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A RESTRICTED WREATH PRODUCT WITH THE PROPERTY R_∞

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Abstract. We prove that for any automorphism ϕ of the restricted wreath product $\mathbb{Z}_2 \mathrm{wr} \mathbb{Z}^2$ the Reidemeister number $R(\phi)$ is infinite.

Introduction

The *Reidemeister number* $R(\phi)$ of an automorphism ϕ of a (countable discrete) group G is the number of its *Reidemeister* or *twisted conjugacy classes*, i.e. the classes of the twisted conjugacy equivalence relation: $g \sim hg\phi(h^{-1})$, $h, g \in G$. Denote by $\{g\}_\phi$ the class of g.

The following two interrelated problems are in the mainstream of the study of Reidemeister numbers.

The first one is the following conjecture by A.Fel'shtyn and R.Hill [9]: $R(\phi)$ is equal to the number of fixed points of the associated homeomorphism $\widehat{\phi}$ of the unitary dual $\widehat{G}$ (the set of equivalence classes of irreducible unitary representations of G), if one of these numbers is finite. The action of $\widehat{\phi}$ on the class of a representation ρ is defined as $[\rho] \mapsto [\rho \circ \phi]$. This conjecture is called TBFT (twisted Burnside-Frobenius theorem). In fact it generalizes to infinite groups and to the twisted case the classical Burnside-Frobenius theorem: the number of conjugacy classes of a finite group is equal to the number of equivalence classes of its irreducible representations.

Some later by A.Fel'shtyn and co-authors the second problem was formulated (see [11] for a historical overview): the problem of description of the class of groups having the R_∞ property. A group has the R_∞ property if $R(\phi) = \infty$ for any automorphism $\phi : G \to G$. Evidently, the second problem is in some sence complementary to the first one: the question about TBFT has no sense for R_∞ groups (formally having a positive answer).

The TBFT conjecture (more precisely some its modification) was proved for polycyclic-by-finite groups in [13, 21]. Preliminary and related results, examples and counter-examples can be found in [9, 12, 14, 10, 40, 19, 23, 41].

The property R_∞ was studied very intensively during the last years and was proved and disproved for many groups (see, in particular [8, 31, 15, 16, 38, 28, 11, 1, 25, 2, 17, 30, 32, 34, 33, 5, 18, 26, 27, 36, 6, 4, 20] and the literature therein). For Jiang type spaces the property R_∞ has some direct topological consequences (see e.g. [27]). Concerning applications of Reidemeister numbers in Dynamics we refer to [29, 7].

In the present paper we prove that the group $\mathbb{Z}_2 \mathrm{~wr~} \mathbb{Z}^2 = (\mathbb{Z}/2\mathbb{Z}) \mathrm{~wr~} (\mathbb{Z} \oplus \mathbb{Z})$ has the property R_∞.

The R_∞ property was proved for some wreath products by $\mathbb{Z}$ and their generalizations in [39, 37]. The case of $\mathbb{Z} \oplus \mathbb{Z}$ is much more complicated, because $\mathbb{Z}$ has only one automorphism

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with finite Reidemeister number, namely $-\mathrm{Id}$, and its square has infinite Reidemeister number. For $\mathbb{Z}\oplus\mathbb{Z}$ we have a lot of automorphisms with finite Reidemeister numbers, and many of them have finite Reidemeister numbers for all their iterations.

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1. Preliminaries

The following easy statement is well known:

Proposition 1.1. *Suppose, H is a ϕ-invariant normal subgroup of G and $\overline{\phi}: G/H \to G/H$ is the induced automorphism. Then ϕ induces an epimorphism of each Reidemeister class of ϕ onto some Reidemeister class of $\overline{\phi}$. In particular, one has $R(\overline{\phi}) \leqslant R(\phi)$.*

Denote by $C(\phi)$ the fixed point subgroup. The following much more non-trivial statement can be extracted from [24] (see also [13]):

Lemma 1.2. *In the above situation $R(\phi|_H) \leqslant R(\phi)\cdot|C(\overline{\phi})|$.*

It is well known (see [12]) the following.

