Pseudo-Differential Calculus in the Fourier-Edge Approach on Non-compact Manifolds

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Contents

Introduction

In recent years the pseudo-differential analysis on manifolds with singularities made enormous progress in constructing specific operator algebras with symbolic structures for expressing parametrices and characterizing regularity of solutions of elliptic equations. This concerns, in particular, manifolds with piece-wise C^∞ geometry, for instance, conical points, edges, corners, ..., or non-compact ends ("exits" to infininity) of different kind, cf. [23],[24]. The main analytical difficulty is the control of symbols and distributions up to the singularities and to manage the variety of new phenomena, such as asymptotics of solutions or the interplay of different (operator-valued) symbolic structures in a transparent way. In the edge and corner theories as they were investigated in [21],[4] the operator-valued symbols live in natural way on manifolds with exits to infinity, namely on infinite cones with base spaces of lower singularity orders. For the spaces of higher singularity orders this is to be expected, too, where the corresponding pseudo-differential algebras have to be the result of some iteration of lower order "cone" and "wedge" theories. In view of the complexity of the cone, wedge, and corner operator algebras with one or two cone axis variables such an iteration can be realistic only by developing suitable axiomatic ideas. The present paper will establish some typical part of the axiomatic approach that is connected with higher non-compact edges going to infinity.

The specific assumptions in our theory at infinity generalize those of the corresponding calculus for scalar symbols, cf. PARENTI [12], [13], CORDES [2], SCHROHE [15], EGOROV, SCHULZE [5]. Moreover, there are imposed the structures inherited from the local edge pseudo-differential calculus with cone-operator-valued symbols, here based on the Fourier transform on the edge. The latter theory has the form of a pseudo-differential calculus with operator-valued symbols in which the symbol estimates as well as the Sobolev space norms contain actions of one-parameter groups of isomorphisms on the corresponding model spaces. Analogous structures may be expected in higher edge theories, i.e., when the base of the model cone of the wedge has more complicated singularities. So it will be natural to formulate the theory mainly in terms of general model Hilbert spaces. Our calculus at infinity is also new in the special case of "ordinary" edges. The notation "Fourier-edge approach" indicates a connection of the design of the pseudo-differential theory adapted to some specific part of an underlying space with piece-wise C^∞ geometry, here for edges without singularities. In applications there also occur edges that have singular points, e.g., conical ones, as it is the case, for instance, for the one-dimensional edges of a cube near the corner points. Then the present theory is to be combined with some kind of Mellin-edge approach that relies on the Mellin transform on $\mathbb{R}_+$ in the distance variable to the singularities, cf. DORSCHFELDT, SCHULZE [4].

Another ingredient for the calculus near conical or corner singularities is a certain order reduction approach based on the Mellin or Fourier transform, containing global operator-valued symbols along the corresponding base manifolds, cf. SCHULZE [19], [21].

The pseudo-differential operators on manifolds with singularities in general are then to be obtained by glueing together the corresponding local variants.

The strategy of the present article is as follows. First in Section 1.1 we establish general pseudo-differential operators globally in $\mathbb{R}^q$ with weight conditions to the amplitude functions at infinity, where the values of the amplitude functions are operators acting between Banach spaces, with fixed strongly continuous groups of isomorphisms. The basic results are analogous to those of the scalar theory. Here we follow the scheme of CORDES [2]. Moreover, we introduce the weighted abstract wedge Sobolev spaces in $\mathbb{R}^q$ along the lines of the original definitions in SCHULZE [23], here with weights at infinity. For performing the calculus including the continuity in the weighted Sobolev spaces we assume in Section 1.2 that the involved spaces are Hilbert spaces with unitary groups of isomorphisms. On one hand we generalize elements of KUMANO-GO's technique [9] with amplitude functions dependent on the double covariables, on the other hand we generalize the proof of HWANG [8] of the global L_2-continuity to the operator-valued case. Section 1.3 extends the results to the case of arbitrary strongly continuous groups of isomorphisms. First, for the calculus, we may allow Banach spaces, again, whereas for the proof of the continuity in the weighted Sobolev spaces we impose Hilbert spaces and assume the existence of certain order reducing symbols as they are known in concrete applications. Section 1.4 deals with the global ellipticity in $\mathbb{R}^q$ for symbols with compact variation as it was supposed in a simpler situation in LUKE [11]. We obtain parametrices within the class and the Fredholm property. In Section 2.1 we study the invariance of the calculus under natural diffeomorphisms between open sets in $\mathbb{R}^q$ that are conical for large arguments. This extends the results of SCHROHE [14] to the operator-valued case, here for simplicity for the case of diffeomorphisms that are homogeneous of degree 1 for large arguments. Section 2.2 formulates the theory on a non-compact C^∞ manifold with conical exits to infinity, and studies ellipticity, parametrices and Fredholm property on such manifolds. Section 3.1 illustrates the present theory in terms of boundary value problems with the transmission property. This has some relation to SCHROHE [15], though here we develop another aspect; the exit conditions in our context are only of interest tangent to the boundary. Section 3.2 constructs order reducing symbols for the edge pseudo-differential calculus in the sense of [20], [23], [24].

Let us finally note that SEILER [25] studied a concrete global operator algebra in the infinite wedge that is already of interest in boundary value problems without the transmission property.

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1 The global Fourier-Edge Pseudo-Differential Operators

1.1 Weighted operators and abstract wedge Sobolev spaces

Let $E, \widetilde{E}$ be Banach spaces. By $\mathcal{L}(E, \widetilde{E})$ we denote as usual the space of all linear continuous operators $T : E \to \widetilde{E}$. The space $\mathcal{L}(E, \widetilde{E})$ is endowed with the operator norm topology. We set $\mathcal{L}(E) := \mathcal{L}(E, E)$. By $\mathcal{L}_\sigma(E)$ we denote the space $\mathcal{L}(E)$ but equipped with the strong operator topology. In the sequel we fix a strongly continuous group action on E, i.e., a continuous funktion $\kappa : \mathbb{R}_+ \to \mathcal{L}_\sigma(E)$, $\lambda \mapsto \kappa_\lambda$, which satisfies the composition rule $\kappa_\lambda \kappa_\beta = \kappa_{\lambda\beta}$ for all $\lambda, \beta \in \mathbb{R}_+$.

A standard example is given by $E = L_2(\mathbb{R}_+)$ and the group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ defined by $(\kappa_\lambda u)(t) := \lambda^{\frac{1}{2}} u(\lambda t)$ for all $\lambda \in \mathbb{R}_+$ and $u \in L_2(\mathbb{R}_+)$. In this case $\{\kappa_\lambda\}$ is even a group of unitary operators.

Remark 1.1. If $\{\kappa_\lambda\}$ is an arbitrary group action on E, then $\left\{ \|\kappa_\lambda\|_{\mathcal{L}(E)} \ : \ \lambda \in \mathbb{R}_+ \right\}$ cannot be expected to be bounded. For each group action, there are constants $c, M > 0$ such that

$$\|\kappa_\lambda\|_{\mathcal{L}(E)} \leq \begin{cases} c\lambda^M & \text{, for } \lambda \geq 1 \\ c\lambda^{-M} & \text{, for } \lambda < 1. \end{cases} \tag{1.1}$$

Throughout this paper we fix a smoothed norm function

$$[\cdot] : \mathbb{R}^q \to \mathbb{R}_+,$$

i.e., $[\cdot]$ is a smooth and strictly positive function on $\mathbb{R}^q$ and there is a constant $c > 0$ such that $[\eta] = |\eta|$ for $|\eta| > c$. For abbreviation we then set

$$\kappa(\eta) := \kappa_{[\eta]}.$$

Fix two pairs $\left(E, \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}\right), \left(\widetilde{E}, \{\widetilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+}\right)$ of Banach spaces $E, \widetilde{E}$ together with corresponding group actions.

Definition 1.2. Let $\mu, \varrho \in \mathbb{R}$. Then $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ is defined as the space of all $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that

$$\left\| \widetilde{\kappa}^{-1}(\eta) \left\{ D_y^\alpha D_\eta^\beta a(y,\eta) \right\} \kappa(\eta) \right\|_{\mathcal{L}(E,\widetilde{E})} \leq c\, [y]^{\varrho - |\alpha|} [\eta]^{\mu - |\beta|} \tag{1.2}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^q$ and all $y, \eta \in \mathbb{R}^q$, with constants $c > 0$ depending only on α, β.

Moreover, we set

$$S^{\mu,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) := \bigcap_{\varrho \in \mathbb{R}} S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}),$$

$$S^{-\infty,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) := \bigcap_{\mu \in \mathbb{R}} S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}),$$

$$S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q) := \bigcap_{\mu,\varrho \in \mathbb{R}} S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}).$$

Remark 1.3. (i) The best constants c that are possible in the estimates (1.2) define a system of semi-norms on $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, which makes it a Fréchet space.

(ii) Similarly to the scalar theory one can also consider spaces $S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$ of (y, y')-dependent symbols, where the estimate (1.2) has to be replaced by

$$\left\| \widetilde{\kappa}^{-1}(\eta) \left\{ D_y^\alpha D_{y'}^{\alpha'} D_\eta^\beta a(y, y', \eta) \right\} \kappa(\eta) \right\|_{\mathcal{L}(E, \widetilde{E})} \le c\, [y]^{\varrho - |\alpha|} [y']^{\varrho' - |\alpha'|} [\eta]^{\mu - |\beta|}.$$

These symbols, however, will only play a minor role in the calculus, since pseudo-differential operators with (y, y')-dependent amplitude functions can be replaced by operators with y'-independent amplitude functions modulo smoothing error terms.

The following Lemma collects some simple facts about symbol spaces.

Lemma 1.4. *Let $E, \widehat{E}, \widetilde{E}$ be Banach spaces with associated group actions.*

(i) If $\mu \le \nu$, $\varrho \le \tau$, then there is a continuous embedding

$$S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \hookrightarrow S^{\nu,\tau}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}).$$

(ii) If $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, then

$$D_y^\alpha D_\eta^\beta a \in S^{\mu - |\beta|, \varrho - |\alpha|}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}).$$

(iii) We have

$$S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; \widehat{E}, \widetilde{E}) S^{\nu,\tau}(\mathbb{R}^q \times \mathbb{R}^q; E, \widehat{E}) \subseteq S^{\mu+\nu, \varrho+\tau}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}),$$

with the point-wise composition of operator functions.

If $E = \widetilde{E} = \mathbb{C}$ and the group actions are the trival ones, i.e., $\kappa_\lambda = \widetilde{\kappa}_\lambda = \mathrm{id}$ for all $\lambda > 0$, we will write $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q)$ instead of $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Note that for all $\beta, \gamma \in \mathbb{R}$ we have $[\eta]^\beta [y]^\gamma \in S^{\beta,\gamma}(\mathbb{R}^q \times \mathbb{R}^q)$. The multiplication by $[\eta]^\beta [y]^\gamma$ induces isomorphisms

$$[\eta]^\beta [y]^\gamma : S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \to S^{\mu+\beta, \varrho+\gamma}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \qquad (1.3)$$

for all $\mu, \varrho \in \mathbb{R}$.

Proposition 1.5. *Let $a_j \in S^{\mu_j, \varrho_j}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, $j \in \mathbb{N}$, be an arbitrary sequence, and $\mu_j \to -\infty$, $\varrho_j \to -\infty$ as $j \to \infty$. Then there is an $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ with $\mu = \max\{\mu_j\}$, $\varrho = \{\varrho_j\}$ such that for every $\beta \in \mathbb{R}$ there is an $N = N(\beta)$ with*

$$a(y, \eta) - \sum_{j=0}^{N} a_j(y, \eta) \in S^{\mu-\beta, \varrho-\beta}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}).$$

The symbol a is unique modulo $S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$.

Proof. Without loss of generality we may assume $\mu_j < \mu_{j-1}$ and $\varrho_j < \varrho_{j-1}$ for all $j \geq 1$. Let $\chi(y,\eta) \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q)$ be an excision function with $\chi(y,\eta) = 0$ for all $|y|^2 + |\eta|^2 < 1$ and $\chi(y,\eta) = 1$ for all $|y|^2 + |\eta|^2 > 2$. For all $c > 0$ define $h_c(y,\eta) = 0$ for $|y|^2 + |\eta|^2 < c$ and $h_c(y,\eta) = 1$ for $|y|^2 + |\eta|^2 \geq c$. If $c \geq 1$, then the following estimate holds:

$$\begin{aligned}
&\left|(D_y^{\alpha_1} D_\eta^{\beta_1} \chi)\left(\frac{y}{c},\frac{\eta}{c}\right) \frac{(1+|y|)^{|\beta_1|}}{c^{|\beta_1|}} \frac{(1+|\eta|)^{|\alpha_1|}}{c^{|\alpha_1|}}\right| \\
&\leq \sup\left\{(D_y^{\alpha_1} D_\eta^{\beta_1} \chi)\left(\frac{y}{c},\frac{\eta}{c}\right) \;:\; |y|^2 + |\eta|^2 \geq c^2\right\} 4^{|\alpha_1|+|\beta_1|} \\
&=: M_{\alpha_1,\beta_1,c}
\end{aligned}$$

By using the symbol estimates for a_j we get

$$\begin{aligned}
&\left\|\widetilde{\kappa}^{-1}(\eta)\left\{D_y^{\alpha_2} D_\eta^{\beta_2} a_j(y,\eta)\right\} \kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \\
&\quad \leq p_{\alpha_2,\beta_2}^{\mu_j,\varrho_j}(a_j)(1+|y|)^{\varrho_j - |\alpha_2|}(1+|\eta|)^{\mu_j - |\beta_2|},
\end{aligned}$$

where $p_{\alpha,\beta}^{\mu_j,\varrho_j}(a_j)$ denote the best constants in the symbol estimates of a_j.

Now, employing the Leibniz rule, we obtain

$$\begin{aligned}
&(1+|y|)^{|\alpha|-\varrho_{j-1}}(1+|\eta|)^{|\beta|-\mu_{j-1}} \\
&\quad \cdot \left\|\widetilde{\kappa}^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta \chi\left(\frac{y}{c},\frac{\eta}{c}\right) a_j(y,\eta)\right\} \kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \\
&\leq K_{\alpha,\beta,c}(a_j) h_{c^2}(y,\eta)(1+|y|)^{\varrho_j - \varrho_{j-1}}(1+|\eta|)^{\mu_j-\mu_{j-1}} \\
&\leq K_{\alpha,\beta,c}(a_j) \sup\left\{(1+|y|)^{\varrho_j-\varrho_{j-1}}(1+|\eta|)^{\mu_j-\mu_{j-1}} \;:\; |y|^2 + |\eta|^2 \geq c^2\right\} \\
&=: C_{\alpha,\beta,c}(a_j),
\end{aligned}$$

where

$$K_{\alpha,\beta,c}(a_j) := \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \frac{\alpha!\beta!}{\alpha_1!\alpha_2!\beta_1!\beta_2!} M_{\alpha_1,\beta_1,c} p_{\alpha_2,\beta_2}^{\mu_j,\varrho_j}(a_j).$$

Note that $C_{\alpha,\beta,c}(a_j) \to 0$ as $c \to \infty$. Choose $c_j > 0$ such that $C_{\alpha,\beta,c_j}(a_j) \leq 2^{-j}$ for all $|\alpha| + |\beta| \leq j$. Now let

$$\widetilde{a}_j(y,\eta) := \chi\left(\frac{y}{c_j},\frac{\eta}{c_j}\right) a_j(y,\eta).$$

This yields

$$\begin{aligned}
&\left\|\widetilde{\kappa}^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta \widetilde{a}_j(y,\eta)\right\} \kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \\
&\quad \leq 2^{-j}(1+|y|)^{\varrho_{j-1}-|\alpha|}(1+|\eta|)^{\mu_{j-1}-|\beta|}
\end{aligned}$$

for all $|\alpha| + |\beta| \leq j$. So $\sum_{j=N+1}^\infty \widetilde{a}_j$ converges for all $N \in \mathbb{N}$ in $S^{\mu_N,\varrho_N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Since $\widetilde{a}_j \in S^{\mu_N,\varrho_N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ also $\sum_{j=N}^\infty \widetilde{a}_j$ converges in the space $S^{\mu_N,\varrho_N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$.

Now define $a := \sum_{j=0}^{\infty} \widetilde{a}_j$. The sum is finite in each point $(y,\eta) \in \mathbb{R}^q \times \mathbb{R}^q$ and converges in $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Moreover,

$$\begin{aligned}
a(y,\eta) &- \sum_{j=0}^{N-1} a_j(y,\eta) \\
&= a(y,\eta) - \sum_{j=0}^{N-1} \widetilde{a}_j(y,\eta) + \sum_{j=0}^{N-1} (\widetilde{a}_j(y,\eta) - a_j(y,\eta)) \\
&= \sum_{j=N}^{\infty} \widetilde{a}_j(y,\eta) + \sum_{j=0}^{N-1} \left(1 - \chi\left(\frac{y}{c_j}, \frac{\eta}{c_j}\right)\right) a(y,\eta) \\
&\in S^{\mu_N,\varrho_N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}).
\end{aligned}$$

If $a' \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ is another symbol that satisfies $a' - \sum_{j=0}^{N-1} a_j \in S^{\mu_N,\varrho_N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ for all $N \in \mathbb{N}$, then

$$a - a' = \left(a - \sum_{j=0}^{N-1} a_j\right) - \left(a' - \sum_{j=0}^{N-1} a_j\right) \in S^{\mu_N,\varrho_N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$$

for any $N \in \mathbb{N}$ and hence

$$a - a' \in S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}).$$

□

As usual we write $a(y,\eta) \sim \sum_{j=0}^{\infty} a_j(y,\eta)$, and call a the *asymptotic sum* of the sequence $(a_j)_{j\in\mathbb{N}}$.

Proposition 1.6. *(i) Let $p_j(y,\eta) \in S^{\mu_j,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, $j \in \mathbb{N}$, be an arbitrary sequence with $\mu_j \to -\infty$ as $j \to \infty$. Then there is a $p(y,\eta) \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ with $\mu = \max\{\mu_j\}$ such that for every $\beta \in \mathbb{R}$ there is an $N = N(\beta)$ with*

$$p(y,\eta) - \sum_{j=0}^{N} p_j(y,\eta) \in S^{\mu-\beta,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$$

and $p(y,\eta)$ is unique modulo $S^{-\infty,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$.

(ii) Let $r_j(y,\eta) \in S^{\mu,\varrho_j}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, $j \in \mathbb{N}$, be an arbitrary sequence with $\varrho_j \to -\infty$ as $j \to \infty$. Then there is an $r(y,\eta) \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ with $\varrho = \max\{\varrho_j\}$ such that for every $\beta \in \mathbb{R}$ there is an $N = N(\beta)$ with

$$r(y,\eta) - \sum_{j=0}^{N} r_j(y,\eta) \in S^{\mu,\varrho-\beta}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$$

and $r(y,\eta)$ is unique modulo $S^{\mu,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$.

Proof. The proof follows by simple modifications of the idea of the proof of Proposition 1.5. If $\chi(\eta)$ is an excision function in $\mathbb{R}^q$ then $p(y,\eta)$ can be obtained as a convergent sum

$$p(y,\eta) = \sum_{j=0}^{\infty} \chi\left(\frac{\eta}{c_j}\right) p_j(y,\eta).$$

Analogously $r(y,\eta)$ follows as a convergent sum

$$r(y,\eta) = \sum_{j=0}^{\infty} \chi\left(\frac{y}{c_j}\right) r_j(y,\eta).$$

In both cases c_j are constants tending to ∞ sufficiently fast as $j \to \infty$. □

With a symbol $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ we associate a pseudo-differential operator $\mathrm{Op}(a)$ via

$$\mathrm{Op}(a)u(y) := \iint e^{i(y-y')\eta} a(y,\eta)u(y)\, dy đ\eta,$$

for $u \in C_0^\infty(\mathbb{R}^q, E)$, where $đ\eta := (2\pi)^{-q} d\eta$. Then $\mathrm{Op}(a)u \in C^\infty(\mathbb{R}^q, \widetilde{E})$.

Definition 1.7. For every $\mu, \varrho \in \mathbb{R}$ we set

$$L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E}) := \left\{ \mathrm{Op}(a) \ : \ a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \right\}.$$

Moreover, we set

$$L^{-\infty,-\infty}(\mathbb{R}^q; E, \widetilde{E}) := \bigcap_{\mu,\varrho \in \mathbb{R}} L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E}).$$

Proposition 1.8. *If $A \in L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$, $\mu, \varrho \in \mathbb{R}$, then A induces a continuous operator*

$$A : \mathcal{S}(\mathbb{R}^q, E) \to \mathcal{S}(\mathbb{R}^q, \widetilde{E}).$$

Proof. Write $A = \mathrm{Op}(a)$ with $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Let $u \in \mathcal{S}(\mathbb{R}^q, E)$, $\varphi \in C_0^\infty(\mathbb{R}^q)$, and consider $\varphi(y)Au(y)$. Put $b(y,\eta) := \varphi(y)a(y,\eta)\widehat{u}(\eta)$. Since $\widehat{u} \in \mathcal{S}(\mathbb{R}^q, E)$, it follows that $b \in \mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q, E)$. Hence $\mathcal{F}^{-1}_{\eta\to y} b(y,\eta) \in \mathcal{S}(\mathbb{R}^q, \widetilde{E})$. Since φ was arbitrary, we get $Au \in C^\infty(\mathbb{R}^q, \widetilde{E})$.

