, \tilde{f}').$$

If $M_1 \subsetneqq M$ then $d_{M_1}^G(M,G) = 0$ and the right-hand side vanishes by the induction assumption. Taking the inverse Fourier transform then implies that $\hat{I}_M(\rho_\lambda, X, \tilde{f}')$ vanishes as well. We may therefore suppose that $M_1 = M$, in which case $\rho = \pi_\nu$. If $v$ is nonarchimedean the infinitesimal character of $\pi_\nu$ is given by a finite set of cuspidal pairs $(\rho_1, M_1), \ldots, (\rho_s, M_s)$ where $M_1, \ldots, M_s \in \mathcal{L}^M$ and $\rho_1 \in \Sigma(M_1(F_v)), \ldots, \rho_s \in \Sigma(M_s(F_v))$. If $v$ is an archimedean valuation, then the infinitesimal character of $\pi_\nu$ also determines a set of cuspidal pairs $(\rho_1, M_0), \ldots, (\rho_s, M_0)$ as above. It follows from the remark immediately preceding Theorem 27.3 [FK86] and Theorem 26 [FK86] in the nonarchimedean case; and §3 [Mezb] in the archimedean case, that these supercuspidal (and hence elliptic) representations are not metic. The discussion of §7 [Art89b] tells us that for $\nu \in \mathfrak{a}_M^*$ in general position we may express $\hat{I}_M(\pi_\nu, X, \tilde{f}')e^{-\nu(X)}$ as

$$\sum_{\{L_1, L \in \mathcal{L}(M) : L_1 \supset L \supsetneqq M\}} \int_{\nu_L + i\mathfrak{a}_{L,v}^*/i\mathfrak{a}_{M_1,v}^*} \sum_{i=1}^{t} \sum_{j=1}^{s} \left(\Delta_{ij}(\pi)\Phi\right)(\lambda_i, \rho_j) d\lambda.$$

Here, $\Delta_{ij}(\pi)$ is a differential operator on $\mathfrak{a}_{M,\mathbf{C}}^* \times \mathfrak{a}_{M_j,\mathbf{C}}^*$ and $\Phi(\lambda_i, \rho_j)$ is equal to the product of

$$\lim_{\nu_1 \to 0} \sum_{Q \in \mathcal{P}^{L_1}(L)} \tilde{r}_{Q|Q_0}(\pi^L_{\lambda_i+\lambda}, \rho_{j,\lambda})^{-1} \tilde{r}_{Q|Q_0}(\pi^L_{\lambda_i+\lambda+\nu_1}, \rho_{j,\lambda+\nu_1}) \theta_Q^{L_1}(\nu_1)^{-1}$$

with $\hat{I}_{L_1}(\rho^L_{j,\lambda}, h_{L_1}(X), \tilde{f}')$. It is obvious from this equation that $\hat{I}_M(\pi_\nu, X, \tilde{f}')$ vanishes if

$$\hat{I}_{L_1}(\rho^L_{j,\lambda}, h_{L_1}(X), \tilde{f}') = 0, \ L_1 \supset L \supsetneqq M.$$

Since $L_1 \supsetneqq M$ and $\rho_{j,\lambda}$ is not metic, the earlier descent argument shows that this is indeed true. We have thus succeeded in showing that $\hat{I}_M(\pi_\nu, X, \tilde{f}') = 0$ if $\nu \in \mathfrak{a}_M^*$ is in general position. However, the mean value property (Lemma 3.2 [Art88a]) expresses the value of $\hat{I}_M(\pi_\nu, X, \tilde{f}')$ at any $\nu \in \mathfrak{a}_M^*$ in terms of nearby points. In consequence, $\hat{I}_M(\pi_\nu, X, \tilde{f}')$ vanishes for all $\nu \in \mathfrak{a}_M^*$.□

The following lemma gives an expansion of the local distributions occurring in $I_t^\Sigma(\tilde{f})$ in terms of the distributions $I_M^\Sigma(\tilde{\pi})$.

**Lemma 17.1 (12.1)** *Suppose that* $\pi \in \Pi_{\text{unit}}(M(\mathbf{A})^1)$ *and* $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. *Then* $n^{-1}\hat{I}_M(\pi, \tilde{f}')$ *is equal to*

$$\sum_{L \in \mathcal{L}(M)} \sum_{\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A})^1)} \int_{\varepsilon_M + i\mathfrak{a}_M^*/i\mathfrak{a}_L^*} r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda'}, \tilde{\pi}_\lambda) I_L^\Sigma(\tilde{\pi}_\lambda, \tilde{f}) d\lambda. \tag{47}$$

*Proof.* This proof is almost identical to the proof of Lemma 12.1 II [AC89]. It is included so that the reader may feel a sense of continuity. Any statements which seem unjustified may be compared to the analogous statements in Lemma 12.1, where the details are given. To begin, relabel the summation index $L$ in (47) by $L_1$. We then replace $I^{\Sigma}_{L_1}(\tilde{\pi}_\lambda, \tilde{f})$ with expression (36),

$$\sum_{Q\in\mathcal{P}(L_1)} \omega_Q \sum_{L\in\mathcal{L}(L_1)} \sum_{\tilde{\rho}\in\Pi(\tilde{M}(\mathbf{A})^1)} \int_{\varepsilon_Q + i\mathfrak{a}^*_{L_1}/i\mathfrak{a}^*_L} r^{\tilde{L}}_{\tilde{L}_1}(\tilde{\pi}_{\lambda+\mu}, \tilde{\rho}_{\lambda+\mu}) I^{\Sigma}_L(\tilde{\rho}_{\lambda+\mu}, \tilde{f}) d\mu.$$

We deform the contour of integration in $\mu$ so that (47) is equal to the sum over $L_1, L \in \mathcal{L}(M)$, with $L_1 \subset L$, of

$$\sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \sum_{\tilde{\rho}\in\Sigma(\tilde{M}(\mathbf{A})^1)} \int_{\varepsilon_M + i\mathfrak{a}^*_M/i\mathfrak{a}^*_L} r^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_\lambda) r^{\tilde{L}}_{\tilde{L}_1}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) I^{\Sigma}_L(\tilde{\rho}_\lambda, \tilde{f}) d\lambda.$$

Taking the sums over $L_1$ and $\tilde{\pi}$ inside the integral we find that

$$\begin{aligned}
&\sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \sum_{L_1\in\mathcal{L}^L(M)} r^{\tilde{L}_1}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_\lambda) r^{\tilde{L}}_{\tilde{L}_1}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) \\
&= \tilde{r}^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) \sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \delta(\pi, \tilde{\pi}) \Delta(\tilde{\pi}, \tilde{\rho}) \\
&= \tilde{r}^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) \sum_{\tilde{\rho}_1\in\Sigma(\tilde{M}(\mathbf{A})^1)} \Delta(\pi, \tilde{\rho}_1) \sum_{\tilde{\pi}\in\Pi(\tilde{M}(\mathbf{A})^1)} \Gamma(\tilde{\rho}_1, \tilde{\pi}) \Delta(\tilde{\pi}, \tilde{\rho}) \\
&= \tilde{r}^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) \Delta(\pi, \tilde{\rho}) \\
&= r^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\rho}_\lambda).
\end{aligned}$$

Expansion (47) is therefore equal to

$$\sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\rho}\in\Sigma(\tilde{M}(\mathbf{A})^1)} \int_{\varepsilon_M + i\mathfrak{a}^*_M/i\mathfrak{a}^*_L} r^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) I^{\Sigma}_L(\tilde{\rho}_\lambda, \tilde{f}) d\lambda. \tag{48}$$

By Lemma 16.2 and the spectral vanishing property, Proposition 17.1, we have

$$\sum_{\tilde{\rho}\in\Sigma(\tilde{M}(\mathbf{A})^1)} r^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\rho}_\lambda) I^{\Sigma}_L(\tilde{\rho}_\lambda, \tilde{f})$$

$$
\begin{aligned}
&= n^{\dim(A_M/A_L)} \sum_{\tilde\rho\in\Sigma(M(\mathbf{A})^1)} r_M^L(\pi_{\lambda'},\tilde\rho'_{\lambda'})\left(n^{-1}\hat I_L(\tilde\rho'_{\lambda'},\tilde f')\right)\\
&= n^{\dim(A_M/A_L)} \sum_{\rho\in\Sigma(M(\mathbf{A})^1)} r_M^L(\pi_{\lambda'},\rho_{\lambda'})\left(n^{-1}\hat I_L(\rho_{\lambda'},\tilde f')\right).
\end{aligned}
$$

Substituting back into (48) and noting that $d\lambda' = n^{\dim(A_M/A_L)}d\lambda$, we obtain

$$
\sum_{L\in\mathcal{L}(M)}\sum_{\rho\in\Sigma(M(\mathbf{A})^1)}\int_{\varepsilon_M+i\mathfrak{a}_M^*/i\mathfrak{a}_L^*} r_M^L(\pi_\lambda,\rho_\lambda)\left(n^{-1}\hat I_L(\rho_\lambda,\tilde f')\right)d\lambda.
$$

Since $\pi$ is unitary, this expression is equal to $n^{-1}\hat I_M(\pi,\tilde f')$.□

**Proposition 17.2 (12.2)** *Suppose that* $t\geq 0$ *and* $\tilde f\in\mathcal{H}(G(\mathbf{A}))$. *Then*

$$
I_t^\Sigma(\tilde f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}\int_{\Pi^\Sigma(M,t)} a^{M,\Sigma}(\tilde\pi)I_M^\Sigma(\tilde\pi,\tilde f)d\pi. \tag{49}
$$

*Proof.* From §13 (cf. (38) we know that $I_t^\Sigma(\tilde f)$ equals

$$
\begin{aligned}
&\sum_{\{M_1,M\in\mathcal{L}:M\supset M_1\}} |W_0^{M_1}||W_0^G|^{-1}\\
&\times \sum_{\pi\in\Pi_{\mathrm{disc}}(M_1,t)}\int_{i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*} a_{\mathrm{disc}}^M(\pi)r_{M_1}^M(\pi_{\lambda'})n^{-1}\hat I_M(\pi_{\lambda'},\tilde f')d\lambda'.
\end{aligned}
$$

Substituting expression (47), deforming the contour of integration appropriately, and noting that $d\lambda' = n^{\dim(A_{M_1}/A_M)}d\lambda$, we find that $I_t^\Sigma(\tilde f)$ equals

$$
\sum_{\{M_1,M,L\in\mathcal{L}:M_1\subset M\subset L\}}\sum_{\tilde\pi_1\in\Pi(\tilde M_1(\mathbf{A})^1,t)} |W_0^{M_1}||W_0^G|^{-1}n^{\dim(A_{M_1}/A_M)}
$$

$$
\times\int_{\varepsilon_M+i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_L^*}\sum_{\pi\in\Pi_{\mathrm{disc}}(M_1,t)} a_{\mathrm{disc}}^{M_1}(\pi)r_{M_1}^M(\pi_{\lambda'})r_{\tilde M}^{\tilde L}(\pi_{\lambda'},\tilde\pi_{1,\lambda})I_L^\Sigma(\tilde\pi_{1,\lambda},f)d\lambda. \tag{50}
$$

The term $r_{\tilde M}^{\tilde L}(\pi_{\lambda'},\tilde\pi_{1,\lambda})$ in the above sum vanishes unless $\delta(\pi,\tilde\pi_1)\neq 0$. Fix some $\pi_1\in\Pi_{\mathrm{disc}}(M_1,t)$ such that $\delta(\pi_1,\tilde\pi_1)\neq 0$. Then for any other $\pi\in\Pi_{\mathrm{disc}}(M_1,t)$ with $\delta(\pi,\tilde\pi_1)\neq 0$ we may write

$$
\begin{aligned}
&n^{\dim(A_{M_1}/A_M)}r_{M_1}^M(\pi_{\lambda'})\\
&\quad= \sum_{\{L_1:M_1\subset L_1\subset M\}} n^{\dim(A_{M_1}/A_{L_1})}r_{M_1}^{L_1}(\pi_{1,\lambda'})n^{\dim(A_{L_1}/A_M)}\tilde r_{L_1}^M(\pi_{1,\lambda'},\pi_{\lambda'})\\
&\quad= \sum_{\{L_1:M_1\subset L_1\subset M\}} n^{\dim(A_{M_1}/A_{L_1})}r_{M_1}^{L_1}(\pi_{1,\lambda'})\tilde r_{\tilde L_1}^{\tilde M}(\pi_{1,\lambda'},\pi_{\lambda'}).
\end{aligned}
$$

We substitute this expression into (50) and deform the contour of integration from $\varepsilon_M + i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$ to $\varepsilon_{L_1} + i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$ for some small regular point $\varepsilon_{L_1}$ in $\mathfrak{a}^*_{L_1}$. We then bring the sum over $M$ inside the integral. Notice that

$$\sum_{\{M: L_1 \subset M \subset L\}} \tilde{r}^{\tilde{M}}_{\tilde{L}_1}(\pi_{1,\lambda'}, \pi_{\lambda'}) r^{\tilde{L}}_{\tilde{M}}(\pi_{\lambda'}, \tilde{\pi}_{1,\lambda}) = \delta(\pi, \tilde{\pi}_1) \tilde{r}^{\tilde{L}}_{\tilde{L}_1}(\pi_{1,\lambda'}, \tilde{\pi}_{1,\lambda})$$

by equation (46). Observe also that

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(M_1, t)} a^{M_1}_{\mathrm{disc}}(\pi) \delta(\pi, \tilde{\pi}_1) = a^{M_1, \Sigma}_{\mathrm{disc}}(\tilde{\pi}_1).$$