Lemma 1.3. *For an abelian group G the Reidemeister class of the unit element is a subgroup, and the other classes are corresponding cosets.*

The following statement is very useful in the field.

Lemma 1.4. *A right shift by $g\in G$ maps Reidemeister classes of ϕ onto Reidemeister classes of $\tau_{g^{-1}}\circ\varphi$, where τ_g is the inner automorphism: $\tau_g(x) = gxg^{-1}$. In particular, $R(\tau_g\circ\phi) = R(\phi)$.*

Proof. Indeed,

$$xy\varphi(x^{-1})g = x(yg)g^{-1}\varphi(x^{-1})g = x(yg)(\tau_{g^{-1}}\circ\varphi)(x^{-1}).$$

□

Also we need the following statement ([22], [19, Prop. 3.4]):

Lemma 1.5. *Let $\phi: G\to G$ be an automorphism of a finitely generated residually finite group G with $R(\phi)<\infty$ (in particular, G can be a finitely generated abelian group). Then the subgroup of fixed elements is finite: $|C(\phi)|<\infty$.*

Note, that this is not correct for infinitely generated groups, see [41].

2. The main result

Let $\Gamma := \mathbb{Z}_2 \,\mathrm{wr}\, \mathbb{Z}^2$ be the restricted wreath product. In other words,

$$\Gamma = \oplus_{(m.k)\in\mathbb{Z}^2}(\mathbb{Z}_2)_{(m,k)} \rtimes_\alpha \mathbb{Z}^2, \qquad (\mathbb{Z}_2)_{(m,k)} \cong \mathbb{Z}_2, \qquad \alpha(s,t)(\delta_{m,k}) := \delta_{m+s,k+t},$$

where $(s,t)\in\mathbb{Z}^2$ and $\delta_{m,k}$ is a unique non-trivial element of $(\mathbb{Z}_2)_{(m,k)}$. The direct sum supposes only finitely many non-trivial components for each element (in contrast with the direct product corresponding to the (unrestricted) wreath product).

The group Γ is a finitely generated metabelian group, in particular, residually finite (see e.g. [35]).

Let $\phi : \Gamma \to \Gamma$ be an automorphism. We will prove that $R(\phi) = \infty$. Denote $\Sigma := \oplus_{(m.k)\in\mathbb{Z}^2}(\mathbb{Z}_2)_{(m,k)} \subset \Gamma$. Then Σ is a characteristic subgroup as the torsion subgroup. Denote the restriction of ϕ by $\phi' : \Sigma \to \Sigma$, and the quotient automorphism by $\overline{\phi} : \mathbb{Z}^2 \to \mathbb{Z}^2$.

If $R(\phi) < \infty$, then $R(\overline{\phi}) < \infty$ by Proposition 1.1; and by Lemma 1.5, $\overline{\phi}$ has finitely many fixed elements. Thus, by Lemma 1.2, $R(\phi') < \infty$. Hence, to prove that $R(\phi) = \infty$, it is sufficient to verify that $R(\phi') = \infty$.

Since Σ is abelian, the results of e.g. [3] imply that

$$\phi'(\alpha(g)(h)) = \alpha(\overline{\phi}(g))(\phi'(h)), \qquad h \in \Sigma, \quad g \in \mathbb{Z}^2. \tag{1}$$

Any element of Σ is a finite sum of some elements $\delta_{m,k}$. Let

$$\phi'(\delta_{0,0}) = \delta_{i(1),j(1)} + \cdots + \delta_{i(n),j(n)}. \tag{2}$$

Lemma 2.1. *In (2) one has $n = 1$. Moreover, ϕ' is a permutation of $\delta_{m,k}$'s.*

Proof. First of all, apply (1) to $h = \delta_{0,0}$, $g = (m, k)$. We have:

$$\phi'(\delta_{m,k}) = \phi'(\alpha(g)(h)) = \alpha(\overline{\phi}(g))(\phi'(\delta_{0,0})). \tag{3}$$

Thus, for any $(m, n) \in \mathbb{Z}^2$, the element $\phi'(\delta_{m,n})$ is obtained by the appropriate shift of indexes in the right side expression in (2).