Next, let us estimate $\sup_{y \in \mathbb{R}^q} \left\| y^\beta D_y^\alpha Au(y) \right\|_{\widetilde{E}}$, for given multi-indices $\alpha, \beta \in \mathbb{N}^q$. We have

$$D_y^\alpha e^{iy\eta} a(y,\eta) = e^{iy\eta} \sum_{|\gamma| \le |\alpha|} c_{\alpha,\gamma} \eta^\gamma D_y^{\alpha-\gamma} a(y,\eta), \tag{1.4}$$

with suitable constants $c_{\alpha,\gamma} \in \mathbb{C}$. By (1.1) there are constants $c, M, \widetilde{M} > 0$ such that $\|\kappa_\lambda\|_{\mathcal{L}(E)} \le c \max\{\lambda^M, \lambda^{-M}\}$ and $\|\widetilde{\kappa}_\lambda\|_{\mathcal{L}(\widetilde{E})} \le c \max\left\{\lambda^{\widetilde{M}}, \lambda^{-\widetilde{M}}\right\}$. Choose $k, l \in \mathbb{N}$ such that $2k > |\beta| + \varrho + q$, $2l > |\alpha| + \mu + q + M + \widetilde{M}$. Writing

$$(1 + |\eta|^2)^{-l} (1 - \Delta_{y'})^l e^{i(y-y')\eta} = e^{i(y-y')\eta}, \tag{1.5}$$

$$(1 + |y - y'|^2)^{-k} (1 - \Delta_\eta)^k e^{i(y-y')\eta} = e^{i(y-y')\eta}, \tag{1.6}$$

an integration by parts gives

$$
\begin{aligned}
&y^\beta D_y^\alpha Au(y)\\
&= \sum_{|\gamma|\le|\alpha|} c_{\alpha,\gamma} \iint e^{i(y-y')\eta}\eta^\gamma(1+|\eta|^2)^{-l} y^\beta D_y^{\alpha-\gamma} a(y,\eta)(1-\Delta_{y'})^l u(y')\,dy'\,đ\eta\\
&= \sum_{|\gamma|\le|\alpha|} c_{\alpha,\gamma} \iint e^{i(y-y')\eta} y^\beta (1+|y-y'|^2)^{-k}\\
&\quad\cdot \left\{(1-\Delta_\eta)^k \eta^\gamma (1+|\eta|^2)^{-l} D_y^{\alpha-\gamma} a(y,\eta)\right\}(1-\Delta_{y'})^l u(y')\,dy'\,đ\eta.
\end{aligned}
$$

Here we have used the notation $\Delta_\eta = \sum_{j=1}^q \partial^2/\partial\eta_j^2$. Using $(1+|y-y'|^2)^{-k} \le c_k(1+|y'|^2)^k(1+|y|^2)^{-k}$, we estimate

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}^q} \left\|y^\beta D_y^\alpha Au(y)\right\|_{\widetilde{E}}\\
&\quad\le \sup_{y\in\mathbb{R}^q} c_\alpha c_k \iint |y|^\beta (1+|y'|^2)^k (1+|y|^2)^{-k} \|\widetilde{\kappa}(\eta)\|_{\mathcal{L}(\widetilde{E})}\\
&\qquad\cdot \left\|\widetilde{\kappa}^{-1}(\eta)\left\{(1-\Delta_\eta)^k\eta^\gamma(1+|\eta|^2)^{-l} D_y^{\alpha-\gamma} a(y,\eta)\right\}\kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})}\\
&\qquad\cdot \left\|\kappa^{-1}(\eta)\right\|_{\mathcal{L}(E)} \left\|(1-\Delta_{y'})^l u(y')\right\|_{\mathcal{L}(E)}\,dy'\,đ\eta\\
&\quad\le \sup_{y\in\mathbb{R}^q} c_{\alpha,\beta,k} \iint |y^\beta|\,(1+|y|^2)^{-k}\,[y]^\varrho\,[\eta]^{M+\widetilde{M}}\\
&\qquad\cdot [\eta]^{\mu+|\alpha|-2l} p(a) \left\|(1+|y'|^2)^k(1-\Delta_{y'})^l u(y')\right\|_{\mathcal{L}(E)}\,dy'\,đ\eta\\
&\quad\le c'_{\alpha,\beta,k}\, p(a) \int \left\|(1+|y'|^2)^k(1-\Delta_{y'})^l u(y')\right\|_{\mathcal{L}(E)}\,dy',
\end{aligned}
$$

where p is some semi-norm on $S^{\mu,\varrho}(\mathbb{R}^q\times\mathbb{R}^q; E,\widetilde{E})$ and $c'_{\alpha,\beta,k}$ is a constant which only depends on $\alpha,\beta\in\mathbb{N}^q, k,l\in\mathbb{N}$. To given $k,l\in\mathbb{N}$ there is a semi-norm π_ι on $\mathcal{S}(\mathbb{R}^q,E)$ such that

$$
\sup_{y\in\mathbb{R}^q} \left\|(1+|y|^2)^k(1-\Delta_y)^l u(y)\right\|_E \le (1+|y|^2)^{-q}\pi_\iota(u)
$$

for all $u\in\mathcal{S}(\mathbb{R}^q,E)$. Hence

$$
\sup_{y\in\mathbb{R}^q} \left\|y^\beta D_y^\alpha Au(y)\right\|_{\widetilde{E}} \le c_{\alpha,\beta,k,l}\, p(a)\pi_\iota(u),
$$

proving the continuity of A as an operator from $\mathcal{S}(\mathbb{R}^q,E)$ to $\mathcal{S}(\mathbb{R}^q,\widetilde{E})$. $\square$

By $C_b^\infty(\mathbb{R}^q,E)$ we denote the space of all $u\in C^\infty(\mathbb{R}^q,E)$ with

$$
\sup_{y\in\mathbb{R}^q} \|D^\alpha u(y)\|_E < \infty
$$

for all multi-indices $\alpha\in\mathbb{N}^q$. Note that $C_b^\infty(\mathbb{R}^q,E)$ is a Fréchet space in the corresponding semi-norm system.

Theorem 1.9. *Every $A \in L^{\mu,0}(\mathbb{R}^q; E, \widetilde{E})$, $\mu \in \mathbb{R}$, induces a continuous operator*

$$A : C_b^\infty(\mathbb{R}^q, E) \to C_b^\infty(\mathbb{R}^q, \widetilde{E}).$$

Proof. Let $A = \mathrm{Op}(a)$ with $a \in S^{\mu,0}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Using (1.4), (1.5), (1.6), we get for $u \in C_0^\infty(\mathbb{R}^q, E)$ and $\alpha \in \mathbb{N}^q$

$$\begin{aligned} &D_y^\alpha Au(y) \\ &= \sum_{|\gamma|\le|\alpha|} c_{\alpha,\gamma} \iint e^{i(y-y')\eta}(1+|y-y'|^2)^{-k} \\ &\quad \cdot \left\{(1-\Delta_\eta)^k \eta^\gamma (1+|\eta|^2)^{-l} D_y^{\alpha-\gamma} a(y,\eta)\right\} (1-\Delta_{y'})^l u(y')\, dy'd\eta. \end{aligned}$$

Choose $k, l \in \mathbb{N}$ such that $k > q$ and $2l > |\alpha| + \mu + q + M + \widetilde{M}$. It then follows easily that the last integral also converges for $u \in C_b^\infty(\mathbb{R}^q, E)$. We have

$$\begin{aligned} &\|D^\alpha Au(y)\|_{\widetilde{E}} \\ &\le \sum_{|\gamma|\le|\alpha|} c_{\alpha,\gamma} \iint (1+|y-y'|^2)^{-k} \|\widetilde{\kappa}(\eta)\|_{\mathcal{L}(\widetilde{E})} \\ &\quad \cdot \left\|\widetilde{\kappa}^{-1}(\eta)\left\{(1-\Delta_\eta)^k \eta^\gamma (1+|\eta|^2)^{-l} D_y^{\alpha-\gamma} a(y,\eta)\right\}\kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \\ &\quad \cdot \left\|\kappa^{-1}(\eta)\right\|_{\mathcal{L}(E)} \left\|(1-\Delta_{y'})^l u(y')\right\|_{\mathcal{L}(E)} dy'd\eta \\ &\le c_\alpha \iint (1+|y-y'|^2)^{-k} [\eta]^{M+\widetilde{M}} [\eta]^{\mu-2l} p(a)\pi_\iota(u)\, dy'd\eta \end{aligned}$$

for all $y \in \mathbb{R}^q$, where p is some suitable semi-norm on $S^{\mu,0}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, π_ι is a semi-norm on $C_b^\infty(\mathbb{R}^q, E)$, and c_α is a constant which only depends on α. Hence

$$\begin{aligned} \left\|D_y^\alpha Au(y)\right\|_{\widetilde{E}} &\le c_\alpha p(a)\pi_\iota(u) \int (1+|y-y'|^2)^{-k}\, dy' \\ &\le c'_\alpha p(a)\pi_\iota(u), \end{aligned}$$

for all $y \in \mathbb{R}^q$, with a constant c'_α which only depends on $\alpha \in \mathbb{N}^q$. This proves that $A \in \mathcal{L}(C_b^\infty(E), C_b^\infty(\widetilde{E}))$. □

For $y, \eta \in \mathbb{R}^q$ define $e_\eta(y) := e^{iy\eta}$.

Proposition 1.10. *Let $a(y,\eta) \in S^{\mu,0}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, $A = \mathrm{Op}(a)$. Then, for every $f \in E$,*

$$a(y,\eta)f = e_{-\eta}(y)A(e_\eta(y)f). \tag{1.7}$$

Proof. Let $f \in E$. Then

$$\begin{aligned}(\mathrm{Op}(a)e_\eta f)(y) &= \iint e^{i(y-y')\zeta} a(y,\zeta) e_\eta(y') f \, dy' đ\zeta \\ &= \iint e^{iy'(\eta-\zeta)} e^{iy\zeta} a(y,\zeta) f \, dy' đ\zeta \\ &= \iint e^{-iy'\eta'} e^{iy(\eta+\eta')} a(y,\eta+\eta') f \, dy' đ\eta' \\ &= e^{iy\eta} \iint e^{i(y-y')\eta'} a(y,\eta+\eta') f \, dy' đ\eta'.\end{aligned}$$

The assertion now follows, using that $\iint e^{-iy'\eta'} a(y,\eta+\eta') f \, dy' đ\eta' = a(y,\eta) f$. □

Proposition 1.10 shows that the mapping

$$\mathrm{Op} : S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \to L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E}), \quad a \mapsto \mathrm{Op}(a)$$

is bijective, provided that $\varrho \leq 0$.

Proposition 1.11. *The mapping*

$$\mathrm{Op} : S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \to L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E}), \quad a \mapsto \mathrm{Op}(a)$$

is bijective for all $\mu, \varrho \in \mathbb{R}$.

Proof. Let $A = \mathrm{Op}(a)$ with $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q)$. Notice that the function

$$c^\varrho(y) := (1 + |y|^2)^{-\varrho/2}$$

induces isomorphisms

$$M_{c^\varrho} : S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \to S^{\mu,0}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$$

for all $\mu \in \mathbb{R}$, where $(M_{c^\varrho} a)(y,\eta) := c^\varrho(y) a(y,\eta)$. The mapping $R^\varrho := \mathrm{Op}(c^\varrho)$ induces isomorphisms

$$R^{-\varrho} : L^{\mu,0}(\mathbb{R}^q; E, \widetilde{E}) \to L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$$

for all $\mu \in \mathbb{R}$. Writing now $\mathrm{Op} = R^{-\varrho} \circ \mathrm{Op} \circ M_{c^\varrho}$, the assertion follows, using that $\mathrm{Op} : S^{\mu,0}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \to L^{\mu,0}(\mathbb{R}^q; E, \widetilde{E})$ is an isomorphism. □

Proposition 1.12. *The following conditions are equivalent:*

(i) An operator $C \in \mathcal{L}(\mathcal{S}(\mathbb{R}^q, E), \mathcal{S}(\mathbb{R}^q, \widetilde{E}))$ *is an integral operator with kernel in* $\mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$.

(ii) $C = \mathrm{Op}(a)$ *for some* $a \in S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$.

(iii) $C \in L^{-\infty,-\infty}(\mathbb{R}^q; E, \widetilde{E})$.

Proof. Let $C = \mathrm{Op}(a)$, where $a \in S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Put $k(y,y') := \int e^{i(y-y')\eta} a(y,\eta)\, d\eta$. Using (1.5) and (1.6) and integrating by parts, we see that

$$\sup_{y,y' \in \mathbb{R}^q} \left\| (1+|y|^2)^N (1+|y-y'|^2)^M D_y^\alpha D_{y'}^\beta k(y,y') \right\|_{\mathcal{L}(E,\widetilde{E})} < \infty$$

for all $\alpha, \beta \in \mathbb{N}^q$ and all $M, N \in \mathbb{N}$. An application of Peetre's inequality

$$(1+|y|^2)^M \le C_M (1+|y-y'|^2)^M (1+|y'|^2)^M$$

now shows that $k \in \mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$. To prove the converse, let $k \in \mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$ be given and let $Cu(y) := \int k(y,y')u(y')\, dy'$. Put $a(y,\eta) := (2\pi)^q \int e^{i(y'-y)\eta} k(y,y')\, dy'$. By analogous considerations as above, it follows that $a \in S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Moreover, we have $\mathrm{Op}(a)u(y) = Cu(y)$, since

$$\begin{aligned}
\mathrm{Op}(a)u(y) &= \iint e^{i(y-y')\eta} e^{-iy\eta} (\mathcal{F}^{-1}_{x\to\eta} k(y,x)) u(y')\, dy'd\eta \\
&= \int (\mathcal{F}u)(\eta)(\mathcal{F}^{-1}_{x\to\eta} k(y,x))\, d\eta \\
&= (2\pi)^{-q} \int \mathcal{F}_{x\to\eta}(\widetilde{u}(x) * k(y,x))(\eta)\, d\eta,
\end{aligned}$$

where $\widetilde{u}(x) := u(-x)$. The last integral equals $\int u(x'-x)k(y,x')\, dx'\big|_{x=0}$, and hence $\mathrm{Op}(a)u(y) = Cu(y)$. We thus have shown the equivalence of (i) and (ii). Clearly (ii) implies (iii). On the other hand, let $C \in L^{-\infty,-\infty}(\mathbb{R}^q; E, \widetilde{E})$. By definition, for all $m \in \mathbb{N}$, there is a symbol $a_m \in S^{-m,-m}(\mathbb{R}^q \times \mathbb{R}^q, E, \widetilde{E})$ such that $C = \mathrm{Op}(a_m)$. In view of Proposition 1.11, the symbol a_m is uniquely determined. Since

$$S^{-(m+1),-(m+1)}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \hookrightarrow S^{-m,-m}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$$

for all $m \in \mathbb{N}$, it follows that $a_m = a_{m+1}$ for all $m \in \mathbb{N}$. Put $a := a_1$. Then $a \in S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ and $C = \mathrm{Op}(a)$. □

It is a natural question whether operators in $L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$ can be extended to continuous mappings on Sobolev spaces. The adequate spaces in the operator-valued set-up are the abstract wedge spaces.

Definition 1.13. Let E be a Banach space with an associated fixed strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$.

(i) For $s \in \mathbb{R}$ the space $\mathcal{W}^s(\mathbb{R}^q, E)$ is the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,E)} = \left\{ \int [\eta]^{2s} \left\| \kappa^{-1}(\eta)(\mathcal{F}_{y\to\eta}u)(\eta) \right\|_E^2 d\eta \right\}^{1/2}.$$

(ii) For $s, \delta \in \mathbb{R}$, we define weighted wedge spaces by

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, E) := \left\{ u \in \mathcal{S}'(\mathbb{R}^q, E) \;:\; \langle y\rangle^\delta u \in \mathcal{W}^s(\mathbb{R}^q, E) \right\}.$$

(iii) The space $L_2(\mathbb{R}^q, E)$ is the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\|u\|_{L_2(\mathbb{R}^q,E)} = \left\{ \int \|u(y)\|_E^2 \, dy \right\}^{1/2} .$$

In general, the spaces $L_2(\mathbb{R}^q, E)$ and $\mathcal{W}^0(\mathbb{R}^q, E)$ are not equal, even if $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a group of unitary operators. In order to have equality in this case, we have to assume that E is a Hilbert space.

1.2 The case of Hilbert spaces and unitary actions

Throughout this section we assume that E and $\widetilde{E}$ are separable Hilbert spaces and that, moreover, the associated group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ are unitary.

Definition 1.14. Let $\mu, \mu', \varrho, \varrho' \in \mathbb{R}$. Then $S^{\mu,\mu',\varrho,\varrho'}(\mathbb{R}^{2q}\times\mathbb{R}^{2q}; E, \widetilde{E})$ is defined as the space of all $a \in C^\infty(\mathbb{R}^{2q} \times \mathbb{R}^{2q}, \mathcal{L}(E, \widetilde{E}))$ such that

$$\left\| D_y^\alpha D_{y'}^{\alpha'} D_\eta^\beta D_{\eta'}^{\beta'} a(y, y', \eta, \eta') \right\| \leq c\,[\eta]^{\mu-|\beta|}\,[\eta']^{\mu'-|\beta'|}\,[y]^{\varrho-|\alpha|}\,[y']^{\varrho'-|\alpha'|}$$

for all multi-indices $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^q$ and all $y, y', \eta, \eta' \in \mathbb{R}^q$, with constants $c > 0$ depending only on $\alpha, \alpha', \beta, \beta'$.

With each symbol $a \in S^{\mu,\mu',\varrho,\varrho'}(\mathbb{R}^{2q}\times\mathbb{R}^{2q}; E, \widetilde{E})$ we associate an operator $\mathbf{Op}(a)$ by

$$\mathbf{Op}(a)u(y) := \iiiint e^{-i(z\eta+z'\eta')} a(y, y + z, \eta, \eta')u(y + z + z')\, dz dz' đ\eta đ\eta'$$

for $u \in C_0^\infty(\mathbb{R}^q)$. Then $\mathbf{Op}(a) : C_0^\infty(\mathbb{R}^q) \to C^\infty(\mathbb{R}^q)$.

Theorem 1.15. *Let $a \in S^{\mu,\mu',\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^{2q}; E, \widetilde{E})$ and put*

$$b(y, \eta) := \iint e^{-iz\zeta} a(y, y + z, \eta + \zeta, \eta)\, dz đ\zeta, \qquad (1.8)$$

where the integral is interpreted as an oscillatory integral.
Then $b \in S^{\mu+\mu',\varrho+\varrho'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, and

$$\mathbf{Op}(a) = \mathrm{Op}(b). \qquad (1.9)$$

The symbol b admits the asymptotic expansion

$$b(y, \eta) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} \left(\partial_\eta^\alpha D_{y'}^\alpha a\right)(y, y', \eta, \eta')\Big|_{y'=y,\eta'=\eta} . \qquad (1.10)$$

Proof. Let $0 \leq \theta \leq 1$, let $\widetilde{a} \in S^{\mu,\mu',\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^{2q}; E, \widetilde{E})$, and put

$$\widetilde{b}_\theta(y,\eta) := \iint e^{-iz\zeta}\widetilde{a}(y, y+z, \eta+\theta\zeta, \eta)\, dz d\zeta. \tag{1.11}$$

To given multi-indices $\alpha, \beta \in \mathbb{N}^q$ choose $k, l \in \mathbb{N}$ such that $2k > q + |\mu| + |\beta|$ and $2l > q + |\varrho'| + |\alpha|$. Put

$$\begin{aligned} &r_\theta(y,z,\eta,\zeta) \\ &\quad := (1+|\zeta|^2)^{-k}(1-\Delta_z)^k(1+|z|^2)^{-l}(1-\Delta_\zeta)^l\widetilde{a}(y, y+z, \eta+\theta\zeta, \eta). \end{aligned}$$

Integrating by parts we see that

$$\widetilde{b}_\theta(y,\eta) = \iint e^{-iz\zeta} r_\theta(y,z,\eta,\zeta)\, dz d\zeta.$$

The symbol estimates yield

$$\begin{aligned} \left\| D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\|_{\mathcal{L}(E,\widetilde{E})} \leq [\zeta]^{-2k}[z]^{-2l} \sum_{\substack{|\gamma|\leq|\alpha| \\ |\delta|\leq|\beta|}} c_{\gamma,\delta}\,[y]^{\varrho-|\gamma|}\,[y+z]^{\varrho'-(|\alpha|-|\gamma|)} \\ \cdot [\eta+\theta\zeta]^{\mu-|\delta|}\,[\eta]^{\mu'-(|\beta|-|\delta|)}\,\pi_{\alpha,\beta}(\widetilde{a}), \end{aligned}$$

where $\pi_{\alpha,\beta}$ is a semi-norm on $S^{\mu,\mu',\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^{2q}; E, \widetilde{E})$ that only depends on $\alpha, \beta \in \mathbb{N}^q$, and $c_{\gamma,\delta}$, $|\gamma| \leq |\alpha|$, $|\delta| \leq |\beta|$, are constants that only depend on $\alpha, \beta, \gamma, \delta$, but not on $\theta \in [0,1]$. Using Peetre's inequality $[x+x']^s \leq c_s\,[x]^s\,[x']^{|s|}$, we get

$$\begin{aligned} &\left\| D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\|_{\mathcal{L}(E,\widetilde{E})} \\ &\quad \leq c_{\alpha,\beta}\,[y]^{\varrho+\varrho'-|\alpha|}\,[z]^{|\varrho'|+|\alpha|-2l}\,[\eta]^{\mu+\mu'-|\beta|}\,[\zeta]^{|\mu|+|\beta|-2k}\,\pi_{\alpha,\beta}(\widetilde{a}), \end{aligned}$$

with a constant $c_{\alpha,\beta}$ that only depends on $\alpha, \beta \in \mathbb{N}^q$, but not on $\theta \in [0,1]$. Hence

$$\begin{aligned} \left\| D_y^\alpha D_\eta^\beta \widetilde{b}_\theta(y,\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} &\leq c_{\alpha,\beta}\pi_{\alpha,\beta}(\widetilde{a})\,[y]^{\varrho+\varrho'-|\alpha|}\,[\eta]^{\mu+\mu'-|\beta|} \\ &\quad \cdot \iint [z]^{|\varrho'|+|\alpha|-2l}\,[\zeta]^{|\mu|+|\beta|-2k}\, dz d\zeta \\ &\leq c'_{\alpha,\beta}\pi_{\alpha,\beta}(\widetilde{a})\,[y]^{\varrho+\varrho'-|\alpha|}\,[\eta]^{\mu+\mu'-|\beta|}, \end{aligned}$$

with a constant $c'_{\alpha,\beta}$ that only depend on $\alpha, \beta \in \mathbb{N}^q$ and on $k, l \in \mathbb{N}$, but not on the symbol $\widetilde{a}$ and the parameter $\theta \in [0,1]$. Thus $\widetilde{b}_\theta \in S^{\mu+\mu',\varrho+\varrho'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ for all $\theta \in [0,1]$, and the set $\left\{\widetilde{b}_\theta \,:\, 0 \leq \theta \leq 1\right\}$ is bounded in $S^{\mu+\mu',\varrho+\varrho'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Now let $a = \widetilde{a}$. Then $b(y,\eta) = \widetilde{b}_1(y,\eta)$. For $u \in C_0^\infty(\mathbb{R}^q, E)$, we have

$$\operatorname{Op}(b)u(y) = \iint e^{i(y-y')\eta}\left\{\iint e^{-iz\zeta} a(y, y+z, \eta+\zeta, \eta)\, dz d\zeta\right\} u(y')\, dy' d\eta.$$

Substituting $\eta' = \eta + \zeta$ and $z' = y' - y - z$ in the oscillatory integral, we see that

$$\begin{aligned}
&\mathrm{Op}(b)u(y) \\
&= \iiiint e^{-i(z'+z)\eta} e^{-iz(\eta'-\eta)} a(y, y+z, \eta', \eta) u(y+z+z')\, dz dz' d\eta' d\eta \\
&= \iiiint e^{-i(z'\eta + z\eta')} a(y, y+z, \eta', \eta) u(y+z+z')\, dz' dz d\eta' d\eta \\
&= \mathbf{Op}(a)u(y).
\end{aligned}$$

Hence $\mathrm{Op}(b) = \mathbf{Op}(a)$. It remains to prove (1.10). A Taylor expansion around $\zeta = 0$ yields

$$\begin{aligned}
&a(y, y+z, \eta+\zeta, \eta) \\
&\quad = \sum_{|\alpha|<N} \frac{\zeta^\alpha}{\alpha!} \partial_\xi^\alpha a(y, y+z, \xi, \eta)\big|_{\xi=\eta} \\
&\quad + N \sum_{|\alpha|=N} \frac{\zeta^\alpha}{\alpha!} \int_0^1 (1-\theta)^{N-1}\, \partial_\xi^\alpha a(y, y+z, \xi, \eta)\big|_{\xi=\eta+\theta\zeta}\, d\theta.
\end{aligned}$$

Integrating by parts and using the formula $\iint e^{-iz\zeta} f(z)\, dz d\zeta = f(0)$ shows that

$$\begin{aligned}
&\iint e^{-iz\zeta} \frac{\zeta^\alpha}{\alpha!}\, \partial_\xi^\alpha a(y, y+z, \xi, \eta)\big|_{\xi=\eta}\, dz d\zeta \\
&\quad = \iint e^{-iz\zeta} \frac{1}{\alpha!}\, D_{y'}^\alpha \partial_\xi^\alpha a(y, y', \xi, \eta)\big|_{y'=y+z, \xi=\eta}\, dz d\zeta \\
&\quad = \frac{1}{\alpha!}\, \partial_\xi^\alpha D_{y'}^\alpha a(y, y', \xi, \eta)\big|_{y'=y, \xi=\eta}.
\end{aligned}$$

Hence

$$b(y, \eta) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\, (\partial_\eta^\alpha D_{y'}^\alpha a)(y, y', \eta, \eta')\big|_{y'=y, \eta'=\eta} + r_N(y, \eta),$$

where

$$\begin{aligned}
&r_N(y, \eta) \\
&= \sum_{|\alpha|=N} \frac{N}{\alpha!} \int_0^1 (1-\theta)^{N-1} \iint e^{-iz\zeta}\, \partial_\xi^\alpha a(y, y+z, \xi, \eta)\big|_{\xi=\eta+\theta\zeta}\, dz d\zeta d\theta \\
&= \sum_{|\alpha|=N} \frac{N}{\alpha!} \int_0^1 (1-\theta)^{N-1} \iint e^{-iz\zeta}\, \partial_\xi^\alpha D_{y'}^\alpha a(y, y', \xi, \eta)\big|_{y'=y+z, \xi=\eta+\theta\zeta}\, dz d\zeta d\theta.
\end{aligned}$$

Write $\widetilde{a}(y, y', \xi, \eta) = \partial_\xi^\alpha D_{y'}^\alpha a(y, y', \xi, \eta)$.
Then $\widetilde{a} \in S^{\mu-|\alpha|, \mu', \varrho, \varrho'-|\alpha|}(\mathbb{R}^{2q} \times \mathbb{R}^{2q}; E, \widetilde{E})$. With $\widetilde{b}_\theta$ as in (1.11), we have

$$\begin{aligned}
&\int_0^1 (1-\theta)^{N-1} \iint e^{-iz\zeta} \widetilde{a}(y, y+z, \eta+\theta\zeta, \eta)\, dz d\zeta d\theta \\
&\quad = \int_0^1 (1-\theta)^{N-1} \widetilde{b}_\theta(y, \eta)\, d\theta,
\end{aligned}$$

and $\left\{\widetilde{b}_\theta \;:\; 0 \le \theta \le 1\right\}$ is bounded in $S^{\mu+\mu'-N,\varrho+\varrho'-N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Since $\widetilde{b}_\theta$ depends continuously on θ, it follows that

$$\int_0^1 (1-\theta)^{N-1}\widetilde{b}_\theta(y,\eta)\, d\theta \in S^{\mu+\mu'-N,\varrho+\varrho'-N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}),$$

and thus $r_N \in S^{\mu+\mu'-N,\varrho+\varrho'-N}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Since $N \in \mathbb{N}\setminus\{0\}$ was arbitrary, (1.10) follows. □

In the next Proposition $\widetilde{\widetilde{E}}$ is a further separable Hilbert space with associated unitary group action $\left\{\widetilde{\widetilde{\kappa}}_\lambda\right\}_{\lambda\in\mathbb{R}_+}$.