Now (50) may be written as

$$\sum_{\{M_1, L \in \mathcal{L}: M_1 \subset L\}} |W^{M_1}_0||W^G_0|^{-1} \sum_{\tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1, t)} \sum_{\{L_1: M_1 \subset L_1 \subset L\}} n^{\dim(A_{M_1}/A_{L_1})}$$

$$\times \int_{\varepsilon_{L_1} + i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L} a^{M_1,\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1) r^{L_1}_{M_1}(\pi_{1,\lambda'}) \tilde{r}^{\tilde{L}}_{\tilde{L}_1}(\pi_{1,\lambda'}, \tilde{\pi}_{1,\lambda}) I^{\Sigma}_L(\tilde{\pi}_{1,\lambda}, \tilde{f}) d\lambda. \tag{51}$$

The summand indexed by $M_1 = G$ in (51) reduces to

$$\sum_{\tilde{\pi}_1 \in \Pi^{\Sigma}_{\mathrm{disc}}(\tilde{G}, t)} a^{\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1) I^{\Sigma}_G(\tilde{\pi}_1, \tilde{f}). \tag{52}$$

If $M_1 \neq G$ then the induction hypothesis stated after the definition of $a^{M_1,\Sigma}_{\mathrm{disc}}$ implies that

$$a^{M_1,\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1) = a^{\tilde{M}_1}_{\mathrm{disc}}(\tilde{\pi}_1), \ \tilde{\pi}_1 \in \Pi(\tilde{M}_1(\mathbf{A})^1, t).$$

It is immediate from this equation that $a^{M_1,\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1)$ vanishes unless $\tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}_1, t)$. Thus, by a variant of Lemma 16.1, the integrand of (51) is analytic for $\lambda$ near $i\mathfrak{a}^*_{M_1}$, and we may deform the contour of integration from $\varepsilon_{L_1} + i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$ to $i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$. This allows us to take the sum over $L_1$ inside the integral. It is a simple exercise in $(G, M)$ families (cf. proof of Lemma 16.1 (b)) to show that

$$\sum_{\{L_1: M_1 \subset L_1 \subset L\}} n^{\dim(A_{M_1}/A_{L_1})} r^{L_1}_{M_1}(\pi_{1,\lambda'}) \tilde{r}^{\tilde{L}}_{\tilde{L}_1}(\pi_{1,\lambda'}, \tilde{\pi}_{1,\lambda}) = r^{\tilde{L}}_{\tilde{M}_1}(\tilde{\pi}_{1,\lambda}).$$

Therefore the summand indexed by $M_1 \neq G$ in (51) is equal to

$$|W^{M_1}_0||W^G_0|^{-1} \sum_{L \in \mathcal{L}(M_1)} \sum_{\tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}_1, t)} \int_{i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L} a^{M_1,\Sigma}_{\mathrm{disc}}(\tilde{\pi}_1) r^{\tilde{L}}_{\tilde{M}_1}(\tilde{\pi}_{1,\lambda}) I^{\Sigma}_L(\tilde{\pi}_{1,\lambda}, \tilde{f}) d\lambda.$$

Combining this expression with (52) we obtain

$$I_t^{\Sigma}(\tilde{f}) = \sum_{L\in\mathcal{L}} |W_0^L||W_0^G|^{-1} \int_{\Pi^{\Sigma}(\tilde{L},t)} a^{L,\Sigma}(\tilde{\pi}) I_L^{\Sigma}(\tilde{\pi},\tilde{f}) d\tilde{\pi}. \square$$

We can now apply Proposition 17.2 to obtain a striking comparison between the spectral sides of the trace formulas.

**Lemma 17.2 (12.3)** *Suppose that $t \geq 0$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$. Then*

$$I_t^{\Sigma}(\tilde{f}) - I_t(\tilde{f}) = \sum_{\tilde{\pi}\in\Pi(\tilde{G}(\mathbf{A})^1,t)} \left(a_{\mathrm{disc}}^{\Sigma}(\tilde{\pi}) - a_{\mathrm{disc}}(\tilde{\pi})\right) \mathrm{tr}(\tilde{\pi}(\tilde{f}^1)),$$

*where $\tilde{f}^1$ is the restriction of $\tilde{f}$ to $\tilde{G}(\mathbf{A})^1$.*

*Proof.* This proof is almost identical to Lemma 12.3 II [AC89] and is included solely for the sake of continuity. Consider the difference of (49) and

$$I_t(\tilde{f}) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(\tilde{M},t)} a^M(\tilde{\pi}) I_{\tilde{M}}(\tilde{\pi},\tilde{f}) d\tilde{\pi}.$$

If $M_1 \in \mathcal{L}$ and $M_1 \subsetneqq M \subset G$ then the induction hypothesis of §14 implies that

$$a^{M,\Sigma}(\tilde{\pi}) = a^{\tilde{M}}(\tilde{\pi}),\ \tilde{\pi} \in \Pi_{\tilde{M}_1}(\tilde{M},t),$$

and $\Pi^{\Sigma}_{\tilde{M}_1}(\tilde{M},t) = \Pi_{\tilde{M}_1}(\tilde{M},t)$. If $\tilde{\pi}$ is not unitary both $a^{M,\Sigma}(\tilde{\pi})$ and $a^M(\tilde{\pi})$ vanish. When $\tilde{\pi}$ is unitary we know from the discussion at the end of §15 that

$$I_M^{\Sigma}(\tilde{\pi},\tilde{f}) = I_{\tilde{M}}(\tilde{\pi},\tilde{f}).$$

Therefore, the only terms which remain in the difference, $I_t^{\Sigma}(\tilde{f}) - I_t(\tilde{f})$, are those parameterized by $M_1 = M = G$. The lemma now follows from

$$I_G^{\Sigma}(\tilde{\pi},\tilde{f}) = \mathrm{tr}(\tilde{\pi}(\tilde{f}^1)) = I_{\tilde{G}}(\tilde{\pi},\tilde{f}),\ \tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{G},t). \square$$

# 18 The map $\varepsilon_M$

Having simplified the comparison of the spectral sides, we do the same for the geometric sides. We lighten the burden of this task by adding yet another induction hypothesis. From this point on, fix $M \in \mathcal{L}$ such that $M \neq G$. The additional induction hypothesis is that

$$I_L^{\mathcal{M}}(\gamma, \tilde{f}) = I_L^{\Sigma}(\gamma, \tilde{f}),\ \gamma \in L_{\mathrm{comp}}(F_S),\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S)),$$

where $L \in \mathcal{L}(M)$ with $L \neq M$.

The following proposition brings us closer to the proof of Theorem A (i) in that it expresses the difference of our invariant distributions in terms of orbital integrals, which are easier to manage.

**Proposition 18.1 (13.2)** *There exists a unique map*

$$\varepsilon_M : \mathcal{H}(\tilde{G}(F_S))^0 \to \mathcal{I}_{\mathrm{ac}}(M(F_S)),$$

*such that*

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) = \hat{I}_M^M(\gamma, \varepsilon_M(\tilde{f})),\ \gamma \in M_{\mathrm{comp}}(F_S),\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_S))^0.$$

*The map satisfies the splitting property*

$$\varepsilon_M(\tilde{f}) = \varepsilon_M(\tilde{f}_1)\tilde{f}'_{2,M} + \tilde{f}'_{1,M}\varepsilon_M(\tilde{f}_2),$$

*where $S$ is the disjoint union of $S_1$ and $S_2$ (each satisfying the closure property) and $\tilde{f} = \tilde{f}_1\tilde{f}_2$ is a corresponding decomposition.*

*Proof.* If $\varepsilon_M$ satisfies the first property of the proposition then the earlier splitting properties (Proposition 6.2, (23)) imply that the splitting property for $\varepsilon_M$. Therefore, we only need to show that the map $\varepsilon_M$ exists. In other words, must show that

$$\varepsilon_M(\gamma, \tilde{f}) = I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}),\ \gamma \in M_{\mathrm{comp}}(F_S),$$

is an orbital integral at $\gamma$ of a function in $\mathcal{H}_{\mathrm{ac}}(M(F_S))$. The splitting properties (Proposition 6.2, (23)) allow us to restrict our proof to the case that $S$ is comprised of a single valuation $v$. Also, Lemma 11.3 allows us to restrict to the case that $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$. Suppose first that $v$ is archimedean. By our assumptions on $F$, this means that $F_v = \mathbf{C}$. We may therefore apply Jordan canonical form to conclude that every element of $M(\mathbf{C}) \cap G_{\mathrm{oreg}}(\mathbf{C})$ is $G(\mathbf{C})$-conjugate to an element in $M_0(\mathbf{C})$. If $M \supsetneq M_0$ then Lemma 10.2 implies that $\varepsilon_M(\gamma, \tilde{f})$ vanishes. In other words, if $M \supsetneq M_0$ then

$$\varepsilon_M(\tilde{f}) = 0,\ \tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{C})). \tag{53}$$

We therefore need only consider the case $M = M_0$. We proceed by showing that, for fixed $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{C}))$, the function,

$$\gamma \mapsto \varepsilon_{M_0}(\gamma, \tilde{f}), \ \gamma \in M_0(\mathbf{C}) \cap G_{\mathrm{oreg}}(\mathbf{C}),$$

extends to a smooth function of $M_0(\mathbf{C})$. Suppose $\gamma_0$ is an arbitrary element of $M_0(\mathbf{C})$. Then, by the definition of $I^{\mathcal{M}}_{M_0}(\gamma_0, \tilde{f}) - I^{\Sigma}_{M_0}(\gamma_0, \tilde{f})$ (§§6-7) and Lemma 11.2, the limit,

$$\begin{aligned} &\lim_{\{a \to 1: a \in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} \sum_{L \in \mathcal{L}} r^L_{M_0}(\gamma_0^n, a) \left( I^{\mathcal{M}}_L(a\gamma_0, \tilde{f}) - I^{\Sigma}_L(a\gamma_0, \tilde{f}) \right) \\ &= \lim_{\{a \to 1: a \in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} \varepsilon_{M_0}(a\gamma_0, \tilde{f}), \end{aligned}$$

exists. In consequence, $\varepsilon_{M_0}(\cdot, \tilde{f})$ extends to a continuous function on $M_0(\mathbf{C})$. To see that this function is smooth, suppose that $z$ is a differential operator in $\mathcal{Z}_{\mathbf{C}}$, the center of the universal enveloping algebra of the complexified Lie algebra of $G(\mathbf{C})$. This operator passes to a differential operator on $\tilde{G}(\mathbf{C}) \cong G(\mathbf{C}) \times \mu_n$ by virtue of its action on $G(\mathbf{C})$. By Lemma 7.2 [Mezb], we have

$$\begin{aligned} \varepsilon_{M_0}(\gamma_0, z\tilde{f}) &= \lim_{\{a \to 1: a \in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} \varepsilon_{M_0}(a\gamma_0, z\tilde{f}) \\ &= \lim_{\{a \to 1: a \in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} z'_{M_0} \varepsilon_{M_0}(a\gamma_0, \tilde{f}). \end{aligned}$$

Here, $z_{M_0}$ is the image of $z$ under the Harish-Chandra isomorphism and

$$z_{M_0} \mapsto z'_{M_0}$$

is an algebra automorphism (§6 [Mezb]). We can choose a set of generators, $z_1, \ldots, z_{2r}$, of $\mathcal{Z}_{\mathbf{C}}$ and apply a well-known argument involving the fundamental theorem of calculus to show that $z'_{i,M_0} \varepsilon_{M_0}(\gamma_0, \tilde{f})$, $1 \leq i \leq 2r$, exists and is equal to

$$\lim_{\{a \to 1: a \in A_{M_0,\mathrm{reg}}(\mathbf{C})\}} z'_{i,M_0} \varepsilon_{M_0}(a\gamma_0, \tilde{f}) = \varepsilon_{M_0}(\gamma_0, z_i\tilde{f}).$$

A simple induction argument on the number of generators occurring in $z$ then implies that $z'_{M_0} \varepsilon_{M_0}(\gamma_0, \tilde{f})$ exists and

$$z'_{M_0} \varepsilon_{M_0}(\gamma_0, \tilde{f}) = \varepsilon_{M_0}(\gamma_0, z\tilde{f}). \tag{54}$$

