Stability of L^p- spectrum of generalized Schrödinger operators and equivalence of Green's functions

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§ 1

Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open (unbounded) set and let $d \cdot a \cdot d$ be a differential expression, where $a(\cdot)$ is a locally integrable function on Ω with values in the strictly positive real symmetric matrices.

We consider at least three realizations of $-d \cdot a \cdot d$ in $L^2(\Omega)$: A_D, A_i, A_n - the Dirichlet, intermediate Dirichlet and generalized Neumann symmetric Markov generators. It follows from the Beurling-Deny criterion that there exist positivity preserving contraction consistent semigroups on $L^p(\Omega), 1 \leq p < \infty$, with generators $-A_p$ such that $A_2 = A$, where A denotes one of the operators A_D, A_i or A_N.

We shall prove the spectral p-independence of A for all $p \in [1, \infty[$ under the following assumptions on $a(\cdot)$:

$$
\begin{aligned}
&a(\cdot) \in L^1(\Omega_R) \text{ for some } R < \infty, \\
&a(x)(1+x^2)^{-1} \ln^\nu(1+x^2) \in L^\infty(\Omega \setminus \Omega_R) \text{ for some } \nu > 0,
\end{aligned}
$$

where $\Omega_R =: \{x \in \Omega : |x| \leq R\}$.

In the course of proof we show that A_D is local and that $C_0^1(\Omega)$ is a form core of $A_D + V$, assuming only that $a(\cdot)$ and $0 \leq V$ belong to $L^1_{\text{loc}}(\Omega)$.

Next, we consider the generalized Schrödinger operator $\Lambda = A + V, V = V_+ - V_-, V_\pm \in L^1_{\text{loc}}(\Omega)$ with the form small negative part V_-:

$$
V_- \leq \beta A + V_+ + c(\beta) \text{ for some } 0 < \beta < 1 \text{ and } c(\beta) \in \mathbb{R}^1.
$$

Now $-\Lambda_p$ can be defined as a generator of a strongly continuous consistent semigroup in $L^p(\Omega)$ only for $p_0 \leq p \leq p_0'$ with appropriate $1 < p_0 < 2$. We shall prove that for all $z \in \varrho(\Lambda_2)$ the resolvent $(z - \Lambda_2)^{-1}$ can be extended by continuity to a bounded map on $L^p(\Omega)$ for all $p \in]p(\beta), p'(\beta)[$ where $p'(\beta) =: \frac{2}{1-\sqrt{1-\beta}} \cdot \frac{d}{d-2}$, $d \geq 3$ and $p(\beta) =: (p'(\beta))'$.

If $\|e^{-t\Lambda_2} f\|_{p_0} \leq M e^{\omega t} \|f\|_{p_0}, f \in L^2 \cap L^{p_0}(\Omega)$ for some $p_0 \in]p(\beta), 2[$ (so that Λ_p is well-defined for all $p \in [p_0, p_0']$) then $\varrho(\Lambda_2) = \varrho(\Lambda_p)$ for all $p \in [p_0, p_0']$. In particular, we shall see that this is always the case for $p_0 = t(\beta) =: \frac{2}{1+\sqrt{1-\beta}}$. For the Schrödinger operator $\Lambda = -\Delta + V$ in $L^2(\mathbb{R}^d)$ we obtain the equality $\varrho(\Lambda_2) = \varrho(\Lambda_p)$ for all $p \in]p(\beta), p'(\beta)[$ if, in addition, $V_- = V_1^- + V_2^-, V_1^- \in K_d, V_2^- \in L^{d/2,\infty}(\mathbb{R}^d)$, $d \geq 3$ where

$$
K_d = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^d) : \lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \varepsilon} |x-y|^{2-d} |f(y)| \, dy = 0 \right\}
$$

is the Kato class and $L^{q,\infty}$ is a weak L^q space.

We should emphasize that there are fairly simple examples of potentials $V = -V_2^- \in L^{d/2,\infty}(\mathbb{R}^d)$ for which $\Delta - V$ cannot be defined as a generator of a strongly continuous semigroup on $L^p(\mathbb{R}^d)$ if $p \le p(\beta)$ or $p \ge p'(\beta)$. Therefore, $]p(\beta), p'(\beta)[$ is the maximal interval of "bounded solvability" for $-\Delta + V$, and in this sense the very last statement on p-independence of $\sigma(\Lambda_p) = \mathbb{C} \setminus \varrho(\Lambda_p)$ cannot be improved.

The stability of the L^p-spectrum has been studied in [HV1-3], [Sh], [St1], [ScV], [Are], [D2]. The present work is based on ideas developed in [Are], [ScV] and [Se].

The problem of the equivalence of the Green functions G_A of A^{-1} and G_Λ of Λ^{-1} was discussed by many authors (see [Pi], [Ra], [Zh] and papers quoted there). Our treatment of the problem rests on applying the fact that the spectrum of Λ_p is independent of p for a wide class of coefficients and that the spectral bound of $-\Lambda_p$ and the growth bound of $e^{-t\Lambda_p}$ coinside ([Na], [W1]). Our approach leads to general and, more importantly, to natural for unbounded Ω conditions on V. In particular, the following will be proven. Let $\Omega = \mathbb{R}^d$ and let $\Lambda = -\Delta + V$.

If $V \in K_d, \|(-\Delta)^{-1}V_+\|_\infty < \infty$ and $-\beta\Delta + V \ge 0$ for some $0 < \beta < 1$ then there exists a constant $0 < c < 1$ such that for all $x, y \in \mathbb{R}^d$

$$cG_0(x,y) \le G_\Lambda(x,y) \le c^{-1}G_0(x,y)$$

where $G_0(x,y) = c_d|x-y|^{2-d}, d \ge 3$.

It would be mentioned that we do not impose any "optimal" decay assumptions on V except for "$\|(-\Delta)^{-1}V_+\|_\infty < \infty$". The latter is a necessary condition for $V = V_+$.

§ 2

Construction and properties of "free" Markov generators

Let $\Omega \subset \mathbb{R}^d$ be an open set and let $a : \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d$ be a measurable, symmetric matrix-valued function which satisfies the ellipticity condition

$$I \leq a(\cdot) \leq a_v(\cdot) I \quad \text{a. e. for some } a_v : \Omega \to \mathbb{R}^1_+$$

in the sense of non-negative definite matrices. Set

$$du \cdot a \cdot dv =: \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_j}, \quad \langle f \rangle =: \int_\Omega f \, dx.$$

We will be assuming that $a_{ij}, a_v \in L^1_{\text{loc}}(\Omega)$.

Let us consider the family $\mathcal{T}$ of all closed, symmetric non-negative quadratic forms in $L^2(\Omega)$. As a general reference we use [K1, Chapters VI, VIII]. If $\tau \in \mathcal{T}$, then there exists the unique self-adjoint operator $T \geq 0$ such that

$$\begin{aligned} \tau[u, v] &= \langle T^{1/2} u, T^{1/2} v \rangle, \quad \mathcal{D}(\tau) = Q(T) \times Q(T), \\ \tau[u, v] &= \langle T u, v \rangle, \quad u \in \mathcal{D}(T), v \in Q(T) =: \mathcal{D}(T^{1/2}). \end{aligned}$$

In this case we shall write $T \leftrightarrow \tau$.

Let $\tau_1, \tau_2 \in \mathcal{T}$ and assume that $D(\tau_1 + \tau_2) =: \mathcal{D}(\tau_1) \cap \mathcal{D}(\tau_2)$ is dense in $L^2(\Omega)$, then $\tau_1 + \tau_2 \in \mathcal{T}$. If $T_\nu \leftrightarrow \tau_\nu$, $\nu = 1, 2$ and if $T \leftrightarrow \tau_1 + \tau_2$, then T is called the form sum of T_1 and T_2 and denoted by $T_1 \dot{+} T_2$.

Let P^t be a C_0-semigroup on $L^2(\Omega)$. We say that it is a Markov semigroup if for all $t > 0$

$$0 \leq P^t u \leq 1 \quad \text{a. e. whenever } u \in L^2(\Omega), 0 \leq u \leq 1 \text{ a. e.}$$

We define

$$\mathcal{T}_M = \{\tau \in \mathcal{T} : e^{-tT} \text{ is a symmetric Markov semigroup}, T \leftrightarrow \tau\}.$$

We put

$$\begin{aligned} \varepsilon[u, v] &=: \langle d\bar{u} \cdot a \cdot dv \rangle, \\ \mathcal{D}(\varepsilon) &= C_0^1(\Omega) \times C_0^1(\Omega) \end{aligned}$$

and define

$$\mathcal{T}(\varepsilon) = \{\tau \in \mathcal{T} : \tau \supset \varepsilon\}$$

and

$$\mathcal{T}_M(\varepsilon) = \{\tau \in \mathcal{T}_M : \tau \supset \varepsilon\}.$$

We say that $\tau \in \mathcal{T}_M$ is local if

$$\tau[f,g] = 0 \text{ whenever } 0 \le f, g, f \wedge g = 0 \quad (f, g \in \mathcal{D}(\tau)).$$

We then define the following extensions of ε:

$$\begin{aligned}
&\tau_D = \varepsilon^{\sim} \text{ (the closure of } \varepsilon),\\
&\tau_i \supset \tau_D, \mathcal{D}(\tau_i) = \mathcal{D}_i \times \mathcal{D}_i, \mathcal{D}_i = \{u \in H_0^1(\Omega) : \langle d\overline{u} \cdot a \cdot du\rangle < \infty\},\\
&\tau_N \supset \tau_i, \mathcal{D}(\tau_N) = \mathcal{D}_N \times \mathcal{D}_N, \mathcal{D}_N = \{u \in H^1(\Omega) : \langle d\overline{u} \cdot a \cdot du\rangle < \infty\}.
\end{aligned}$$

Lemma 2.1. $\tau_D, \tau_i, \tau_N \in \mathcal{T}_M(\varepsilon)$ and are local.