Proposition 1.16. *Let* $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{\widetilde{E}})$ *and* $b \in S^{\nu,\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$.

(i) Set $c(y,\eta) := \iint e^{-iz\zeta} a(y,\eta+\zeta) b(y+z,\eta)\, dz đ\zeta$.
Then $c \in S^{\mu+\nu,\varrho+\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{\widetilde{E}})$ *and* $\mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(c)$.

(ii) The symbol c admits the asymptotic expansion

$$c(y,\eta) \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\eta^\alpha a(y,\eta)) D_y^\alpha b(y,\eta) \tag{1.12}$$

in $S^{\mu+\nu,\varrho+\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{\widetilde{E}})$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^q, E)$. Then

$$\begin{aligned}&\mathrm{Op}(a)\mathrm{Op}(b)u(y)\\ &\quad = \iiiint e^{i(y-y')\eta} e^{i(y'-y'')\eta'} a(y,\eta) b(y'\eta') u(y'')\, dy'' đ\eta' dy' đ\eta.\end{aligned}$$

Substituting $x = y' - y$ and $x' = y'' - y - x$, we see that the integral on the right hand side equals

$$\iiint e^{-i(x\eta + x'\eta')} a(y,\eta) b(y+x,\eta') u(y+x+x')\, dx' đ\eta' dx đ\eta.$$

Set $q(y,y',\eta,\eta') := a(y,\eta)b(y',\eta')$. Then $q \in S^{\mu,\nu,\varrho,\sigma}(\mathbb{R}^{2q} \times \mathbb{R}^{2q}; E, \widetilde{\widetilde{E}})$ and $\mathrm{Op}(a)\mathrm{Op}(b) = \mathbf{Op}(q)$. By Theorem 1.15, there is a symbol $c \in S^{\mu+\nu,\varrho+\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{\widetilde{E}})$ such that $\mathbf{Op}(q) = \mathrm{Op}(c)$. Moreover,

$$\begin{aligned} c(y,\eta) &\sim \sum_\alpha \frac{1}{\alpha!}\,(\partial_\eta^\alpha D_{y'}^\alpha q)(y,y',\eta,\eta')\big|_{y'=y,\eta'=\eta} \\ &\sim \sum_\alpha \frac{1}{\alpha!}(\partial_\eta^\alpha a(y,\eta)) D_y^\alpha b(y,\eta). \end{aligned}$$

□

Remark 1.17. (i) Formula (1.12) is the Leibniz summation. As usual, we write

$$a\sharp b \sim \sum_{\alpha} \frac{1}{\alpha!}(\partial_\eta^\alpha a(y,\eta))D_y^\alpha b(y,\eta).$$

This relation holds modulo $S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{\tilde{E}})$. In fact, we may make $a\sharp b$ a uniquely determined symbol by setting $\mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(a\sharp b)$, i.e.,

$$(a\sharp b)(y,\eta) = \iint e^{-iz\zeta} a(y, \eta + \zeta) b(y + z, \eta)\, dz đ\zeta,$$

see (1.8). In view of Proposition 1.16 and Proposition 1.11, $a\sharp b$ is then well-defined.

(ii) If $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; \tilde{E}, \tilde{\tilde{E}})$, $b \in S^{\nu,\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E})$, and b is independent of $y \in \mathbb{R}^q$, i.e., $b(y,\eta) = b(\eta)$ for all $y, \eta \in \mathbb{R}^q$, then $a\sharp b = ab$. This follows directly from the composition formula.

The $L_2(\mathbb{R}^q, E)$ scalar product is denoted by $(\cdot,\cdot)_{L_2(\mathbb{R}^q,E)}$, the scalar products of E and $\tilde{E}$ are denoted by $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_{\tilde{E}}$, respectively. If $A : \mathcal{S}(\mathbb{R}^q, E) \to \mathcal{S}(\mathbb{R}^q, \tilde{E})$ is a continuous operator, then the formal adjoint A^* of A is defined by

$$(Au, v)_{L_2(\mathbb{R}^q,\tilde{E})} = (u, A^* v)_{L_2(\mathbb{R}^q,E)}$$

for all $u \in \mathcal{S}(\mathbb{R}^q, E)$, $v \in \mathcal{S}(\mathbb{R}^q, \tilde{E})$. Moreover, if $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E})$, then we denote the point-wise adjoint of a by $a^{(*)}$, i.e., $(a(y,\eta)f, e)_{\tilde{E}} = (f, a^{(*)}(y,\eta)e)_E$ for all $f \in E$, $e \in \tilde{E}$, and $y, \eta \in \mathbb{R}^q$.

Proposition 1.18. *Let $A \in L^{\mu,\varrho}(\mathbb{R}^q; E, \tilde{E})$, $A = \mathrm{Op}(a)$ with $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E})$. Then $A^* \in L^{\mu,\varrho}(\mathbb{R}^q; \tilde{E}, E)$ and $A^* = \mathrm{Op}(a^*)$, where $a^*(y,\eta) = \iint e^{-iz\zeta} a^{(*)}(y + z, \eta + \zeta)\, dz đ\zeta$. The symbol a^* admits the asymtotic expansion*

$$a^*(y,\eta) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\eta^\alpha D_y^\alpha a^{(*)}(y,\eta) \tag{1.13}$$

in $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; \tilde{E}, E)$.

Proof. For $u \in \mathcal{S}(\mathbb{R}^q, E)$, $v \in \mathcal{S}(\mathbb{R}^q, \tilde{E})$, we have

$$\begin{aligned}
(Au, v)_{L_2(\mathbb{R}^q,\tilde{E})} &= \int \left(\iint e^{i(y-y')\eta} a(y,\eta) u(y')\, dy' đ\eta, v(y) \right)_{\tilde{E}} dy \\
&= \iiint (e^{i(y-y')\eta} a(y,\eta) u(y'), v(y))_{\tilde{E}}\, dy' đ\eta dy \\
&= \iiint (e^{i(y-y')\eta} (u(y'), a^{(*)}(y,\eta) v(y))_E\, dy đ\eta dy' \\
&= \int \left(u(y'), \iint e^{i(y'-y)\eta} a^{(*)}(y,\eta) v(y)\, dy đ\eta \right)_E dy' \\
&= (u, \mathrm{Op}(a^{(*)}(y,\eta)) v)_E .
\end{aligned}$$

Hence $A^*v = \iint e^{i(y-y')\eta} a^{(*)}(y', \eta) v(y')\, dy' đ\eta$. Writing $a^{(*)}(y', \eta) = \iint e^{-ix'\zeta} a^{(*)}(y' - x', \eta) u(y')\, dx' đ\zeta$, we get

$$A^*v(y) = \iiiint e^{i(y-y')\eta} e^{-ix'\zeta} a^{(*)}(y' - x', \eta) u(y')\, dx' đ\zeta dy' đ\eta.$$

Substituting $\eta' = \eta + \zeta$ and $y' = y + x' + x$ yields

$$\begin{aligned} A^*v(y) &= \iiiint e^{-i(x'+x)\eta} e^{-ix'(\eta'-\eta)} a^{(*)}(y + x, \eta) u(y + x' + x)\, dx' dx đ\eta' đ\eta \\ &= \iiiint e^{-i(x\eta + x'\eta')} a^{(*)}(y + x, \eta) u(y + x + x')\, dx' dx đ\eta đ\eta'. \end{aligned}$$

Theorem 1.15 now shows that $A^* = \mathrm{Op}(a^*)$, where $a^*(y, \eta) = \iint e^{-ix\zeta} a^{(*)}(y + x, \eta + \zeta)$. The asymptotic expansion (1.13) follows from (1.10). $\square$

The next thing we shall do is to prove that operators in $L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$ act continuously on the weighted Sobolev spaces, i.e.,

$$L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E}) \subseteq \mathcal{L}(\mathcal{W}^{s,\delta}(\mathbb{R}^q, E), \mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \widetilde{E})).$$

The main step here is the corresponding version of the Calderón-Vaillancourt Theorem for operators with operator-valued symbols. The proof of Theorem 1.19 generalizes an idea of HWANG [8].

Theorem 1.19. *Let $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ and suppose that*

$$\pi(a) := \sup_{\substack{(y,\eta) \in \mathbb{R}^q \times \mathbb{R}^q \\ \alpha, \beta \le (1, \dots, 1)}} \left\| D_y^\alpha D_\eta^\beta a(y, \eta) \right\|_{\mathcal{L}(E, \widetilde{E})} < \infty. \tag{1.14}$$

Then $A = \mathrm{Op}(a)$ induces a continuous operator

$$A : L_2(\mathbb{R}^q, E) \to L_2(\mathbb{R}^q, \widetilde{E}).$$

In the proof of Theorem 1.19 we shall find it convenient to write $(i + \eta)^\delta = (i + \eta_1)^{\delta_1} \cdots (i + \eta_q)^{\delta_q}$ and $(i + D_\eta)^\delta = (i + D_{\eta_1})^{\delta_1} \cdots (i + D_{\eta_q})^{\delta_q}$. With this notation we have

$$(i + D_\eta)^\delta e^{i(y-y')\eta} = (i + y - y')^\delta e^{i(y-y')\eta}.$$

Moreover, we write $\widehat{u}(\eta) = (\mathcal{F}_{y\to\eta} u)(\eta)$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^q, E)$ and $v \in C_0^\infty(\mathbb{R}^q, \widetilde{E})$ be given.

Step 1. We first check two norm estimates. Let $\beta, \delta \in \mathbb{N}^q$, where $\delta \ge (1, \dots, 1)$ and $\beta \le \delta$. Put

$$\begin{aligned} f_{\delta,\beta}(y, \eta) &:= \int e^{-iy'\eta} (-D_y)^\beta \frac{u(y')}{(i + y - y')^\delta} dy', \\ g_\delta(y, \eta) &:= \int e^{iy'\eta} \overline{((i + \eta - \eta')^\delta)}^{-1} \widehat{v}(\eta')\, đ\eta. \end{aligned}$$

We want to verifty that

$$\|f_{\delta,\beta}\|_{L_2(\mathbb{R}^{2q},E)} \leq c_{\delta,\beta}\,\|u\|_{L_2(\mathbb{R}^q,E)}\,, \tag{1.15}$$

$$\|g_\delta\|_{L_2(\mathbb{R}^{2q},\widetilde{E})} \leq c_\delta\,\|v\|_{L_2(\mathbb{R}^q,\widetilde{E})}\,, \tag{1.16}$$

with constants $c_\delta, c_{\beta,\delta}$, that only depends on $\delta, \beta \in \mathbb{N}^q$, but not on u, v. For g_δ, we calculate

$$\begin{aligned}
\|g_\delta\|^2_{L_2(\mathbb{R}^{2q},\widetilde{E})} &= \int \|g_\delta(\cdot,\eta)\|^2_{L_2(\mathbb{R}^q_y,\widetilde{E})}\,d\eta \\
&= \int \left\|\mathcal{F}^{-1}_{y\to\eta'} g_\delta(\cdot,\eta)\right\|^2_{L_2(\mathbb{R}^q_{\eta'},\widetilde{E})}\,d\eta \\
&= \iint \left|(i+\eta-\eta')^\delta\right|^{-2} \|\widehat{v}(\eta')\|^2_{\widetilde{E}}\,d\eta d\eta' \\
&= \|\widehat{v}\|^2_{L_2(\mathbb{R}^q,\widetilde{E})} \int \left|(i+\zeta)^\delta\right|^{-2} d\zeta \\
&= \|v\|^2_{L_2(\mathbb{R}^q,\widetilde{E})} \prod_{j=1}^q \int (1+\zeta_j^2)^{-\delta_j}\,d\zeta_j \\
&\leq c_\delta\,\|v\|^2_{L_2(\mathbb{R}^q,\widetilde{E})}\,,
\end{aligned}$$

provided that $d_j \geq 1$ for $j = 1, \ldots, q$. It was used that the Plancherel identity holds for functions $u \in L_2(\mathbb{R}^q, H)$, if H is a Hilbert space. Hence (1.16) is proved. Analogously, it follws that

$$\|f_{\delta,\beta}\|^2_{L_2(\mathbb{R}^{2q},E)} = \|u\|^2_{L_2(\mathbb{R}^q,E)} \int \left|(i-D_x)^\beta (i+x)^{-\delta}\right|^2 dx.$$

If $\beta \leq \delta$, then there are constants $\widetilde{c}_{\delta,\beta} > 0$, that only depends on δ, β, such that

$$\left|(i-D_x)^\beta (i+x)^{-\delta}\right| \leq \widetilde{c}_{\delta,\beta}\left|(i+x)^{-\delta}\right|.$$

Thus

$$\begin{aligned}
\|f_{\delta,\beta}\|^2_{L_2(\mathbb{R}^{2q},E)} &\leq \|v\|^2_{L_2(\mathbb{R}^q,E)}\,\widetilde{c}_{\delta,\beta} \int \left|(i+x)^{-\delta}\right|^2 dx \\
&\leq c_{\delta,\beta}\,\|v\|^2_{L_2(\mathbb{R}^q,E)}\,.
\end{aligned}$$

provided that $\delta \geq (1, \ldots, 1)$. This proves (1.15).

Step 2. Assume that $a \in C_0^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ and set $A = \mathrm{Op}(a)$. We have

$$\begin{aligned}
(Au, v)_{L_2(\mathbb{R}^q,\widetilde{E})} &= \iiint (e^{i(y-y')\eta} a(y,\eta) u(y'), v(y))_{\widetilde{E}}\,dy' đ\eta dy \\
&= \iiiint (e^{i(y-y')\eta - iy\eta'} a(y,\eta) u(y'), \widehat{v}(\eta'))_{\widetilde{E}}\,dy' đ\eta dy đ\eta'.
\end{aligned}$$

An integration by parts gives

$$
\begin{aligned}
&(Au, v)_{L_2(\mathbb{R}^q,\widetilde{E})}\\
&= \iiiint \Big(\frac{(i+D_\eta)^\delta}{(i+y-y')^\delta}\frac{(i+D_y)^\delta}{(i+\eta-\eta')^\delta} e^{i(y-y')\eta - iy\eta'} a(y,\eta)u(y'), \widehat{v}\Big)_{\widetilde{E}}\, dy'\, d\eta\, d\eta'\, dy\\
&= \iiiint \Big(e^{i(y-y')\eta - iy\eta'}(i-D_y)^\delta \left\{(i-D_\eta)^\delta a(y,\eta)\frac{u(y')}{(i+y-y')^\delta}\right\},\\
&\quad \frac{\widehat{v}(\eta')}{(i+\eta-\eta')^\delta}\Big)_{\widetilde{E}}\, dy'\, d\eta\, d\eta'\, dy\\
&= \sum_{\beta+\gamma=\delta} c_{\delta,\beta,\gamma} \iint \Big(e^{iy\eta}\left\{(i-D_y)^\gamma (i-D_\eta)^\delta a(y,\eta)\right\}\\
&\quad \cdot \int e^{-iy'\eta}(i-D_y)^\beta \frac{u(y')}{(i+y-y')^\delta}\, dy', \int e^{iy\eta'}\overline{(i+\eta-\eta')^{-\delta}}\widehat{v}(\eta')\, d\eta'\Big)_{\widetilde{E}}\, dy\, d\eta\\
&= \sum_{\beta+\gamma=\delta} c_{\delta,\beta,\gamma} \iint \Big(e^{iy\eta}\left\{(i-D_y)^\gamma (i-D_\eta)^\delta a(y,\eta)\right\} f_{\delta,\beta}(y,\eta),\\
&\quad g_\delta(y,\eta)\Big)_{\widetilde{E}}\, dy\, d\eta,
\end{aligned}
$$

with suitable constants $c_{\delta,\beta,\gamma}$. Now the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
&\left|(Au,v)_{L_2(\mathbb{R}^q,\widetilde{E})}\right|\\
&\quad \le \sum_{\beta+\gamma=\delta} c_{\delta,\beta,\gamma}\left\{\iint \left\|\left\{(i-D_y)^\gamma (i-D_\eta)^\delta a(y,\eta)\right\} f_{\delta,\beta}(y,\eta)\right\|_{\widetilde{E}}^2\, dy\, d\eta\right\}^{1/2}\\
&\qquad \cdot \left\{\iint \|g_\delta(y,\eta)\|_{\widetilde{E}}^2\, dy\, d\eta\right\}^{1/2}\\
&\quad \le \widetilde{c}_\delta \sup_{\substack{y,\eta\in\mathbb{R}^q\\ \alpha,\beta\le\delta}} \left\|D_y^\alpha D_\eta^\beta a(y,\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \|f_{\delta,\beta}\|_{L_2(\mathbb{R}^{2q},E)}\, \|g_\delta\|_{L_2(\mathbb{R}^{2q},\widetilde{E})}\,.
\end{aligned}
$$

Using (1.15) and (1.16), we see that there is a constant $c_\delta > 0$ such that

$$
\left|(Au,v)_{L_2(\mathbb{R}^q,\widetilde{E})}\right| \le c_\delta \sup_{\substack{y,\eta\in\mathbb{R}^q\\ \alpha,\beta\le\delta}} \left\|D_y^\alpha D_\eta^\beta a(y,\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \|u\|_{L_2(\mathbb{R}^q,E)}\, \|v\|_{L_2(\mathbb{R}^q,\widetilde{E})}
$$

provided that $\delta \ge (1,\ldots,1)$. In particular,

$$
\left|(Au,v)_{L_2(\mathbb{R}^q,\widetilde{E})}\right| \le c\pi(a)\, \|u\|_{L_2(\mathbb{R}^q,E)}\, \|v\|_{L_2(\mathbb{R}^q,\widetilde{E})}\,, \tag{1.17}
$$

with some $c > 0$.

Step 3. Next, let $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$ and suppose that $\pi(a) < \infty$. Let $\omega \in C_0^\infty(\mathbb{R}^q \times \mathbb{R}^q)$ be a cut-off function, i.e., $\omega(y,\eta)$ equals 1 in some neighbourhood of $(y,\eta) = 0$. Put $a_\varepsilon(y,\eta) := \omega(\varepsilon y, \varepsilon\eta)a(y,\eta)$. We then have

$$
\lim_{\varepsilon\to 0+} (\mathrm{Op}(a_\varepsilon)u, v)_{L_2(\mathbb{R}^q,\widetilde{E})} = (\mathrm{Op}(a)u, v)_{L_2(\mathbb{R}^q,\widetilde{E})}.
$$

Moreover,

$$\begin{aligned}\left|(Au, v)_{L_2(\mathbb{R}^q,\tilde{E})}\right| &= \lim_{\varepsilon\to 0+}\left|(\mathrm{Op}(a_\varepsilon)u, v)_{L_2(\mathbb{R}^q,\tilde{E})}\right| \\ &\le \tilde{c}\lim_{\varepsilon\to 0+}\pi(a_\varepsilon)\,\|u\|_{L_2(\mathbb{R}^q,E)}\,\|v\|_{L_2(\mathbb{R}^q,\tilde{E})} \\ &\le c\pi(a)\,\|u\|_{L_2(\mathbb{R}^q,E)}\,\|v\|_{L_2(\mathbb{R}^q,\tilde{E})}\,,\end{aligned}$$

and c is independent of u, v, and of a. Since $C_0^\infty(\mathbb{R}^q, \tilde{E})$ is dense in $L_2(\mathbb{R}^q, \tilde{E})$, it follows that

$$\|Au\|_{L_2(\mathbb{R}^q,\tilde{E})} \le c\pi(a)\,\|u\|_{L_2(\mathbb{R}^q,E)}$$

for all $u \in C_0^\infty(\mathbb{R}^q, E)$. Extension by continuity completes the proof. □

Since the spaces $E, \tilde{E}$ are supposed to be Hilbert spaces with associated unitary group actions, we have $\mathcal{W}^{0,0}(\mathbb{R}^q, E) = L_2(\mathbb{R}^q, E)$ and $\mathcal{W}^{0,0}(\mathbb{R}^q, \tilde{E}) = L_2(\mathbb{R}^q, \tilde{E})$. We note a simple consequence of Theorem 1.19.