This proves that $\varepsilon_{M_0}(\cdot,\tilde{f})$ extends to a smooth function on $M_0(\mathbf{C})$. Combining this fact with estimate (5.7) [Art94] and the inductive definitions of our invariant distributions we find that $\varepsilon_{M_0}(\cdot,\tilde{f})$ is actually in the Schwartz space of $M_0(\mathbf{C})$. Let $\hat{\varepsilon}_{M_0}(\tilde{f})$ be its Fourier transform. If $\hat{\varepsilon}_{M_0}(\tilde{f})$ belongs to $\mathcal{I}_{\mathrm{ac}}(M_0(\mathbf{C}))$ then we may define $\varepsilon_{M_0}(\tilde{f})$ by

$$(\varepsilon_{M_0}(\tilde{f}))_{M_0} = \hat{\varepsilon}_{M_0}(\tilde{f})$$

and the proposition is proven at the archimedean valuations. The only requirement that $\hat{\varepsilon}_{M_0}(\tilde{f})$ does not obviously satisfy for it to be in $\mathcal{I}_{\mathrm{ac}}(\tilde{M}_0(\mathbf{C}))$ is the $K_v \cap M_0(\mathbf{C})$-finiteness requirement. More precisely, we must show that there exists a finite set $\Gamma_{M_0}$ of (equivalence classes of) irreducible admissible representations of $K_v \cap M_0(\mathbf{C})$ such that $\hat{\varepsilon}_{M_0}(\tilde{f})(\pi)$ vanishes unless the restriction of $\pi \in \Pi_{\mathrm{temp}}(M_0(\mathbf{C}))$ to $K_v \cap M_0(\mathbf{C})$ contains a representation in $\Gamma_{M_0}$. Since $\tilde{f}$ is $\tilde{K}_v$-finite, there exists a finite set $\tilde{\Gamma}$ of (equivalence classes of) irreducible admissible representations of $\tilde{K}_v$ such that $\tilde{f}_{\tilde{M}_0}(\tilde{\pi})$, $\tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}_0(\mathbf{C}))$, vanishes unless the restriction of $\tilde{\pi}^{\tilde{G}}$ to $\tilde{K}_v$ contains a representation in $\tilde{\Gamma}$. Clearly, $\tilde{\Gamma}$ determines a finite set of (equivalence classes of) irreducible admissible representations $\Gamma$ of $K_v$ such that $\tilde{f}'_{M_0}(\pi)$, $\pi \in \Pi_{\mathrm{temp}}(M_0(\mathbf{C}))$, vanishes unless the restriction of $\pi^G$ to $K_v$ contains a representation in $\Gamma$. Let $\Gamma_{M_0}$ be the finite set of irreducible constituents of restrictions of representations in $\Gamma$ to $K_v \cap M_0(\mathbf{C})$. It is a straightforward consequence of equation (54) that

$$\tilde{f} \mapsto \hat{\varepsilon}_{M_0}(\tilde{f})(\tilde{\pi}'),\ \tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}_0(\mathbf{C})),$$

is an invariant eigendistribution of $\mathcal{Z}_{\mathbf{C}}$. It then follows from Harish-Chandra's work ([HC65]) that there exists a smooth function $c$ on $\Pi_{\mathrm{temp}}(\tilde{M}_0(\mathbf{C}))$ such that

$$(55)\qquad \hat{\varepsilon}_{M_0}(\tilde{f})(\tilde{\pi}') = c(\tilde{\pi})\tilde{f}_{\tilde{M}_0}(\tilde{\pi}) = c(\tilde{\pi})\tilde{f}'_{M_0}(\tilde{\pi}'),\ \tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}_0(\mathbf{C})).$$

Moreover, if $\pi \in \Pi_{\mathrm{temp}}(M_0(\mathbf{C}))$ is not metic then it is not $\mu_n^{M_0}$-invariant (Lemma 3.1 [Mezb]). Since $\varepsilon_{M_0}(\cdot,\tilde{f})$ is $\mu_n^{M_0}$-invariant, we must have $\hat{\varepsilon}_{M_0}(\tilde{f})(\pi) = 0$. This vanishing property together with equation (55) imply that $\Gamma_{M_0}$ satisfies the $K_v \cap M_0(\mathbf{C})$-finiteness requirement for $\hat{\varepsilon}_{M_0}(\tilde{f})$.

Now suppose $v$ is nonarchimedean. For $\gamma \in M(F_v) \cap G_{\mathrm{oreg}}(F_v)$ define

$${}^c\varepsilon_M(\gamma,\tilde{f}) = {}^cI_M^{\mathcal{M}}(\gamma,\tilde{f}) - {}^cI_M^{\Sigma}(\gamma,\tilde{f}),\ \tilde{f} \in \mathcal{H}(\tilde{G}(F_v))^0,$$

(cf. §15). By Lemma 4.4 [Art88a], this function has compact support as a function of $M(F_v)$-conjugacy classes of $M(F_v)$. It is related to $\varepsilon_M(\gamma, \tilde{f})$ by

$$\varepsilon_M(\gamma, \tilde{f}) - {}^c\varepsilon_M(\gamma, \tilde{f}) \;=\; \hat{I}^M_M(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_M(\tilde{f}')).$$

To see this, recall that by the induction assumption after Theorem A, Theorem 15.1, equations (39)-(40) and their analogues for $I^\Sigma_M(\gamma)$, we have

$$\begin{aligned}
&\varepsilon_M(\gamma, \tilde{f}) - {}^c\varepsilon_M(\gamma, \tilde{f}) \\
&= \sum_{\{L \in \mathcal{L}(M): L \subsetneq G\}} {}^c\hat{I}^{L,\mathcal{M}}_M(\gamma, \theta_{\tilde{L}}(\tilde{f})) - \sum_{\eta_1 \in \mu_n^M/\mu_n^L} \hat{I}^L_M(\eta_1\gamma, \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_L(\tilde{f}')) \\
&= \sum_{\{L \in \mathcal{L}(M): M \subsetneq L \subsetneq G\}} {}^c\hat{I}^{L,\mathcal{M}}_M(\gamma, \theta_{\tilde{L}}(\tilde{f})) - {}^c\hat{I}^{L,\Sigma}_M(\gamma, \theta_{\tilde{L}}(\tilde{f})) \\
&+ \hat{I}^M_M(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_M(\tilde{f}')) \\
&= \hat{I}^M_M(\gamma, \theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_M(\tilde{f}')).
\end{aligned}$$

Since $\theta_{\tilde{M}}(\tilde{f})' - \sum_{\eta \in \mu_n^M/\mu_n^G} {}^\eta\hat{\theta}_M(\tilde{f}')$ belongs to $\mathcal{I}_{\mathrm{ac}}(M(F_v))$ (§4 [Art88a]), this equation implies that ${}^c\varepsilon_M(\gamma, \tilde{f})$ is an orbital integral of a function in $\mathcal{H}_{\mathrm{ac}}(M(F_v))$ if and only if $\varepsilon_M(\gamma, \tilde{f})$ is as well. More generally, given a semisimple element $\sigma \in M(F_v)$, it implies that

$${}^c\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0,\; \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v)$$

if and only if

$$\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0,\; \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

We shall complete the proposition by showing that

$$\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0,\; \gamma \in M_\sigma(F_v) \cap G_{\mathrm{oreg}}(F_v)$$

and then showing that this implies that ${}^c\varepsilon_M(\gamma, \tilde{f})$ is an orbital integral of a function in $\mathcal{H}_{\mathrm{ac}}(M(F_v))$. If $\sigma$ is not $F_v$-elliptic in $M$, that is, if $\mathfrak{a}_{M_\sigma} \neq \mathfrak{a}_M$, then there is a proper Levi subgroup $M_1 \in \mathcal{L}^M$ of $M$ such that $M_\sigma(F_v) \subset M_1(F_v)$. Lemma 10.2 then implies that

$$\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0, \ \gamma \in M_\sigma(F_v) \cap G_{\text{oreg}}(F_v).$$

Suppose that $\sigma$ is $F_v$-elliptic in $M$. We may assume that $\sigma$ is $F_v$-elliptic in $L \in \mathcal{L}(M)$ if and only if $\sigma^n$ is $F_v$-elliptic in $L$. Indeed, we know by Lemma 12.1 that a translate of $\sigma$ by an element in $\mu_n^M$ satisfies this property and we also know that

$$\varepsilon_M(\eta\gamma, \tilde{f}) = \varepsilon_M(\gamma, \tilde{f}), \ \eta \in \mu_n^M.$$

We may therefore apply Lemma 12.3 and Proposition 12.1 to see that $\varepsilon_M(\gamma, \tilde{f})$ is $(M, \sigma)$-equivalent to

$$\sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma, \delta) \left( I_L^{\mathcal{M}}(\delta, \tilde{f}) - I_L^{\Sigma}(\delta, \tilde{f}) \right), \ \gamma \in M_\sigma(F_v) \cap G_{\text{oreg}}(F_v).$$

Since $g_M^M(\gamma, \delta) = 0$, we may apply the induction hypothesis of this section to conclude that the right-hand side vanishes. From our earlier remark we now have

$${}^c\varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} \varepsilon_M(\gamma, \tilde{f}) \overset{(M,\sigma)}{\sim} 0, \ \gamma \in M_\sigma(F_v) \cap G_{\text{oreg}}(F_v),$$

for any semisimple element $\sigma \in M(F_v)$. We use a partition of unity argument on the compact support of ${}^c\varepsilon_M(\gamma, \tilde{f})$ in the space of $M(F_v)$-conjugacy classes to conclude that ${}^c\varepsilon_M(\gamma, \tilde{f})$ is in fact an orbital integral.□

Suppose $v$ is a valuation for which $|n|_v = 1$. Then it is an immediate consequence of Lemma 11.4 that $\varepsilon_M(\tilde{f}_v^0) = 0$. It therefore follows from the splitting property of Proposition 18.1 that the map $\varepsilon_M$ extends to a map on $\mathcal{H}(\tilde{G}(\mathbf{A}))^0$.

# 19 Comparison for $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)$

We now give a sketch of the proof that $I(\tilde{f}) = I^{\Sigma}(\tilde{f})$ for $\tilde{f}$ in a certain subspace $\mathcal{H}(\tilde{G}(\mathbf{A}), M)$ of $\mathcal{H}(\tilde{G}(\mathbf{A}))$. The train of reasoning in this section is based entirely on §§15-16 II [AC89]. We shall outline the arguments found there and leave the confirmation of the details to the reader.

Recall that $M \in \mathcal{L}$ was fixed in the previous section. Let $\mathcal{H}(\tilde{G}(\mathbf{A}), M)$ be the subspace of $\mathcal{H}(\tilde{G}(\mathbf{A}))$ spanned by functions

$$\tilde{f} = \prod_v \tilde{f}_v, \ \tilde{f}_v \in \mathcal{H}(\tilde{G}(F_v)),$$

which satisfy the following property. For two nonarchimedean places $v_1$ and $v_2$, which are not in $\{v : |n|_v \neq 1\}$,

$$\tilde{f}_{v_i,\tilde{L}} = 0, \quad L \in \mathcal{L}, \ i = 1, 2,$$

unless $L$ contains a conjugate of $M$. If $S$ contains $\{v : |n|_v \neq 1\}$ and at least two other nonarchimedean places, we define $\mathcal{H}(\tilde{G}(F_S), M)$ in the same way.

The proof of Lemma 13.1 II [AC89] may be imitated to obtain the following lemma.

**Lemma 19.1 (13.1)** *For $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)$, the distribution*

$$I(\tilde{f}) - I^{\Sigma}(\tilde{f})$$

*equals the sum of*

$$|W(\mathfrak{a}_M)|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S, \gamma) \left( I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) \right)$$

*and*

$$\sum_{\delta \in A_G(F)\backslash \mu_n^G} \sum_{u \in (\mathcal{U}_G(F))_{G,S}} \left( a^{\tilde{M}}(S, u') - a(S, u) \right) I_G^{\mathcal{M}}(\delta u, \tilde{f}).$$

*Proof.* The lemma follows from the splitting properties and the properties of $\tilde{f}$. See Lemma 13.1 II [AC89] for details.□

Let $\mathcal{H}(\tilde{G}(\mathbf{A}), M)^0$ be the space of functions $\tilde{f}$ in

$$\mathcal{H}(G(\mathbf{A}), M) \cap \mathcal{H}(G(\mathbf{A}))^0$$

which satisfy one additional condition, namely, that $\tilde{f}$ vanishes at any element in $\tilde{G}(\mathbf{A})$ whose component at each nonarchimedean place $v$ belongs to $A_G(F_v)$. This ensures that $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)^0$ vanishes at

$$(\delta u, \zeta), \ \delta \in A_G(F), \ u \in \mathcal{U}_G(F), \ \zeta \in \mu_n.$$

**Lemma 19.2 (15.1)** *Suppose that $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)^0$. Then*

$$I(\tilde{f}) - I^{\Sigma}(\tilde{f}) = n^{\dim(A_M)} |W(\mathfrak{a}_M)|^{-1} \hat{I}^M(\varepsilon_M(\tilde{f})),$$