Consider

$$p : \Sigma \to \mathbb{Z}_2, \qquad p(\delta_{m,k}) = 1.$$

Its kernel L is a subgroup of index 2. Suppose, n is even. Then the image of ϕ' is contained in L, by the definition. But ϕ' is an isomorphism. A contradiction.

Now suppose that n is odd, $n \neq 1$, and $\phi'(h) = \delta_{0,0}$ for some $h = \delta_{r(1),s(1)} + \cdots + \delta_{r(t),s(t)}$. Let

$$\pi_1 : \oplus_{(m.k)\in\mathbb{Z}^2}(\mathbb{Z}_2)_{(m,k)} \to \oplus_{m\in\mathbb{Z}}(\mathbb{Z}_2)_{(m)}, \qquad \pi_2 : \oplus_{(m.k)\in\mathbb{Z}^2}(\mathbb{Z}_2)_{(m,k)} \to \oplus_{k\in\mathbb{Z}}(\mathbb{Z}_2)_{(k)}$$

be natural epimorphisms (vertical and horizontal summation). The images $\pi_1(\phi'(\delta_{r(u),s(u)}))$, $u = 1, \dots, t$, have the same odd number of non-zero summands (and moreover, these images can be obtained from each other by index shifts over $\mathbb{Z}$). The same is true for π_2. At least for one of π_1 and π_2 this odd number is > 1, e.g. for π_1. After cancellation of equal images this means that there is several distinct elements $\pi_1(\phi'(\delta_{r(u),s(u)}))$, obtained from each other by index shifts over $\mathbb{Z}$, and having $\delta_0 = \pi_1(\delta_{0,0})$ as their sum. In particular, they have all left-end elements distinct and all right-end elements distinct. Thus, their sum needs to have at least two non-trivial components (the most left of the left ends and the most right of the right ends). So, it cannot be equal to δ_0.

Thus, $n = 1$. Together with the argument at the beginning of the proof, this gives the second statement. □

By this lemma, we can define (x_0, y_0) by $\phi'(\delta_{0,0}) =: \delta_{x_0,y_0}$. The equation (3) can be written now as

$$\phi'(\delta_{m,k}) = \delta_{m',k'}, \qquad (m', k') := \overline{\phi}(m, k) + (x_0, y_0) \in \mathbb{Z}^2. \tag{4}$$

Lemma 2.2. *If δ_{x_1,y_1} and δ_{x_2,y_2} belong to the same Reidemeister class of ϕ', then*

$$\overline{\phi}^t(x_1, y_1) + \overline{\phi}^{t-1}(x_0, y_0) + \cdots + \overline{\phi}(x_0, y_0) + (x_0, y_0) = (x_2, y_2) \tag{5}$$

for some integer t.

Proof. By Lemma 1.3, the elements δ_{x_1,y_1} and δ_{x_2,y_2} belong to the same Reidemeister class of ϕ' if and only if $\delta_{x_1,y_1} - \delta_{x_2,y_2} = h - \phi'(h)$ for some $h \in \Sigma$. Representing h as $h = \delta_{u(1),v(1)} + \dots + \delta_{u(t),v(t)}$ (with distinct summands) and applying (4) one has

$$\delta_{x_1,y_1} - \delta_{x_2,y_2} = h - \phi'(h) = \sum_{j=1}^{t} [\delta_{u(j),v(j)} - \delta_{u(j)',v(j)'}].$$