Corollary 1.20. *Let $A \in L^{0,0}(\mathbb{R}^q; E, \tilde{E})$. Then A induces a continuous operator*

$$A : \mathcal{W}^{0,0}(\mathbb{R}^q, E) \to \mathcal{W}^{0,0}(\mathbb{R}^q, \tilde{E}).$$

Define operators

$$\begin{aligned}R^{s,\delta} &:= \mathrm{Op}([\eta]^s) \circ \mathrm{Op}([y]^\delta), \\ P^{s,\delta} &:= \mathrm{Op}([y]^\delta) \circ \mathrm{Op}([\eta]^s).\end{aligned}$$

The operators $R^{s,\delta}$ and $P^{s,\delta}$ are called *reductions of orders*.

Proposition 1.21. *Let $s, \delta \in \mathbb{R}$. Then*

$$R^{s,\delta} : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \to \mathcal{W}^{0,0}(\mathbb{R}^q, E)$$

is an isometrical isomorphism with inverse $(R^{s,\delta})^{-1} = P^{-s,-\delta}$.

Proof. Let $u \in \mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$.

$$\begin{aligned}\left\|R^{s,\delta}u\right\|^2_{\mathcal{W}^{0,0}(\mathbb{R}^q,E)} &= \int \left\|\mathrm{Op}([\eta]^s)\mathrm{Op}([y]^\delta)u\right\|_E^2 dy \\ &= \int \left\|\kappa^{-1}(\eta)\mathcal{F}_{y\to\eta}\mathrm{Op}([\eta]^s)\mathrm{Op}([y]^\delta)u\right\|_E^2 d\eta \\ &= \int [\eta]^{2s}\left\|\kappa^{-1}(\eta)\mathcal{F}_{y\to\eta}([y]^\delta\, u(y))\right\|_E^2 d\eta \\ &= \left\|[y]^\delta\, u\right\|^2_{\mathcal{W}^{s,0}(\mathbb{R}^q,E)} = \|u\|^2_{\mathcal{W}^{s,\delta}(\mathbb{R}^q,E)}\,.\end{aligned}$$

Moreover, $R^{s,\delta} \circ P^{-s,-\delta} = \mathrm{id}$ on $\mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$. Hence $R^{s,\delta}$ is isometrical and surjective. □

Proposition 1.22. *Let $A \in L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$. Then A induces continuous operators*

$$A : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \widetilde{E})$$

for all $s, \delta \in \mathbb{R}$.

Proof. Put $B := R^{s-\mu,\delta-\varrho} \circ A \circ (R^{s,\delta})^{-1}$. By Proposition 1.21 and Proposition 1.16, $B \in L^{0,0}(\mathbb{R}^q; E, \widetilde{E})$, i.e., $B = \mathrm{Op}(b)$ with some $b \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Now Corollary 1.20 implies that there is a constant $c > 0$ such that

$$\|Bu\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,\widetilde{E})} \leq c\,\|u\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,E)}\,.$$

Hence

$$\begin{aligned}\|Au\|_{\mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q,\widetilde{E})} &= \left\|R^{s-\mu,\delta-\varrho}Au\right\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,\widetilde{E})} \\ &= \left\|BR^{s,\delta}u\right\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,\widetilde{E})} \leq c\left\|R^{s,\delta}u\right\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,E)} \\ &= c\,\|u\|_{\mathcal{W}^{s,\delta}(\mathbb{R}^q,E)}\,.\end{aligned}$$

□

1.3 The case of arbitrary group actions

In this section, we consider again Banach spaces E_0, E_1, E_2 with associated strongly continuous group actions $\{\kappa_{0,\lambda}\}_{\lambda\in\mathbb{R}_+}$, $\{\kappa_{1,\lambda}\}_{\lambda\in\mathbb{R}_+}$, and $\{\kappa_{2,\lambda}\}_{\lambda\in\mathbb{R}_+}$. It is not assumed that the group actions $\{\kappa_{i,\lambda}\}_{\lambda\in\mathbb{R}_+}$, $i = 0, 1, 2$, are unitary. Recall that by (1.1), there are constants $c > 0$, $M_i > 0$ ($i = 0, 1, 2$), such that $\|\kappa_{i,\lambda}\|_{\mathcal{L}(E_i)} \leq c\max\{\lambda^{-M_i}, \lambda^{M_i}\}$.

Proposition 1.23. *Let $a \in S^{\mu,\varrho}(\mathbb{R}^q\times\mathbb{R}^q; E_1, E_2)$ and $b \in S^{\nu,\sigma}(\mathbb{R}^q\times\mathbb{R}^q; E_0, E_1)$.*

(i) Set $c(y,\eta) := \iint e^{-iz\zeta}a(y,\eta+\zeta)b(y+z,\eta)\,dz\,đ\zeta$. Then $c \in S^{\mu+\nu,\varrho+\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2)$ and $\mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(c)$.

(ii) The symbol c admits the asymptotic expansion

$$c(y,\eta) \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\eta^\alpha a(y,\eta))D_y^\alpha b(y,\eta)$$

in $S^{\mu+\nu,\varrho+\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2)$.

Proof. Let $0 \leq \theta \leq 1$ and put

$$\widetilde{c}_\theta(y,\eta) := \iint e^{-iz\zeta}a(y,\eta+\theta\zeta)b(y+z,\eta)\,dz\,đ\zeta.$$

To given multi-indices $\alpha, \beta \in \mathbb{N}^q$ choose $k, l \in \mathbb{N}$ such that $2k > q + |\mu| + |\beta| + M_1 + M_2$ and $2l > q + |\sigma| + |\alpha|$. Put

$$\begin{aligned}&r_\theta(y,z,\eta,\zeta) := \\ &\quad(1+|\zeta|^2)^{-k}(1-\Delta_z)^k(1+|z|^2)^{-l}(1-\Delta_\zeta)^l a(y,\eta+\theta\zeta)b(y+z,\eta).\end{aligned}$$

Integrating by parts, we see that

$$\widetilde{c}_\theta(y,\eta) = \iint e^{-iz\zeta} r_\theta(y,z,\eta,\zeta)\, dz d\!\!^-\zeta.$$

We want to show that $\{\widetilde{c}_\theta \,:\, 0 \le \theta \le 1\}$ is a bounded set in $S^{\mu+\nu,\varrho+\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_2)$. To this end, we consider

$$\begin{aligned}
&\kappa_2^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\}\kappa_0(\eta)\\
&= \kappa_2^{-1}(\eta)\kappa_2(\eta+\theta\zeta)\sum_{\substack{\gamma\le\alpha\\ \delta\le\beta}} \frac{\alpha!\beta!}{(\alpha-\gamma)!\gamma!(\beta-\delta)!\delta!}\kappa_2^{-1}(\eta+\theta\zeta)\\
&\quad\cdot\left\{(1+|\zeta|^2)^{-k}(1-\Delta_\zeta)^l D_\eta^\gamma D_y^\delta a(y,\eta+\theta\zeta)\right\}\kappa_1(\eta+\theta\zeta)(\kappa_1^{-1}(\eta+\theta\zeta)\kappa_1(\eta))\\
&\quad\cdot\kappa_1^{-1}(\eta)\left\{(1-\Delta_z)^k(1+|z|^2)^{-l} D_\eta^{\alpha-\gamma} D_y^{\beta-\delta} b(y+z,\eta)\right\}\kappa_0(\eta),
\end{aligned}$$

and estimate

$$\begin{aligned}
&\left\|\kappa_2^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\}\kappa_0(\eta)\right\|_{\mathcal{L}(E_0,E_2)}\\
&\quad\le \left\|\kappa_2^{-1}(\eta)\kappa_2(\eta+\theta\zeta)\right\|_{\mathcal{L}(E_2)}\sum_{\substack{\gamma\le\alpha\\ \delta\le\beta}} c_{\alpha,\beta,\gamma,\delta}\,[\zeta]^{-2k}\,\widetilde{\pi}_{\alpha,\beta,\gamma,\delta}(a)\\
&\quad\cdot[y]^{\varrho-|\delta|}\,[\eta+\theta\zeta]^{\mu-|\gamma|}\left\|\kappa_1^{-1}(\eta+\theta\zeta)\kappa_1(\eta)\right\|_{\mathcal{L}(E_1)}[z]^{-2l}\,\pi_{\alpha,\beta,\gamma,\delta}(b)\\
&\quad\cdot[y+z]^{\sigma-|\beta|+|\delta|}\,[\eta]^{\nu-|\alpha|+|\gamma|},
\end{aligned}$$

where $\widetilde{\pi}_{\alpha,\beta,\gamma,\delta}$ is a suitable semi-norm on $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2)$ and $\pi_{\alpha,\beta,\gamma,\delta}$ is a suitable semi-norm on $S^{\nu,\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$. In view of (1.1) and Peetre's inequality $[\xi] \le c'\,[\xi+\xi']\,[\xi']$, we can estimate

$$\begin{aligned}
\left\|\kappa_2^{-1}(\eta)\kappa_2(\eta+\theta\zeta)\right\|_{\mathcal{L}(E_2)} &= \left\|\kappa_{([\eta+\theta\zeta]/[\eta]),2}\right\|_{\mathcal{L}(E_2)}\\
&\le c'\left\|\kappa_{[\theta\zeta],2}\right\|_{\mathcal{L}(E_2)} \le c''\,[\zeta]^{M_2}
\end{aligned}$$

and $\left\|\kappa_1^{-1}(\eta+\theta\zeta)\kappa_1(\eta\right\|_{\mathcal{L}(E_1)} \le c''\,[\zeta]^{M_1}$ for $0 \le \theta \le 1$. Hence

$$\begin{aligned}
&\left\|\kappa_2^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\}\kappa_0(\eta)\right\|_{\mathcal{L}(E_0,E_2)}\\
&\quad\le [\zeta]^{M_2}\sum_{\substack{\gamma\le\alpha\\ \delta\le\beta}} c_{\alpha\beta\gamma\delta}\,[\zeta]^{-2k}\,\widetilde{\pi}_{\alpha,\beta,\gamma,\delta}(a)\,[y]^{\varrho-|\delta|}\,[\eta]^{\mu-|\gamma|}\\
&\quad\cdot[\theta\zeta]^{|\mu|+|\gamma|}\,[\zeta]^{M_1}\,[z]^{-2l}\,\pi_{\alpha,\beta,\gamma,\delta}(b)\,[y]^{\sigma-|\beta|+|\delta|}\,[z]^{|\sigma|+|\beta|}\,[\eta]^{\nu-|\alpha|+|\gamma|}\\
&\quad\le c_{\alpha,\beta}\,[\zeta]^{-2k+M_1+M_2+|\mu|+|\alpha|}\,[z]^{-2l+|\sigma|+|\beta|}\\
&\quad\cdot[y]^{\varrho+\sigma-|\beta|}\,[\eta]^{\nu+\mu-|\alpha|}\,\widetilde{\pi}_{\alpha,\beta}(a)\pi_{\alpha,\beta}(b),
\end{aligned}$$

with suitable semi-norms $\widetilde{\pi}_{\alpha,\beta}$, $\pi_{\alpha,\beta}$ on $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E_1, E_2)$ and $S^{\nu,\sigma}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E_1)$, respectively. Thus

$$\begin{aligned}
&\left\|\kappa_2^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta \widetilde{c}_\theta(y,\eta)\right\}\kappa_0(\eta)\right\|_{\mathcal{L}(E_0,E_2)}\\
&\quad\le c_{\alpha,\beta}\widetilde{\pi}_{\alpha,\beta}(a)\pi_{\alpha,\beta}(b)\,[y]^{\varrho+\sigma-|\alpha|}\,[\eta]^{\mu+\nu-|\beta|},
\end{aligned}$$

with a constant $c_{\alpha,\beta}$ that only depends on $\alpha, \beta \in \mathbb{N}$ and on $k, l \in \mathbb{N}$, but not on the symbols a, b and on the parameter $\theta \in [0,1]$. For the rest, we can follow the proofs of Theorem 1.15 and of Proposition 1.16. For $u \in C_0^\infty(\mathbb{R}^q, E_0)$ we compute that

$$\begin{aligned}
&\mathrm{Op}(a)\mathrm{Op}(b)u(y) \\
&= \iiiint e^{-i(x\eta + x'\eta')} a(y,\eta) b(y+x,\eta') u(y+x+x')\, dx' đ\eta' dx đ\eta \\
&= \mathrm{Op}(\widetilde{c}_1)u(y).
\end{aligned}$$

Setting $c := \widetilde{c}_1$, assertion (i) is proved. The asymptotic expansion (ii) follows in complete analogy to the proof of (1.10). □

Remark 1.24. As usual, we write $\mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(a \sharp b)$, i.e., $(a\sharp b)(y,\eta) = \iint e^{-iz\zeta} a(y, \eta + \zeta) b(y+z, \eta)\, dz đ\zeta$. By Proposition 1.16, the symbol $a \sharp b$ is uniquely determined.

In Remark 1.3(ii), we introduced the symbol class $S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$ for Banach spaces with arbitrary group actions $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, $\{\widetilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+}$. As we did in Section 1.2 for symbols in $S^{\mu,\mu',\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^{2q}; E, \widetilde{E})$, we associate with each symbol $a \in S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$ an operator $\mathbf{Op}(a)$ by

$$\mathbf{Op}(a)u(y) := \iiiint e^{-i(z\eta + z'\eta')} a(y, y+z, \eta) u(y+z+z')\, dz dz' đ\eta đ\eta' \quad (1.18)$$

for $u \in C_0^\infty(\mathbb{R}^q)$. While it is possible to write a definition of $S^{\mu,\mu',\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^{2q}; E, \widetilde{E})$ in the more general case (in Definition 1.14, E and $\widetilde{E}$ were supposed to be Hilbert spaces with unitary group actions), doing so, it is not seen to be of interest. The essential use of double symbols in Section 1.2, namely to show that composition and adjunction of operators are possible within the class, is not applicable here because of the presence of additional group actions. We therefore proved Proposition 1.23 without referring to double symbols, but using in fact the same technique. Next, we give the analogue of Theorem 1.15 for symbols in $S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$.

Theorem 1.25. *Let $a \in S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$ and put*

$$b(y,\eta) := \iint e^{-iz\zeta} a(y, y+z, \eta+\zeta)\, dz đ\zeta, \quad (1.19)$$

where the integral is interpreted as an oscillatory integral. Then $b \in S^{\mu,\varrho+\varrho'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, and

$$\mathbf{Op}(a) = \mathrm{Op}(b). \quad (1.20)$$

The symbol b admits the asymtotic expansion

$$b(y,\eta) \sim \sum_\alpha \frac{1}{\alpha!} \left(\partial_\eta^\alpha D_{y'}^\alpha a\right)(y, y', \eta)\Big|_{y'=y}. \quad (1.21)$$

Proof. Let $0 \leq \theta \leq 1$, let $\widetilde{a} \in S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$, and put

$$\widetilde{b}_\theta(y,\eta) := \iint e^{-iz\zeta}\widetilde{a}(y, y+z, \eta+\theta\zeta)\, dz đ\zeta.$$

To given multi-indices $\alpha, \beta \in \mathbb{N}^q$ choose $k, l \in \mathbb{N}$ such that $2k > q + |\mu| + |\beta| + M_1 + M_2$ and $2l > q + |\varrho'| + |\alpha|$. Put

$$r_\theta(y,z,\eta,\zeta) := (1+|\zeta|^2)^{-k}(1-\Delta_z)^k(1+|z|^2)^{-l}(1-\Delta_\zeta)^l\widetilde{a}(y, y+z, \eta+\theta\zeta).$$

Integrating by parts, we see that

$$\widetilde{b}_\theta(y,\eta) = \iint e^{-iz\zeta} r_\theta(y,z,\eta,\zeta)\, dz đ\zeta.$$

As in the proof of Theorem 1.15, we want to show that $\left\{\widetilde{b}_\theta \ : \ 0 \leq \theta \leq 1\right\}$ is a bounded set in $S^{\mu,\varrho+\varrho'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. To this end, we consider

$$\widetilde{\kappa}^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\}\kappa(\eta)$$

and estimate

$$\begin{aligned}
&\left\|\widetilde{\kappa}^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\}\kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \\
&\quad \leq \left\|\widetilde{\kappa}^{-1}(\eta)\widetilde{\kappa}(\eta+\theta\zeta)\right\|_{\mathcal{L}(\widetilde{E})} \\
&\qquad \cdot \left\|\widetilde{\kappa}^{-1}(\eta+\theta\zeta)\left\{D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\}\kappa(\eta+\theta\zeta)\right\|_{\mathcal{L}(E,\widetilde{E})} \\
&\qquad \cdot \left\|\kappa^{-1}(\eta+\theta\zeta)\kappa(\eta)\right\|_{\mathcal{L}(E)} \\
&\quad \leq \left\|\widetilde{\kappa}_{[\eta+\theta\zeta]/[\eta]}\right\|_{\mathcal{L}(\widetilde{E})}\left\|\kappa_{[\eta]/[\eta+\theta\zeta]}\right\|_{\mathcal{L}(E)}[\zeta]^{-2k}[z]^{-2l} \\
&\qquad \cdot \sum_{|\gamma|\leq|\alpha|} c_{\alpha,\beta,\gamma}[y]^{\varrho-|\gamma|}[y+z]^{\varrho'-(|\alpha|-|\gamma|)}[\eta+\theta\zeta]^{\mu-|\beta|}\pi_{\alpha,\beta}(\widetilde{a}),
\end{aligned}$$

where $\pi_{\alpha,\beta}$ is a semi-norm on $S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$ that only depends on $\alpha, \beta \in \mathbb{N}^q$, and $c_{\alpha,\beta,\gamma}$, $|\gamma| \leq |\alpha|$, are constants that only depend on $\alpha, \beta, \gamma \in \mathbb{N}^q$, but not on $\theta \in [0,1]$. Using Peetre's inequality, we see that

$$\begin{aligned}
\left\|\widetilde{\kappa}_{[\eta+\theta\zeta]/[\eta]}\right\|_{\mathcal{L}(\widetilde{E})} &\leq c'\left\|\widetilde{\kappa}_{[\theta\zeta]}\right\|_{\mathcal{L}(\widetilde{E})} \leq c''[\zeta]^{M_2}, \\
\left\|\kappa_{[\eta]/[\eta+\theta\zeta]}\right\|_{\mathcal{L}(E)} &\leq c'\left\|\kappa_{[\theta\zeta]}\right\|_{\mathcal{L}(E)} \leq c''[\zeta]^{M_1}.
\end{aligned}$$

Furthermore, we have $[\eta+\theta\zeta]^{\mu-|\beta|} \leq c\,[\eta]^{\mu-|\beta|}[\zeta]^{|\mu|+|\beta|}$ and $[y+z]^{\varrho'-(|\alpha|-|\gamma|)} \leq c\,[y]^{\varrho'-(|\alpha|-|\gamma|)}[z]^{|\varrho'|+|\alpha|}$. Hence

$$\begin{aligned}
&\left\|\widetilde{\kappa}^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta r_\theta(y,z,\eta,\zeta)\right\}\kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \\
&\quad \leq c_{\alpha,\beta}[y]^{\varrho+\varrho'-|\alpha|}[z]^{|\varrho'|+|\alpha|-2l}[\eta]^{\mu-|\beta|}[\zeta]^{|\mu|+|\beta|+M_1+M_2-2k}\pi_{\alpha,\beta}(\widetilde{a})
\end{aligned}$$

with a constant $c_{\alpha,\beta}$ that only depends on $\alpha, \beta \in \mathbb{N}^q$, but not on $\theta \in [0,1]$. Thus

$$\left\|\widetilde{\kappa}^{-1}(\eta)\left\{D_y^\alpha D_\eta^\beta \widetilde{b}_\theta(y,\eta)\right\}\kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \leq c'_{\alpha,\beta}\pi_{\alpha,\beta}(\widetilde{a})[y]^{\varrho+\varrho'-|\alpha|}[\eta]^{\mu-|\beta|},$$

and it follows that $\widetilde{b}_\theta \in S^{\mu,\varrho+\varrho'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ for all $\theta \in [0,1]$ and that, moreover, the set $\left\{\widetilde{b}_\theta \ : \ 0 \le \theta \le 1\right\}$ is bounded in $S^{\mu,\varrho+\varrho'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Now let $a = \widetilde{a}$. Then $b(y,\eta) = \widetilde{b}_1(y,\eta)$. The relations (1.20) and (1.21) now follow as in the proof of Theorem 1.15. □

Corollary 1.26. *Let $\mu, \varrho, \varrho' \in \mathbb{R}$. If $a \in S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$, then $\mathrm{Op}(a) \in L^{\mu,\varrho+\varrho'}(\mathbb{R}^q; E, \widetilde{E})$.*

Proof. Writing $a(y,y',\eta) = \iint e^{-ix'\zeta} a(y, y' - x', \eta)\, dx' d\zeta$, we see that for all $u \in C_0^\infty(\mathbb{R}^q, E)$, we have

$$\begin{aligned}\mathrm{Op}(a)u(y) &= \iint e^{i(y-y')\eta} a(y,y',\eta) u(y')\, dy' d\eta \\ &= \iiiint e^{i(y-y')\eta} e^{-ix'\zeta} a(y, y' - x', \eta) u(y')\, dy' dx' d\eta d\zeta.\end{aligned}$$

Substituting $\eta' = \eta + \zeta$ and $x = y' - y - x'$, in the oscillatory integral, we see that

$$\begin{aligned}&\mathrm{Op}(a)u(y) \\ &= \iiiint e^{-i(x\eta + x'\eta')} a(y, y+x, \eta) u(y + x + x')\, dx dx' d\eta' d\eta \\ &= \mathbf{Op}(a)u(y).\end{aligned}$$

Now Theorem 1.26 shows that $\mathrm{Op}(a) \in L^{\mu,\varrho+\varrho'}(\mathbb{R}^q; E, \widetilde{E})$. □

In Section 1.2, we proved that if $E, \widetilde{E}$ are Hilbert spaces with unitary actions, then

$$L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E}) \subseteq \bigcap_{s,\delta \in \mathbb{R}} \mathcal{L}(\mathcal{W}^{s,\delta}(\mathbb{R}^q, E), \mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \widetilde{E})),$$

see Proposition 1.22. In the proof, it was used that there exist operators $R^{s,\delta} \in L^{s,\delta}(\mathbb{R}^q; E, E)$ that induce isometrical isomorphisms $R^{s,\delta} : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \to L_2(\mathbb{R}^q, E)$, and that, moreover, $L^{0,0}(\mathbb{R}^q; E, \widetilde{E}) \subseteq \mathcal{L}(L_2(\mathbb{R}^q, E), L_2(\mathbb{R}^q, \widetilde{E}))$. The proof was based on Hilbert space techniques. For Banach spaces $E, \widetilde{E}$, the situation is much more difficult. Theorem 1.35 presents a result on mapping properties in this case. In most applications, however, one deals with the following situation.