*where $I^M$ is the analogue for $M$ of $I = I^G$.*

*Proof.* By the properties of $\tilde{f}$ Lemma 19.1 and Proposition 18.1, we see that $I(\tilde{f}) - I^{\Sigma}(\tilde{f})$ is equal to

$$n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S,\gamma)\hat{I}^M_M(\gamma, \varepsilon_M(\tilde{f})),$$

for a large set of valuations $S$. By the descent property (Corollary 8.3 [Art88a]) and Lemma 10.2 it follows that

$$\hat{I}^{M_1}_{M_1}(\gamma, \varepsilon_M(\tilde{f})_{M_1}) = \hat{I}^M_M(\gamma, \varepsilon_M(\tilde{f})) = 0, \ \gamma \in M_1(F_S) \cap G_{\mathrm{oreg}}(F_S),$$

for any $M_1 \in \mathcal{L}$ such that $M_1 \subsetneqq M$. Therefore, for such $M_1$, $\varepsilon_M(\tilde{f})_{M_1} = 0$. Combining this with the splitting property (Proposition 9.1 [Art88a]) applied to $I^M_{M_1}(\gamma)$, we find that

$$\hat{I}^M_{M_1}(\gamma, \varepsilon_M(f)) = 0, \ \gamma \in M_1(F) \cap G_{\mathrm{oreg}}(F),$$

for any $M_1 \in \mathcal{L}^M$ such that $M_1 \subsetneqq M$. Thus by (25),

$$\begin{aligned}
& n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S,\gamma)\hat{I}^M_M(\gamma, \varepsilon_M(\tilde{f})) \\
= \ & n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1} \sum_{M_1 \in \mathcal{L}^M} |W_0^{M_1}||W_0^M|^{-1} \sum_{\gamma \in (M_1(F))_{M_1,S}} a^{M_1}(S,\gamma)\hat{I}^M_{M_1}(\gamma, \varepsilon_M(\tilde{f})) \\
= \ & n^{\dim(A_M)}|W(\mathfrak{a}_M)|^{-1}\hat{I}^M(\varepsilon_M(\tilde{f})).\square
\end{aligned}$$

We now give an outline of the method of separation by infinitesimal characters, which is to be found in §15 II [AC89]. Let $S_\infty$ denote the set of archimedean (in our case complex) valuations of $F$. Then $G(F_{S_\infty})$ may be regarded as a real Lie group. Let $\mathfrak{h}_{\mathbf{C}}$ denote the standard Cartan subalgebra of its complexified Lie algebra. Let $\mathfrak{h}$ be the real form of $\mathfrak{h}_{\mathbf{C}}$ associated to the split real form of $G(F_{S_\infty})$. Then $\mathfrak{h}$ contains $\mathfrak{a}_{M_0}$ as a vector space and in turn contains all vector spaces of the form $\mathfrak{a}_L$, $L \in \mathcal{L}$. Let $\mathfrak{h}^1$ be the orthogonal complement of $\mathfrak{a}_G$ in $\mathfrak{h}$. We recall the theory of multipliers ([Art83]). Let $\alpha$ belong to $\mathcal{E}(\mathfrak{h}^1)^W$, the convolution algebra of compactly supported, distributions on $\mathfrak{h}^1$, which are invariant under the complex Weyl group $W$ of $G(F_{S_\infty})$. Then there is an action, $\tilde{f} \mapsto \tilde{f}_\alpha$, on $\mathcal{H}(\tilde{G}(\mathbf{A}))$ such that

$$\tilde{f}_{\alpha,\tilde{M}}(\tilde{\pi}) = \hat{\alpha}(\nu_{\tilde{\pi}})\tilde{f}_{\tilde{M}}(\tilde{\pi}), \ \tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A})).$$

As usual, $\nu_{\tilde\pi}$ is taken to be the infinitesimal character of the archimedean factor of $\tilde\pi$. This action of $\mathcal{E}(\mathfrak{h}^1)^W$ on $\mathcal{H}(\tilde G(\mathbf{A}))$ affects only the archimedean factor of $\tilde f$, and is supported on characters. The map,

$$\alpha \mapsto \alpha',\ \ \alpha \in \mathcal{E}(\mathfrak{h}^1)^W,$$

given by

$$\alpha'(\nu) = n^{-\dim(\mathfrak{h}^1)}\alpha(n^{-1}\nu), \nu \in \mathfrak{h}^1, \tag{56}$$

is compatible with map (13). More precisely, given $\alpha \in \mathcal{E}(\mathfrak{h}^1)^W$ and $\tilde\pi \in \Pi_{\mathrm{temp}}(\tilde G(F_S))$, we have $\nu_{\tilde\pi} = \nu_{n^{-1}\tilde\pi'}$ (§3 [Mezb]) and in turn

$$(\tilde f_\alpha)'(\tilde\pi') = \hat\alpha(n^{-1}\nu_{\tilde\pi'})\tilde f_{\tilde G}(\tilde\pi) = \hat\alpha'(\nu_{\tilde\pi'})\tilde f'(\tilde\pi') = \tilde f'_{\alpha'}(\tilde\pi').$$

This map is also compatible with the map $\varepsilon_M$ of Proposition 18.1.

**Lemma 19.3 (14.4)** *Suppose $\tilde f \in \mathcal{H}(\tilde G(\mathbf{A}))$, and $\alpha \in \mathcal{E}(\mathfrak{h}^1)^W$. Then $\varepsilon_M(\tilde f_\alpha) = \varepsilon_M(f)_{\alpha'}$.*

*Proof.* Suppose $\tilde f = \prod_v \tilde f_v \in \mathcal{H}(\tilde G(\mathbf{A}))$ and suppose $S$ contains all valuations at which $\tilde f_v \neq \tilde f_v^0$. If $v$ is nonarchimedean then $\alpha$ acts trivially on $\tilde f_v$, that is $\tilde f_{v,\alpha} = \tilde f_v$. If $v$ is complex, it follows from equations (53) and (55) that $\varepsilon_M(\tilde f_{v,\alpha}) = \varepsilon_M(\tilde f)_{\alpha'}$. Repeated applications of the splitting property of Proposition 18.1 yield

$$\begin{aligned}
\varepsilon_M(\tilde f_\alpha) &= \sum_{v_1\in S}\varepsilon_M(f_{v_1,\alpha})\prod_{w\in S-v_1}(\tilde f_{w,\alpha})'_M\prod_{v\notin S}(\tilde f_v^0)'_M\\
&= \sum_{v_1\in S}\varepsilon_M(f_{v_1})_{\alpha'}\prod_{w\in S-v_1}((\tilde f_w)'_{\alpha'})_M\prod_{v\notin S}(\tilde f_v^0)'_M\\
&= \varepsilon_M(\tilde f)_{\alpha'}.\Box
\end{aligned}$$

Let $\mathfrak{h}_u^*$ be the set of points $\nu$ in $\mathfrak{h}^*_{\mathbf{C}}/i\mathfrak{a}^*_G$ such that the complex conjugate of $\nu$ with respect to $\mathfrak{h}^*$ is equal to $-s\nu$ for some element $s\in W$ of order two. The archimedean infinitesimal character of $\nu_{\tilde\pi}$ associated to any $\tilde\pi \in \Pi_{\mathrm{unit}}(\tilde G(\mathbf{A})^1)$ belongs to $\mathfrak{h}_u^*$. Given $\nu_1 \in \mathfrak{h}_u^*$ define

$$\Pi_{\nu_1}(\tilde G(\mathbf{A})^1) = \{\tilde\pi \in \Pi(\tilde G(\mathbf{A})^1) : \nu_{\tilde\pi} = s\nu_1 \text{ for some } s\in W\}.$$

We can use multipliers to prove the following lemma.

**Lemma 19.4 (15.4)** *For each $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)^0$ and $\nu_1 \in \mathfrak{h}_u^*$, we have*

$$\sum_{\tilde{\pi} \in \Pi_{\nu_1}(\tilde{G}(\mathbf{A})^1)} \left(a_{\mathrm{disc}}(\tilde{\pi}) - a_{\mathrm{disc}}^{\Sigma}(\tilde{\pi})\right) \operatorname{tr}\left(\tilde{\pi}(\tilde{f}^1)\right) = 0.$$

As the proof of Lemma 19.4 is almost identical to that of Lemma 15.4 II [AC89], we shall only sketch the proof. Given $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)^0$, $T \geq 0$ and $\nu_1 \in \mathfrak{h}_u^*$ we choose $\alpha_1 \in \mathcal{E}(\mathfrak{h}^1)^W$ as in Lemma 15.2 II [AC89]. The multiplier $\alpha_1$ has the property that $0 \leq \hat{\alpha}_1(\nu_{\tilde{\pi}}) \leq 1$ for any $\tilde{\pi} \in \Pi_{\mathrm{unit}}(\tilde{G}(\mathbf{A}))$ whose $K$-type is the same as that of $\tilde{f}$. Furthermore the inverse image of 1 under $\hat{\alpha}_1$ is $\{s\nu_1 : s \in W\}$. Following Corollaries 14.2-14.3 II [AC89], we find that $\varepsilon_M(\tilde{f})$ is a moderate function in $\mathcal{I}_{\mathrm{ac}}(M(\mathbf{A}))$ in the sense of §6 [Art88b]. We may then apply Corollary 6.5 [Art88b] (cf. (15.5) II [AC89]) to obtain

$$\left| \sum_{t \leq T} I_t(\tilde{f}_{\alpha_1^m}) - I_t^{\Sigma}(\tilde{f}_{\alpha_1^m}) - n^{\dim(A_M)} |W(\mathfrak{a}_M)|^{-1} \hat{I}_t^M(\varepsilon_M(\tilde{f})_{(\alpha_1')^m}) \right| \leq Ce^{-kNm}$$

for some positive constants $C$, $k$ and $N$. Thus

$$\sum_{t \leq T} I_t(\tilde{f}_{\alpha_1^m}) - I_t^{\Sigma}(\tilde{f}_{\alpha_1^m}) \tag{57}$$

approaches

$$\sum_{t \leq T} n^{\dim(A_M)} |W(\mathfrak{a}_M)|^{-1} \hat{I}_t^M(\varepsilon_M(\tilde{f})_{(\alpha_1')^m}) \tag{58}$$

as $m$ approaches infinity. According to Lemma 15.3 II [AC89] we may write (58) as

$$\sum_{t \leq T} n^{\dim(A_M)} |W(\mathfrak{a}_M)|^{-1} \sum_{\pi \in \Pi_{\mathrm{disc}}(M,t)} a_{\mathrm{disc}}^M(\pi) \int_{i\mathfrak{a}_M^*/i\mathfrak{a}_G^*} \varepsilon_M(\tilde{f}^1, \pi, \lambda) \hat{\alpha}_1'(\nu_\pi + \lambda)^m d\lambda,$$

for some Schwartz function

$$\lambda \mapsto \varepsilon_M(\tilde{f}^1, \pi, \lambda),\ \lambda \in i\mathfrak{a}_M^*/i\mathfrak{a}_G^*.$$

The multiplier $\alpha_1$ was chosen so that

$$0 \leq \hat{\alpha}_1'(\nu_\pi + \lambda) < 1,$$

for all but finitely many $\lambda \in i\mathfrak{a}_M^*/i\mathfrak{a}_G^*$ in the above integral. Thus, by the dominated convergence theorem, the integral approaches zero as $m$ approaches infinity. Expression (57) therefore also approaches zero as $m$ approaches infinity. Moreover, by Lemma 17.2 it is equal to

$$\sum_{t \leq T} \sum_{\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1, t)} (a_{\text{disc}}(\tilde{\pi}) - a_{\text{disc}}^{\Sigma}(\tilde{\pi})) \mathrm{tr}\left(\tilde{\pi}(\tilde{f}^1)\right) \hat{\alpha}_1(\nu_{\tilde{\pi}})^m.$$

By our choice of $\alpha_1$, the limit of this expression as $m$ approaches infinity is the expression in Lemma 19.4.

In §16 II [AC89] it is shown, using the Plancherel formula, how to extend Lemma 19.4 to functions in $\mathcal{H}(\tilde{G}(\mathbf{A}), M)$. We shall not repeat the arguments here. We merely translate the final result of that section into the following proposition.