Proof. Define

$$a^n(\cdot) = I + (a(\cdot) - I)(1 + \frac{1}{n} a_u(\cdot))^{-1}, n \in \mathbb{N}.$$

Evidently $I \le a^n(\cdot) \le (n+1)I$ and $a^n(\cdot) \le a^{n+1}(\cdot) \le a(\cdot)$ a. e. Let $E = H_0^1(\Omega)$ or $H^1(\Omega)$. Let

$$\begin{aligned}
\tau^n[u,v] &=: \langle d\overline{u} \cdot a^n \cdot dv\rangle, \quad \mathcal{D}(\tau^n) = E \times E\\
\varepsilon^n[u,v] &=: \langle d\overline{u} \cdot a^n \cdot dv\rangle, \quad \mathcal{D}(\varepsilon^n) = \mathcal{D}(\varepsilon).
\end{aligned}$$

Then $\tau^n \in \mathcal{T}_M(\varepsilon^n), \tau_D \in \mathcal{T}_M(\varepsilon)$ and τ^n are local (see [Fu]). Define τ by

$$\begin{aligned}
\tau[u,v] &=: \lim_n \tau^n[u,v], \mathcal{D}(\tau) = \mathcal{D} \times \mathcal{D},\\
D &=: \{u \in E : \sup_n \varepsilon^n[u] < \infty\}.
\end{aligned}$$

Then $\tau \supset \varepsilon^{\sim}$ by definition, and $\tau \in \mathcal{T}(\varepsilon)$ by the limit theorem for an increasing sequence of closed, symmetric non-negative quadratic forms ([K1, Ch. VIII, Th. 3.13]). The Markov property of $e^{-tA}(A \leftrightarrow \tau)$ and the local property of τ follow immediately. Since $\tau_D \subset \tau$, one concludes that τ_D is local. □

It should be mentioned that τ_D is the maximal element of $\mathcal{T}_M(\varepsilon)$ endowing with the semi-order $\prec$:

$$\tau_1 \prec \tau_2 \iff \mathcal{D}(\tau_1) \supset \mathcal{D}(\tau_2) \text{ and } \tau_1[u] \le \tau_2[u], u \in \mathcal{D}(\tau_2).$$

One can show (we will not do this here) that τ_N is the minimal element of $\mathcal{T}_M(\varepsilon)$ if $a_u(\cdot) \in L^\infty(\Omega)$. This is particulary known for $a(\cdot) = I$ [Fu]. Also, in the case $H^1(\Omega) = H_0^1(\Omega)$ it is natural to describe the class of $a(\cdot)$'s for which the Markov uniqueness ($\tau_D = \tau_{\min}$) holds true.

Let $0 \le V \in L^1_{\text{loc}}(\Omega)$ and $\tau \in \mathcal{T}(\varepsilon)$. If $A \leftrightarrow \tau$ then $Q(V) \cap Q(A)$ is dense in $L^2(\Omega)$ and $A \dot{+} V$ is well-defined. It is easy to see using the Trotter-Kato product formula [K2] that if $\tau \in \mathcal{T}_M(\varepsilon)$ then $e^{-t(A \dot{+} V)}$ is a symmetric Markov semigroup. If $\tau \leftrightarrow T, t \leftrightarrow L$ we write $T \le L$ iff $\tau \prec t$.

§ 3
Weights compatible with $A\dot{+}V$

Definition 3.1. Let $\varrho : \Omega \to \mathbb{R}^1_+$ and

$$\varrho_n(x) = \begin{cases} \varrho(x) & \text{if} \quad 1/n \le \varrho(x) \le n, \\ n & \text{if} \quad \varrho(x) \ge n, \\ 1/n & \text{if} \quad 1/n \ge \varrho(x), \end{cases} \qquad (n \in \mathbb{N}).$$

We say that a weight ϱ and the operator $H = A\dot{+}V$ are compatible if

a) $\varrho, \varrho^{-1} \in W^{1,\infty}_{\text{loc}}(\Omega)$. $(W^{1,2}(\Omega) \equiv H^1(\Omega))$;

b) $\varrho^{-2} d\varrho \cdot a \cdot d\varrho \le c_0 H + c_1$ for some constants $0 \le c_0, c_1 < \infty$;

c) $u \in Q(H)$ implies $u\varrho_n^\delta \in Q(H)$ and $\langle H^{1/2}\varrho_n^{-\delta}u, H^{1/2}\varrho_n^\delta v\rangle = \langle H^{1/2}u, H^{1/2}v\rangle - \delta \cdot k(\varrho_n)[u,v]$, $u, v \in Q(H)$ for all $\delta \in \mathbb{R}^1$ and all $n \in \mathbb{N}$,

where $k(\varrho) = k_1(\varrho) + k_2(\varrho) + \delta k_3(\varrho)$, $k_1(\varrho)[u,v] = \langle u, \varrho^{-1} d\varrho \cdot a \cdot dv\rangle$, $k_2(\varrho)[u,v] = -\overline{k_1(\varrho)[v,u]}$, $k_3(\varrho)[u,v] = \langle u\varrho^{-2} d\varrho \cdot a \cdot d\varrho, v\rangle$, $D(k(\varrho_n)) = D(k_\nu(\varrho(n)) = Q(H) \times Q(H)$, $\nu = 1,2,3$.

Lemma 3.2.

1. If $A = A_D$ or A_i then
$$\text{a)} + \text{b)} \Longrightarrow \text{c)}.$$

2. If $A = A_N$ then
$$\text{a)} + \text{b)} + \text{e}_\Omega) \Longrightarrow \text{c)}$$
where
$$(\text{e}_\Omega) \qquad (C^1_0(\overline{\Omega}) \cap H^1(\Omega))^{\sim}_{H^1(\Omega)} = H^1(\Omega).$$

Proof.

1. Let $H = A_i\dot{+}V$. Since $Q(A_i) \subset H^1_0(\Omega)$ and $\varrho_n^\delta \in W^{1,\infty}(\Omega)$ one has
$$u \in Q(H) \text{ implies } \varrho_n^\delta u \in H^1_0(\Omega),$$
$$d\varrho_n^\delta u \equiv d(\varrho_n^\delta u) = \varrho_n^\delta du + \delta u \varrho_n^{\delta-1} d\varrho_n$$
$$d\varrho_n^\delta \overline{u} \cdot a \cdot d\varrho_n^\delta u \le 2\delta^2 n^{2|\delta|}|u|^2 d\varrho \cdot a \cdot d\varrho + 2n^{2|\delta|} d\overline{u} \cdot a \cdot du.$$
Hence by b)
$$\langle d\varrho_n^\delta \overline{u} \cdot a \cdot d\varrho_n^\delta u\rangle \le (1 + c_0\delta^2) 2n^{2|\delta|}\langle H^{1/2}u, H^{1/2}u\rangle + 2c_1\delta^2 n^{2|\delta|}\|u\|_2^2 \tag{3.1}$$
so that $\varrho_n^\delta u \in Q(A_i)$ and
$$\langle A_i^{1/2}\varrho_n^\delta u, A_i^{1/2}\varrho_n^\delta u\rangle = \langle A_i^{1/2}u, A_i^{1/2}v\rangle - \delta \cdot k(\varrho_n)[u,v],\ u, v \in Q(H).$$
Since $u \in Q(H)$ implies $\varrho_n^\delta u \in Q(V)$, the case $A = A_i$ is proved.

2. The same proof works for $A = A_N$. We recall only that assumption (e_Ω) is valid if e. g. Ω has the extension property [Ste, p. 181].

3. Let $H = A_D\dot{+}V$. Since $\tau_D \subset \tau_i$, we need only to prove that $u \in Q(H)$ implies $u\varrho_n^\delta \in Q(H)$. Taking into account the fact that H is a Markov generator and that $\varrho_n^\delta \geq n^{-|\delta|} > 0$, without loss we restrict $u \in Q(H)$ to $0 \leq u \in Q(H)$. Since $\varrho_n^\delta u \in Q(V)$, we have only to show that $\varrho_n^\delta u \in Q(A_D)$. To do this it is sufficient ([K1, Ch. VI, Th. 1.16]) to find $v_m \in Q(A_D)$ with

$$\sup_m \tau_D[v_m] < \infty \quad \text{and} \quad \|\varrho_n^\delta u - v_m\|_2 \to 0 \quad \text{as } m \to \infty \tag{3.2}$$

for $0 \leq u \in Q(H)$.

Since $\tau_D \subset \tau_i$, it is clear that $\varrho_n^\delta u \in H_0^1(\Omega)$ and $d\varrho_n^\delta u \cdot a \cdot d\varrho_n^\delta u \in L^1(\Omega)$ and the same is true for $u \wedge \ell, \ell \in \mathbb{N}$. Moreover, since $\|H^{1/2}u_\ell\|_2 \leq \|H^{1/2}u\|_2$, one obtains (see (3.1))

$$\sup_k \langle d\varrho_n^\delta w_k \cdot a \cdot d\varrho_n^\delta w_k \rangle < \infty \quad \text{and} \quad \|\varrho_n^\delta(u - w_k)\|_2 \to 0 \quad \text{as} \quad k \to \infty, \tag{3.3}$$

where $w_k = u \wedge k$.