Suppose, E and $\widetilde{E}$ are Hilbert spaces with associated group actions $\{\kappa_\lambda\}$, $\{\widetilde{\kappa}_\lambda\}$. Moreover, suppose there are

- Hilbert spaces $E_0, \widetilde{E}_0$ with associated unitary group actions $\{\kappa_{0,\lambda}\}$, $\{\widetilde{\kappa}_{0,\lambda}\}$, and
- symbols $r^s(\eta)$ in $S^{s,0}(\mathbb{R}^q \times \mathbb{R}^q; E, E_0)$ and $\widetilde{r}^s(\eta)$ in $S^{s,0}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{E}_0)$, for all $s \in \mathbb{R}$, that induce point-wise isomorphisms, i.e., $r^s(\eta) : E \xrightarrow{\cong} E_0$, $\widetilde{r}^s(\eta) : \widetilde{E} \xrightarrow{\cong} \widetilde{E}_0$, for all $\eta \in \mathbb{R}^q$.

Under these assumptions, we have the following result.

Proposition 1.27. *Let $A \in L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$. Then A induces continuous operators*

$$A : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \widetilde{E})$$

for all $s, \delta \in \mathbb{R}$.

Before turning to the proof, let us notice that if we put $p^{-s}(\eta) := (r^s(\eta))^{-1}$, $\widetilde{p}^{-s}(\eta) := (\widetilde{r}^s(\eta))^{-1}$, for all $\eta \in \mathbb{R}^q$, then $p^{-s} \in S^{-s,0}(\mathbb{R}^q \times \mathbb{R}^q; E_0, E)$ and $\widetilde{p}^{-s} \in S^{-s,0}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}_0, \widetilde{E})$. Since $\mathrm{Op}(p^{-s})\mathrm{Op}(r^s) = \mathrm{id}_{\mathcal{W}^{0,0}(\mathbb{R}^q,E)}$ and $\mathrm{Op}(r^s)\mathrm{Op}(p^{-s}) = \mathrm{id}_{\mathcal{W}^{0,0}(\mathbb{R}^q,E_0)}$, it follows that $\mathrm{Op}(r^s)$ induces an isomorphism

$$\mathrm{Op}(r^s) : \mathcal{W}^{s,0}(\mathbb{R}^q, E) \to \mathcal{W}^{0,0}(\mathbb{R}^q, E_0).$$

Analogously, it is seen that $\mathrm{Op}(\widetilde{r}^s)$ induces an isomorphism

$$\mathrm{Op}(\widetilde{r}^s) : \mathcal{W}^{s,0}(\mathbb{R}^q, \widetilde{E}) \to \mathcal{W}^{0,0}(\mathbb{R}^q, \widetilde{E}_0).$$

Proof of Proposition 1.27. Put $R^{s,\delta} := \mathrm{Op}(r^s) \circ \mathrm{Op}([y]^\delta)$, $\widetilde{R}^{s,\delta} := \mathrm{Op}(\widetilde{r}^s) \circ \mathrm{Op}([y]^\delta)$. By Proposition 1.23 we have $R^{s,\delta} \in L^{s,\delta}(\mathbb{R}^q; E, E_0)$ and $\widetilde{R}^{s,\delta} \in L^{s,\delta}(\mathbb{R}^q; \widetilde{E}, \widetilde{E}_0)$. The operators $R^{s,\delta}$, $\widetilde{R}^{s,\delta}$ induce isomorphisms $R^{s,\delta} : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \to \mathcal{W}^{0,0}(\mathbb{R}^q, E_0)$ and $\widetilde{R}^{s,\delta} : \mathcal{W}^{s,\delta}(\mathbb{R}^q, \widetilde{E}) \to \mathcal{W}^{0,0}(\mathbb{R}^q, \widetilde{E}_0)$. Moreover, $(R^{s,\delta})^{-1} \in L^{-s,-\delta}(\mathbb{R}^q; E_0, E)$ and $(\widetilde{R}^{s,\delta})^{-1} \in L^{-s,-\delta}(\mathbb{R}^q; \widetilde{E}_0, \widetilde{E})$. Put $B := \widetilde{R}^{s-\mu,\delta-\varrho} \circ A \circ (R^{s,\delta})^{-1}$. By Proposition 1.23, we have $B \in L^{0,0}(\mathbb{R}^q; E_0, \widetilde{E}_0)$. Now Corollary 1.20 implies that there is a $c > 0$ such that

$$\|Bu\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,\widetilde{E}_0)} \leq c\, \|u\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,E_0)} .$$

Hence

$$\begin{aligned} \|Au\|_{\mathcal{W}^{s-\mu,\delta-\varrho}(R^q,\widetilde{E})} &\leq c_1 \left\| \widetilde{R}^{s-\mu,\delta-\varrho} A u \right\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,\widetilde{E}_0)} \\ &= \left\| B R^{s,\delta} u \right\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,E_0)} \leq c_2 \left\| R^{s,\delta} u \right\|_{\mathcal{W}^{0,0}(\mathbb{R}^q,E_0)} \leq c_3 \|u\|_{\mathcal{W}^{s,\delta}(\mathbb{R}^q,E)} . \end{aligned}$$

□

Remark 1.28. Note that if there exists an isomorphism $a_0 : E \to E_0$ such that $\kappa_{0,\lambda} a_0 \kappa_\lambda^{-1} \in C^\infty(\mathbb{R}_+, \mathcal{L}(E, E_0))$, then we find an order reducing symbol $r^s(\eta)$ as above by putting $r^s(\eta) = [\eta]^s\, \kappa_{0,[\eta]} a_0 \kappa_{[\eta]}^{-1}$.

1.4 Ellipticity and parametrices

For Banach spaces $E, \widetilde{E}$, we denote by $\mathcal{K}(E, \widetilde{E})$ the space of all operators $A \in \mathcal{L}(E, \widetilde{E})$ that are compact.

Definition 1.29. (i) A symbol $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ is said to have compact variation, if for all $y, \eta \in \mathbb{R}^q$

$$a(y,\eta) - a(y,0) \in \mathcal{K}(E, \widetilde{E}).$$

The space of all such symbols is denoted by $S^{\mu,\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. The space of all operators $A \in L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$ for which there is a symbol $a \in S^{\mu,\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ such that $A = \mathrm{Op}(a)$ is denoted by $L^{\mu,\varrho}_{cv}(\mathbb{R}^q; E, \widetilde{E})$.

(ii) By $S^{\mu,\varrho}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, we denote the space of all symbols $a \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ such that $a(y,\eta) \in \mathcal{K}(E, \widetilde{E})$ for all $y, \eta \in \mathbb{R}^q$. The space of all operators $A \in L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$ for which there is a symbol $a \in S^{\mu,\varrho}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ such that $A = \mathrm{Op}(a)$, is denoted by $L^{\mu,\varrho}_{c}(\mathbb{R}^q; E, \widetilde{E})$.

The applications we have in view will deal with symbols having compact variation. Notice that

$$S^{\mu,\varrho}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \subseteq S^{\mu,\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}).$$

Lemma 1.30. *Let $a \in S^{\mu,\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$.*

(i) Then $D^\alpha_\eta a \in S^{\mu-|\alpha|,\varrho}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, for any multi-index $\alpha \neq 0$.

(ii) If $b \in S^{\mu',\varrho'}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{\widetilde{E}})$, then $ba \in S^{\mu+\mu',\varrho+\varrho'}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{\widetilde{E}})$.

Proof. The proof is clear. Use that

$$\begin{aligned} &b(y,\eta)a(y,\eta) - b(y,0)a(y,0) \\ &\quad = b(y,\eta)(a(y,\eta) - a(y,0)) + (b(y,\eta) - b(y,0))a(y,0). \end{aligned}$$

□

Proposition 1.31. *Let $a_j \in S^{\mu_j,\varrho_j}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, $j \in \mathbb{N}$, be an arbitrary sequence, where $\mu_j \to -\infty$, $\varrho_j \to -\infty$ as $j \to \infty$. Put $\mu := \max\{\mu_j\}$, $\varrho := \max\{\varrho_j\}$. Then there is an $a \in S^{\mu,\varrho}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ such that for every $k \in \mathbb{N}$ there is an $N = N(k)$ with*

$$a - \sum_{j=0}^{N} \in S^{\mu-k,\varrho-k}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}).$$

The symbol a is unique modulo $S^{-\infty,-\infty}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$.

Proof. Following the proof of Proposition 1.5, we obtain a as a convergent series $a = \sum_{j \geq 0} \widetilde{a}_j$, where $\widetilde{a}_j(y,\eta) = \chi(c_j^{-1}y, c_j^{-1}\eta)a_j(y,\eta)$, $j \in \mathbb{N}$. Since $\widetilde{a}_j \in S^{\mu_j,\varrho_j}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ for all $j \in \mathbb{N}$, and since the series $\sum_{j\geq 0} \widetilde{a}_j$ is actually finite in each point $(y,\eta) \in \mathbb{R}^q \times \mathbb{R}^q$, it follows that $a \in S^{\mu,\varrho}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Moreover, if a' is another symbol in $S^{\mu,\varrho}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ satisfying $a' - \sum_{j=0}^{N(k)} a_j \in S^{\mu_N,\varrho_N}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, then $a - a' \in \bigcap_{N\in\mathbb{N}} S^{\mu_N,\varrho_N}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, and hence $a - a' \in S^{-\infty,-\infty}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. □

Lemma 1.32. *Let* $a \in S^{\mu,\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ *and* $b \in S^{\mu',\varrho'}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{\widetilde{E}})$. *Then*

$$b\sharp a - ba \in S^{\mu+\mu'-1,\varrho+\varrho'-1}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{\widetilde{E}}).$$

Proof. By Proposition 1.23, we have

$$b\sharp a \sim \sum_{\alpha \in \mathbb{N}^q} \frac{1}{\alpha!}(D^\alpha_\eta b(y,\eta))\partial^\alpha_y a(y,\eta).$$

If $\alpha \neq 0$, then Lemma 1.30 shows that

$$(D^\alpha_\eta b)\partial^\alpha_y a \in S^{\mu+\mu'-|\alpha|,\varrho+\varrho'-|\alpha|}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{\widetilde{E}}).$$

By Proposition 1.31, the sum $\sum_{\alpha\neq 0} 1/\alpha!(D^\alpha_\eta b)\partial^\alpha_y a$ can be carried out in the space $S^{\mu+\mu'-1,\varrho+\varrho'-1}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{\widetilde{E}})$. □

Definition 1.33. (i) A symbol $a \in S^{\mu,\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, $\mu, \varrho \in \mathbb{R}$, is called *elliptic*, if there are symbols $b_r \in S^{-\mu,-\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, E)$, $b_l \in S^{-\mu,-\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, E)$ such that $b_l a = \mathrm{id}_E + r$, $ab_r = \mathrm{id}_{\widetilde{E}} + r'$, where $r \in S^{-1,-1}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, E)$ and $r' \in S^{-1,-1}_{c}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{E})$.

(ii) An operator $A \in L^{\mu,\varrho}_{cv}(\mathbb{R}^q; E, \widetilde{E})$ is called elliptic, if $A = \mathrm{Op}(a)$, where $a \in S^{\mu,\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ is an elliptic symbol.

Proposition 1.34. *Let* $A \in L^{\mu,\varrho}_{cv}(\mathbb{R}^q; E, \widetilde{E})$. *Then* A *is elliptic if and only if there is an operator* $B = \mathrm{Op}(b) \in L^{-\mu,-\varrho}_{cv}(\mathbb{R}^q; \widetilde{E}, E)$ *such that* $BA - \mathrm{id} \in L^{-\infty,-\infty}_{c}(\mathbb{R}^q; E, E)$ *and* $AB - \mathrm{id} \in L^{-\infty,-\infty}_{c}(\mathbb{R}^q; \widetilde{E}, \widetilde{E})$.

Proof. Let $A = \mathrm{Op}(a)$ be elliptic. Then there is a symbol $b' \in S^{-\mu,-\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, E)$ such that $b'a - \mathrm{id}_E = r \in S^{-1,-1}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, E)$. By Lemma 1.32 it follows that

$$b'\sharp a - \mathrm{id}_E = (b'\sharp a - b'a) + (b'a - \mathrm{id}_E) \in S^{-1,-1}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, E).$$

Hence $\mathrm{Op}(b')\mathrm{Op}(a) = 1 + \mathrm{Op}(r')$, with $r' \in S^{-1,-1}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, E)$. Write $r'_{[j]} := r'\sharp \ldots \sharp r'$ (j-times). By Proposition 1.31, $r'' \sim \sum_{j\geq 1}(-1)^j r'_{[j]}$ can be carried out in $S^{-1,-1}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, E)$. Put $b := (1+r'')\sharp b'$. Then $b \in S^{-\mu,-\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, E)$, and

$$\mathrm{Op}(b)\mathrm{Op}(a) = \mathrm{Op}(b\sharp c) = \mathrm{Op}((1+r'')\sharp(1+r')) = 1 + \mathrm{Op}(r_l),$$

where $r_l \in S^{-\infty,-\infty}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, E)$. Analogously, it is shown that there are symbols $b_r \in S^{-\mu,-\varrho}_{cv}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, E)$ and $r_r \in S^{-\infty,-\infty}_{c}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{E})$ such that $\mathrm{Op}(a)\mathrm{Op}(b_r) = 1 + \mathrm{Op}(r_r)$. Now

$$\begin{aligned}\mathrm{Op}(b_r) + \mathrm{Op}(r_l)\mathrm{Op}(b_r) &= (1 + \mathrm{Op}(r_l))\mathrm{Op}(b_r)\\ &= \mathrm{Op}(b)\mathrm{Op}(a)\mathrm{Op}(b_r) = \mathrm{Op}(b) + \mathrm{Op}(b)\mathrm{Op}(r_r).\end{aligned}$$

Hence $\mathrm{Op}(b) - \mathrm{Op}(b_r) \in L^{-\infty,-\infty}_{c}(\mathbb{R}^q; \widetilde{E}, E)$. It now follows that $\mathrm{Op}(b)\mathrm{Op}(a) - \mathrm{id} \in L^{-\infty,-\infty}_{c}(\mathbb{R}^q \times \mathbb{R}^q; E, E)$ and $\mathrm{Op}(a)\mathrm{Op}(b) - \mathrm{id} \in L^{-\infty,-\infty}_{c}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{E})$.

Conversely, suppose that $B = \mathrm{Op}(b) \in L_{cv}^{-\mu,-\varrho}(\mathbb{R}^q; \widetilde{E}, E)$ is given such that $AB - \mathrm{id} \in L_c^{-\infty,-\infty}(\mathbb{R}^q; \widetilde{E}, \widetilde{E})$ and $BA - \mathrm{id} \in L_c^{-\infty,-\infty}(\mathbb{R}^q; E, E)$. Write $ba - \mathrm{id}_E = (ba - b\sharp a) + (b\sharp a - \mathrm{id}_E)$ and $ab - \mathrm{id}_{\widetilde{E}} = (ab - a\sharp b) + (a\sharp b - \mathrm{id}_{\widetilde{E}})$. Using Lemma 1.32, we see then that $ba - \mathrm{id}_E \in S_c^{-1,-1}(\mathbb{R}^q \times \mathbb{R}^q; E, E)$ and $ab - \mathrm{id}_{\widetilde{E}} \in S_c^{-1,-1}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{E})$. But this means that a is elliptic, and thus $A = \mathrm{Op}(a)$ is elliptic. □

Theorem 1.35. *Let $\delta < -q/2 - 1$.*

(i) If $a \in S^{\mu,\delta}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, then $\mathrm{Op}(a)$ induces continuous operators

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{E})$$

for all $s \in \mathbb{R}$.

(ii) If $\nu < 0$ and $a \in S_c^{\nu,\delta}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, then

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^s(\mathbb{R}^q, \widetilde{E})$$

is compact for all $s \in \mathbb{R}$.

Proof. See SCHULZE [18]. □

Proposition 1.34 and Theorem 1.35 yield the following result.

Theorem 1.36. *Let $A \in L_{cv}^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$ be elliptic and suppose moreover that $A \in \mathcal{L}(\mathcal{W}^{s,\delta}(\mathbb{R}^q, E), \mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \widetilde{E}))$. Then*

$$A : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \widetilde{E})$$

is a Fredholm operator for all $s, \delta \in \mathbb{R}$.

In Section 1.3 we considered Hilbert spaces $E, \widetilde{E}$ for which there exists Hilbert spaces $E_0, \widetilde{E}_0$ with associated unitary group actions and symbols $r^s(\eta) \in S^{s,0}(\mathbb{R}^q \times \mathbb{R}^q; E, E_0)$, $\widetilde{r}^s(\eta) \in S^{s,0}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{E}_0)$ for all $s \in \mathbb{R}$ that induce point-wise isomorphisms.
If $A \in L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$, then $A \in \mathcal{L}(\mathcal{W}^{s,\delta}(\mathbb{R}^q, E), \mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \widetilde{E}))$ by Proposition 1.27. Under these assumptions, we also have the following theorem.

Theorem 1.37. *Let $A \in L_{cv}^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$ be elliptic. If $u \in \mathcal{W}^{-\infty,-\infty}(\mathbb{R}^q, E)$, then $Au \in \mathcal{W}^{s,\delta}(\mathbb{R}^q, \widetilde{E})$ implies that $u \in \mathcal{W}^{s+\mu,\delta+\varrho}(\mathbb{R}^q, E)$. Moreover, we have $\ker A \subseteq \mathcal{S}(\mathbb{R}^q, E)$ and there is a finite-dimensional subspace N_- in $\mathcal{S}(\mathbb{R}^q, \widetilde{E})$ such that*

$$A(\mathcal{W}^{s+\mu,\delta+\varrho}(\mathbb{R}^q, E)) \oplus N_- = \mathcal{W}^{s,\delta}(\mathbb{R}^q, \widetilde{E}).$$

Proof. By Proposition 1.27 we have $A \in \mathcal{L}(\mathcal{W}^{s+\mu,\delta+\varrho}(\mathbb{R}^q, \widetilde{E}), \mathcal{W}^{s,\delta}(\mathbb{R}^q, \widetilde{E}))$ for all $s, \delta \in \mathbb{R}$, and by Theorem 1.36, A is a Fredholm operator for all $s, \delta \in \mathbb{R}$. Since A is elliptic, there exits a parametrix $B \in L_{cv}^{-\mu,-\varrho}(\mathbb{R}^q; \widetilde{E}, E)$ to A. Hence, if $f := Au \in \mathcal{W}^{s,\delta}(\mathbb{R}^q, \widetilde{E})$ for some $u \in \mathcal{W}^{-\infty,-\infty}(\mathbb{R}^q, E)$, then $Bf - u \in \bigcap_{s,\delta\in\mathbb{R}} \mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$, and hence $u \in \mathcal{W}^{s+\mu,\delta+\varrho}(\mathbb{R}^q, E)$. It follows that if $u \in$

$\ker A$, then $u \in \bigcap_{s,\delta\in\mathbb{R}} \mathcal{W}^{s,\delta}(\mathbb{R}^q, E) = \mathcal{S}(\mathbb{R}^q, E)$. In particular, the kernel is independent of $s, \delta \in \mathbb{R}$. Next, put $B := \widetilde{R}^{s,\delta} \circ A \circ (R^{s+\mu,\delta+\varrho})^{-1}$, with $R^{s,\delta}$, $\widetilde{R}^{s,\delta}$ as in the proof of Proposition 1.27. Then $B \in \mathcal{L}(L_2(\mathbb{R}^q, E_0), L_2(\mathbb{R}^q, \widetilde{E}_0))$, and B is clearly elliptic. Hence B^* is elliptic (this follows easily by Proposition 1.18), and it follows that $\ker B^* \subseteq \mathcal{S}(\mathbb{R}^q, \widetilde{E}_0)$. Now there is a finite-dimensional subspace $\widetilde{N}_- \subseteq \mathcal{S}(\mathbb{R}^q, \widetilde{E}_0)$ such that $B(L_2(\mathbb{R}^q, E_0)) \oplus \widetilde{N}_- = L_2(\mathbb{R}^q, \widetilde{E}_0)$. Writing $A = (\widetilde{R}^{s,\delta})^{-1} \circ B \circ R^{s+\mu,\delta+\varrho}$, we see that there exits a finite-dimensional subspace N_- of $\mathcal{S}(\mathbb{R}^q, \widetilde{E})$ such that $A(\mathcal{W}^{s+\mu,\delta+\varrho}(\mathbb{R}^q, E)) \oplus N_- = \mathcal{W}^{s,\delta}(\mathbb{R}^q, \widetilde{E})$. □

2 Operators on manifolds with conical exits

In this section, we discuss the model situation of a non-compact manifold having a conical exit to infinity. A detailed examination of the special features in this case serves to illuminate the general character of the calculus discussed in the previous section.

2.1 Invariance under diffeomorphisms

We say that an open set $U \subseteq \mathbb{R}^q$ is *conical in the large*, if there is a constant $c > 0$ such that

$$x \in U \Rightarrow \lambda x \in U \text{ for all } \lambda \geq 1 \text{ and } |x| \geq c.$$

Let $E, \widetilde{E}$ be Banach spaces with associated group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, $\{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$.

Definition 2.1. Let $U \in \mathbb{R}^q$ be open and conical in the large.

(i) Then $S^{\mu,\varrho}(U \times \mathbb{R}^q; E, \widetilde{E})$ is defined as the space of all $a \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that for each open set $U_0 \subsetneq \mathbb{R}^q$ that is also conical in the large and that satisfies $\overline{U_0} \subseteq U$, there exists a symbol $a_0 \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ with $a_0|_{U_0\times\mathbb{R}^q} = a|_{U_0\times\mathbb{R}^q}$.

(ii) The space $\mathcal{S}(U \times U, \mathcal{L}(E, \widetilde{E}))$ is defined as the space of all $c \in C^\infty(U \times U, \mathcal{L}(E, \widetilde{E}))$ such that for each open set $U_0 \subsetneq \mathbb{R}^q$ that is conical in the large and satisfies $\overline{U_0} \subset U$, there exists a $c_0 \in \mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ with $c_0|_{U_0\times U_0} = c|_{U_0\times U_0}$.

(iii) By $L^{-\infty,-\infty}(U; E, \widetilde{E})$, we denote the space of all integral operators with kernel in $\mathcal{S}(U \times U, \mathcal{L}(E, \widetilde{E}))$.

(iv) By $L^{\mu,\varrho}(U; E, \widetilde{E})$, we denote the space of all operators $A = \mathrm{Op}(a) + C$, where $a \in S^{\mu,\varrho}(U \times \mathbb{R}^q; E, \widetilde{E})$ and $C \in L^{-\infty,-\infty}(U; E, \widetilde{E})$.

Lemma 2.2. *(i) If $a \in S^{-\infty,-\infty}(U \times \mathbb{R}^q; E, \widetilde{E})$, then* $\mathrm{Op}(a) \in L^{-\infty,-\infty}(U; E, \widetilde{E})$.

(ii) $L^{-\infty,-\infty}(U; E, \widetilde{E}) = \bigcap_{\mu,\varrho} L^{\mu,\varrho}(U; E, \widetilde{E})$.