**Proposition 19.1 (16.2)** *For any $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)$, we have*

$$I(\tilde{f}) = I^{\Sigma}(\tilde{f}).$$

*Proof.* By Lemma 17.2 and the extension of Lemma 19.4 to $\mathcal{H}(\tilde{G}(\mathbf{A}), M)$, the difference $I_t(\tilde{f}) - I_t^{\Sigma}(\tilde{f})$, $t \geq 0$, is equal to

$$\sum_{\{\nu_1 : \|\mathrm{Im}(\nu_1)\| = t\}} \sum_{\tilde{\pi} \in \Pi_{\nu_1}(\tilde{G}(\mathbf{A})^1)} \left(a_{\text{disc}}(\tilde{\pi}) - a_{\text{disc}}^{\Sigma}(\tilde{\pi})\right) \mathrm{tr}\left(\tilde{\pi}(\tilde{f}^1)\right) = 0.$$

It is then obvious that

$$I(\tilde{f}) = \sum_t I_t(\tilde{f}) = \sum_t I_t^{\Sigma}(\tilde{f}) = I^{\Sigma}(\tilde{f}). \square$$

# 20 Completion of the proofs of Theorems A and B

We first prove Theorem A (i). The structure of the proof is as follows. We begin by showing that for any valuation $v$ of $F$ and each $\gamma_v \in M(F_v) \cap G_{\text{oreg}}(F_v)$ there exists a constant $\varepsilon_M(\gamma_v)$ such that

$$I_M^{\mathcal{M}}(\gamma_v, \tilde{f}) - I_M^{\Sigma}(\gamma_v, \tilde{f}) = \varepsilon_M(\gamma_v) I_G^{\mathcal{M}}(\gamma_v, \tilde{f}), \;\; \tilde{f} \in \mathcal{H}(\tilde{G}(F_v)).$$

This part of the proof follows §17 II [AC89] almost exactly. It then follows from the splitting properties that for any $S_0$ containing $\{v : |n|_v \neq 1\}$ and

$$\gamma = \prod_{v\in S_0} \gamma_v \in M(F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0}),$$

we have

$$(59)\quad I_M^{\mathcal{M}}(\gamma, \tilde{f}) - I_M^{\Sigma}(\gamma, \tilde{f}) = \left(\sum_{v\in S_0} \varepsilon_M(\gamma_v)\right) I_G^{\mathcal{M}}(\gamma, \tilde{f}),\ \ \tilde{f} \in \mathcal{H}(\tilde{G}(F_{S_0})).$$

Using the strong approximation theorem, we show that $\sum_{v\in S_0} \varepsilon_M(\gamma_v)$ vanishes unless $M = M_0$.[1] The induction hypothesis of §18 and Corollary 11.1 then imply that $\varepsilon_M(\gamma_v) = 0$ unless $M = M_0$. We then take care of the case $M = M_0$ by using the local trace formula as in the proof of Corollary 11.1.

Suppose $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}), M)$. According to Lemma 19.1 and Proposition 19.1 the sum of

$$(60)\qquad |W(\mathfrak{a}_M)|^{-1} \sum_{\gamma\in(M(F))_{M,S}/\mu_n^M} a^M(S,\gamma)\left(I_M^{\mathcal{M}}(\gamma,\tilde{f}) - I_M^{\Sigma}(\gamma,\tilde{f})\right)$$

and

$$(61)\qquad \sum_{\delta\in A_G(F)\backslash\mu_n^G}\ \sum_{u\in(\mathcal{U}_G(F))_{G,S}} \left(a^{\mathcal{M}}(S,u) - a(S,u)\right) I_G^{\mathcal{M}}(\delta u, \tilde{f})$$

vanishes. Recall that $S$ is a large finite set of valuations which depends on $\tilde{f}$ and contains $\{v : |n|_v \neq 1\}$. We may assume that $S = \{v_1, \ldots, v_k\}$, where $v_1$ and $v_2$ are any two distinct valuations satisfying

$$|n|_{v_1} = |n|_{v_2} = 1.$$

Take $\gamma$ to be an element in $M(F) \cap G_{\mathrm{oreg}}(F)$ such that $\gamma$ is $F_{v_i}$-elliptic in $M(F_{v_i})$, $i = 1, 2$. Choose $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$ such that it is supported on a very small neighborhood of $\gamma'\mathbf{i}(\mu_n)$ in $\tilde{G}(F_{v_i})$ and

$$\hat{I}_M^{M,\mathcal{M}}(\gamma, \tilde{f}_{i,\tilde{M}}) = I_G^{\mathcal{M}}(\gamma, \tilde{f}_i) = 1,\ \ i = 1, 2.$$

For the remaining $3 \leq i \leq k$, let $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$ be arbitrary and set $\tilde{f} = \tilde{f}_1 \cdots \tilde{f}_k$. This function lies in $\mathcal{H}(\tilde{G}(F_S), M)$ by the construction of $\tilde{f}_1$

[1] I am indebted to J. Arthur for providing the underlying ideas for this portion of the proof.

and $\tilde{f}_2$. If we shrink the support of $\tilde{f}_1$ and $\tilde{f}_2$ around $\gamma'\mathbf{i}(\mu_n)$ the set $S$ is not affected. Bearing in mind that $\tilde{M}(F)$ is a discrete subgroup of $\tilde{M}(\mathbf{A})$, we may then assume that the support of $\tilde{f}_1$ and $\tilde{f}_2$ is so small that (61) vanishes and the only contribution to the sum in (60) comes from $\tilde{G}(F_S)$-conjugates of $\gamma'$. The distributions in (60) are constant on $\tilde{G}(F_S)$-conjugacy classes. By Theorem 8.2 [Art86] we have

$$a^M(S,\gamma) = \mathrm{vol}(M_\gamma(F)\backslash M_\gamma(\mathbf{A})^1),$$

for $S$ large enough (in a sense depending only on $\gamma$). Consequently,

$$I_M^{\mathcal{M}}(\gamma,\tilde{f}) - I_M^{\Sigma}(\gamma,\tilde{f}) = 0. \tag{62}$$

We apply the splitting properties (Proposition 6.2, equation (23)) and the induction hypothesis of §10 to this equation repeatedly in order to obtain

$$\sum_{i=1}^{k}\left(I_M^{\mathcal{M}}(\gamma,\tilde{f}_i) - I_M^{\Sigma}(\gamma,\tilde{f}_i)\right)\prod_{j\neq i}\hat{I}_M^{M,\mathcal{M}}(\gamma,\tilde{f}_{j,\tilde{M}}) = 0. \tag{63}$$

Suppose $3 \leq i \leq k$. Choose $\gamma \in M(F)$ as in (63), $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$ such that

$$\hat{I}_M^{M,\mathcal{M}}(\gamma,\tilde{f}_{i,\tilde{M}}) = I_G^{\mathcal{M}}(\gamma,\tilde{f}_i) = 0,$$

and the remaining $\tilde{f}_j \in \mathcal{H}(\tilde{G}(F_{v_j}))$, $j \neq i$, such that

$$\hat{I}_M^{M,\mathcal{M}}(\gamma,\tilde{f}_{j,\tilde{M}}) \neq 0.$$

In this case, the left-hand side of (63) is a nonzero multiple of

$$I_M^{\mathcal{M}}(\gamma,\tilde{f}_i) - I_M^{\Sigma}(\gamma,\tilde{f}_i).$$

Consequently, this distribution vanishes for any $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$ such that $I_G^{\mathcal{M}}(\gamma,\tilde{f}_i)$ vanishes. This implies that there exists a constant $\varepsilon_M(\gamma)$ such that

$$I_M^{\mathcal{M}}(\gamma,\tilde{f}_i) - I_M^{\Sigma}(\gamma,\tilde{f}_i) = \varepsilon_M(\gamma) I_G^{\mathcal{M}}(\gamma,\tilde{f}_i), \tag{64}$$

for any $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$. Observe that if $M \supsetneqq M_0$ and $F_{v_i} = \mathbf{C}$ then the left-hand side of this equation vanishes by Jordan canonical form and Lemma 10.2. This implies that $\varepsilon_M(\gamma) = 0$ at the archimedean places whenever $M \supsetneqq M_0$.

Let $V_i = \{v_i, v_1, v_2\}$. Suppose that $v_i$ is nonarchimedean. The set

$$\left\{\prod_{v\in V_i} \gamma_v \in M(F_{V_i}) \cap G_{\mathrm{oreg}}(F_{V_i}) : \gamma_{v_j} \text{ is } F_{v_j}\text{-elliptic in } M(F_{v_j}),\ j = 1, 2\right\}$$

is open in $M(F_{V_i})$. Since $M(F)$ is dense in $M(F_{V_i})$, it is also dense in the above open subset. It follows that we may approximate any $\gamma_i \in M(F_{v_i})$ by an element $\gamma$ occurring in (64). Since all of the distributions in (64) are smooth on $M(F_{v_i}) \cap G_{\mathrm{oreg}}(F_{v_i})$, the function $\varepsilon_M$ also extends to a smooth function on this space and we have

$$(65) I_M^{\mathcal{M}}(\gamma_i, \tilde{f}_i) - I_M^{\Sigma}(\gamma_i, \tilde{f}_i) = \varepsilon_M(\gamma_i) I_G^{\mathcal{M}}(\gamma_i, \tilde{f}_i),\ \gamma_i \in M(F_{v_i}) \cap G_{\mathrm{oreg}}(F_{v_i}).$$

Now suppose that $F_{v_i} = \mathbf{C}$. If $M \supsetneqq M_0$ then, as we have already remarked, equation (65) holds with $\varepsilon_M(\gamma_i) = 0$. Otherwise, it is obvious that one can approximate $\gamma_i \in M_0(\mathbf{C}) \cap G_{\mathrm{oreg}}(\mathbf{C})$ by an element $\gamma \in M(F) \cap G_{\mathrm{oreg}}(F)$ which is $F_{v_j}$-elliptic in $M_0(F_{v_j})$, for $j = 1, 2$. After all, the latter condition is satisfied for any element in $M_0(F_{v_j})$, $j = 1, 2$. As a result, equation (65) holds for the archimedean places as well. Finally, notice that since our choice of $v_1$ and $v_2$ was arbitrary as long as $|n|_{v_1} = |n|_{v_2} = 1$, equation (65) holds for any $1 \leq i \leq k$.

Suppose $M \supsetneqq M_0$ and $S_0 \supset \{v : |n|_v \neq 1\}$. We may assume that $S \supset S_0$. By approximating elements of $M(F_S)$ with elements of $M(F)$ as above, repeated use of the splitting properties (Proposition 6.2, equation (23)) and the induction hypothesis of §10, we obtain equation (59) from (62). We would like to show that $\sum_{v\in S_0} \varepsilon_M(\gamma_v)$ vanishes and then apply Corollary 11.1.

Suppose that $v \notin S_0$ and $\gamma_v$ lies in $K_v \cap M(F_v) \cap G_{\mathrm{oreg}}(F_v)$. It is obvious that $I_G^{\mathcal{M}}(\gamma_v, \tilde{f}_v^0) \neq 0$ ((27)). However, according to Lemma 11.4 the left-hand side of equation (65) vanishes. Therefore,

$$(66) \qquad \varepsilon_M(\gamma_v) = 0,\ \gamma_v \in K_v \cap M(F_v) \cap G_{\mathrm{oreg}}(F_v),\ v \notin S_0.$$

Recall decomposition (4),

$$M = \prod_{i=1}^{\ell} M(i) \cong \prod_{i=1}^{\ell} \mathrm{GL}(r_i).$$

We consider the subgroup $\prod_{i=1}^{\ell} \mathrm{SL}(r_i)$ of $\prod_{i=1}^{\ell} \mathrm{GL}(r_i)$ and identify it with its image in $M$ using the above isomorphism. We shall show that

$$(67) \qquad \sum_{v \in S_0} \varepsilon_M(\gamma_v) = 0, \ \gamma_0 = \prod_{v \in S_0} \gamma_v \in \prod_{i=1}^{\ell} \mathrm{SL}(r_i, F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0}).$$

One should keep in mind that the summands parameterized by archimedean valuations all vanish, as we are assuming that $M \supsetneqq M_0$. We may suppose that $v_1, v_2$ are as earlier and $v_1, v_2 \notin S_0$. For $i = 1, 2$ we choose $\gamma_{v_i}$ to be $F_{v_i}$-elliptic elements in $M(F_{v_i})$ which also lie in $\prod_{j=1}^{\ell} \mathrm{SL}(r_j, F_{v_i})$, $K_{v_i}$ and $G_{\mathrm{oreg}}(F_{v_i})$. To see that this is possible, consider a finite Galois extension $E$ of $F_{v_i}$. Let $x_1$ be an element of $E$ such that $F_{v_i}(x_1^n) = E$. Choose a non-trivial element $\sigma$ in $\mathrm{Gal}(E/F_{v_i})$ and set $x = x_1/\sigma(x_1)$. Then $|x|_E = 1$, $N_{E/F_{v_i}}(x) = 1$ and $F(x^n) = E$. These properties of $x$ can be translated respectively into the above context as $\gamma_{v_i} \in K_{v_i}$, $\gamma_{v_i} \in \prod_{i=1}^{\ell} \mathrm{SL}(r_i, F_{v_i})$ and $\gamma_{v_i} \in G_{\mathrm{oreg}}(F_{v_i})$ being $F_{v_i}$-elliptic in $M(F_{v_i})$. Now suppose $\gamma_0 \in \prod_{i=1}^{\ell} \mathrm{SL}(r_i, F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0})$. The the strong approximation theorem ([Kne65]) allows us to choose $\gamma \in \prod_{i=1}^{\ell} \mathrm{SL}(r_i, F) \cap G_{\mathrm{oreg}}(F)$ such that $\gamma$ is close to $\gamma_0$ at the nonarchimedean valuations in $S_0$, $\gamma$ is close to $\gamma_{v_i}$ in $M(F_{v_i})$ for $i = 1, 2$, and $\gamma \in K_v$ for any $v \notin S_0 \cup \{v_1, v_2\}$. Choose $\tilde{f}_i \in \mathcal{H}(\tilde{G}(F_{v_i}))$, $1 \leq i \leq k$, such that $\tilde{f}_1, \tilde{f}_2$ are supported on very small neighborhoods of $\gamma' \mathbf{i}(\mu_n)$ and $\tilde{f}_i = \tilde{f}_{v_i}^0$ if $v_i \notin S_0$. It then follows from equation (63), equation (66) and our earlier observations that equation (67) holds.