Thus, it suffices to prove (3.2) for $0 \leq u \in Q(H) \cap L^\infty(\Omega)$. By the definition of τ_D for such an u there exist $u_k \in C_0^1(\Omega)$ with

$$[u - u_k] =: \tau_D[u - u_k] + \|u - u_k\|_2^2 \to 0 \quad \text{as } k \to \infty.$$

Since $\tau_D \in \mathcal{T}_M(\varepsilon)$, we may suppose without loss that u_k are real. Then, since τ_D is local, one has $[u - u_k \vee 0] \leq \tau_D[u - u_k] + \|u - u_k \vee 0\|_2^2 \to 0$ as $k \to \infty$. In particular, $\sup_k \tau_D[u_k^+] < \infty, u_k^+ = u_k \vee 0$. Again, since $\tau_D \in \mathcal{T}_M(\varepsilon), \tau_D[u \wedge u_k^+] \leq \tau_D[u] + \tau_D[u_k^+]$ and $\|V^{1/2}(u \wedge u_k^+)\|_2 \leq \|V^{1/2}u\|_2$. Hence (see (3.1)) (3.3) holds with $w_k = u \wedge u_k^+$. The latter means that it suffices to prove (3.2) for $0 \leq u \in Q(H) \cap L^\infty_{\text{com}}(\Omega)$. Once more, for such an u there exist $f_k \in C_0^1(\Omega)$ with $f_k = \text{Re } f_k$ and $[u - f_k] \to 0$ as $k \to \infty$. If $\|u\|_\infty = \ell$ then there exist $h_k \in C^1(\mathbb{R}^1)$ with $0 \leq h_k \leq 2\ell, 0 \leq h_k' \leq 1$ and $h_k(0) = 0$ such that $h_k \circ f_k \in C_0^1(\Omega), 0 \leq h_k \circ f_k \leq 2\ell$ and $\tau_D[h_k \circ f_k] \leq \tau_D[u]$ and $\|u - h_k \circ f_k\|_2 \to 0$ as $k \to \infty$. Let $u_k = (h_k \circ f_k) \cdot \phi$, where $\phi \in C_0^1(\Omega)$, $\phi(x) = 1$ if $x \in \operatorname{supp} u, 0 \leq \phi \leq 1$. Then $\tau_D[u_k] \leq 2\tau[u] + 8\ell^2 \langle d\phi \cdot a \cdot d\phi \rangle$. $\|V^{1/2}u_k\|_2 \leq 2\ell\|V^{1/2}\phi\|_2$ and therefore (3.3) holds with $w_k = u_k \in C_0^1(\Omega)$. Thus, one needs only to prove (3.2) for $0 \leq u \in C_0^1(\Omega)$. The latter can be checked easily. □

Remark 3.3. A slight change in the proof actually shows that $C_0^1(\Omega)$ is a form core of $H = A_D\dot{+}V$. (cf. [D1, Th. 1.8.1]).

We next prove the following proposition which will be a crucial ingredient in our analysis of the spectral p-independence.

Propositon 3.4. Let a weight ϱ and the operator H are compatible. Define the quadratic forms $t, k, k_\nu, \nu = 1, 2, 3$ in $L^2(\Omega)$ by

$$t[u, v] = \langle H^{1/2}u, H^{1/2}v \rangle - \delta \cdot k[u, v], \delta \in \mathbb{R}^1,$$

$$k = k_1 + k_2 + \delta k_3, \mathcal{D}(t) = \mathcal{D}(k_\nu) = Q(H) \times Q(H),$$

$$k[u, v] =: k(\varrho)[u, v], k_\nu[u, v] =: k_\nu(\varrho)[u, v] \text{ (see Def. 3.1.).}$$

Assume that

b') $d\varrho \cdot a \cdot d\varrho \leq \varrho^2(c_0 + c_1 V^{1-\gamma})$ a. e. for some $\gamma \in]0,1]$ and $0 < c_0, c_1 < \infty$.

Then the following assertions hold

(i) For any $\delta \in \mathbb{R}^1$ the form t is quasi m-sectorial,

$$t[u,v] = \langle H_{2,\delta}u, v\rangle, u \in \mathcal{D}(H_{2,\delta}) \subset Q(H), v \in Q(H)$$

where $H_{2,\delta}$ is quasi m-accretive operator associated with t.

(ii) Fix $\delta_0 > 0$, then fix $\lambda_0 > 0$ by the condition $\delta_0\kappa < 1$ where

$$\kappa = \|(c_0 + c_1 V^{1-\gamma})^{1/2}(\lambda_0 + V)^{-1/2}\|_{2\to 2}.$$

For all $\delta \in \mathbb{R}^1$ with $|\delta| \leq \delta_0$ there exists $\omega = \omega(\delta_0, \lambda_0) > 0$ such that $\|(z+H_{2,\delta})^{-1}\|_{2\to 2} \leq |z-\lambda_0|^{-1}, |\arg(z-\lambda_0)| \leq \frac{\pi}{2} + \omega$.

(iii) Let $F \subset \varrho(-H)$ be compact, $\overset{\circ}{F}$ connected, $F = \overline{\overset{\circ}{F}}$ and $\lambda_0 \in \overset{\circ}{F}$. There exist $\delta_1 \in]0, \delta_0]$ and a constant c_2 such that

$$F \subset \varrho(-H_{2,\delta}), \|(z + H_{2,\delta}^{-1})\|_{2\to 2} \leq c_2$$

for all $\delta \in \mathbb{R}^1$ with $|\delta| \leq \delta_1$ and all $z \in F$.

(iv) For all $\lambda > \lambda_0$ and all δ with $|\delta| < \delta_0$

$$\varrho^\delta(\lambda + H)^{-1}\varrho^{-\delta}f = (\lambda + H_{2,\delta})^{-1}f, \quad f \in L^2_{\text{com}}(\Omega).$$

(v) If F and δ_1 are given in (iii) then

$$\varrho^\delta(z + H)^{-1}\varrho^{-\delta}f = (z + H_{2,\delta})^{-1}f, \quad f \in L^2_{\text{com}}(\Omega)$$

for all δ with $|\delta| \leq \delta_1$ and all $z \in F$.

Remarks 3.5.

1. The condition b') has been introduced by T. Kato [K3].
2. Since $\gamma \neq 0$ in b'), $\lim_{\lambda\to\infty} \|(c_0 + c_1 V^{1-\gamma})^{1/2}(\lambda + V)^{-1/2}\|_{2\to 2} = 0$.
3. Proposition 3.4 holds true in the case $\gamma = 0$ with the following additional assumptions:

   in (i) $\quad \delta^2 < c_1^{-1}$,

   in (ii)-(v) $\quad \delta_0^2 < c_1^{-1}$ and $\lambda_0 > c_0 \wedge \frac{c_0}{c_1}$.

The proof of Propositon 3.4. Define the (complex) Hilbert spaces $\mathcal{H}_+ \subset L^2(\Omega) \subset \mathcal{H}_-$ setting $\mathcal{H}_+ = (Q(H), \|\cdot\|_+), \|u\|_+ = \|(\lambda_0 + H)^{1/2}u\|_2, \mathcal{H}_- = (\mathcal{H}_+)^*$. Let $u, v \in \mathcal{H}_+$. One has

$$\begin{aligned}
0 \leq k_\delta[u] &\leq \|(c_0 + c_1 V^{1-\gamma})^{1/2}u\|_2^2 \leq \kappa^2\|u\|_+^2,\\
|k_1[u,v]| &= |k_2[v,u]| \leq \langle\sqrt{d\varrho\cdot a\cdot d\varrho}\varrho^{-1}|u|, \sqrt{d\bar{v}\cdot a\cdot dv}\rangle\\
&\leq \|(c_0 + c_1 V^{1-\gamma})^{1/2}u\|_2\|H^{1/2}v\|_2 \leq \kappa\|u\|_+\cdot\|v\|_+,\\
\text{Re } t[u] &= \|u\|_+^2 - \delta^2 k_3[u] - \lambda_0\|u\|_2^2,\\
\text{Im } t[u] &= \sqrt{-1}\delta\cdot(\overline{k_1[u]} - k_2[u]),\\
(1-\delta^2\kappa^2)\|u\|_+^2 &\leq \text{Re } t[u] + \lambda_0\|u\|_2^2 \leq \|u\|_+^2.
\end{aligned}$$

The above proves (i) and (ii) (see [K1, Ch. VI]). It is clear also that $\langle H^{1/2}u, H^{1/2}v\rangle = \langle \widehat{H}u, v\rangle$, where $\widehat{H}: \mathcal{H}_+ \to \mathcal{H}_-$ and $\mathcal{D}(H) = \{f \in \mathcal{H}_+ : \widehat{H}f \in L^2\}$, $\widehat{H} \restriction \mathcal{D}(H) = H$, $K_\nu[u,v] = \langle \widehat{K}_\nu u, v\rangle$, $k[u,v] = \langle \widehat{K}u, v\rangle, \widehat{K}_\nu, \widehat{K} : \mathcal{H}_+ \to \mathcal{H}_-$, $H_{2,\delta} = \widehat{H} - \delta\widehat{K} \restriction \mathcal{D}(H_{2,\delta})$, $\mathcal{D}(H_{2,\delta}) = \{f \in \mathcal{H}_+ : \widehat{H}f - \delta\widehat{K}f \in L^2\}$. Set $\widehat{B} =: \lambda + \widehat{H} - \delta^2\widehat{K}_3, \lambda > \lambda_0$ and define $B =: \widehat{B} \restriction \{f \in \mathcal{H}_+ : \widehat{B}f \in L^2\}$ - the form sum of $\lambda + H$ and $-\delta^2\varrho^{-2}d\varrho \cdot a \cdot d\varrho$. Since $\kappa^2\delta^2 < 1$, $B^{-1/2} : \mathcal{H}_- \to L^2, B^{1/2} : L^2 \to \mathcal{H}_+$. Define $\Phi =: -\sqrt{-1}B^{-1/2}(\widehat{K}_1 + \widehat{K}_2)B^{-1/2}$ so that $\Phi = \Phi^* : L^2 \to L^2$ and

$$(3.4) \qquad (\lambda + H_{2,\delta})^{-1} = B^{-1/2}(1 - \sqrt{-1}\Phi)^{-1}B^{-1/2}.$$