Proof. Let $a \in S^{-\infty,-\infty}(U \times \mathbb{R}^q; E, \widetilde{E})$ and put $k(y,y') := \int e^{i(y-y')\eta} a(y,\eta)\, d\eta$. Then $\mathrm{Op}(a)u(y) = \int k(y,y')u(y')\, dy'$, and clearly $k \in \mathcal{S}(U \times U, \mathcal{L}(E, \widetilde{E}))$. This proves (i). To prove (ii), let $A \in \bigcap_{\mu,\varrho} L^{\mu,\varrho}(U; E, \widetilde{E})$. Let U_0 be an arbitrary open set that is conical in the large such that $\overline{U_0} \subset U$. Choose an open set U_1 that is conical in the large such that $\overline{U_0} \subset U_1$ and $\overline{U_1} \subset U$. Moreover, let $\varphi, \varphi_0 \in C^\infty(\mathbb{R}^q)$ such that $0 \le \varphi \le 1$, $0 \le \varphi_0 \le 1$, $\varphi|_{\overline{U_0}} \equiv 1$ and $\varphi_0|_{\mathbb{R}^q \setminus \overline{U_1}} \equiv 0$ and $\varphi\varphi_0 \equiv \varphi$. Then $\varphi A \varphi = \varphi_0 \varphi A \varphi \varphi_0$, and $\varphi A \varphi \in \bigcap_{\mu,\varrho} L^{\mu,\varrho}(\mathbb{R}^q; E, \widetilde{E})$. Hence by Corollary 1.26 and Proposition 1.12, there is a symbol $a \in S^{-\infty,-\infty}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ such that $\varphi A \varphi = \mathrm{Op}(a)$. Then $\varphi_0 \mathrm{Op}(a) \varphi_0 \in L^{-\infty,-\infty}(U; E, \widetilde{E})$, and thus $\varphi A \varphi \in L^{-\infty,-\infty}(U; E, \widetilde{E})$. Since φ, φ_0 are arbitrary, it follows that the distributional kernel of A is in $\mathcal{S}(U \times U, \mathcal{L}(E, \widetilde{E}))$. Hence $A \in L^{-\infty,-\infty}(U; E, \widetilde{E})$. □

Definition 2.3. We define $S^{\mu,\varrho,\varrho'}(U \times U \times \mathbb{R}^q; E, \widetilde{E})$ as the space of all $a \in C^\infty(U \times U \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that for each open set $U_0 \subseteq \mathbb{R}^q$ that is also conical in the large and that satisfies $\overline{U_0} \subset U$, there exists a symbol $a_0 \in S^{\mu,\varrho,\varrho'}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$ with $a_0|_{U_0 \times U_0 \times \mathbb{R}^q} = a|_{U_0 \times U_0 \times \mathbb{R}^q}$.

Proposition 2.4. *Let $a \in S^{\mu,\varrho,\varrho'}(U \times U \times \mathbb{R}^q; E, \widetilde{E})$. Then $\mathrm{Op}(a) \in L^{\mu,\varrho+\varrho'}(U; E, \widetilde{E})$. If*

$$\underline{a}(y,\eta) \sim \sum_\alpha \frac{1}{\alpha!} \left(\partial_\eta^\alpha D_{y'}^\alpha a\right)(y,y',\eta)\big|_{y'=y} \tag{2.1}$$

in $S^{\mu,\varrho+\varrho'}(U \times \mathbb{R}^q; E, \widetilde{E})$, then $\mathrm{Op}(a) - \mathrm{Op}(\underline{a}) \in L^{-\infty,-\infty}(U; E, \widetilde{E})$.

Proof. If $\underline{a}(y,\eta) \sim \sum_\alpha \frac{1}{\alpha!} \left(\partial_\eta^\alpha D_{y'}^\alpha a\right)(y,y',\eta)\big|_{y'=y}$ then $\underline{a}$ is uniquely determined modulo $S^{-\infty,-\infty}(U \times \mathbb{R}^q; E, \widetilde{E})$. Let U_0 be an open set that is conical in the large such that $\overline{U_0} \subset U$. Choose an open set U_1 that is conical in the large such that $\overline{U_0} \subset U_1$ and $\overline{U_1} \subset U$. Moreover, let $\varphi \in C^\infty(\mathbb{R}^q)$ such that $0 \le \varphi \le 1$, $\varphi|_{\overline{U_0}} \equiv 1$ and $\varphi|_{\mathbb{R}^q \setminus \overline{U_1}} \equiv 0$. Then Theorem 1.25, in particular, formula (1.21), shows that $\varphi(y)(\mathrm{Op}(a) - \mathrm{Op}(\underline{a}))\varphi(y') \in L^{-\infty,-\infty}(\mathbb{R}^q; E, \widetilde{E})$. Since U_0 isarbitrary, it follows that $\mathrm{Op}(a) - \mathrm{Op}(\underline{a})$ is an integral operator with kernel in $\mathcal{S}(U \times U, \mathcal{L}(E, \widetilde{E}))$. □

Let $V \subseteq \mathbb{R}^q$ be another open set that is conical in the large. We consider a diffeomorphism

$$\chi : U \to V$$

for which there is a $c > 0$ such that $\chi(\lambda x) = \lambda \chi(x)$ for all $\lambda \ge 1$ and $|x| \ge c$. For $x \in V$ let $\varepsilon(x) > 0$ such that $\{x' : |x - x'| \le \varepsilon(x)\} \subseteq V$. For $x, x' \in V$ satisfying $|x - x'| \le \varepsilon(x)$ put

$$M(x,x') := \int_0^1 d\chi^{-1}(x' + t(x - x'))\, dt. \tag{2.2}$$

The mean value theorem then shows that

$$\chi^{-1}(x) - \chi^{-1}(x') = M(x,x')(x - x'). \tag{2.3}$$

Lemma 2.5. *There is a symbol $s \in S^{0,1}(V \times \mathbb{R}^q)$ that is independent of $\eta \in \mathbb{R}^q$, i.e., $s = s(x)$, such that the following holds:*

(i) For each $V_0 \subset V$ which is conical in the large and satisfies $\overline{V_0} \subset V$, there is a $c' > 0$ such that

$$c' [x] \leq s(x) \text{ for all } x \in \overline{V_0}.$$

(ii) If $x \in V$, then $|x - x'| \leq s(x)$ implies that $M(x, x')$ is invertible.

Proof. Since $M(x, x) = d\chi^{-1}(x)$, the matrix $M(x, x)$ is invertible for all $x \in V$. If we construct a function $s = s(x)$ such that $|x - x'| \leq s(x)$ implies $|M(x, x') - M(x, x)| \leq \frac{1}{2} |M(x, x)|$, then $M(x, x')$ is invertible. We have

$$\begin{aligned} &M^{-1}(x, x)(M(x, x') - M(x, x)) \\ &\quad = \int_0^1 ((d\chi^{-1})^{-1}(x)(d\chi^{-1})(x' + t(x - x')) - 1)\, dt. \end{aligned}$$

Writing $d\chi^{-1} = ((d\chi^{-1})_{i,j})_{i,j=1,\dots,q}$, we get

$$\begin{aligned} &(d\chi^{-1})(x' + t(x - x')) \\ &\quad = (d\chi^{-1})(x) + \int_0^1 (\nabla (d\chi^{-1})_{i,j}(x + s(1 - t)(x - x')), x - x')_{i,j=1,\dots,q}\, ds. \end{aligned}$$

Since $(D_x^\alpha \chi^{-1})_{i,j} \in S^{0,0}(V \times \mathbb{R}^q)$ for all $i, j = 1, \dots, q$, and for $|\alpha| = 1$, we see that there is a symbol $\varphi \in S^{0,0}(V \times \mathbb{R}^q)$ that only depends on x, i.e., $\varphi = \varphi(x)$, and is strictly positive, i.e., $\varphi(x) \geq 1$ for all $x \in V$, such that

$$\begin{aligned} &\left| \int_0^1 (d\chi^{-1})^{-1}(x)(\nabla (d\chi^{-1})_{i,j}(x + s(1 - t)(x' - x), x - x')_{i,j=1,\dots,q}\, ds \right| \\ &\quad \leq \int_0^1 \varphi(x) [x + s(1 - t)(x' - x)]^{-1} |x - x'|\, ds. \end{aligned}$$

Notice that if $|x - x'| \leq \varepsilon [x]$, where $\varepsilon > 0$ is some small constant, then $[x + s(1 - t)(x' - x)]^{-1} \leq c_\varepsilon [x]^{-1}$ for $0 \leq s, t \leq 1$, with some constant $c_\varepsilon > 0$. Hence, for $|x - x'| \leq \varepsilon [x]$,

$$\int_0^1 \varphi(x) [x + s(1 - t)(x' - x)]^{-1} |x - x'|\, ds \leq c_\varepsilon \varphi(x) [x]^{-1} |x - x'|.$$

Put $s(x) := \frac{1}{\varepsilon \varepsilon' \varphi(x)} [x]$, where $0 < \varepsilon' < \frac{1}{c_\varepsilon} q^{-2}$. Then $s \in S^{0,1}(V \times \mathbb{R}^q)$, and $|x - x'| \leq s(x)$ implies that $|x - x'| \leq \varepsilon [x]$ and

$$\int_0^1 \varphi(x) [x + s(1 - t)(x' - x)]^{-1} |x - x'|\, ds \leq \frac{1}{2} q^{-2}.$$

It then follows that

$$\left| M^{-1}(x, x)(M(x, x') - M(x, x)) \right| \leq \frac{1}{2}.$$

□

Lemma 2.6. *There is a function $\omega \in S^{0,0,0}(V \times V \times \mathbb{R}^q)$, $\omega = \omega(x, x')$, such that $\omega(x, x') = 1$ for $|x - x'| \leq \frac{1}{2}s(x)$ and $\omega(x, x') = 0$ for $|x - x'| \geq \frac{2}{3}s(x)$. Here s is the function from Lemma 2.5.*

Proof. Choose a funcion $\psi \in C^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t \leq 1/2$ and $\psi(t) = 0$ for $t > 2/3$. Define $\omega(x, x') := \psi(|x - x'|/s(x))$. Then ω is as required. The only point to verify is that $\omega \in S^{0,0,0}(V \times V \times \mathbb{R}^q)$. But this follows easily since $s \in S^{0,1}(V \times \mathbb{R}^q)$. □

Let $A \in L^{\mu,\varrho}(U; E, \widetilde{E})$ and consider the push-forward $\chi_* A$, i.e.,

$$(\chi_* A)v(y) = (\chi^*)^{-1} A(\chi^* v)(y), \quad v \in C_0^\infty(V),$$

where $(\chi^* v)(x) = v(\chi(x))$.

Lemma 2.7. *The push-forward χ_* induces a bijection $\chi_* : L^{-\infty,-\infty}(U; E, \widetilde{E}) \to L^{-\infty,-\infty}(V; E, \widetilde{E})$.*

Proof. Let $C \in L^{-\infty,-\infty}(U; E, \widetilde{E})$ and write $Cu(y) = \int_U c(y, y')u(y')\, dy'$. For $v \in C_0^\infty(V)$ we compute

$$\begin{aligned}(\chi_* C)v(x) &= \int_V c(\chi^{-1}(x), x')v(\chi(x'))\, dx' \\ &= \int_V c(\chi^{-1}(x), \chi^{-1}(x'))\left|\det d\chi^{-1}(x')\right| v(x')\, dx'.\end{aligned}$$

By assumption on χ, we have $\left|\det d\chi^{-1}(\cdot)\right| \in C_b^\infty(V)$, and hence the kernel $c'(x, x') := c(\chi^{-1}(x), \chi^{-1}(x'))\left|\det d\chi^{-1}(x')\right|$ is in $\mathcal{S}(V \times V, \mathcal{L}(E, \widetilde{E}))$. Thus $\chi_* C \in L^{-\infty,-\infty}(V; E, \widetilde{E})$. Analogously, it follows that

$$(\chi^{-1})_* : L^{-\infty,-\infty}(V; E, \widetilde{E}) \to L^{-\infty,-\infty}(U; E, \widetilde{E}).$$

Since $(\chi_*)^{-1} \circ \chi_* = \mathrm{id}$, the assertion follows. □

Theorem 2.8. *The push-forward χ_* induces bijections*

$$\chi_* : L^{\mu,\varrho}(U; E, \widetilde{E}) \to L^{\mu,\varrho}(V; E, \widetilde{E})$$

for all $\mu, \varrho \in \mathbb{R}$. If $A = \mathrm{Op}(a) + C$ with $a \in S^{\mu,\varrho}(U \times \mathbb{R}^q; E, \widetilde{E})$ and $C \in L^{-\infty,-\infty}(U; E, \widetilde{E})$, then $\chi_* A = \mathrm{Op}(b) + C'$, where $C' \in L^{-\infty,-\infty}(V; E, \widetilde{E})$ and $b \in S^{\mu,\varrho}(V \times \mathbb{R}^q; E, \widetilde{E})$. We have

$$b(x, \eta) = a(\chi^{-1}(x), ({}^t d\chi)(\chi^{-1}(x))\eta) \tag{2.4}$$

modulo $S^{\mu-1,\varrho-1}(V \times \mathbb{R}^q; E, \widetilde{E})$.

Proof. In view of Lemma 2.7 we may assume that $A = \mathrm{Op}(a)$ with $a \in S^{\mu,\varrho}(U \times \mathbb{R}^q; E, \widetilde{E})$. Then

$$\begin{aligned}(\chi_* A)v(x) = \int_{\mathbb{R}^q} \int_V & e^{i(\chi^{-1}(x) - \chi^{-1}(x'))\eta} a(\chi^{-1}(x), \eta) \\ & \cdot \left|\det d\chi^{-1}(x')\right| v(x')\, dx' d\eta\end{aligned}$$

for $v \in C_0^\infty(V,E)$. Let ω be as in Lemma 2.6 and put
$a_0(y,y',\eta) := \omega(\chi(y),\chi(y'))a(y,\eta)$ and
$a_1(y,y',\eta) := (1-\omega(\chi(y),\chi(y')))a(y,\eta)$. Since

$$\sum_\alpha \frac{1}{\alpha!}\,(\partial_\eta^\alpha D_{y'}^\alpha a_1)(y,y',\eta)\big|_{y'=y} \sim 0$$

in $S^{\mu,\varrho}(U\times\mathbb{R}^q;E,\widetilde{E})$, Proposition 2.4 implies that $\mathrm{Op}(a_1)$ belongs to $L^{-\infty,-\infty}(U;E,\widetilde{E})$. Now Lemma 2.7 shows that $\chi_*\mathrm{Op}(a_1) \in L^{-\infty,-\infty}(V;E,\widetilde{E})$. It thus suffices to consider $\chi_*\mathrm{Op}(a_0)$. We have

$$\begin{aligned}\chi_*\mathrm{Op}(a_0)v(x) &= \int_{\mathbb{R}^q}\int_V e^{i(\chi^{-1}(x)-\chi^{-1}(x'))\eta}\omega(x,x')a(\chi^{-1}(x),\eta)\\ &\quad\cdot\left|\det d\chi^{-1}(x')\right| v(x')\,dx'd\eta\\ &= \int_{\mathbb{R}^q}\int_V e^{i(x-x')^tM(x,x')\eta}\omega(x,x')a(\chi^{-1}(x),\eta)\\ &\quad\cdot\left|\det d\chi^{-1}(x')\right| v(x')\,dx'd\eta\\ &= \int_{\mathbb{R}^q}\int_V e^{i(x-x')\eta}\omega(x,x')a(\chi^{-1}(x),({}^tM)^{-1}(x,x')\eta)\\ &\quad\cdot\left|\det d\chi^{-1}(x')\right|\cdot\left|\det({}^tM)^{-1}(x,x')\right| v(x')\,dx'd\eta.\end{aligned}$$

Thus $\chi_*\mathrm{Op}(a_0) = \mathrm{Op}(d)$ with

$$\begin{aligned}d(x,x',\eta) &:= \omega(x,x')a(\chi^{-1}(x),({}^tM)^{-1}(x,x')\eta)\\ &\quad\cdot\left|\det d\chi^{-1}(x')\right|\cdot\left|({}^tM)^{-1}(x,x')\right|.\end{aligned}$$

It remains to show that $d \in S^{\mu,\varrho,0}(V\times V\times\mathbb{R}^q;E,\widetilde{E})$. Once this is shown, the theorem follows by Proposition 2.4 writing

$$b(x,\eta) \sim \sum_\alpha \frac{1}{\alpha!}\,(\partial_\eta^\alpha D_{x'}^\alpha d)(x,x',\eta)\big|_{x=x'}.$$

In particular, (2.4) then follows, since

$$b(x,\eta) - a(\chi^{-1}(x),({}^td\chi)(\chi^{-1}(x))\eta) = \sum_{\alpha\neq 0}\frac{1}{\alpha!}\,(\partial_\eta^\alpha D_{x'}^\alpha d)(x,x',\eta)\big|_{x=x'},$$

and the right hand side is in $S^{\mu-1,\varrho-1}(V\times\mathbb{R}^q;E,\widetilde{E})$. To prove that $d \in S^{\mu,\varrho,0}(V\times V\times\mathbb{R}^q;E,\widetilde{E})$, let $V_0 \subset V$ be an open set that is conical in the large and satisfies $\overline{V_0} \subset V$. Choose then an open set V_1 that is conical in the large, where $\overline{V_1} \subset V$ and $\overline{V_0} \subset V_1$. Moreover, let $\varphi \in C^\infty(\mathbb{R}^q)$ such that $0 \le \varphi \le 1$ and $\varphi|_{\overline{V_0}} \equiv 1$ and $\varphi|_{\mathbb{R}^q\setminus\overline{V_1}} \equiv 0$. To V_0, V_1 there exist open sets $U_0, U_1 \subset U$ that are conical in the large with $\overline{U_0} \subset U_1$, $\overline{U_1} \subset U$ and such that $\chi^{-1}(x) \in U_0$ for all $x \in V_0$ and $\chi^{-1}(x) \in U_1$ for all $x \in V_1$. What we want to show is that there exists a symbol $d_1 \in S^{\mu,\varrho,0}(\mathbb{R}^{2q}\times\mathbb{R}^q;E,\widetilde{E})$ satisfying $d_1|_{\overline{V_0}\times\overline{V_0}\times\mathbb{R}^q} \equiv d|_{\overline{V_0}\times\overline{V_0}\times\mathbb{R}^q}$. Set $\varphi_0(x,x') := \varphi(x)\varphi(x')$. Let $x,x' \in \overline{V_0}$.

$$\begin{aligned}d(x,x',\eta) &= \omega(x,x')a(\varphi_0(x,x')\chi^{-1}(x),\varphi_0(x,x')({}^tM)^{-1}(x,x')\eta)\\ &\quad\cdot\left|\varphi_0(x,x')\det d\chi^{-1}(x')\right|\cdot\left|\varphi_0(x,x')\det({}^tM)^{-1}(x,x')\right|.\end{aligned}$$

By definition, there is a symbol $a_1 \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ such that $a|_{\overline{U_1} \times \mathbb{R}^q} \equiv a_1|_{\overline{U_1} \times \mathbb{R}^q}$. Hence, for $x, x' \in \overline{V_0}$, we have

$$\begin{aligned} &a(\varphi_0(x,x')\chi^{-1}(x), \varphi_0(x,x')({}^tM)^{-1}(x,x')\eta) \\ &\quad = a_1(\varphi_0(x,x')\chi^{-1}(x), \varphi_0(x,x')({}^tM)^{-1}(x,x')\eta). \end{aligned}$$

Since $\chi^{-1}(\lambda x) = \lambda \chi^{-1}(x)$ for $\lambda \geq 1$ and $|x| \geq c'$, for some constant $c' > 0$, we see that $\varphi_0(x,x') \left|\det d\chi^{-1}(x')\right| \in S^{0,0,0}(\mathbb{R}^{2q} \times \mathbb{R}^q)$ and

$$\varphi_0(x,x')\chi^{-1}(x) \in \bigtimes_{j=1}^{q} S^{0,1,0}(\mathbb{R}^{2q} \times \mathbb{R}^q)$$

Analogously, we get by (2.2) that $\varphi_0(x,x') \left|\det({}^tM)^{-1}(x,x')\right| \in S^{0,0,0}(\mathbb{R}^{2q} \times \mathbb{R}^q)$ and

$$\varphi_0(x,x')({}^tM)^{-1}(x,x')\eta \in \bigtimes_{j=1}^{q} S^{1,0,0}(\mathbb{R}^{2q} \times \mathbb{R}^q).$$

Moreover, a simple calculation shows that $b_0(x,x',\eta) \in \bigtimes_{j=1}^q S^{0,1,0}(\mathbb{R}^{2q} \times \mathbb{R}^q)$, $b_1(x,x',\eta) \in \bigtimes_{j=1}^q S^{1,0,0}(\mathbb{R}^{2q} \times \mathbb{R}^q)$, and $a_1 \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, imply $a_1(b_0(x,x',\eta), b_1(x,x',\eta)) \in S^{\mu,\varrho,0}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$. Thus, if we define

$$\begin{aligned} d_1(x,x',\eta) &:= \omega(x,x')a_1(\varphi_0(x,x')\chi^{-1}(x), \varphi_0(x,x')({}^tM)^{-1}(x,x')\eta) \\ &\quad \cdot \left|\varphi_0(x,x')\det d\chi^{-1}(x')\right| \cdot \left|\varphi_0(x,x')\det({}^tM)^{-t}(x,x')\right|, \end{aligned}$$

then $d_1 \in S^{\mu,\varrho,0}(\mathbb{R}^{2q} \times \mathbb{R}^q; E, \widetilde{E})$ and $d_1|_{\overline{V_0} \times \overline{V_0} \times \mathbb{R}^q} \equiv d|_{\overline{V_0} \times \overline{V_0} \times \mathbb{R}^q}$. Hence $d \in S^{\mu,\varrho,0}(V \times V \times \mathbb{R}^q; E, \widetilde{E})$. □

Remark 2.9. Equation (2.4) gives the first term of the asymptotic expansion

$$b(x,\eta) \sim \sum_{\alpha} (\partial_\eta^\alpha a)(\chi^{-1}(x), ({}^td\chi)(\chi^{-1}(x))\eta)\varphi_\alpha(\chi^{-1}(x),\eta), \tag{2.5}$$

where $\varphi_\alpha(y,\eta) := D_z^\alpha e^{i\delta(z,y)\eta}\big|_{z=y}$, $\alpha \in \mathbb{N}^q$, and $\delta(z,y) = \chi(z) - \chi(x) - d\chi(y)(z-y)$. The expansion (2.5) can be carried out in $S^{-\infty,-\infty}(V \times \mathbb{R}^q; E, \widetilde{E})$. Notice that for $\alpha \in \mathbb{N}^q$, we have $(\partial_\eta^\alpha a)(y, ({}^td\chi)(y)\eta) \in S^{\mu-|\alpha|,\varrho}(U_y \times \mathbb{R}_\eta^q; E, \widetilde{E})$ and that

$$\begin{aligned} \varphi_\alpha(y,\lambda\eta) &= D_z^\alpha e^{i\delta(z,y)\lambda\eta}\Big|_{z=y} = D_z^\alpha e^{i\delta(\lambda z,\lambda y)\eta}\Big|_{z=y} \\ &= \lambda^{|\alpha|}\, D_z^\alpha e^{i\delta(z',\lambda y)\eta}\Big|_{z'=y} = \lambda^{|\alpha|}\varphi_\alpha(\lambda y,\eta). \end{aligned}$$

Thus $\varphi_\alpha(\lambda y,\eta) = \lambda^{-|\alpha|}\varphi_\alpha(y,\eta)$. Since φ_α is a polynomial in η of degree $\leq |\alpha|/2$, it follows that $(\partial_\eta^\alpha a)(y, ({}^td\chi)(y)\eta)\varphi_\alpha(\chi^{-1}(x),\eta) \in S^{\mu-|\alpha|/2,\varrho-|\alpha|/2}(V \times \mathbb{R}^q; E, \widetilde{E})$.