Suppose $\delta = \prod_{v \in S_0} \delta_v \in M(F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0})$ and

$$\gamma_0 = \prod_{v \in S_0} \gamma_v \in \prod_{i=1}^{\ell} \mathrm{SL}(r_i, F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0})$$

such that $\delta\gamma_0 \in M(F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0})$. Our next step is to show that

$$(68) \qquad \sum_{v \in S_0} \varepsilon_M(\delta_v \gamma_v) = \sum_{v \in S_0} \varepsilon_M(\delta_v).$$

It is easily seen that we may choose $\gamma_{v_i}$, $i = 1, 2$, as in the previous paragraph, but with the additional requirement that $\gamma_{v_i}^2$ satisfies the same conditions as $\gamma_{v_i}$. Weak approximation allows us to choose $\gamma_1 \in M(F) \cap G_{\mathrm{oreg}}(F)$ such that $\gamma_1$ is close to $\gamma_{v_i}$ in $M(F_{v_i})$ for $i = 1, 2$, and $\gamma_1$ is close to $\delta$ at the nonarchimedean valuations of $S_0$. Let $V$ be the finite set of valuations

$\{v \notin S_0 : \gamma_1 \notin K_v\}$. Choosing $S$ large enough and $\tilde{f}_1, \ldots, \tilde{f}_k$ appropriately in equation (63), we may conclude that

$$0 = \sum_{v \in S_0 \cup V} \varepsilon_M(\gamma_1) = \sum_{v \in S_0} \varepsilon_M(\delta_v) + \sum_{v \in V} \varepsilon_M(\gamma_1). \tag{69}$$

(In this equation we are abusing notation as $\varepsilon_M(\gamma_1)$ depends on $v \in S_0 \cup V$.) As earlier, by strong approximation, we may choose $\gamma_2 \in \prod_{i=1}^{\ell} \mathrm{SL}(r_i, F) \cap G_{\mathrm{oreg}}(F)$ such that it is close to $\gamma_{v_i}$ in $M(F_{v_i})$ for $i = 1, 2$; it is close to $\gamma_0$ at the nonarchimedean valuations in $S_0$; it is close to the identity in $M(F_V)$; and it lies in $K_v$ for $v \notin S_0 \cup V$. We may assume that the product $\gamma_1\gamma_2 \in M(F)$ is also in $G_{\mathrm{oreg}}(F)$. By construction, $\gamma_1\gamma_2$ is close to $\gamma_{v_i}^2$ in $M(F_{v_i})$ for $i = 1, 2$; close to $\delta\gamma_0$ at the nonarchimedean valuations of $S_0$; and close to $\gamma_1$ in $M(F_V)$. Furthermore $\gamma_1\gamma_2$ lies in $K_v$ for valuations $v \notin S_0 \cup V$. Again, a judicious choice of $S$ and $\tilde{f}_1, \ldots, \tilde{f}_k$ in equation (63) implies that

$$0 = \sum_{v \in S_0 \cup V} \varepsilon_M(\gamma_1\gamma_2) = \sum_{v \in S_0} \varepsilon_M(\delta_v\gamma_v) + \sum_{v \in V} \varepsilon_M(\gamma_1).$$

Comparing this equation with equation (69), we find that (68) is true.

It is a simple exercise to show that any $\delta_0 = \prod_{v \in S_0} \delta_{0,v} \in M(F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0})$ can be written as a product $\delta\gamma_0$, where $\delta \in M_0(F_{S_0}) \cap G_{\mathrm{oreg}}(F_{S_0})$ and $\gamma_0$ is as above. As $M \supsetneq M_0$, equation (68) and Lemma 10.2 imply that

$$\sum_{v \in S_0} \varepsilon_M(\delta_{0,v}) = \sum_{v \in S_0} \varepsilon_M(\delta_v) = 0.$$

By equation (59) and Corollary 11.1 this constitutes a proof of Theorem A (i) for $M \supsetneq M_0$.

To take care of the case $M = M_0$, we must show that

$$\varepsilon_{M_0}(\gamma) = 0, \ \gamma \in M_0(F_v) \cap G_{\mathrm{oreg}}(F_v).$$

Bearing in mind the induction hypothesis of §18, we may argue as in the proof of Corollary 11.1 (cf. equation (33)) to conclude that

$$\int_{M_0(F_v)/\mu_n^M} \varepsilon_{M_0}(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}_1) I_{\tilde{G}}(\gamma', \tilde{f}_2) d\gamma = 0,$$

for any genuine Hecke function $\tilde{f}_1$ on $\tilde{G}(F_v)$ and $\tilde{f}_2 \in \mathcal{H}(\tilde{G}(F_v))$. Fix $\gamma_1 \in M_0(F_v) \cap G_{\mathrm{oreg}}(F_v)$. Since $\mathcal{H}(\tilde{G}(F_v))$ is dense in $C_c^\infty(\tilde{G}(F_v))$ we may let $\tilde{f}_2$

approach the antigenuine analogue of Dirac measure on $\tilde{G}(F_v)$ at $\gamma_1'$. The Weyl integration formula then implies that the function,

$$\tilde{\gamma} \mapsto |D^G(\mathbf{p}(\tilde{\gamma}))|^{1/2} I_{\tilde{G}}(\tilde{\gamma}, \tilde{f}_2), \ \tilde{\gamma} \in \tilde{M}_0(F_v),$$

approaches the antigenuine analogue of Dirac measure on $\tilde{M}_0(F_v)$ at $\gamma_1'$. It then follows from our equation that

$$\varepsilon_{M_0}(\gamma_1)|D^G(\gamma_1^n)|^{-1/2} I_{\tilde{G}}(\gamma_1', \tilde{f}_1) = 0,$$

and in turn that $\varepsilon_{M_0}(\gamma_1) = 0$. This concludes the proof of Theorem A (i).

All that remains to be done now is to prove Theorem A (ii) and Theorem B.

*Proof of Theorem A (ii).* We wish to show that

$$a^{\tilde{M}}(S, \gamma') = a^M(S, \gamma), \ \gamma \in M(F).$$

Suppose first that $\gamma \in M(F)$ has Jordan decomposition $\gamma = \sigma u$, where the semisimple element $\sigma$ is not in $A_G(F)$ if $M = G$. Then

$$\dim(M_\sigma) < \dim(G),$$

so we may apply the induction hypothesis of §10 to decompositions (28) and (29) and the lemma follows. On the other hand, if $M = G$ and $\sigma \in A_G(F)$ then

$$a^{\tilde{G}}(S, \gamma') = a^{\tilde{G}}(S, u')$$

and

$$a^G(S, \gamma) = a^G(S, u),$$

by (28) and (29) respectively. It follows from Theorem A (i) and Lemma 19.1, where we may now take $M = M_0$, that

$$\sum_{\sigma \in A_G(F)} \sum_{u \in (\mathcal{U}_G(F))} \left( a^{\tilde{G}}(S, u') - a^G(S, u) \right) I_G^{\mathcal{M}}(\sigma u, \tilde{f}) = 0,$$

for any $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$. We may choose $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ above so that for a fixed element $u_1 \in \mathcal{U}_G(F)$, we have

$$I_G^{\mathcal{M}}(\sigma u, \tilde{f}) = \begin{cases} 1, & \text{if } \sigma = 1 \text{ and } u = u_1 \\ 0, & \text{otherwise} \end{cases}$$

(§3.3 [Vig81]). This clearly implies that $a^{\tilde{G}}(S, u_1') = a^G(S, u_1)$.□

It has already been shown in §15 that Theorem B (i) follows from Theorem A (i). This leaves us with a single proof to be completed.

*Proof of Theorem B (ii).* By the induction hypothesis of §14, we need only show that

$$a_{\text{disc}}^{G,\Sigma}(\tilde{\pi}) = a_{\text{disc}}^{\tilde{G}}(\tilde{\pi}), \ \ \tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1).$$

Let $\nu_1$ be the infinitesimal character of the archimedean factor of some fixed $\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1)$, and let $K_1$ be a compact open subgroup of $\prod_{v \notin S_\infty} K_v$ such that $\tilde{\pi}$ is bi-$\mathbf{s}(K_1)$-invariant. Let $\Pi_{\nu_1, K_1}(\tilde{G}(\mathbf{A})^1)$ be the set of bi-$\mathbf{s}(K_1)$-invariant representations in $\Pi(\tilde{G}(\mathbf{A})^1)$ with infinitesimal character $\nu_1$. In the process of proving Proposition 19.1 (cf. (16.6) II [AC89]), one obtains

$$\sum_{\tilde{\pi} \in \Pi_{\nu_1, K_1}(\tilde{G}(\mathbf{A})^1)} \left( a_{\text{disc}}^{G,\Sigma}(\tilde{\pi}) - a_{\text{disc}}^{\tilde{G}}(\tilde{\pi}) \right) \text{tr} \left( \tilde{\pi}(\tilde{f}^1) \right) = 0,$$

for any $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A})^1)$ which is bi-$\mathbf{s}(K_1)$-invariant. This sum is finite by Lemma 4.2 [Art88b], and the linear forms,

$$\tilde{f} \mapsto \text{tr} \left( \tilde{\pi}(\tilde{f}^1) \right), \ \ \tilde{\pi} \in \Pi_{\nu_1, K_1}(\tilde{G}(\mathbf{A})^1),$$

on the space of bi-$\mathbf{s}(K_1)$-invariant functions in $\mathcal{H}(\tilde{G}(\mathbf{A}))$ are linearly independent. The result follows.□

# 21 A global correspondence

We derive a global correspondence from a special case of Theorem B (ii). The global datum $a_{\text{disc}}^{\tilde{G}}(\tilde{\pi})$ is defined by the equation (cf. (4.3) [Art88b], (9.2) II [AC89]),

$$\sum_{\tilde{\pi} \in \Pi(\tilde{G}(\mathbf{A})^1, t)} a_{\text{disc}}^{\tilde{G}}(\tilde{\pi}) I_{\tilde{G}}(\tilde{\pi}, \tilde{f})$$

$$(70)= \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{s \in W(\mathfrak{a}_M)_{\text{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \text{tr} \left( \tilde{M}(s, 0) \rho_{\tilde{Q}, t}(0, \tilde{f}) \right).$$

Here, $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$, $Q$ is any element of $\mathcal{P}(M)$ and $\rho_{\tilde{Q},t}(0)$ is the induced representation of $\tilde{G}(\mathbf{A})^1$ obtained from the genuine subrepresentation of $\tilde{M}(\mathbf{A})^1$

on $L^2(\mathbf{s_0}(M(F))\backslash\tilde{M}(\mathbf{A})^1)$ which decomposes into a discrete sum of elements in $\Pi(\tilde{M}(\mathbf{A})^1,t)$. The term $\tilde{M}(s,0)$ is the global intertwining operator associated to an element in

$$W(\mathfrak{a}_M)_{\mathrm{reg}} = \{s \in W(\mathfrak{a}_M) : \det(s-1)_{\mathfrak{a}_M^G} \neq 0\}.$$

By Theorem B (ii) and the definitions of §§13-15, the right-hand side of (70) is equal to

$$\begin{aligned}
&\sum_{\tilde{\pi}\in\Pi(\tilde{G}(\mathbf{A})^1,t)} a_{\mathrm{disc}}^{G,\Sigma}(\tilde{\pi}) I_G^{\Sigma}(\tilde{\pi},\tilde{f}) \\
&= \sum_{\pi\in\Pi(G(\mathbf{A})^1,nt)} a_{\mathrm{disc}}^G(\pi) n^{-1}\hat{I}_G(\pi,\tilde{f}') \\
&= n^{-1}\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s\in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1}\mathrm{tr}\left(M(s,0)\rho_{Q,nt}(0,f)\right),
\end{aligned}$$

where $f \in \mathcal{H}(G(\mathbf{A}))$ is a function such that $f_G = \tilde{f}'$. We would like to convert this identity of representations on $\tilde{G}(\mathbf{A})^1$ and $G(\mathbf{A})^1$ to an identity of representations on $\tilde{G}(\mathbf{A})$ and $G(\mathbf{A})$. We may embed the connected component $A_G(\mathbf{R})^0$ of $A_G(\mathbf{R})$ diagonally into $\prod_{v\in S_\infty} A_G(F_v)$. The map,

$$H_M : A_G(\mathbf{R})^0 \to \mathfrak{a}_G,$$

is an isomorphism which allows us to pull back the Haar measure on $\mathfrak{a}_G$ to $A_G(\mathbf{R})^0$. Given $\lambda \in i\mathfrak{a}_G^*$, we define

$$I_{\mathrm{disc},t,\lambda}(\tilde{f}) = \int_{A_G(\mathbf{R})^0} \sum_{\tilde{\pi}\in\Pi(\tilde{G}(\mathbf{A})^1,t)} a_{\mathrm{disc}}^G(\tilde{\pi}) I_{\tilde{G}}(\tilde{\pi},\tilde{f}_{a'}) e^{\lambda(H_G(\mathbf{p}(a')))}da',$$

where

$$\tilde{f}_{a'}(\tilde{\gamma}) = \tilde{f}(a'\tilde{\gamma}),\ \tilde{\gamma}\in\tilde{G}(\mathbf{A}),\ a \in A_G(\mathbf{R})^0.$$