Although the assertions (iii) and (iv) follow from (3.4) it will be convenient to use a slight modification of it.

$$(3.4') \qquad (\lambda + H_{2,\delta})^{-1} = S^{-1/2}(1-Y)^{-1}S^{-1/2}, \quad \lambda > \tilde{\lambda}_0 \geq \lambda_0,$$

where $S = \lambda + H, Y = \delta S^{-1/2}\widehat{K}S^{-1/2}$; $\tilde{\lambda}_0$ is fixed by the condition $\delta_0\kappa < \sqrt{2} - 1$, which implies

$$\begin{aligned} \|Y\|_{2\to 2} &= |\delta|\|S^{-1/2}(\widehat{K}_1 + \widehat{K}_2 + \delta\widehat{K}_3)S^{-1/2}\|_{2\to 2} \\ &\leq |\delta|(2\kappa + |\delta|\kappa^2) < 1. \end{aligned}$$

By (3.4') one has

$$\|(\lambda + H)^{-1} - (\lambda + H_{2,\delta})^{-1}\|_{2\to 2} \leq \|S^{-1/2}\|^2_{2\to 2} \cdot \|1 - (1-Y)^{-1}\|_{2\to 2} \to 0$$

as $|\delta| \to 0$.

The latter immediately yields (iii) with slightly different F ($\tilde{\lambda}_0 \in \mathring{F}$). See [K1, Ch. IV, Th. 2.2.5 and Remark 3.13]. To justify (iv) we note that (3.4), (3.4') hold for $H_{2,\delta}(\varrho_n), B(\varrho_n)$, $Y(\varrho_n)$ where $H_{2,\delta}(\varrho) \equiv H_{2,\delta}$ etc. Given $\varphi, \psi \in L^2(\Omega)$ one has

$$\begin{aligned} |\langle (Y_1 - Y_1(\varrho_n))\varphi, \psi\rangle| &= |\langle \varrho^{-1}d(\varrho - \varrho_n) \cdot a \cdot dS^{-1/2}\varphi, S^{-1/2}\psi\rangle| \\ &\leq \|H^{1/2}S^{-1/2}\varphi\|_2 \cdot \|\sqrt{d(\varrho - \varrho_n) \cdot a \cdot d(\varrho - \varrho_n)}\varrho^{-1}S^{-1/2}\psi\|_2 \\ &\leq \|\varphi\|_2 \cdot \|(1 - \mathbb{1}_n)(c_0 + c_1V^{1-\gamma})^{1/2}S^{-1/2}\psi\|_2 \end{aligned}$$

where $\mathbb{1}_n$ is the indicator of the set $\{x \in \Omega : \frac{1}{n} \leq \varrho(x) \leq n\}$. Since $(c_0 + c_1V^{1-\gamma})^{1/2}S^{-1/2}\psi \in L^2(\Omega)$ and $\varrho, \varrho^{-1} \in L^\infty_{\text{loc}}(\Omega)$, one obtains

$$Y_1 = w - L^2 - \lim_n Y_1(\varrho_n)$$

and similarly $Y_\nu = w - L^2 - \lim_n Y_\nu(\varrho_n), \nu = 2, 3$. Therefore, by (3.4')

$$(3.5) \qquad (\lambda + H_{2,\delta})^{-1} = w - L^2 - \lim_n(\lambda + H_{2,\delta}(\varrho_n))^{-1}, \quad \forall\lambda > \tilde{\lambda}_0 \geq \lambda_0.$$

Let $t_\varrho[u,v]$ denote $t[u,v]$. Then (i), (ii) imply

$$\begin{aligned} t_{\varrho_n}[u,v] &= \langle H^{1/2}\varrho_n^{-\delta}u, H^{1/2}\varrho_n^{\delta}v\rangle, \quad u, v \in \mathcal{H}_+, \\ t_{\varrho_n}[u,v] &= \langle H_{2,\delta}(\varrho_n)u, v\rangle, \qquad u \in \mathcal{D}(H_{2,\delta}(\varrho_n)), v \in \mathcal{H}_+. \end{aligned}$$

Note that $\varrho_n^\delta e^{-tH}\varrho_n^{-\delta}, t \geq 0$ is a C_0-semigroup on $L^2(\Omega)$ and $\|\varrho_n^\delta e^{-tH}\varrho_n^{-\delta}\|_{2\to 2} \leq n^{2|\delta|}$. Let $-\Gamma$ denote its generator. We claim that $\Gamma = H_{2,\delta}(\varrho_n)$. Indeed for $f \in \mathcal{D}(H_{2,\delta}(\varrho_n)) \subset \mathcal{H}_+$ and $g \in \mathcal{H}_+$ one has

$$\begin{aligned}\frac{1}{t}\langle(1-\varrho_n^\delta e^{-tH}\varrho_n^{-\delta})f,g\rangle &= \frac{1}{t}\int_0^t\langle e^{-sH}H^{1/2}\varrho_n^{-\delta}f, H^{1/2}\varrho_n^\delta g\rangle\, ds \\ &\to \langle H^{1/2}\varrho_n^{-\delta}f, H^{1/2}\varrho_n^\delta g\rangle = \langle H_{2,\delta}(\varrho_n)f,g\rangle \ \text{ as } t\to 0.\end{aligned}$$

Hence $\Gamma \restriction \mathcal{D}(H_{2,\delta}(\varrho_n)) = H_{2,\delta}(\varrho_n)$. Since both $-\Gamma$ and $-H_{2,\delta}(\varrho_n)$ are generators and $\lambda \in \varrho(-\Gamma)\cap\varrho(-H_{2,\delta}(\varrho_n))$ for $\lambda > \tilde{\lambda}_0 > 0$, the last equality means that $\Gamma = H_{2,\delta}(\varrho_n)$. In particular

$$(3.6) \qquad (z+H_{2,\delta}(\varrho_n))^{-1} = \varrho_n^\delta(z+H)^{-1}\varrho_n^{-\delta}, \quad z \in \varrho(-H) = \varrho(-H_{2,\delta}(\varrho_n)).$$

Given $f,g \in L^2_{\text{com}}(\Omega)$, choose $n_0 = n_0(\operatorname{supp} f, \operatorname{supp} g)$ such that

$$\langle\varrho^\delta(\lambda+H)^{-1}\varrho^{-\delta}f,g\rangle = \langle\varrho_n^\delta(\lambda+H)^{-1}\varrho_n^{-\delta}f,g\rangle$$

for all $\lambda \in \varrho(-H)$ and $n \geq n_0$ (due to $\varrho, \varrho^{-1} \in L^\infty_{\text{loc}}(\Omega)$). Now (3.5) and (3.6) combined lead to (iv) (for all $\lambda > \tilde{\lambda}_0 \geq \lambda_0$). We are now in a position to prove (v). First note that if $\lambda > \lambda_0$, $|\delta| < \delta_0$ then $\lambda \in \varrho(-H_{2,\delta})$ and by (3.4) $[(\lambda+H_{2,-\delta})^{-1}]^* = (\lambda+H_{2,\delta})^{-1}$. Let $f,g \in L^2_{\text{com}}(\Omega)$. By (iv) one has

$$\begin{aligned}\langle\varrho^\delta(\lambda+H)^{-2}\varrho^{-\delta}f,g\rangle &= \langle\varrho^\delta(\lambda+H)^{-1}\varrho^{-\delta}f, \varrho^{-\delta}(\lambda+H)^{-1}\varrho^\delta g\rangle \\ &= \langle(\lambda+H_{2,\delta})^{-1}f, (\lambda+H_{2,-\delta})^{-1}g\rangle = \langle(\lambda+H_{2,\delta})^{-2}f,g\rangle.\end{aligned}$$

Thus, $\varrho^\delta(\lambda+H)^{-\ell}\varrho^{-\delta}f = (\lambda+H_{2,\delta})^{-\ell}f$ for $\ell = 2$ and hence for all $\ell \in \mathbb{N}$. Finally, let $\lambda \in \mathring{F}$, $z \in \{\xi \in F : |\xi-\lambda|c_2 < 1\}, |\delta| < \delta_1$. Using (iii) yields

$$\begin{aligned}\langle(z+H_{2,\delta})^{-1}f,g\rangle &= \sum_{k=0}^\infty\langle(z-\lambda)^k(\lambda+H_{2,\delta})^{-k-1}f,g\rangle \\ &= \sum_{k=0}^\infty\langle(z-\lambda)^k(\lambda+H)^{-k-1}\varrho^{-\delta}f,\varrho^\delta g\rangle = \langle(z+H)^{-1}\varrho^{-\delta}f,\varrho^\delta g\rangle \\ &= \langle\varrho^\delta(z+H)^{-1}\varrho^{-\delta}f,g\rangle.\end{aligned}$$

Since F is compact, (v) is proved. □

§ 4

(L^p, L^q) estimates for "weighted" resolvents

Before proving the crucial for the whole approach (L^p, L^q) estimates we need the following technical lemma.