2.2 Pseudo-differential operators on manifolds with conical exits

Definition 2.10. A manifold with conical exits in our set-up is a smooth q-dimensional manifold M which is a union

$$M = X_0 \cup X_1 \cup \ldots \cup X_N \cup \partial X_1 \cup \ldots \cup \partial X_N,$$

where

- $X_0, \ldots, X_N$ are q-dimensional smooth submanifolds with compact smooth boundaries,
- $\partial X_1, \ldots, \partial X_N$ are connected $(q-1)$-dimensional submanifolds,
- X_0 is compact,
- $\partial X_0 = \partial X_1 \cup \ldots \cup \partial X_N$,
- and X_j is diffeomorphic to $\partial X_j \times [1, \infty)$ for $j + 1, \ldots, N$, where the corresponding diffeomorphisms are homogeneous for large arguments.

Manifolds with conical exits are a special case of the SG-compatible manifolds in the sense of SCHROHE [14]. Notice that, in particular, every compact manifold is a manifold with conical exits, namely where $N = 0$. Another example is the case $M = \mathbb{R}^q$.

Every manifold with conical exits has a finite atlas. Assume for simplicity that $N = 1$.

The set $X_0 \cup \{\partial X_0 \times [1, 2)\}$ can be covered by finitely many open sets which are relatively compact. To introduce coordinates on $\partial X_0 \times [1, \infty)$, choose an open cover of ∂X_0, say $\{Y_j : j = 1, \ldots, l\}$. Denote the corresponding coordinate maps by $\alpha_j : Y_j \to V_j \subseteq \mathbb{R}^{q-1}$, $j = 1, \ldots, l$. As a cover of $\partial X_0 \times [1, \infty)$ choose $X_j = Y_j \times [1, \infty)$ with the $\varphi_j : X_j \to U_j \subseteq \mathbb{R}^q$ given by $\varphi_j(v, t) = (t\alpha_j(v), t)$. Notice that the sets U_j are conical in the large and that the coordinate maps φ_j, $j = 1, \ldots, l$, as well as the corresponding changes of coordinates are diffeomorphisms that are homogeneous for large arguments.

In the sequel, we fix this atlas and denote the open cover by $\{M_j : j = 1, \ldots, n\}$ and the coordinate mappings by $\varphi_j : M_j \to U_j$, $j = 1, \ldots, n$.

Lemma 2.11. *(i) There exists a partition of unity on M subordinated to the open cover $\{M_j : j = 1, \ldots, n\}$, i.e., mappings $\Phi_j : M \to \mathbb{R}$ with $0 \leq \Phi_j \leq 1$, $\operatorname{supp}\Phi_j \subseteq M_j$, and $\sum_{j=1}^n \Phi_j = 1$, such that $\Phi_j \circ \varphi_j^{-1} \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q)$.*

(ii) If $\{\Phi_j : j = 1, \ldots, n\}$ is the partition of unity of (i), then there exists a partition of unity $\{\Psi_j : j = 1, \ldots, n\}$ on M subordinated to the open cover $\{M_j : j = 1, \ldots, n\}$, such that $\Psi_j \circ \varphi_j^{-1} \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q)$ and $\Psi_j \Phi_j = \Phi_j$ for $j = 1, \ldots, n$.

Proof. For simplicity, let us assume that $N = 1$ and that $M_1 = \partial X_0 \times [1, \infty)$. Let $\widetilde{\Phi}_j : M_j \to U_j$, $j = 1, \ldots, n$, be a partition of unity subordinated to the open cover $\{M_j : j = 1, \ldots, n\}$. Assume that $M_j \cap (\partial X_0 \times [1, \infty)) \neq \emptyset$ for $j = 1, \ldots, k$, and $M_j \cap (\partial X_0 \times [1, \infty)) = \emptyset$ for $j = k + 1, \ldots, n$. Let

$\{Y_j : j = 1, \ldots, k\}$ be the open cover on X_0 induced by $\{M_j : j = 1, \ldots, k\}$, and denote by $\alpha_j : Y_j \to V_j$, $j = 1, \ldots, k$, the corresponding coordinate maps. For $(v,t) \in X_0 \times [1,\infty)$ define $\widetilde{\Psi}_j(v,t) := \alpha_j(v)$. Then $\sum_{j=1}^{k} \widetilde{\Psi}_j = 1$ on $\partial X_0 \times [1,\infty)$. Let $\omega \in C^\infty(M)$ be such that $0 \leq \omega \leq 1$ and such that ω equals 1 on $X_0 \cup \partial X_0 \cup (\partial X_0 \times [1,2))$ and that ω vanishes on $\partial X_0 \times (3,\infty)$. Put $\Phi_j = \widetilde{\Phi}_j$ if $j \geq k+1$, i.e., if $M_j \subset X_0$, and put $\Phi_j = \omega\widetilde{\Phi}_j + (1-\omega)\widetilde{\Psi}_j$, if $1 \leq j \leq k$. Then $0 \leq \Phi_j \leq 1$ for $j = 1, \ldots, n$, and $\sum_{j=1}^{n} \Phi_j = 1$. Moreover, $\Phi_j \circ \varphi_j^{-1} \in S^{0,0}(\mathbb{R}^q \times \mathbb{R}^q)$. To prove (ii), make the same construction as in (i), starting with partitions of unity $\{\theta_j : j = 1, \ldots, n\}$ on M and $\left\{\widetilde{\theta}_j : j = 1, \ldots, k\right\}$ on ∂X_0 that satisfy $\theta_j \widetilde{\Phi}_j = \widetilde{\Phi}_j$ for $j = 1, \ldots, n$, and $\widetilde{\theta}_j \widetilde{\Psi}_j = \widetilde{\Psi}_j$ for $j = 1, \ldots, k$. □

Using the partition of unity $\{\Phi_j : j = 1, \ldots, n\}$ of Lemma 2.11, we define pseudo-differential operators on M. First, let us notice that a function $f \in C^\infty(M,E)$ is in $\mathcal{S}(M,E)$, if $(\Phi_k^{-1})^*(\varphi_j f) \in \mathcal{S}(\mathbb{R}^n, E)$, for $j, k = 1, \ldots, n$.

Definition 2.12. (i) An operator $A : \mathcal{S}(M,E) \to \mathcal{S}(M,\widetilde{E})$ is in $L^{-\infty,-\infty}(M;E,\widetilde{E})$, if in local coordinates, A has a kernel in $\mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$.

(ii) We write $A \in L_c^{-\infty,-\infty}(M;E,\widetilde{E})$, if the kernel of $\Psi_j A \Phi_k$, $j,k = 1, \ldots, n$, in local coordinates takes its values in $\mathcal{K}(E,\widetilde{E})$, for all $(y,y') \in \mathbb{R}^{2q}$.

(iii) We write $A \in L^{\mu,\varrho}(M;E,\widetilde{E})$ if there are symbols $a_j \in S^{\mu,\varrho}(U_j \times \mathbb{R}^q; E, \widetilde{E})$ and operators $A_j \in L^{-\infty,-\infty}(M;E,\widetilde{E})$, $j = 1, \ldots, n$, such that $\varphi_{j,*}(\Psi_j A \Phi_j) = \mathrm{Op}(a_j)$ and $(1-\Psi_j)A\Phi_j \in L^{-\infty,-\infty}(M;E,\widetilde{E})$, for all $j = 1, \ldots, n$.

(iv) We write $A \in L_{cv}^{\mu,\varrho}(M;E,\widetilde{E})$, if there are symbols $a_j \in S_{cv}^{\mu,\varrho}(U_j \times \mathbb{R}^q; E, \widetilde{E})$ and operators $A_j \in L_c^{-\infty,-\infty}(M;E,\widetilde{E})$, $j = 1, \ldots, n$, such that $\varphi_{j,*}(\Psi_j A \Phi_j) = \mathrm{Op}(a_j)$ and $(1-\Psi_j)A\Phi_j \in L^{-\infty,-\infty}(M;E,\widetilde{E})$, for all $j = 1, \ldots, n$.

Remark 2.13. In view of Theorem 2.8, the definition of $L^{\mu,\varrho}(M;E,\widetilde{E})$ makes sense. Notice that if $\left\{\widetilde{\Psi}_j : j = 1, \ldots, n\right\}$ is another partition of unity as $\{\Psi_j : j = 1, \ldots, n\}$ in Lemma 2.11, then $\varphi_{j,*}(\Psi_j A \Phi_j - \widetilde{\Psi}_j A \Phi_j) = \varphi_{j,*}((\Psi_j - \widetilde{\Psi}_j)A\Phi_j) \in L^{-\infty,-\infty}(U_j;E,\widetilde{E})$, since $\Psi_j - \widetilde{\Psi}_j$ vanishes on $\mathrm{supp}\Phi_j$. In particular, if $a_j \in S_{cv}^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}$, then

$$\begin{aligned}&\varphi_{j,*}((\Psi_j - \widetilde{\Psi}_j)(\varphi_{j,*}^{-1}\mathrm{Op}(a_j))\Phi_j)\\&\quad = \varphi_{j,*}((\Psi_j - \widetilde{\Psi}_j)(\varphi_{j,*}^{-1}\mathrm{Op}(|y-y'|^{-2}\,\Delta_\eta a_j))\Phi_j) \in L_c^{-\infty,-\infty}(U_j;E,\widetilde{E}).\end{aligned}$$

Hence the definition of $L_{cv}^{\mu,\varrho}(M;E,\widetilde{E})$ makes sense.

Let $\widetilde{\widetilde{E}}$ be a further Banach space with associated group action.

Proposition 2.14. *If $A \in L^{\mu,\varrho}(M;\widetilde{E},\widetilde{\widetilde{E}})$ and $B \in L^{\nu,\sigma}(M;E,\widetilde{E})$, then $AB \in L^{\mu+\nu,\varrho+\sigma}(M;E,\widetilde{\widetilde{E}})$.*

Proof. Writing $AB = (\sum_{j=1}^{n} \Psi_j A \Phi_j)(\sum_{j=1}^{n} \Psi_j B \Phi_j)$ the assertion follows easily using Proposition 1.23 in local coordinates. □

Next we turn to the action of operators in $L^{\mu,\varrho}(M; E, \widetilde{E})$ on wedge Sobolev spaces.

Definition 2.15. Let $U \subseteq \mathbb{R}^q$ be an open set that is conical in the large. The space $\mathcal{W}^{s,\delta}(U, E)$, $s, \delta \in \mathbb{R}$, is defined as the space of all $u \in \mathcal{D}'(U, E)$ such that for each open set $U_0 \subseteq \mathbb{R}^q$ that is also conical in the large and satisfies $\overline{U_0} \subseteq U$, there exists a function $u_0 \in \mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$ with $u_0|_{U_0} = u|_{U_0}$.

In the sequel, we assume that E is a Hilbert space with associated group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$. Furthermore suppose that there exists a Hilbert space E_0 with unitary group action $\{\kappa_{0,\lambda}\}_{\lambda \in \mathbb{R}_+}$ and for all $s \in \mathbb{R}$ a symbol $r^s = r^s(\eta)$ in $S^{s,0}(\mathbb{R}^q \times \mathbb{R}^q; E, E_0)$ that induces point-wise isomorphisms, i.e., $r^s(\eta) : E \xrightarrow{\cong} E_0$ for all $\eta \in \mathbb{R}^q$. The operators $R^{s,\delta} := \operatorname{Op}(r^s) \circ \operatorname{Op}([y]^\delta)$ then induce isomorphisms $R^{s,\delta} : \mathcal{W}^{s,\delta}(\mathbb{R}^q, E) \to \mathcal{W}^{0,0}(\mathbb{R}^q, E_0)$ for all $s, \delta \in \mathbb{R}$.

Let $U, V \subseteq \mathbb{R}^q$ be open sets that are conical in the large and consider again a diffeomorphism $\chi; U \to V$ for which there is a $c > 0$ such that $\chi(\lambda x) = \lambda \chi(x)$ for all $\lambda \geq 1$ and $|x| \geq c$.

Proposition 2.16. *Let $s, \delta \in \mathbb{R}$. If $v \in \mathcal{W}^{s,\delta}(V, E)$, then $\chi^* v \in \mathcal{W}^{s,\delta}(U, E)$.*

Proof. If $v \in \mathcal{D}'(V, E)$, then $\chi^* v \in \mathcal{D}(U, E)$. Let $U_0 \subset U$ be open and conical in the large such that $\overline{U_0} \subset U$. We have to prove that $\chi^* v|_{U_0}$ can be extended to an element in $\mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$. To U_0 we can find an open set $V_0 \subset V$, also conical in the large, that satisfies $\overline{V_0} \subset V$, such that $\chi^{-1}(\overline{U_0}) \subseteq \overline{V_0}$. By definition, there is a $v_0 \in \mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$ such that $v_0|_{V_0} = v|_{V_0}$. Moreover, let $\chi_0 : \mathbb{R}^q \to \mathbb{R}^q$ be a diffeomorphism such that $\chi_0|_{U_0} = \chi|_{U_0}$ and such that $0 \leq c' \leq |\det d\chi^{-1}(x)| \leq c$ for suitable constants $c', c > 0$.

(i) Assume first that $E = E_0$ and $s, \delta = 0$. An application of the Plancherel formula shows that

$$\begin{aligned} \|v_0 \circ \chi_0\|^2_{\mathcal{W}^0(\mathbb{R}^q, E_0)} &= (2\pi)^q \int_{\mathbb{R}^q} \|v_0(x)\|^2_{E_0} \left|\det(d\chi_0)^{-1}(\chi_0(x))\right| \, dx \\ &\leq c(\chi_0)(2\pi)^q \int_{\mathbb{R}^q} \|v_0(x)\|^2_{E_0} \, dx = c(\chi_0) \|v_0\|^2_{\mathcal{W}^0(\mathbb{R}^q, E_0)} . \end{aligned}$$

Since $v_0 \circ \chi_0|_{U_0} = v \circ \chi|_{U_0}$, it follows that $\chi^* v \in \mathcal{W}^{0,0}(U, E_0)$.

(ii) If $v \in \mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$, we write $\chi_0^* v_0 = ((R^{s,\delta})^{-1} \circ \chi_0^*) \circ (\chi_{0,*} R^{s,\delta}) v_0$. We have $R^{s,\delta} \in L^{s,\delta}(\mathbb{R}^q; E, E_0)$. By Theorem 2.8 we have $\chi_{0,*} R^{s,\delta} \in L^{s,\delta}(\mathbb{R}^q; E, E_0)$, and hence Proposition 1.27 implies that $\chi_{0,*} R^{s,\delta} v_0 \in \mathcal{W}^{0,0}(\mathbb{R}^q, E_0)$. By step (i), we then have $\chi_0^*(\chi_{0,*} R^{s,\delta} v_0) \in \mathcal{W}^{0,0}(\mathbb{R}^q, E_0)$.
Moreover, $(R^{s,\delta})^{-1} \in L^{-s,-\delta}(\mathbb{R}^q; E_0, E)$. Using Proposition 1.27, again, it follows that $\chi_0^* v_0 \in \mathcal{W}^{s,\delta}(\mathbb{R}^q, E)$. Since $\chi_0^* v_0|_{U_0} = \chi^* v|_{U_0}$, we get $\chi^* v \in \mathcal{W}^{s,\delta}(U, E)$. □

Definition 2.17. $\mathcal{W}^{s,\delta}(M, E)$ is the space of all $u \in \mathcal{W}^{s,\delta}_{loc}(M, E)$ for which $(\varphi_j^{-1})^*(\Phi_j u) \in \mathcal{W}^{s,\delta}(U_j, E)$, for $j = 1, \dots, n$.

Remark 2.18. On $\mathcal{W}^{s,\delta}(M,E)$, we introduce a norm by

$$\|u\|_{s,\delta} := \left\{ \sum_{j=1}^{n} \left\| (\varphi_j^{-1})^* (\Phi_j u) \right\|^2_{\mathcal{W}^{s,\delta}(\mathbb{R}^q,E)} \right\}^{1/2}.$$

In fact, $\mathcal{W}^{s,\delta}(M,E)$ is a Hilbert space.

Notice that by Proposition 2.16 the definition of $\mathcal{W}^{s,\delta}(M,E)$ is independent of the particular choice of coordinates.

Suppose that $\widetilde{E}$ is a further Hilbert space with associated group action $\{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, that $\widetilde{E}_0$ is a Hilbert space with associated unitary group action $\{\widetilde{\kappa}_{0,\lambda}\}_{\lambda\in\mathbb{R}_+}$, and that for all $s \in \mathbb{R}$ there exist symbols $\widetilde{r}^s = \widetilde{r}^s(\eta)$ in $S^{s,0}(\mathbb{R}^q \times \mathbb{R}^q; \widetilde{E}, \widetilde{E}_0)$ that induce point-wise isomorphisms, i.e., $\widetilde{r}^s(\eta) : \widetilde{E} \xrightarrow{\cong} \widetilde{E}_0$ for all $\eta \in \mathbb{R}^q$. Under these assumptions, we have the following result.

Note that every $A \in L^{\mu,\varrho}(M; E, \widetilde{E})$ induces a continuous operator

$$A : \mathcal{S}(M,E) \to \mathcal{S}(M,\widetilde{E}).$$

Theorem 2.19. *Let $A \in L^{\mu,\varrho}(M; E, \widetilde{E})$. Then A extends to continuous operators*

$$A : \mathcal{W}^{s,\delta}(M,E) \to \mathcal{W}^{s-\mu,\delta-\varrho}(M,\widetilde{E})$$

for all $s, \delta \in \mathbb{R}$.

Proof. If $A \in L^{-\infty,-\infty}(M; E, \widetilde{E})$, then the assertion is obvious. It thus suffices to consider $\Psi_j A \Phi_j$, for $j = 1, \dots, n$. By Proposition 1.27 we have

$$(\varphi_j^{-1})^*(\Psi_j A \Phi_j) \in \mathcal{L}(\mathcal{W}^{s,\delta}(\mathbb{R}^q, E), \mathcal{W}^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \widetilde{E}))$$

for all $s, \delta \in \mathbb{R}$ and $j = 1, \dots, n$. The proposition now follows. □

Definition 2.20. An operator $A \in L^{\mu,\varrho}_{cv}(M; E, \widetilde{E})$ is called *elliptic*, if $\varphi_{j,*}(\Psi_j A \Phi_j)$ is elliptic for $j = 1, \dots, n$.

In Remark 2.13 we noted that if $A \in L^{\mu,\varrho}(M; E, \widetilde{E})$ and if $k \neq j$, then $\Psi_k A \Phi_j \in L^{-\infty,-\infty}_c(M; E, \widetilde{E})$. Proposition 1.36 therefore yields the following result.

Theorem 2.21. *Let $A \in L^{\mu,\varrho}_{cv}(M; E, \widetilde{E})$. Then A is elliptic if and only if there is an operator $B \in L^{-\mu,-\varrho}_{cv}(M; \widetilde{E}, E)$ such that*
$BA - \mathrm{id} \in L^{-\infty,-\infty}_c(M; E, E)$ and $AB - \mathrm{id} \in L^{-\infty,-\infty}_c(M; \widetilde{E}, \widetilde{E})$.

A simple consequence of Theorem 1.35 (ii) is that if $C \in L^{-\infty,-\infty}_c(M; E, \widetilde{E})$, then

$$C : \mathcal{W}^{s,\delta}(M,E) \to \mathcal{W}^{s,\delta}(M,\widetilde{E})$$

is a compact operator for all $s, \delta \in \mathbb{R}$. Proposition 2.19 and Proposition 2.21 thus yield the following result.

Theorem 2.22. *Let $A \in L_{cv}^{\mu,\varrho}(M; E, \widetilde{E})$ be elliptic. Then*

$$A : \mathcal{W}^{s,\delta}(M, E) \to \mathcal{W}^{s-\mu,\delta-\varrho}(M, \widetilde{E})$$

is a Fredholm operator for all $s, \delta \in \mathbb{R}$. Moreover, if $u \in \mathcal{W}^{-\infty,-\infty}(M, E)$, then $Au \in \mathcal{W}^{s,\delta}(M, \widetilde{E})$ implies that $u \in \mathcal{W}^{s+\mu,\delta+\varrho}(M, E)$.

3 Examples and Remarks

3.1 Operator-valued symbols for boundary value problems

This section will study necessary elements for pseudo-differential boundary value problems in the infinite half space

$$\mathbb{R}_+^{1+q} = \left\{(t, y) \in \mathbb{R}^{1+q} \ : \ t > 0, y \in \mathbb{R}^q\right\}$$

under the aspect of the operator-valued symbols with exit behaviour for $|y| \to \infty$. The full algebra of boundary value problems with the transmission property on a manifold with exits in the sense of the results of Chapter 2 will not be formulated here. More details will be presented in the book SCHULZE [18]. We shall start with scalar pseudo-differential symbols that have the exit behavior for $|y| \to \infty$.

Let us denote by $S^{\mu,\varrho}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{R}^{1+q})$ for $\mu, \varrho \in \mathbb{R}$ the space of all $a(t, y, \tau, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{R}^{1+q})$ such that

$$\left|D_{t,y}^\alpha D_{\tau,\eta}^\beta a(t, y, \tau, \eta)\right| \leq c\,[\tau, \eta]^{\mu-|\beta|}\,[y]^{\varrho-|\alpha|}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^{1+q}$ and all $t \in [0, t_0]$, $y \in \mathbb{R}^q$, $(\tau, \eta) \in \mathbb{R}^{1+q}$, with constants $c = c(\alpha, \beta, t_0) > 0$, and for arbitrary $t_0 > 0$. By subscript cl we indicate the corresponding subspace of classical elements in (τ, η).

Choose a function $f(\xi) \in \mathcal{S}(\mathbb{R})$ with supp$f \subset \mathbb{R}_-$ and $\int f(\xi)\,d\xi = 1$. Set $\chi(\tau) = \int e^{-i\tau\xi} f(\xi)\,d\xi$. Then $\chi(\tau) \in \mathcal{S}(\mathbb{R})$ and $\chi(0) = 1$. Let $\delta = \sup|\partial_\tau \chi(\tau)|$ and fix a constant $\sigma > \delta$ so large that $\chi(\tau/(\sigma\langle\eta\rangle))\langle\eta\rangle - i\tau \neq 0$ for all $(\tau, \eta) \in \mathbb{R}^{1+q}$. Then

$$r_-^\mu(\tau, \eta) = \left(\chi\left(\frac{\tau}{\sigma\langle\eta\rangle}\right)\langle\eta\rangle - i\tau\right)^\mu \tag{3.1}$$

for any $\mu \in \mathbb{R}$ is a classical symbol in $\mathbb{R}^{1+q}$, elliptic of order μ, cf. SCHROHE, SCHULZE [17]. In particular, it can be interpreted as an element of $S_{cl}^{\mu,0}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{R}^{1+q})$ though it has constant coefficients.