Since $\tilde{f}'_{a'} = \tilde{f}'_a$, $\lambda(H_G(\mathbf{p}(a'))) = \lambda'(H_G(a))$ and $da' \circ \mathbf{s} = n da$ our earlier identity becomes

(71) $$I_{\mathrm{disc},t,\lambda}(\tilde{f}) = I_{\mathrm{disc},nt,\lambda'}(f).$$

This identity may be rewritten as

$$\begin{aligned}
&\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s\in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1}\mathrm{tr}\left(\tilde{M}(s,0)\rho_{\tilde{Q},t,\lambda}(0,\tilde{f})\right) \\
&= \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{s\in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1}\mathrm{tr}\left(M(s,0)\rho_{Q,nt,\lambda'}(0,f)\right).
\end{aligned}$$

Here

$$\rho_{\tilde{Q},t,\lambda}(0) = \bigoplus_{\tilde{\pi}\in\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)} \mathrm{Ind}_{\tilde{Q}}^{\tilde{G}}\tilde{\pi},$$

where $\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)$ is the set of irreducible, $\lambda$-equivariant and genuine subrepresentations of $\tilde{M}(\mathbf{A})$ on $L^2(\mathbf{s_0}(M(F))\backslash\tilde{M}(\mathbf{A}))$ whose infinitesimal characters' imaginary parts have norm equal to $t\geq 0$. Since $\tilde{M}(s,0)$ intertwines $\rho_{\tilde{Q},t,\lambda}(0)$ with itself, Schur's lemma implies that there exist complex numbers $c_{s,\tilde{\pi}}$ such that

$$\mathrm{tr}\left(\tilde{M}(s,0)\rho_{\tilde{Q},t,\lambda}(0,\tilde{f})\right) = \sum_{\tilde{\pi}\in\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)} c_{s,\tilde{\pi}}\mathrm{tr}\left((\mathrm{Ind}_{\tilde{Q}}^{\tilde{G}}\tilde{\pi})(\tilde{f})\right).$$

Set

$$c_{\tilde{\pi}} = |W_0^M||W_0^G|^{-1}\sum_{s\in W(\mathfrak{a}_M)_{\mathrm{reg}}}|\det(s-1)_{\mathfrak{a}_M^G}|^{-1}c_{s,\tilde{\pi}},\ \tilde{\pi}\in\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda).$$

Identity (71) now has the form

$$(72)\sum_{M\in\mathcal{L}}\sum_{\tilde{\pi}\in\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)} c_{\tilde{\pi}}\mathrm{tr}\left(\tilde{\pi}^{\tilde{G}}(\tilde{f})\right) - \sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbf{A}),nt,\lambda')} c_{\pi}\mathrm{tr}\left(\pi^{G}(f)\right) = 0.$$

Suppose $\tilde{\pi}=\otimes_v\tilde{\pi}_v\in\Pi(\tilde{G}(\mathbf{A}))$ and $\pi=\otimes_v\pi_v\in\Pi(G(\mathbf{A}))$. We say that $\tilde{\pi}$ corresponds (or lifts) weakly to $\pi$ if

$$\mathrm{tr}\left(\tilde{\pi}_v(\tilde{f}_v)\right) = \mathrm{tr}\left(\pi_v(f_v)\right), \tag{73}$$

for any matching functions $\tilde{f}_v\in\mathcal{H}(\tilde{G}(F_v))$, $f_v\in\mathcal{H}(G(F_v))$ and almost every valuation $v$ of $F$ (§3). This relation is referred to as a *quasi-lift* in §28 [FK86]. If (73) holds for all valuations $v$ then we say that $\tilde{\pi}$ corresponds (or lifts) to $\pi$. We say that $\pi$ is metic if $\pi_v$ is metic (§2) for all valuations $v$ and that $f=\prod_v f_v\in\mathcal{H}(G(\mathbf{A}))$ matches $\tilde{f}=\prod_v\tilde{f}_v\in\mathcal{H}(\tilde{G}(\mathbf{A}))$ if $f_v$ matches $\tilde{f}_v$ for all valuations $v$.

**Theorem 21.1** *(i) Suppose $t\geq 0$, $\lambda\in i\mathfrak{a}_G^*$ and $\tilde{\pi}_0=\otimes_v\tilde{\pi}_{0,v}$ belongs to $\Pi_{\mathrm{disc}}(\tilde{G}(\mathbf{A}),t,\lambda)$. Then there exist a unique Levi subgroup $L\in\mathcal{L}$ and a unique representation $\pi_0\in\Pi_{\mathrm{disc}}(L(\mathbf{A}),nt,\lambda')$ such that $\pi_0$ is metic and $\tilde{\pi}_0$ corresponds weakly to $\pi_0^G$.*

*(ii) Suppose $t \geq 0$, $\lambda \in i\mathfrak{a}_G^*$ and $\pi_0 = \pi_{0,v} \in \Pi_{\mathrm{disc}}(G(\mathbf{A}), nt, \lambda')$ is metic. Then there exists $\tilde{\pi}_0 \in \Pi_{\mathrm{disc}}(\tilde{G}(\mathbf{A}), t, \lambda)$ such that $\tilde{\pi}_0$ corresponds weakly to $\pi_0$. Moreover, if $\tilde{\pi} \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ and $\tilde{\pi}^{\tilde{G}}$ corresponds weakly to $\pi_0$ then $M = G$.*

*(iii) Suppose the representation $\pi_0$ of (ii) is cuspidal. Then there exists a unique representation $\tilde{\pi}_0 \in \Pi_{\mathrm{disc}}(\tilde{G}(\mathbf{A}), t, \lambda)$ which corresponds to $\pi_0$. Moreover, if $\tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ and $\tilde{\pi}_1^{\tilde{G}}$ corresponds weakly to $\pi_0$ then $\tilde{\pi}_1 = \tilde{\pi}_0$.*

*Proof of (i).* We assume inductively that (i) holds if $G$ is replaced by a proper Levi subgroup in $\mathcal{L}$. Let $V$ be the finite set of valuations at which $\tilde{\pi}_0$ is ramified. Suppose $w \notin V$. By §§11, 16, 17 [FK86], we can assign to each unramified representation in $\Pi(\tilde{G}(F_w))$ a Satake parameter. Recall that a Satake parameter corresponds to an element of $(\mathbf{C}^r)^{S_r}$, the set of equivalence classes of $\mathbf{C}^r$ under permutation. Given a valuation $v$ such that $|n|_v = 1$, let $\mathcal{H}(\tilde{G}(F_v), K_v)$ be the subset of $\mathcal{H}(\tilde{G}(F_v))$ of $\mathbf{s}(K_v)$ bi-invariant functions. Suppose $\tilde{f} = \prod_v \tilde{f}_v \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ such that $\tilde{f}_w \in \mathcal{H}(\tilde{G}(F_w), K_w)$, $f = \prod_v f_v \in \mathcal{H}(G_v)$ matches $\tilde{f}$, and $t_w \in (\mathbf{C}^r)^{S_r}$. Set

$$a(t_w) = \sum_{M \in \mathcal{L}} \left( \sum_{\tilde{\pi}} c_{\tilde{\pi}} \prod_{v \neq w} \mathrm{tr}\left(\tilde{\pi}_v^{\tilde{G}}(\tilde{f}_v)\right) - \sum_{\pi} c_{\pi} \prod_{v \neq w} \mathrm{tr}\left(\pi_v^{G}(f_v)\right) \right),$$

where the first sum is parameterized by the representations $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ belonging to $\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ such that $\tilde{\pi}_w$ has Satake parameter $t_w$ and the second sum is parameterized by the representations $\pi = \otimes_v \tilde{\pi}_v$ belonging to $\Pi_{\mathrm{disc}}(M(\mathbf{A}), nt, \lambda')$ such that $\pi_w$ has Satake parameter $nt_w$. Equation (72) implies that

$$\sum_{t_w \in (\mathbf{C}^r)^{S_r} - \{0\}} a(t_w) \tilde{f}_w^{\vee}(t_w) = 0,$$

where $\tilde{f}_w^{\vee}$ is the Satake transform (§11 [FK86]) of $\tilde{f}_w$. It follows from the Satake isomorphism that

$$a(t_w) = 0, \ t_w \in (\mathbf{C}^r)^{S_r} - \{0\}.$$

Suppose $t_{\tilde{\pi}_0,w}$ is the Satake parameter of $\pi_{0,w}$. We can combine the earlier argument with an induction argument on the number of valuations outside of $V$ to conclude that

$$(74) \sum_{M \in \mathcal{L}} \sum_{\tilde{\pi} \in \Pi(\tilde{M}, \tilde{\pi}_0, V)} c_{\tilde{\pi}} \prod_{v \in V} \mathrm{tr}\left(\tilde{\pi}_v^{\tilde{G}}(\tilde{f}_v)\right) = \sum_{M \in \mathcal{L}} \sum_{\pi \in \Pi(M, \tilde{\pi}_0, V)} c_{\pi} \prod_{v \in V} \mathrm{tr}\left(\pi_v^{G}(f_v)\right).$$

Here, $\Pi(\tilde{M}, \tilde{\pi}_0, V)$ is the set of representations

$$\tilde{\pi} = \otimes_v \tilde{\pi}_v \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$$

which satisfy

$$\mathrm{tr}\left(\tilde{\pi}_w^{\tilde{G}}(\tilde{h})\right) = \tilde{h}^\vee(t_{\tilde{\pi}_0,w}),\ \tilde{h} \in \mathcal{H}(\tilde{G}(F_w), K_w),\ w \notin V,$$

and $\Pi(M, \tilde{\pi}_0, V)$ is the set of representations

$$\pi = \otimes_v \pi_v \in \Pi_{\mathrm{disc}}(M(\mathbf{A}), nt, \lambda')$$

which satisfy

$$\mathrm{tr}\left(\pi_w^G(h)\right) = h^\vee(nt_{\tilde{\pi}_0,w}),\ h \in \mathcal{H}(G(F_w), K_w),\ w \notin V.$$

Of course, $\tilde{\pi}_0$ is an element of set $\Pi(\tilde{G}, \tilde{\pi}_0, V)$ and $c_{\tilde{\pi}_0} = 1$. If $\Pi(\tilde{M}, \tilde{\pi}_0, V)$ is empty for every $M \neq G$ then Lemma 16.1.1 [JL70], applied to equation (74), implies that $\Pi(M, \tilde{\pi}_0, V)$ is not empty for some $M \in \mathcal{L}$. If $\Pi(\tilde{M}, \tilde{\pi}_0, V)$ is not empty for some $M \neq G$ then our induction assumption implies that $\Pi(M, \tilde{\pi}, V_1)$ is not empty for some finite set $V_1 \supset V$. In any case, we have shown the existence of $\pi_0$ as in the theorem. The uniqueness of this representation follows from the strong multiplicity one property for the cuspidal representations of general linear groups (Theorem 4.4 [JS81]) and the construction of the discrete spectrum of general linear groups in terms of the cuspidal representations ([MW89]).
*Proof of (ii).* Suppose $V$ is the finite set of valuations at which $\pi_0$ is ramified. Given $w \notin V$, set $t_{\pi_0,w}$ to be the Satake parameter of $\pi_{0,w}$. Given a finite set $V_1 \supset V$, define $\Pi(\tilde{M}, \pi_0, V_1)$ to be the set of representations

$$\tilde{\pi} = \otimes_v \tilde{\pi}_v \in \Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$$

which satisfy

$$\mathrm{tr}\left(\tilde{\pi}_w^{\tilde{G}}(\tilde{h})\right) = \tilde{h}^\vee(nt_{\pi_0,w}),\ \tilde{h} \in \mathcal{H}(\tilde{G}(F_w), K_w),\ w \notin V.$$

Arguing as in (i) and applying the strong multiplicity property for general linear groups, we obtain

$$\prod_{v \in V_1} \mathrm{tr}\left(\pi_{0,v}^G(f_v)\right) = \sum_{M \in \mathcal{L}} \sum_{\tilde{\pi} \in \Pi(\tilde{M}, \pi_0, V_1)} c_{\tilde{\pi}} \prod_{v \in V_1} \mathrm{tr}\left(\tilde{\pi}_v^{\tilde{G}}(\tilde{f}_v)\right). \tag{75}$$