Lemma 4.1. Let $\tau \in \mathcal{T}_M(\varepsilon)$ be such one that

$$\tau_i \supset \tau \supset \tau_D$$

or

$$\tau_N \supset \tau \supset \tau_D \quad \text{if the condition } (\mathrm{e}_\Omega) \text{ of Lemma 3.2 holds.}$$

If $0 \le u, u^{p-1}$ and $u^{p/2}$ belong to $Q(H)$ for some $p > 1$, $H = A \dot{+} V, A \leftrightarrow \tau$, then $\tau[u, u^{p-1}] = 4\frac{p-1}{p^2}\tau[u^{p/2}]$ and

$$\langle H^{1/2}\varrho_n^\delta u, H^{1/2}\varrho_n^\delta u^{p-1}\rangle = 4\frac{p-1}{p^2}\tau[u^{p/2}] + \|V^{1/p}u\|_p^p + 2\frac{p-2}{p}\delta\cdot k_1[u^{p/2}] - \delta^2 k_3[u^{p/2}], \quad \delta \in \mathbb{R}^1$$

where $k_1[v] = \langle v, \varrho^{-1}d\varrho_n \cdot a \cdot dv\rangle, k_3[v] = \langle v\varrho^{-2}d\varrho_n \cdot a \cdot d\varrho, v\rangle$.

Proof. We consider e. g. the case $\tau_i \supset \tau \supset \tau_D$. If $a_u \in L^\infty(\Omega)$ then the statements are evident for $u \in C_0^1(\Omega)$ and, since $H_0^1(\Omega) = (C_0^1(\Omega))^\sim_{H_0^1(\Omega)}$, for $u \in H_0^1(\Omega)$ too. Let τ^m, k^m, k_ν^m be built by $a^m(\cdot)$ (see the proof of Lemma 2.1). Because of $\tau \subset \tau_i$ one has

$$\begin{aligned}
\tau[u, u^{p-1}] &= \tau_i[u, u^{p-1}] = \lim_m \tau^m[u, u^{p-1}]. \\
\tau^m[u, u^{p-1}] &= 4\frac{p-1}{p^2}\tau^m[u^{p/2}], \quad \tau^m[u^{p/2}] \to \tau_i[u^{p/2}] = \tau[u^{p/2}], \\
k_1^m[u, u^{p-1}] + k_2^m[u, u^{p-1}] &= 2\frac{p-2}{p}k_1^m[u^{p/2}] \to 2\frac{p-2}{p}k_1[u^{p/2}], \\
k_3^m[u, u^{p-1}] &= k_3^m[u^{p/2}] \to k_3[u^{p/2}] \text{ as } m \to \infty.
\end{aligned}$$

These completes the prove of the Lemma. □

Proposition 4.2. Let $H = A \dot{+} V, A \leftrightarrow \tau, \tau_i \supset \tau \supset \tau_D$ (or $\tau_N \supset \tau \supset \tau_D$ if Ω satisfies (e_Ω)). Let the assumptions of Proposition 3.4 be hold. For all $1 < p \le q < \infty$ with $1/p - 1/q = 2/d$ and $0 < 1/p - 1/q < 2/d$ if $q = \infty$ there exist $0 < \lambda_p, \delta_p < \infty$ such that the operator $\varrho^\delta(\lambda + H)^{-1}\varrho^{-\delta} : L^\infty_{\mathrm{com}}(\Omega) \to L^1_{\mathrm{loc}}(\Omega)$, $\lambda > 0, \delta \in \mathbb{R}^1$, can be extended by continuity to a bounded map from $L^p(\Omega)$ into $L^q(\Omega)$ as soon as $\lambda > \lambda_p$ and $|\delta| < \delta_p$.

Proof. Let $u_n = \varrho_n^\delta(\lambda + H)^{-1}\varrho_n^{-\delta}f, 0 \le f \in L^\infty_{\mathrm{com}}(\Omega)$, $\lambda > 0$, $\delta \in \mathbb{R}^1$. It is clear that $0 \le u_n \in L^\infty \cap L^1(\Omega)$ and, since $e^{-tH_r}g = e^{-tH}g, g \in L^r \cap L^2(\Omega)$, $1 \le r < \infty$, one has $(\lambda + H_r)^{-1}\varrho_n^{-\delta}f = \varrho_n^{-\delta}u_n$. Therefore $\varrho_n^{-\delta}u_n \in \mathcal{D}(H_r)$ and

$$\langle(\lambda + H_r)\varrho_n^{-\delta}u_n, \varrho_n^{-\delta}u_n^{\nu-1}\rangle = \langle f, u_n^{\nu-1}\rangle \qquad \forall \nu > 1. \tag{4.1}$$

We need now the following general result [LSe, Th. 2.1]. If B is a symmetric Markov generator and if $h \in \mathcal{D}(B_r)$ for some $1 < r < \infty$ then $h|h|^{r-2/2} \in Q(B)$.
The above leads to $(\varrho_n^{-\delta} u_n)^{r/2} \in Q(H)$. Putting consequently $r = 2, 2(\nu - 1), \nu$ with $\nu > 3/2$, and using the fact that $Q(H)$ is invariant under multiplication by ϱ_n^{δ} one has $u_n, u_n^{\nu-1}, u_n^{\nu/2}$ all belong to $Q(H)$. Hence Lemma 4.1 is applicable to (4.1) so that

$$4\frac{\nu-1}{\nu^2}\tau[v] + \lambda\|v\|_2^2 + \langle v, Vv\rangle = \langle f, u_n^{\nu-1}\rangle - 2\frac{\nu-2}{\nu}\delta k_1[v] + \delta^2 k_3[v]$$

where $v =: u_n^{\nu/2}$.
The inequality $2k_1[v] \le \mu k_3[v] + \frac{1}{\mu}\tau[v]$ with $\mu = \frac{|\nu-2|}{\nu-1}\nu|\delta|$ and the condition $k_3[v] \le c_0\|v\|_2^2 + c_1\langle v, V^{1-\gamma}v\rangle$ give

$$2\frac{\nu-1}{\nu^2}\tau[v] + (\lambda - c_0|\delta|s)\|\gamma\|_2^2 \le \langle f, u_n^{\nu-1}\rangle + c_1|\delta|s\langle v, V^{1-\gamma}v\rangle - \langle v, Vv\rangle$$

where $s = |\delta| + \nu\frac{|\nu-2|}{\nu-1}$.
By the Young inequality $c_1|\delta|sV^{1-\gamma} \le \gamma(c_1|\delta|s)^{1/\gamma} + (1-\gamma)V$ a. e., so that

$$2\frac{\nu-1}{\nu^2}\tau[v] + (\lambda - \widetilde{\lambda}_\nu)\|v\|_2^2 \le \langle f, u_n^{\nu-1}\rangle - \gamma\langle v, Vv\rangle \tag{4.2}$$

where $\widetilde{\lambda}_\nu = c_0\delta_\nu(\delta_\nu + \nu\frac{|\nu-2|}{\nu-1}) + (c_1\delta_\nu(\delta_\nu + \nu\frac{|\nu-2|}{\nu-1}))^{1/\gamma}$.
Since $\langle f, u_n^{\nu-1}\rangle \le \|f\|_\nu\|u_n\|_\nu^{\nu-1}$, one has by (4.2)

$$(\lambda - \widetilde{\lambda}_\nu)\|u_n\|_\nu \le \|f\|_\nu,\ \lambda > \widetilde{\lambda}_\nu,\ |\delta| \le \delta_\nu. \tag{4.3}$$

Similarly,

$$\langle f, u_n^{\nu-1}\rangle \le \|f\|_p\|u_n\|_{\nu_j}^{\nu-1} \le c_1(\nu,d)\|f\|_p^\nu + c_2(\nu,d)\|u_n\|_{\nu_j}^\nu$$

where $\frac{1}{p} - \frac{1}{\nu_j} = \frac{2}{d}, j = \frac{d}{d-2}, d \ge 3, p > \frac{3}{2}\frac{d}{d+1}$. Now (4.2) and the Sobolev imbedding theorem combined give

$$\|u_n\|_{\nu_j} \le c(\nu,d)\|f\|_p. \tag{4.4}$$

The case $\frac{3}{2}\frac{d}{d+1} < p < q < \infty, \frac{1}{p} - \frac{1}{q} = \frac{2}{d}$ or $\frac{3}{2} < p \le q < \infty, \frac{1}{p} - \frac{1}{q} \le \frac{2}{d}$ follows now by applying (3.5) to (4.3), (4.4) with $\lambda > \lambda_p = \lambda_0 \vee \widetilde{\lambda}_p$, $|\delta| < \delta_0 \wedge \delta_p$. By duality the same holds for all $1 < p$.