So one of δ's with "+" on the right should be equal to δ_{x_1,y_1}, one of δ's with "-" on the right should be equal to δ_{x_2,y_2} (or vice versa), and the remaining δ's should annihilate. Since all $\delta_{u(j),v(j)}$ are distinct, all $\delta_{u(j)',v(j)'}$ are distinct too, by Lemma 2.1. So the cancellation can be only as $\delta_{u(j),v(j)} = \delta_{u(i)',v(i)'}$. Thus, after the appropriate renumbering of $1, \dots, t$, we have:

$$(x_1, y_1) = (u(1), v(1)), \quad (u(1)', v(1)') = (u(2), v(2)), \quad \dots$$

$$(u(t-1)', v(t-1)') = (u(t), v(t)), \quad (u(t)', v(t)') = (x_2, y_2),$$

or

$$\begin{aligned}
\overline{\phi}(x_1, y_1) + (x_0, y_0) &= (u(2), v(2)),\\
\overline{\phi}^2(x_1, y_1) + \overline{\phi}(x_0, y_0) + (x_0, y_0) &= (u(2)', v(2)') = (u(3), v(3)),\\
\overline{\phi}^3(x_1, y_1) + \overline{\phi}^2(x_0, y_0) + \overline{\phi}(x_0, y_0) + (x_0, y_0) &= (u(3)', v(3)') = (u(4), v(4)),\\
\dots \quad \dots & \quad \dots\\
\overline{\phi}^t(x_1, y_1) + \overline{\phi}^{t-1}(x_0, y_0) + \dots + \overline{\phi}(x_0, y_0) + (x_0, y_0) &= (u(t)', v(t)') = (x_2, y_2).
\end{aligned}$$

□

Theorem 2.3. *The group $\Gamma = \mathbb{Z}^2 \operatorname{wr} \mathbb{Z}^2$ has the property R_∞.*

Proof. One can reduce the proof of $R(\phi) = \infty$ to the case $(x_0, y_0) = (0, 0)$. Indeed, consider the element $w := (-x_0, -y_0) \in \mathbb{Z}^2 \subset \Gamma$ and the corresponding inner automorphism $\tau_w : \Gamma \to \Gamma$. Then by Lemma 1.4, $R(\tau_w \circ \phi) = R(\phi)$. On the other hand,

$$(\tau_w \circ \phi)'(\delta_{0,0}) = \alpha(w)(\phi'(\delta_{0,0})) = \alpha(-x_0, -y_0)(\delta_{x_0,y_0}) = \delta_{0,0}.$$

So, suppose $(x_0, y_0) = (0, 0)$. Then (5) takes the form $\overline{\phi}^t(x_1, y_1) = (x_2, y_2)$ for some integer t. Thus, it is sufficient to prove that $\overline{\phi} : \mathbb{Z}^2 \to \mathbb{Z}^2$ has infinitely many orbits.

For this purpose denote by $A \in GL_2(\mathbb{Z})$ the matrix of $\overline{\phi}$. Let us show that each orbit intersects the first coordinate axis not more than in 2 points. Denote by $(x, 0)$, $x \neq 0$, one point from the intersection and suppose that

$$A^n \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$$

is the next intersection. Evidently, $z \neq 0$, and

$$A^n = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with integer entries and $\det A^n = \pm 1$. Thus, $a = \pm 1$. Hence, for $a = 1$ we have only one intersection point, namely $(x, 0)$, and for $a = -1$ we have two intersection points: $(\pm x, 0)$. □

Remark 2.4. In fact, it was sufficient for our purposes to use the following well-known exercise-level fact: the set $\{(x,0) \mid x \in \mathbb{Z},\ x \geqslant 1\}$ parametrize orbits of the entire $GL_2(\mathbb{Z})$ on $\mathbb{Z} \oplus \mathbb{Z}$ because for any matrix $B \in GL_2(\mathbb{Z})$ the greatest common divisor of coordinates of $B \begin{pmatrix} x \\ 0 \end{pmatrix}$ is equal to x.

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