Since the left-hand side is not zero there exists a representation $\tilde{\pi}_0$ belonging to $\Pi_{\text{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ such that $\tilde{\pi}^{\tilde{G}}$ corresponds weakly to $\pi_0$. Furthermore, any $\tilde{\pi} \in \Pi_{\text{disc}}(\tilde{M}(\mathbf{A}), t, \lambda)$ such that $\tilde{\pi}^{\tilde{G}}$ corresponds weakly to $\pi_0$ occurs as a summand on the right-hand side for some $V_1 \supset V$. By (i) (with $G$ replaced by $M$) there exist $M_1 \in \mathcal{L}^M$ and $\pi_1 \in \Pi_{\text{disc}}(M_1, nt, \lambda')$ such that $\tilde{\pi}_0$ corresponds weakly to $\pi_1^M$. In particular, $\pi_{1,v}$ and $\pi_{0,v}$ have the same Satake parameters at almost every valuation $v$. By the strong multiplicity one property mentioned earlier, it follows that $\pi_1 = \pi_0$ and in turn that $M_1 = M = G$.
*Proof of (iii).* According to Theorem 27.3 [FK86], for every valuation $v$ of $F$, there exists a unitary representation $\tilde{\pi}_{0,v} \in \Pi(\tilde{G}(F_v))$ such that

$$\operatorname{tr}\left(\tilde{\pi}_{0,v}(\tilde{f}_v)\right) = \operatorname{tr}\left(\pi_{0,v}(f_v)\right),$$

for any matching functions $\tilde{f}_v \in \mathcal{H}(\tilde{G}(F_v))$ and $f_v \in \mathcal{H}(G(F_v))$. Together with the second assertion of (ii) this allows us to rewrite equation (75) as

$$\prod_{v \in V_1} \operatorname{tr}\left(\tilde{\pi}_{0,v}(\tilde{f})\right) = \sum_{\tilde{\pi} \in \Pi(\tilde{G}, \pi_0, V_1)} \prod_{v \in V_1} \operatorname{tr}\left(\tilde{\pi}_v(\tilde{f})\right).$$

It follows from Lemma 16.1.1, applied to this equation, that there exists a unique representation $\tilde{\pi}_0 \in \Pi(\tilde{G}, \pi_0, V_1)$ which corresponds to $\pi_0$. The second assertion of (iii) follows from the second assertion of (ii) and the fact that the finite set $V_1 \supset V$ is arbitrary.□

Theorem 21.1 (iii) yields the multiplicity one and strong multiplicity one property for the representations of $\tilde{G}(\mathbf{A})$ which lift to cuspidal representations of $G(\mathbf{A})$. We expect the same properties to hold for representations of $\tilde{G}(\mathbf{A})$ which lift to any representation in the discrete spectrum of $G(\mathbf{A})$. One could attempt to prove this by showing that the local metaplectic correspondence preserves character identities for the local factors of representations in the residual spectrum of $G(\mathbf{A})$ and then argue as in Theorem 21.1 (iii). Let us describe the residual spectrum of $G(\mathbf{A})$ in more detail so that we can state these character identities as a precise conjecture.

Recall decomposition (4),

$$M = M(1) \times \cdots \times M(\ell) \cong \mathrm{GL}(r_1) \times \cdots \mathrm{GL}(r_\ell).$$

Suppose that $r_1 = \cdots = r_\ell$, so that $r = \ell r_1$. Suppose further that $\pi = \otimes_v \pi_v$ is a unitary, cuspidal and metic representation of $\mathrm{GL}(r_1, \mathbf{A})$. Let

$\otimes_{i=1}^{\ell}\pi|\det(\cdot)|^{(\ell(2i-1))/2}$ be the representation of $M(\mathbf{A})$ given by

$$\begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_\ell \end{pmatrix} \mapsto \prod_{i=1}^{\ell} \pi(\gamma_i)|\det(\gamma_i)|^{\frac{\ell-(2i-1)}{2}},\ \gamma_1,\ldots,\gamma_\ell \in \mathrm{GL}(r_1,\mathbf{A}),$$

where $|\cdot| = \otimes_v|\cdot|_v$, and let $P \in \mathcal{P}(M)$ be the unique parabolic subgroup containing the group of upper-triangular matrices. The induced representation $\mathrm{Ind}_P^G\left(\otimes_{i=1}^{\ell}\pi|\det(\cdot)|^{(\ell-(2i-1))/2}\right)$ has a unique irreducible quotient which belongs to the discrete spectrum of $G(\mathbf{A})$ (§2 [Jac84]). Conversely, any discrete representation of $G(\mathrm{A})$ is of the above form ([MW89]).

**Conjecture 1** *Suppose that $v$ is a nonarchimedean valuation of $F$ and $\pi_v \in \Pi(\mathrm{GL}(r_1,F_v))$ and $P \in \mathcal{P}(M)$ are as above. Suppose further that $\pi_v$ is metic and that $\pi_1$ is the unique irreducible quotient of $\mathrm{Ind}_P^G\left(\pi_v|\det(\cdot)|_v^{(\ell-(2i-1))/2}\right)$. Then there exists $\tilde{\pi}_v \in \Pi(\widetilde{\mathrm{GL}}(r_1,F_v))$ such that*

$$\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}\left(\tilde{\pi}_v|\det\circ\mathbf{p}(\cdot)|_v^{\frac{\ell-(2i-1)}{2}}\right)$$

*has a unique irreducible quotient $\tilde{\pi}_1$ which satisfies*

$$\mathrm{tr}\left(\tilde{\pi}_1(\tilde{f})\right) = \mathrm{tr}\left(\pi_1(f)\right)$$

*for any matching functions $\tilde{f} \in \mathcal{H}(\tilde{G}(F_v))$ and $f \in \mathcal{H}(G(F_v))$.*

This conjecture can be shown to hold in the case that $\pi_v$ is supercuspidal by using an argument in the proof of Theorem 29.1 [FK86]. Conjecture 1 would follow from a character identity of unitary representations if one could generalize the work of Bernstein, Tadić and Zelevinsky ([BZ77], [Zel80], [Tad95]) to $\tilde{G}(F_v)$ (see also [Hua95]).

Taking the case $r = 2$ into consideration ([Fli80]), we make a second conjecture.

**Conjecture 2** *Suppose $t \geq 0$, $\lambda \in i\mathfrak{a}_G^*$ and $\tilde{\pi}$ belongs to $\Pi_{\mathrm{disc}}(\tilde{G}(\mathbf{A}),t,\lambda)$. Then there exists a unique metic representation $\pi \in \Pi_{\mathrm{disc}}(G(\mathbf{A}),nt,\lambda')$ such that $\tilde{\pi}$ corresponds to $\pi$. Moreover if $\tilde{\pi}_1$ is a representation in $\Pi_{\mathrm{disc}}(\tilde{M}(\mathbf{A}),t,\lambda)$ such that $\tilde{\pi}_1^{\tilde{G}}$ corresponds weakly to $\pi$ then $\tilde{\pi}_1 = \tilde{\pi}$. This correspondence maps cuspidal representations to cuspidal representations and residual representations to residual representations.*

This conjecture holds true given the general strong multiplicity one property, which follows from Conjecture 1, and given a characterization of the residual spectrum of $\tilde{G}(\mathbf{A})$ as in [MW89]. We refer the reader to [Sun] for a discussion of the residual spectrum of $\tilde{G}(\mathbf{A})$. The final assertion of Conjecture 2 is false if **(†)** is not assumed. This can be seen in [Fli80] and §29 [FK86].

# 22 Appendix. Tensor products of metaplectic representations

Suppose $v$ is a nonarchimedean valuation of $F$. In §26.2 [FK86] a method of induction from parabolic subgroups of $\tilde{G}(F_v)$ is delineated. Recall decomposition (4),

$$M = M(i) \times \cdots \times M(\ell).$$

This method of induction relates tensor products of representations of $\tilde{M}(i)(F_v)$ to representations of $\tilde{M}(F_v)$. We describe this relationship and prove all of the claims made in §26.2 [FK86] concerning it under assumption **(†)**. As mentioned earlier, it seems that the claims are not true in general ([Sun97]). For fixed $M$ assumption **(†)** is stronger than needed.

Let $(\cdot,\cdot)_{F_v} : F_v^\times \times F_v^\times \to \mu_n$ be the $n$th Hilbert symbol on $F_v$ and let $B$ be a maximal subgroup of $F_v^\times$ with respect to the property that $(x_1, x_2)_{F_v} = 1$ for all $x_1, x_2 \in B$. For $1 \le i \le \ell$, set

$$\tilde{M}^B(i)(F_v) = \{\tilde{\gamma} \in \tilde{M}(i)(F_v) : \det(\mathbf{p}(\tilde{\gamma})) \in B\}.$$

It is a simple matter to check that $\tilde{M}^B(i)(F_v)$ is a normal subgroup of finite index in $\tilde{M}(i)(F_v)$. Let $\tilde{\pi}_i$ be a genuine irreducible admissible representation of $\tilde{M}(i)(F_v)$. The restriction of $\tilde{\pi}_i$ to $\tilde{M}^B(i)(F_v)$ is the sum of conjugates of some irreducible representation $\tilde{\rho}_i$ of $\tilde{M}^B(i)(F_v)$. More precisely,

$$\tilde{\pi}_i|_{\tilde{M}^B(i)} = \sum_\gamma \tilde{\rho}_i^\gamma,$$

where the sum runs over representatives $\gamma$ of cosets in $\tilde{M}(i)(F_v)/\tilde{M}^B(i)(F_v)$ and

$$\tilde{\rho}_i^\gamma(\gamma_1) = \tilde{\rho}_i(\gamma\gamma_1\gamma^{-1}),\ \gamma_1 \in \tilde{M}^B(i)(F_v).$$

**Lemma 22.1** *If $\gamma$ is as above then $\tilde{\rho}_i$ is not equivalent to $\tilde{\rho}_i^\gamma$ unless $M(i) \cong$ GL(1) or $\gamma \in \tilde{M}^B(i)(F_v)$.*

*Proof.* If $M(i) = \mathrm{GL}(1)$ then $\tilde{M}(i)(F_v) \cong F_v^\times \times \mu_n$. In particular $\tilde{M}(i)(F_v)$ is abelian and $\tilde{\rho}_i = \tilde{\rho}_i^\gamma$. Suppose that $M(i)$ is not isomorphic to GL(1). By using the Iwasawa decomposition, it is easy to see that representatives of $\tilde{M}(i)(F_v)/\tilde{M}^B(i)(F_v)$ may be taken to be diagonal matrices. Let $\gamma$ be such a representative corresponding to the diagonal element

$$\begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_{r_i} \end{pmatrix} \in \mathrm{GL}(r_i, F_v),$$

and suppose that $\tilde{\rho}_i^\gamma$ is equivalent to $\tilde{\rho}_i$. In other words, suppose that there exists a linear isomorphism $T$ such that

$$T \circ \tilde{\rho}_i^\gamma(\tilde{\gamma}) = \tilde{\rho}_i'(\tilde{\gamma}) \circ T, \ \ \tilde{\gamma} \in \tilde{M}^B(i)(F_v).$$

Suppose $x \in B$ and choose $\tilde{\gamma} \in \tilde{M}^B(i)(F_v)$ such that $\mathbf{p}(\tilde{\gamma})$ corresponds to the scalar matrix

$$\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \in \mathrm{GL}(r_i, F_v).$$

By Proposition 0.1.5 [KP84] and the bilinearity of the Hilbert symbol, we have

$$\begin{aligned} \tilde{\rho}_i(\tilde{\gamma}) &= T \circ \tilde{\rho}_i^\gamma(\tilde{\gamma}) \circ T^{-1} \\ &= T \circ \tilde{\rho}_i(\gamma\tilde{\gamma}\gamma^{-1}) \circ T^{-1} \\ &= \left( (\det(\gamma), \det(\mathbf{p}(\tilde{\gamma})))_{F_v}^{1+2m} / \prod_{j=1}^{r_i} (\gamma_i, x)_{F_v} \right) T \circ \tilde{\rho}_i(\tilde{\gamma}) \circ T^{-1}. \end{aligned}$$

It may be verified by following 0.1.1 [KP84] that $\tilde{\gamma}$ is in the center of $\tilde{M}^B(i)(F_v)$ and so, by Schur's lemma, $\tilde{\rho}_i(\tilde{\gamma})$ is a nonzero scalar operator. Consequently the above identity reduces to

$$(\det(\gamma), x)_{F_v}^{r_i(1+2m)-1} = 1.$$

Since $\gcd(n, r_i(1+2m)-1) = 1$ under assumption **(†)**, we have $(\det(\gamma), x)_{F_v} = 1$. The element $x \in B$ was chosen arbitrarily so this means that $\gamma \in \tilde{M}^B(i)(F_v)$.□

Continuing with the discussion on tensor products, we set $\tilde{\rho} = \otimes_{i=1}^{\ell} \tilde{\rho}_i$. This representation passes to an irreducible representation of the subgroup

$$\tilde{M}^B(F_v) = \{\tilde{\gamma} \in \tilde{M}(F_v) : \det(\mathbf{p}(\tilde{\gamma})) \in B\}.$$

If $M \neq M_0$ then Lemma 22.1 implies that $\tilde{\rho}$ is inequivalent to any of its conjugates by elements in $\tilde{M}(F_v) - \tilde{M}^B(F_v)$. If $M = M_0$, we obtain the same result by Proposition 3 [FK86]. Applying Mackey's criterion, we find that the representation of $\tilde{M}(F_v)$ induced from $\tilde{\rho}$ is irreducible. This process may be reversed without difficulty. Hence, every genuine irreducible admissible representation of $\tilde{M}(F_v)$ corresponds to a unique set of genuine irreducible admissible representations of $\tilde{M}(i)(F_v)$, for $1 \leq i \leq \ell$.

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