To treat the case $p > \frac{d}{2}, q = \infty$, we proceed as follows. Fix $p_0 \in]\frac{d}{2}, \infty[$ and let $\lambda > \widetilde{\lambda}_{p_0}$ so that $\|u_n\|_{p_0} \le (\lambda - \widetilde{\lambda}_{p_0})^{-1}\|f\|_{p_0}$. Let $p \ge p_0$ and $j_1 = \frac{j}{p'}$ so that $1 \le j_1 < j$. By the Sobolev imbedding theorem $\|v\|_2^2 + \|\nabla v\|_2^2 \ge c_d\|v\|_{2j_1}^2$ and by inequalities $\langle f, u_n^{p-1}\rangle \le \|f\|_{p_0}\cdot\|u_n\|_{(p-1)p_0'}^{p-1}$, $\|u_n\|_p^p \le \|u_n\|_{p_0}\cdot\|u_n\|_{(p-1)p_0'}^{p-1}$ we obtain from (4.2)

$$\|u_n\|_{pj_1}^p \le c\cdot(p-1)^\Gamma\|f\|_{p_0}\cdot\|u_n\|_{(p-1)p_0'}^{p-1}$$

where $\Gamma = 1 + \frac{1}{\gamma}$ and $c = c(d, p_0, \delta_{p_0})$. Set $\iota = \frac{j}{p_0'}$. One has

$$\|u_n\|_{(p-1)p_0'\iota} \le \left[c\cdot(p-1)^\Gamma\|f\|_{p_0}\right]^{\frac{1}{p}}\cdot\|u_n\|_{(p-1)p_0'}^{\frac{p-1}{p}}.$$

The latter admits iteration on p. Putting consecutively $p-1 = p_0 - 1, (p_0-1)\iota, (p_0-1)\iota^2, \dots, (p_0-1)\iota^m$ one has

$$\|u_n\|_{p_0\iota^{m+1}} \le \left[c\cdot(p-1)^\Gamma\right]^{\alpha_m} \cdot e^{\Gamma\beta_m}\cdot c^{\delta_m}\|f\|_{p_0} \tag{4.5}$$

where

$$\begin{aligned}
\alpha_m &=: \sum_{k=1}^{m} \frac{1}{1+(p_0-1)\iota^{k-1}} \prod_{\iota=k}^{m} \frac{(p_0-1)\iota^i}{1+(p_0-1)\iota^i},\\
\beta_m &=: \sum_{k=0}^{m-2} \frac{m-k-1}{1+(p_0-1)\iota^{m-k-1}} \prod_{i=0}^{k} \frac{(p_0-1)\iota^{m-i}}{1+(p_0-1)\iota^{m-i}} + \frac{m}{1+(p_0-1)\iota^m},\\
\delta_m &=: \frac{1}{p_0'} \prod_{i=1}^{m} \frac{(p_0-1)\iota^i}{1+(p_0-1)\iota^i}.
\end{aligned}$$

We then have

$$\alpha_m \le \alpha =: \frac{1}{p_0} + \frac{1}{p_0-1}\cdot\frac{1}{\iota-1}, \quad \beta_m \le \beta =: \sum_{i=1}^{\infty} i\iota^i, \quad \delta_m \le \frac{1}{p_0'}.$$

Let $u = \varrho^\delta(\lambda+H)^{-1}\varrho^{-\delta}f$. Applying (3.5) to (4.5) yields

$$\|u\|_\infty \le \lim_m \|u\|_{p_0\iota^{m+1}} \le \left[c\cdot(p_0-1)^\Gamma\right]^\alpha e^{\Gamma\beta} c^{\frac{1}{p_0'}} \|f\|_{p_0}.$$

□

Remarks 4.3.

1. Without further assumptions the resolvent $(z-H_{2,\delta})^{-1}$ even if $V=0$ cannot be extended by continuity to a bounded map on $L^1(\Omega)$ (or $L^\infty(\Omega)$) for some $z \in \varrho(H_{2,\delta})$ and $\delta \neq 0$.
2. The above variant of Moser's iteration process appeared in [Se] and then was applied to related problems in Orlicz spaces in [LP].

We consider now the case of $V = V_+ - V_-, 0 \le V_\pm \in L^1_{\text{loc}}(\Omega)$. L^p theory of $A+V$ can be developed under the following condition on V_-

$$V_- \le \beta A + V_+ + c(\beta) \qquad \text{for some } \beta \le 1 \text{ and } c(\beta) \in \mathbb{R}^1$$

in the sense that

$$\langle f, (V_- \wedge n) f\rangle \le \beta\langle f, Af\rangle + \langle f, V_+ f\rangle + c(\beta)\|f\|_2^2$$

for all $f \in Q(A)\cap Q(V_+)$ and for all $n \in \mathbb{N}$.

Setting $A_{(n)} = A \dot{+} V_+ - V_- \wedge n$ and using semiboundness of $A_{(n)}$ and (pointwise a. e.) inequalities $0 \le e^{-tA_{(n)}}|f| \le e^{-tA_{(n+1)}}|f|$ $(t>0)$ one has:

$$\mathcal{V}_2^t =: s - L^2 - \lim_n e^{-tA_{(n)}}$$

exists and determines a C_0-semigroup. For all $p \in [t(\beta), t'(\beta)]$ $(t(\beta) = 2/1+\sqrt{1-\beta}, t'(\beta) =: 2/1-\sqrt{1-\beta})$

$$\mathcal{V}_p^t =: \left(\mathcal{V}_2^t \restriction [L^2\cap L^p]\right)^{\sim}_{L^p\to L^p} \tag{4.6}$$

is a C_0-semigroup and

$$\|\mathcal{V}_p^t\|_{p\to p} \le e^{tc(\beta)}.$$

Let $-A_p$ denote the generator of $\mathcal{V}_p^t$. Then for all $p \in]t(\beta), t'(\beta)[$ and for all $\lambda > c(\beta)$ and $1 \le j_1 \le j$

$$(4.7) \qquad (\lambda + A_p)^{-1} : L^p(\Omega) \to L^{p_{j_1}}(\Omega).$$

Moreover, $(\lambda + A_p)^{-1}$ is extended by continuity to a map from $L^{q_1}(\Omega)$ into $L^{p_{j_1}}(\Omega)$, $\frac{1}{q_1} = \frac{1}{p_{j_1}} + \frac{1}{j_1}$. The above facts can be easily extracted from the proof of Th. 3.2. in [LSe]. The proof on Proposition 3.4 and 4.2 can be adapted to obtain the following

Proposition 4.4. Let $H^+ = A \dot{+} V_+$ satisfy the hypotheses of Proposition 4.2. In addition, assume that

$$d\varrho \cdot a \cdot d\varrho \le c_0 \varrho^2 \qquad \text{a. e. for some constant } c_0 < \infty.$$

Then for all $p \in]t(\beta), t'(\beta)[$ there exist $0 < \lambda_p, \delta_p < \infty$ such that the operator $\varrho^\delta(\lambda + A)^{-1}\varrho^{-\delta} : L^\infty_{\text{com}}(\Omega) \to L^1_{\text{loc}}(\Omega), \lambda > c(\beta), \delta \in \mathbb{R}^1$ can be extended by continuity to a bounded map

$$\text{from } L^p(\Omega) \text{ into } L^{p_{j_1}}(\Omega)$$

and

$$\text{from } L^{q_1}(\Omega) \text{ into } L^{p_{j_1}}(\Omega), \quad \frac{1}{q_1} = \frac{1}{p_{j_1}} + \frac{1}{j_1}$$

for all $\lambda > \lambda_p$ and $|\delta| \le \delta_p$.

We comment that one can state first all of the claims for $A_{(n)}$ (in order to have the fact: $\varrho_n^\delta(\lambda + A_{(m)})^{-1}\varrho_n^{-\delta} f \in L^\infty \cap L^1(\Omega), f \in L^\infty_{\text{com}}(\Omega)$) and then taking the limit obtain the desized for A.

§ 5

L^p spectral independence

Definition 5.1. We say that $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is L^1-regular if

1) $|\psi(x) - \psi(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^d$ and some constant $L < \infty$.

2) For each $\varepsilon > 0$

$$\sup_{k \in \mathbb{Z}_d} \sum_{i \in \mathbb{Z}_d} e^{-\varepsilon|\psi(k)-\psi(i)|} =: c_\varepsilon < \infty.$$

Lemma 5.2. Let ψ be L^1-regular, $\delta_0 > 0$ and $1 \leq p \leq q \leq \infty$. For each linear operator $\mathcal{N} : L^\infty_{\mathrm{com}}(\Omega) \to L^1_{\mathrm{loc}}(\Omega)$ one has

$$\|\mathcal{N}\|_{r_1 \to r_2} \leq c_{\delta_0} e^{L\delta_0\sqrt{d}} \sup_{|\xi| \leq \delta_0} \|e^{\xi\cdot\psi}\mathcal{N}e^{-\xi\cdot\psi}\|_{p\to q}$$

for all $p \leq r_1 \leq r_2 \leq q$.

Proof. We subdivide $\mathbb{R}^d$ into cubes of unite size length as follows. For $i \in \mathbb{Z}_d$ define $Q_i =: \{x \in \Omega : |x - i|_\infty < \frac{1}{2}\}$. Let $k, i \in \mathbb{Z}_d$, $f \in L^\infty(\Omega)$, $\mathrm{supp} f \subset Q_i$. Putting

$$\xi = \delta_0 \frac{\psi(k) - \psi(i)}{|\psi(k) - \psi(i)|} \text{ if } \psi(k) \neq \psi(i) \text{ and } \xi = 0 \text{ if } \psi(k) = \psi(i).$$

One has

$$\begin{aligned}
\|\mathcal{N}f\|_{Q_k,q} &= \|e^{-\xi\cdot\psi}e^{\xi\cdot\psi}\mathcal{N}e^{-\xi\cdot\psi}e^{\xi\cdot\psi}f\|_{Q_k,q} \\
&\leq ce^{-\xi\cdot\psi(k)}\|e^{\xi\cdot\psi}\mathcal{N}e^{-\xi\cdot\psi}\|_{p\to q}\cdot\|e^{\xi\cdot\psi}f\|_{Q_i,p} \\
&\leq cMe^{-\xi\cdot\psi(k)}\|e^{\xi\cdot\psi}f\|_{Q_i,p} \\
&\leq c^2Me^{-\xi\cdot(\psi(k)-\psi(i))}\|f\|_{Q_i,p} \leq c^2Me^{-\delta_0|\psi(k)-\psi(i)|}\|f\|_{Q_i,p}.
\end{aligned}$$

(where $\sup_{x\in Q_k} e^{\xi\cdot\psi(k)}\cdot e^{-\xi\cdot\psi(x)} \leq e^{\delta_0 L\frac{1}{2}\sqrt{d}} =: c, M =: \sup_{|\xi|\leq\delta_0}\|e^{\xi\cdot\psi}\mathcal{N}e^{-\xi\cdot\psi}\|_{p\to q}$). For arbitrary $f \in L^\infty_{\mathrm{com}}(\Omega)$ one has

$$\begin{aligned}
\|\mathcal{N}f\|^{r_2}_{r_2} &= \sum_{k\in\mathbb{Z}_d}\|\mathcal{N}f\|^{r_2}_{Q_k,r_2} \leq \sum_k \|\mathcal{N}f\|^{r_2}_{Q_k,q} \\
&\leq \sum_k\Big(\sum_i\|\mathcal{N}(\mathbb{1}_{Q_i}f)\|_{Q_k,q}\Big)^{r_2} \\
&\leq c^{2r_2}M^{r_2}\sum_k\Big(\sum_i e^{-\delta_0|\psi(k)-\psi(i)|}\|f\|_{Q_i,p}\Big)^{r_2} \\
&\leq c^{2r_2}M^{r_2}\sum_k\Big(\sum_i e^{-\delta_0|\psi(k)-\psi(i)|}\|f\|_{Q_i,r_1}\Big)^{r_2} \\
&\leq c^{2r_2}M^{r_2}C^{r_2}_{\delta_0}\|f\|^{r_2}_{r_1}.
\end{aligned}$$

(see [DS, Ch. VI, 11.4] for the last step). □

Remark 5.3. Lemma 5.2 is a straightforward generalization of Proposition 3.2 in [ScV], where it was considered the case $\psi(x) = x$ and $r_1 = r_2$.

Theorem 5.4. Let $A = A_D, A_i$ or A_N. (If $A = A_N$ we suppose that Ω has the extension property). Assume that

$$\begin{aligned}
&\psi : \mathbb{R}^d \to \mathbb{R}^d \text{ is } L^1\text{-regular},\\
&d(\alpha \cdot \psi) \cdot a \cdot d(\alpha \cdot \psi) \le c_1(\varepsilon) \text{ for all } \alpha \in \mathbb{R}^d \text{ with } |\alpha| \le \varepsilon, \text{ a. e. } x \in \Omega
\end{aligned}$$

where $c_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $\sigma(A_p) = \sigma(A), \forall p \in [1, \infty[$.

Proof. Since $e^{-tA_p}[L^p(\Omega)] \subset L^p(\Omega) \cap L^\infty(\Omega) \subset L^q(\Omega), q > p$, one has $\sigma(A_p) \supset \sigma(A)$ (see [HV1]).
Put $\varrho(\cdot) =: \varrho_\alpha(\cdot) = e^{\frac{\alpha}{|\alpha|}\cdot\psi(\cdot)}, \alpha \in \mathbb{R}^d \setminus \{0\}$. It is easily seen that ϱ and A are compatible. Now the resolvent equation

$$(z + A_{2,\delta})^{-1} = (\lambda + A_{2,\delta})^{-1} + (\lambda - z)(z + A_{2,\delta})^{-1}(\lambda + A_{2,\delta})^{-1},$$

Proposition 3.4. (iii) and 4.2 imply

$$\|(z + A_{2,\delta})^{-1}\|_{p\to 2} \le \Big(1 + c_2 \sup_{z\in F} |z - \lambda|\Big) c$$

for $z, \lambda \in F, |\delta| \le \delta_1, 0 \le \frac{1}{p} - \frac{1}{2} \le \frac{2}{d}$. Proposition 3.4. (v) and Lemma 5.2 yield

$$\|\varrho^\delta (z + A_2)^{-1} \varrho^{-\delta}\|_{p\to p} \le c^{(1)}, \quad |\delta| \le \frac{\delta_1}{2}.$$

Repeating this procedure s times leads to the bound

$$\|\varrho^\delta (z + A)^{-1} \varrho^{-\delta}\|_{p\to p} \le c^{(s)}, \quad s = \left[\frac{d}{4}\right] + 1$$

for $z \in F, |\delta| \le \delta_s, 1 \le p < 2$ and by duality for all $1 \le p < \infty$. Since e^{-tA_p} are consistent, we obtain the inclusion $F \subset \varrho(-A_p)$ $\forall p \ge 1$. □

Corollary 5.5. The spectral p-independence of A_p, $1 \le p < \infty$ holds if

$$\begin{aligned}
&a_u \in L^1_{\text{loc}}(\Omega),\\
&a_u(x) \le c \cdot (1 + x^2) \ln^{-\nu}(e + |x|) \qquad \text{a. e. } x \in \{y \in \Omega : |y| > R\}
\end{aligned}$$

where $\nu > 0, R < \infty, 0 < c < \infty$ are some constants.

Proof. Set $\psi(x) = x(1 + |x|)^{-1} \ln^{\nu_1}(\frac{|x|}{R} \vee 1), \quad \nu_1 = \frac{\nu}{2} + 1.$ □

Remark 5.6. For $H^+ = A \dot{+} V_+$ the following is valid. If for all $|x|$ sufficiently large

$$\begin{aligned}
c|x|^m &\le V_+(x), c > 0, m > 0,\\
a_u(x) &\le \tilde{c} \cdot (1 + |x|^{\mu+m}), \tilde{c} > 0, \mu < 2,
\end{aligned}$$

then $\sigma(H_p^+) = \sigma(H^+), \forall p \ge 1$.

Theorem 5.7. Let A and ψ satisfy the hypotheses of Theorem 5.4. Then the following is valid:

(I) Let $A(V) \equiv A$. If for some $k > 1$

$$\|e^{-tA(kV)} f\|_1 \le Me^{\omega t}\|f\|_1, \quad f \in L^1(\Omega) \cap L^2(\Omega)$$

then $\sigma(A_p) = \sigma(A), \forall p \in [1, \infty[$.
Moreover, the resolvent $(z - A)^{-1}$ is an integral operator with

$$(5.1) \quad \|(z - A_1)^{-1}\|_{1\to p'} = \operatorname*{ess\,sup}_{y\in\Omega} \left(\int |(z - A)^{-1}(x, y)|^{p'}\, dx \right)^{\frac{1}{p'}} = \|(\overline{z} - A_p)^{-1}\|_{p\to\infty}$$

for all $z \in \varrho(A)$ and $p \in]\frac{d}{2}, \infty[$.

(II) For all $z \in \varrho(A)$ the resolvent $(z - A)^{-1}$ extendes to a bounded map on $L^r(\Omega)$ for all $r \in]p(\beta), p'(\beta)[$.

(III) If for some $p_0 \in]p(\beta), 2[$

$$\|V_2^t f\|_{p_0} \le Me^{\omega t}\|f\|_{p_0}, \quad f \in L^2(\Omega) \cap L^{p_0}(\Omega)$$

then $\sigma(A_p) = \sigma(A), \forall p \in [p_0, p_0']$.
In particular the following is always true

$$\sigma(A_p) = \sigma(A), \quad \forall p \in [t(\beta), t'(\beta)].$$

Proof.

(I) In fact, the proof of Theorem 5.4 gives the bound

$$\|\varrho^\delta(z + A)^{-1}\varrho^{-\delta}\|_{1\to p} \le \widehat{c}^{(s)}, \quad z \in F, |\delta| \le \delta_s, \forall p \in \left[1, \frac{d}{d-2}\right[.$$

Combining with the Dunford-Pettis theorem this yields (5.1) for $A(0) = A$. There are many ways of deriving (L^p, L^q)-estimates for $\varrho^\delta(\lambda + A(V))^{-1}\varrho^{-\delta}$ from the related estimates for $\varrho^\delta(\lambda + A)^{-1}\varrho^{-\delta}$; e. g. one can use the inequalities (6.2). After that the proof of the equality $\sigma(A_p) = \sigma(A)$ can be carried out in the same manner as it has been done for A_p.

(II) The proof follows directly from Proposition 3.4, 4.4 and Lemma 5.2.

(III) By virture of (II) the proof of "$\varrho(A_p) \supset \varrho(A)$" is straightforward. If $p_0 \in]t(\beta), 2[$ then by (4.7) $(z - A_p)^{-1}[L^p(\Omega)] \subset L^q(\Omega)$ for all $z \in \varrho(-A_p)$ and suitable $q > p$, so $\varrho(A_p) \subset \varrho(A)$ for all $p \in [p_0, p_0']$. Thus we have only to treat the case $p_0 \in]p(\beta), t(\beta)]$. Lemma 5.2 with $r_1 < r_2$ applied to $\mathcal{N} = (\lambda + A)^{-1}$ with $\lambda > c(\beta)$ sufficiently large and Proposition 4.4 yield $(\lambda + A)^{-1} : L^p \to L^q$ for all $p \in [p_0, p_0']$ and $q = q_p > p$. Thus, again $\varrho(A_p) \subset \varrho(A)$. The last claim follows from (4.6). □

Remarks 5.8.

1. The hypotheses on V of Theorem 5.7.I can be checked for potentials which non-negative parts belong to the Kato class

$$\widehat{K}_d(H^+) =: \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \inf_{\lambda>0} \|(\lambda + A\dot{+}V_+)^{-1}|f|\|_\infty < 1 \right\}$$

(see [LSe, § 5]). Highly oscillating potentials are considered in [St2].

2. Under the assumptions of Theorem 5.7.II the expected result on integral representability of $(z-A)^{-1}, z \in \varrho(A)$ should be as follows.

If $p \in]p(\beta), p'(\beta)[$ and $z \in \varrho(A)$ then the extension of $(z-A)^{-1}$ to a map on $L^p(\Omega)$ is an integral operator. At present the following is known. If $p \in]p(\beta), p'(\beta)[$ and $\mathrm{Re}\, z > s(-A) =: \sup\{\lambda \in \mathbb{R}^1 : \lambda \in \sigma(-A)\}$ then the extension of $(z+A)^{-1}$ on $L^p(\Omega)$ is a regular integral operator.

To justify the claim we note that

$$|(z+A)^{-1}h| \le (\mathrm{Re}\, z + A)^{-1}|h|, \quad \mathrm{Re}\, z > s(-A),$$
$$(\lambda + A)^{-1}f \le [(\lambda + A(kV))^{-1}f]^{\frac{1}{k}} \cdot [(\lambda + A)^{-1}f]^{\frac{1}{k'}}, f \ge 0, \lambda > c(\beta) \vee 0,$$

with $k-1 \in]0, \frac{1}{\beta} - 1[$ sufficiently small. Thus, if $f_n, f \in L^p(\Omega), |f_n| \le f$ and $f_n \to 0$ a. e., one has

$$|(\lambda + A)^{-1}f_n| \le g \cdot [(\lambda + A_p)^{-1}|f_n|]^{\frac{1}{k'}}$$

where $g^k = (\lambda + A(kV))^{-1}f$.

Since $(\lambda + A_p)^{-1}$ is integral, $(\lambda + A_p)^{-1}|f_n| \to 0$ a. e. The claim follows now from the Bukhvalov criterium [Bu] (see also [ArB], [W2]) for $\lambda > c(\beta) \vee 0$ and, hence, for all z with $\mathrm{Re}\, z > s(-A)$.

Of course, the above arguments work for A_p, $p \in]t(\beta), t'(\beta)[$, with arbitrary $a(\cdot) \ge I, a_u \in L^1_{\mathrm{loc}}(\Omega)$.

In applications it is usually needed more than the bare fact of integrability, e. g. in the theory of eigenfunction expansion one needs Carleman's property of $(\lambda + A)^{-s}$ to hold for some s ($> \frac{d}{4}$) and all $\lambda > 0$ sufficiently large. One can show that in the conditions of Theorem 5.7.II the latter does hold (see also [Se], where considered a slightly different situation).

3. Let $\Omega = \mathbb{R}^d, A = -\Delta, V_- = V_1^- + V_2^-$. If $V_2^- \in L^{\frac{d}{2},\infty}(\mathbb{R}^d), d \ge 3$ with $\|V_2^-\|_{\frac{d}{2},\infty} \le \Omega_d^{\frac{2}{d}}(\frac{d-2}{2})^2\beta, 0 < \beta < 1, \Omega_d = |\{x \in \mathbb{R}^d;\ |x| \le 1\}|$, then according to [KPS]

$$\|e^{-t(-\Delta \dot{-} V_2^-)}f\|_p \le M_p\|f\|_p,\ f \in L^2 \cap L^p, \forall p \in]p(\beta), p'(\beta)[.$$

By (6.2) one has

$$\|e^{-t(-\Delta \dot{+} V)}f\|_p \le \widetilde{M}_p e^{\omega_p t}\|f\|_p,\ f \in L^2 \cap L^p, \forall p \in]p(\beta), p'(\beta)[$$

where $V = V_+ - V_-, V_-^1 \in K_d$. Set $\varrho_\alpha(x) = e^{\frac{\alpha}{|\alpha|}\cdot x}, \alpha \in \mathbb{R}^d \setminus \{0\}$. Then all of the assumptions of Theorem 5.7.III hold and hence $\sigma(A_p) = \sigma(A), A = -\Delta \dot{+} V$.

§ 6

Equivalence of Green's functions

Since local and/or global singularities of $a(\cdot)$ as well as local singularities of V_- such as $c|x - x_0|^{-2}, x_0 \in \Omega$ distroy the property of e^{-tA}, e^{-tA} to admit an upper Gaussian bound, there is not any deep link between this property and the spectral p-independence of A_p, A_p as Theorems 5.4 and 5.7 show.

Nevertheless, we indicate one extremely useful application of Theorem 5.7 to the problem of the equivalence of the Green functions G_A and G_A, which shows that the question of spectral independence presents not only academic value.

Theorem 6.1. Let $A = A_D$ or A_i satisfies the hypotheses of Theorem 5.4. Assume that for some $k > 1$

$$\begin{aligned} &A(kV) \geq 0, \\ &\|e^{-tA(kV)}\|_{1\to 1} \leq Me^{\omega t} \qquad (t > 0, \omega > 0, M \geq 1). \end{aligned} \tag{6.1}$$

Then for any $m \in [1, k[$ there exist finite numbers M_1, M_2 such that

$$\|e^{-tA(mV)}\|_{1\to 1} \leq M_1, \qquad \|e^{-tA(mV)}\|_{1\to\infty} \leq Mt^{-\frac{d}{2}} \quad (t > 0).$$

Furthermore, if

$$\Omega = \mathbb{R}^d, \quad a_u \in L^\infty(\Omega) \quad \text{and} \quad \|A^{-1}V_+\|_\infty < \infty$$

then there exists a constant $0 < c < 1$ such that

$$c|x - y|^{2-d} \leq G_A(x, y) \leq c^{-1}|x - y|^{2-d}, \quad \forall x, y \in \mathbb{R}^d.$$

Proof. Fix $m \in]1, k[$. The inequality

$$e^{-tA(mV)}f \leq \left(e^{-tA(kV)}f\right)^{\frac{m}{k}} \cdot \left(e^{-tA}f\right)^{\frac{k-m}{k}} \quad \text{a. e. } 0 \leq f \in L^1(\Omega), \tag{6.2}$$

which is a consequence of the Trotter-Kato product formula (see [HS]), and (6.1) imply the bound

$$\|e^{-tA(mV)}\|_{1\to 1} \leq M^{\frac{m}{k}} e^{\frac{m}{k}\omega t}$$

and hence by Theorem 5.7.I $\sigma(A_1(m_1V)) = \sigma(A(m_1V))\ \forall m_1 \in]1, m[$. Since $A(m_1V) \geq 0$, we conclude that the type of $e^{-tA_1(m_1V)}$ is non-positive, so that

$$\|e^{-tA(m_1V)}\|_{1\to 1} \leq M_1 \qquad (t \geq 0, M_1 < \infty).$$

Since $A(m_1V) \geq \frac{k-m_1}{k}A$, one has

$$Q(A(m_1V)) \subset L^{2j}(\Omega).$$

The latter is equivalent to the bound

$$\|e^{-tA(m_1V)}\|_{1\to\infty} \le M_2 t^{-\frac{d}{2}} \qquad (t>0, M_2<\infty) \tag{6.3}$$

(see [LSe, Th. 7.1] or [VSC, Ch. II]).

If $a(\cdot) \in L^\infty(\mathbb{R}^d)$ then due to [Aro] (see also [D1], [Str]) there exist constants $0 < M_0, c_0 < 1$ such that

$$M_0 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4c_0 t}} \le e^{-tA}(x,y) \le M_0^{-1} t^{-\frac{d}{2}} e^{-c_0\frac{|x-y|^2}{4t}} \tag{6.4}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

The R.H.S. of (6.4) combined with (6.3) and the inequality

$$e^{-tA(V)}(x,y) \le \left(e^{-tA(m_1V)}(x,y)\right)^{\frac{1}{m_1}} \cdot \left(e^{-tA}(x,y)\right)^{1-\frac{1}{m_1}}$$

give the bound

$$e^{-tA(V)}(x,y) \le M_3^{-1} t^{-\frac{d}{2}} e^{-c\frac{|x-y|^2}{4t}} \qquad (t>0, 0<c, M_3<1). \tag{6.5}$$

Now choose $p_1 > 1$ such that $\|A^{-1}V_+\|_\infty < p_1 - 1$. Put $W = -\frac{1}{p-1}V_+$, $p > p_1$. By [Vo] the operator $-(A_1 + W)$ defined on $\mathcal{D}(A_1)$ generates a bounded C_0-semigroup on $L^1(\Omega)$ and $A_1(W) = A_1 + W$. Next, $A(kW) \ge 0$ and $\|e^{-tA_1(kW)}\|_{1\to1} \le \widetilde{M}_1$ with $k = \frac{p-1}{p_1-1} > 1$. Thus, the preceding leads to (6.5) with W instead of V. The latter, the L.H.S. of (6.4) and the inequality

$$e^{-tA}(x,y) \le \left(e^{-tA(V)}(x,y)\right)^{\frac{1}{p}} \cdot \left(e^{-tA(W)}(x,y)\right)^{\frac{1}{p'}}$$

give the bound

$$M_3 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4ct}} \le e^{-tA(V)}(x,y) \qquad (0<M_3, c<1). \tag{6.6}$$

Now the equivalence $G_A \sim G_A$ follows from (6.5), (6.6). □

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