Let $a(t, y, \tau, \eta) \in S_{cl}^{\mu,\varrho}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{R}^{1+q})$ and set

$$\left(a_{y,\eta}^{(\alpha,\beta)}\right)(t, \tau) := \left(D_{t,y}^\alpha D_{\tau,\eta}^\beta\right) a(t, y, \tau, \eta)$$

for every fixed y, η and arbitrary multi-indices $\alpha, \beta \in \mathbb{N}^{1+q}$. Then $a_{y,\eta}^{(\alpha,\beta)}(t,\tau) \in S_{cl}^{\mu-|\beta|}(\overline{\mathbb{R}}_+ \times \mathbb{R})$, and we get asymptotic expansions

$$a_{y,\eta}^{(\alpha,\beta)}(t,\tau) \sim \sum_{j=0}^{\infty} \theta_j^{\pm}(t)\tau^{\mu-|\beta|-j}$$

for $\tau \to \pm\infty$ with coefficients $\theta_j^{\pm}(t) \in C^\infty(\overline{\mathbb{R}}_+)$. Then $a(t,y,\tau,\eta)$ is said to have the *transmission property* with respect to $t = 0$ if $\theta_j^+(0) = \theta_j^-(0)$ for all j and all α, β.

Note, in particular, that the symbol (3.1) has the transmission property.

Let us set $H^s(\mathbb{R}_+^{1+q}) = H^s(\mathbb{R}^{1+q})|_{\mathbb{R}_+^{1+q}}$, and, in particular, $H^s(\mathbb{R}_+) = H^s(\mathbb{R})|_{\mathbb{R}_+}$ for $s \in \mathbb{R}$. Denote by

$$e^+ : H^s(\mathbb{R}_+) \to \mathcal{S}'(\mathbb{R})$$

for $s > -1/2$ the operator that extends the given distribution by 0 to $t < 0$. Moreover, let r^+ be the operator of restriction to $t > 0$. Then, if $a(t,y,\tau,\eta) \in S^{\mu,\varrho}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{R}^{1+q})$ is given, we can form the operator family

$$\mathrm{op}^+(a)(y,\eta) := r^+\mathrm{op}(a)e^+ : C_0^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}_+) \qquad (3.2)$$

for $\mathrm{op}(a)u(t) = \iint e^{i(t-t')\tau}a(t,y,\tau,\eta)u(t')\,dt'd\tau$. Here we might first extend a to a symbol given for all $t \in \mathbb{R}$. However (3.2) will be independent of the specific extension, so we use the notation without explicit reference to such an extension. Let us assume (for convenience) that the symbols a in question are independent of t for $t > const$. It is well-known that when $a(t,y,\tau,\eta)$ has the transmission property with respect to $t = 0$ the operators (3.2) extend to continuous operators

$$\mathrm{op}^+(a)(y,\eta) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+) \qquad (3.3)$$

for all $s > -1/2$. Moreover, (3.3) induces continuous operators

$$\mathrm{op}^+(a)(y,\eta) : \mathcal{S}(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+)$$

for all y, η.

For $u \in H^s(\mathbb{R}_+)$ we set

$$(\kappa_\lambda u)(t) = \lambda^{1/2}u(\lambda t),$$

$\lambda \in \mathbb{R}_+$, which is a strongly continuous group of isomorphisms on $H^s(\mathbb{R}_+)$ for every s. It is unitary on $H^0(\mathbb{R}_+) = L_2(\mathbb{R}_+)$.

Note that

$$H^s(\mathbb{R}_+^{1+q}) = \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}_+)),$$

cf. SCHULZE [23], and hence we get for the Sobolev spaces with weights for $|y| \to \infty$

$$\mathcal{W}^{s,\delta}(\mathbb{R}^q, H^s(\mathbb{R}_+)) = \langle y\rangle^{-\delta} H^s(\mathbb{R}^{1+q}).$$

The following may be found in SCHROHE, SCHULZE [16].

Theorem 3.1. *We have*

$$\mathrm{op}^+(r_-^s)(\eta) \in S_{cl}^{s,0}(\mathbb{R}^q \times \mathbb{R}^q; H^s(\mathbb{R}_+), H^0(\mathbb{R}_+))$$

for every $s \in \mathbb{R}$, $s > -1/2$; $\mathrm{op}^+(r_-^s)(\eta) : H^s(\mathbb{R}_+) \to H^0(\mathbb{R}_+)$ *is an isomorphism for every* $\eta \in \mathbb{R}^q$, *where* $\mathrm{op}^+(r_-^{-s})(\eta) = \mathrm{op}^+(r_-^s)^{-1}(\eta)$.

Putting $E^s := \{u \in H^s_{loc}(\mathbb{R}_+) : \langle t\rangle^s u \in H^s(\mathbb{R}_+)\}$, $s \in \mathbb{R}$, we get other examples of Hilbert spaces in which $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ act as strongly continuous groups of isomorphisms. We have $\mathcal{S}(\overline{\mathbb{R}}_+) = \varprojlim_{k\in\mathbb{N}} E^k$. Introduce the symbol classes $S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E^l, E^k)$ for every $k, l \in \mathbb{N}$. Then, choosing a function $\theta : \mathbb{N} \to \mathbb{N}$, we get

$$\bigcap_{k\in\mathbb{N}} S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E^{\theta(k)}, E^k).$$

Moreover, we set

$$S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)) := \bigcup_\theta \left\{ \bigcap_{k\in\mathbb{N}} S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; E^{\theta(k)}, E^k) \right\},$$

where the union is taken over all $\theta : \mathbb{N} \to \mathbb{N}$. In an analogous manner we can introduce the subspaces $S_{cl}^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$ of classical (with respect to η) symbols.

Theorem 3.2. *Let* $a(t, y, \tau, \eta) \in S_{cl}^{\mu,\varrho}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{R}^{1+q})$, $\mu \in \mathbb{Z}$, $\varrho \in \mathbb{R}$, *be a symbol with the transmission property that is independent of* t *for* $t > const$ *for a constant* > 0. *Then we have*

$$\mathrm{op}^+(a)(y, \eta) \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+)) \tag{3.4}$$

for every $s \in \mathbb{R}$, $s > -1/2$, *and*

$$\mathrm{op}^+(a)(y, \eta) \in S^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)). \tag{3.5}$$

In particular, if the symbol a *is independent of* t *then we may replace* $S^{\mu,\varrho}$ *by* $S_{cl}^{\mu,\varrho}$, *both in* (3.4) *and* (3.5).

Remark 3.3. Under the conditions of Theorem 3.2 we have more precisely

$$D_y^\alpha D_\eta^\beta \mathrm{op}^+(a)(y, \eta) \in S^{\mu-|\beta|,\varrho-|\alpha|}(\mathbb{R}^q \times \mathbb{R}^q; H^s(\mathbb{R}_+), H^{s-\mu+|\beta|}(\mathbb{R}_+))$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^q$; analogous relations hold with subscript cl for t-independent a. In fact, $D_y^\alpha D_\eta^\beta \mathrm{op}^+(a)(y, \eta) = \mathrm{op}^+(D_y^\alpha D_\eta^\beta a)(y, \eta)$, where $D_y^\alpha D_\eta^\beta a(t, y, \tau, \eta) \in S^{\mu-|\beta|,\varrho-|\alpha|}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{R}^{1+q})$ has the transmission property. Then it suffices to apply Theorem 3.2, again, for the orders $\mu - |\beta|$, $\varrho - |\alpha|$.

Let us set

$$S_{cl}^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; L_2(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+)) = \varprojlim_{k\in\mathbb{N}} S_{cl}^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; L_2(\mathbb{R}_+), E^k), \tag{3.6}$$

$\mu, \varrho \in \mathbb{R}$. If $g(y, \eta)$ belongs to (3.6) we will denote by $g^*(y, \eta)$ the point-wise adjoint with respect to the $L_2(\mathbb{R}_+)$-scalar product. Every $g(y, \eta)$ with

$$g(y, \eta), g^*(y, \eta) \in S_{cl}^{\mu,\varrho}(\mathbb{R}^q \times \mathbb{R}^q; L_2(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$$

will be called a *Green symbol of type* 0 *and order* (μ, ϱ) in the global boundary symbolic calculus for Boutet de Monvel's algebra in $\mathbb{R}_+^{1+q}$. Then, every

$$g(y,\eta) = \sum_{j=0}^{d} g_j(y,\eta) \frac{\partial^j}{\partial t^j}$$

for Green symbols of type 0 and orders $(\mu - j, \varrho)$ will be called *Green symbol of type d and order* (μ, ϱ).

By $B^{-\infty,-\infty,0}(\mathbb{R}_+^{1+q})$ we will denote the subspace of all operators C with kernels in $C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \overline{\mathbb{R}}_+ \times \mathbb{R}^q)$ such that C as well as the formal adjoint C^* with respect to the $L_2(\mathbb{R}_+^{1+q})$-scalar product induce continuous operators

$$C, C^* : \mathcal{S}(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+)) \to \mathcal{S}(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+))$$

which extend to continuous operators

$$C, C^* : \langle y \rangle^k \langle t \rangle^k H^s(\mathbb{R}_+^{1+q}) \to \langle y \rangle^{-l} \langle t \rangle^{-l} H^\infty(\mathbb{R}_+^{1+q})$$

for all $s \in \mathbb{R}$, $s > -1/2$, and all $k, l \in \mathbb{N}$. Moreover, $B^{-\infty,-\infty,d}(\mathbb{R}_+^{1+q})$ for $d \in \mathbb{N}$ will denote the space of all

$$C = \sum_{j=0}^{d} C_j \frac{\partial^j}{\partial t^j}$$

for arbitrary $C_j \in B^{-\infty,-\infty,0}(\mathbb{R}_+^{1+q})$.

Definition 3.4. Let us denote by $B^{\mu,\varrho,d}(\mathbb{R}_+^{1+q})$ for $\mu \in \mathbb{Z}$, $\varrho \in \mathbb{R}$, $d \in \mathbb{N}$ the space of all operators

$$A = \mathrm{Op}(\mathrm{op}^+(a) + g) + C$$

for arbitrary $a(t, y, \tau, \eta) \in S_{cl}^{\mu,\varrho}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{R}^{1+q})$ with the transmission property, independent of t for $t > const$, a Green symbol $g(y, \eta)$ of order (μ, ϱ) and type d, and $C \in B^{-\infty,-\infty,d}(\mathbb{R}_+^{1+q})$.

The following theorems are consequences of the general calculus of Section 1.2, 1.3 and of the fact that the corresponding operator-valued symbols form algebras with the required properties.

Theorem 3.5. *Every $A \in B^{\mu,\varrho,d}(\mathbb{R}_+^{1+q})$ induces continuous operators*

$$A : \mathcal{S}(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+)) \to \mathcal{S}(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+))$$

and

$$A : H^{s,\delta}(\mathbb{R}_+^{1+q}) \to H^{s-\mu,\delta-\varrho}(\mathbb{R}_+^{1+q})$$

for all $s \in \mathbb{R}$, $s > d - 1/2$.

Theorem 3.6. *Let $A \in B^{\mu,\varrho,d}(\mathbb{R}_+^{1+q})$, $B \in B^{\nu,\beta,c}(\mathbb{R}_+^{1+q})$. This implies $AB \in B^{\mu+\nu,\varrho+\beta,h}(\mathbb{R}_+^{1+q})$ for $h = \max\{(d+\nu)^+, c\}$ where $\delta^+ = \max\{\delta, 0\}$.*

For studying ellipticity it will be necessary in general to pass from operators $A \in B^{\mu,\varrho,d}(\mathbb{R}_+^{1+q})$ to block matrices

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{matrix} H^{s,\delta}(\mathbb{R}_+^{1+q}) \\ \oplus \\ H^{s,\delta}(\mathbb{R}^q, \mathbb{C}^{N_-}) \end{matrix} \to \begin{matrix} H^{s-\mu,\delta-\varrho}(\mathbb{R}_+^{1+q}) \\ \oplus \\ H^{s-\mu,\delta-\varrho}(\mathbb{R}^q, \mathbb{C}^{N_+}) \end{matrix}$$

for suitable N_-, N_+. This follows easily by producing the remaining entries (trace, potential operators with respect to $\mathbb{R}^q$ and $N_+ \times N_-$-matrices of scalar pseudo-differential operators on $\mathbb{R}^q$) in terms of a modification of the Green symbols and of the smoothing operators.

The corresponding block matrix Green symbols $(g(y,x)) = (g_{i,j}(y,x))_{i,j=1,2}$ of type 0 are defined by

$$g(y,\eta) \in S_{cl}^{\mu,\varrho}\left(\mathbb{R}^q \times \mathbb{R}^q; L_2(\mathbb{R}_+) \oplus \mathbb{C}^{N_-}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_+}\right)$$

with

$$g^*(y,\eta) \in S_{cl}^{\mu,\varrho}\left(\mathbb{R}^q \times \mathbb{R}^q; L_2(\mathbb{R}_+) \oplus \mathbb{C}^{N_+}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_-}\right)$$

those of type d as symbols in which the left upper coners are of type d in the former sense and, in addition, the trace entry

$$g_{21}(y,\eta) = \sum_{j=0}^{d} g_{21,j}(y,\eta)\frac{\partial^j}{\partial t^j}$$

for trace entries $g_{21,j}(y,\eta)$ of type 0 and orders $(\mu - j, \varrho)$

The class of smoothing elements with additional N_+ trace, N_- potential conditions and type 0, denoted by $\mathcal{B}^{-\infty,-\infty,0}(\mathbb{R}_+^{1+q}; N_-, N_+)$, is defined as the set of all continuous operators

$$\mathcal{C} : \mathcal{S}\left(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_-}\right) \to \mathcal{S}\left(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_+}\right)$$

with

$$\mathcal{C}^* : \mathcal{S}\left(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_+}\right) \to \mathcal{S}\left(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_-}\right)$$

being continuous such that $\mathcal{C}$ and $\mathcal{C}^*$ extend to continuous operators

$$\mathcal{C} : \begin{matrix} \langle y\rangle^k \langle t\rangle^k H^s(\mathbb{R}_+^{1+q}) \\ \oplus \\ \langle y\rangle^k H^s(\mathbb{R}^q, \mathbb{C}^{N_-}) \end{matrix} \to \begin{matrix} \langle y\rangle^{-l} \langle t\rangle^{-l} H^\infty(\mathbb{R}_+^{1+q}) \\ \oplus \\ \langle y\rangle^{-l} H^\infty(\mathbb{R}^q, \mathbb{C}^{N_+}) \end{matrix}$$

and

$$\mathcal{C}^* : \begin{matrix} \langle y\rangle^k \langle t\rangle^k H^s(\mathbb{R}_+^{1+q}) \\ \oplus \\ \langle y\rangle^k H^s(\mathbb{R}^q, \mathbb{C}^{N_+}) \end{matrix} \to \begin{matrix} \langle y\rangle^{-l} \langle t\rangle^{-l} H^\infty(\mathbb{R}_+^{1+q}) \\ \oplus \\ \langle y\rangle^{-l} H^\infty(\mathbb{R}^q, \mathbb{C}^{N_-}) \end{matrix};$$

respectively, for all $s \in \mathbb{R}$, $s > -1/2$, and all $k, l \in \mathbb{N}$.

The class of smoothing elements of type d has the left upper corners in $\mathcal{B}^{-\infty,-\infty,d}(\mathbb{R}^{1+q}_+)$ whereas the condition for the smoothing trace entry is to be replaced by

$$T = \sum_{j=0}^{d} T_j \frac{\partial^j}{\partial t^j}$$

where T_j are in $\mathcal{B}^{-\infty,-\infty,0}(\mathbb{R}^{1+q}_+; N_-, N_+)$ trace operators in the above sense.

Now Theorem 3.6 has an immediate extension to a composition theorem for operators when the number of trace conditions in B equals the number of potential conditions in A.

3.2 Edge pseudo-differential operators

The boundary value problems of the previous section are a very particular case of problems on a manifold with edges, where the half space is replaced by a wedge in which the edge $\cong \mathbb{R}^q$ is the substitute of the boundary. An infinite wedge is of the form $X^\Delta \times \mathbb{R}^q$, where X is a closed compact C^∞ manifold, $n = \dim X$, and $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ the model cone with base X. For $n = 0$ we just obtain the half space with boundary $\mathbb{R}^q$. The edge pseudo-differential problems as they were discussed in SCHULZE [20],[23], EGOROV, SCHULZE [5] correspond for $n = 0$ to pseudo-differential boundary value problems for operators without the transmission property, cf. SCHULZE [24], and they are of independent interest in applications, e.g. for mixed boundary value problems. The theory developed so far concerns the local situation with respect to the edge variables. For dealing with the operator-valued symbolic structures for corners of higher order there also appear symbolic levels operating along infinite wedges in the sense that the edges have (conical) exits to inifinity. The typical model situation for this is the case $\mathbb{R}^q$, and we then have to solve the problem of establishing the pseudo-differential calculus in this case. If we want to apply the general ideas from the Chapters 1 and 2 we have to specify the chosen spaces E and $\widetilde{E}$ and to construct the corresponding order reductions that are compatible wih the operator algebras. This is the program of the present section.

Instead of X^Δ we will take the open infinite stretched cone $X^\wedge = \mathbb{R}_+ \times X \ni (t, x)$. Let us first remind of the weighted Sobolev spaces on $X^\wedge$. Choose a classical parameter-dependent elliptic family $R^\mu(\lambda)$ of pseudo-differential operators of order μ on X, with the parameter $\lambda \in \mathbb{R}^l$. For instance, in local coordinates on X we can start with symbols of the form $(|\xi|^2 + |\lambda|^2 + d)^{\mu/2}$, $d > 0$, and then form the corresponding global operators, using a system of charts on X and a partition of unity. It is well-known that then there is a constant $c_1 > 0$ such that

$$R^\mu(\lambda) : H^s(X) \to H^{s-\mu}(X)$$

is an isomorphism for all $|\lambda| > c_1$ and all $s \in \mathbb{R}$. In particular, by choosing d sufficiently large we obtain isomorphisms for all $\lambda \in \mathbb{R}^l$. Now let us consider the Mellin transform

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\, dt,$$

first for $u(t) \in C_0^\infty(\mathbb{R}_+)$, $z \in \mathbb{C}$, and then extended to suitable distribution spaces (also vector-valued ones), where we allow at the same time z to vary only on a corresponding subset of $\mathbb{C}$. In particular, let

$$\Gamma_\beta = \{z \in \mathbb{C} \ : \ \mathrm{Re}\, z = \beta\}, \quad \beta \in \mathbb{R}.$$

Then M extends to an isomorphism $M_\gamma : t^\gamma L_2(\mathbb{R}_+) \to L_2(\Gamma_{\frac{1}{2}-\gamma})$.

Let us now define weighted Sobolev spaces on $X^\wedge$ based on the Mellin transform in t both globally in $t \in \mathbb{R}_+$ and then only for $t \to 0$. The latter variant will be formulated in terms of charts $\chi_1 : U_1 \to V_1$, where U_1 is a coordinate neighbourhood on X and V_1 an open set in $S^n = \{\tilde{x} \in \mathbb{R}^{n+1} \ : \ |\tilde{x}| = 1\}$. This induces a diffeomorphism

$$\chi : \mathbb{R}_+ \times U_1 \to V = \{\tilde{x} \in \mathbb{R}^{n+1} \ : \ \tilde{x}/|\tilde{x}| \in V_1\} \tag{3.7}$$

by $\chi(t,x) = t\chi_1(x)$. A cut-off function in this section will be any real function $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega(t) = 1$ for $0 \leq t \leq const$ for a constant > 0.

Definition 3.7. $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s,\gamma \in \mathbb{R}$ denotes the closure of $C_0^\infty(\mathbb{R}_+, C^\infty(X))$ with respect to the norm

$$\left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} ||R^s(\mathrm{Im}\, z)(Mu)(z)||^2_{L_2(X)}\, dz \right\}^{1/2},$$

$n = \dim X$, where $R^s(\lambda)$ is an operator family as mentioned, here for $l = 1$. The norm in $L_2(X)$ refers to a corresponding fixed Riemannian metric on X and to the associated measure dx. Moreover, $\mathcal{K}^{s,\gamma}(X^\wedge)$ denotes the subspace of all $u \in H^s_{loc}(X^\wedge)$ such that $\omega u \in \mathcal{H}^{s,\gamma}(X^\wedge)$ for any cut-off function $\omega(t)$ and $(\chi^{-1})^*(1-\omega)u\varphi \in H^s(\mathbb{R}^{n+1})$ for every chart $\chi_1 : U_1 \to V_1$, the associated diffeomorphism (3.7), and $\varphi \in C_0^\infty(U_1)$.

These definitions are correct in the sense that $\mathcal{H}^{s,\gamma}(X^\wedge)$ is independent of the specific choice of $R^s(\lambda)$, furthermore $\mathcal{H}^{s,\gamma}(X^\wedge) \subset H^s_{loc}(X^\wedge)$ ensures that $\mathcal{K}^{s,\gamma}(X^\wedge)$ is independent of $\omega(t)$. Moreover, standard invariance properties of the $H^s(\mathbb{R}^{n+1})$ spaces over conical subsets of $\mathbb{R}^{n+1}$ show that $\mathcal{K}^{s,\gamma}(X^\wedge)$ is independent of the choice of charts χ_1.

The spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ only play an auxiliary role here, whereas the $\mathcal{K}^{s,\gamma}(X^\wedge)$ are the adequate cone Sobolev spaces. If we set

$$(\kappa_\lambda u)(t,x) = \lambda^{\frac{n+1}{2}} u(\lambda t, x) \text{ for } \lambda \in \mathbb{R}_+,$$

then $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}}$ induces a strongly continuous group of isomorphisms on $\mathcal{K}^{s,\gamma}(X^\wedge)$ for every $s,\gamma \in \mathbb{R}$. In particular, it is a unitary group on $\mathcal{K}^{0,0}(X^\wedge) = t^{-n/2}L_2(X^\wedge)$, where L_2 refers to $dtdx$. In other words, the spaces

$$E = \mathcal{K}^{s,\gamma}(X^\wedge), \quad E_0 = \mathcal{K}^{0,0}(X^\wedge) \tag{3.8}$$

are in such a relation as it was assumed above in Section 1.3 in the abstract set-up.

Now we can form the scale of weighted wedge Sobolev spaces on (the open stretched) wedge $X^\wedge \times \mathbb{R}^q$

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) \text{ for } s, \gamma \in \mathbb{R}. \tag{3.9}$$

They were introduced in SCHULZE [20]. The spaces (3.9) have many natural properties, cf SCHULZE [23], [24]. Note, in particular, that in spite of the anisotropic description with respect to the role of the (t, x, y)-variables in the definition we have

$$H^s_{comp}(X^\wedge \times \mathbb{R}^q) \subset \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) \subset H^s_{loc}(X^\wedge \times \mathbb{R}^q)$$

for all $s, \gamma \in \mathbb{R}$.

Let $k^\varrho(t)$ for $\varrho \in \mathbb{R}$ be a strictly positive function in $C^\infty(\mathbb{R}_+)$ with $k^\varrho(t) = t^\varrho$ for $0 < t < c_0$, $k^\varrho(t) = 1$ for $c_1 < t < \infty$ with certain $0 < c_0, c_1 < \infty$, then the multiplication by $k^\varrho(t)$ induces isomorphisms

$$k^\varrho : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma+\varrho}(X^\wedge)$$

for all $s, \gamma \in \mathbb{R}$.

Theorem 3.8. *For every $s, \gamma \in \mathbb{R}$ there exists an operator-valued symbol*

$$r^{s,\gamma}(\eta) \in S^s_{cl}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{0,0}(X^\wedge))$$

such that $r^{s,\gamma}(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{0,0}(X^\wedge)$ is an isomorphism for every $\eta \in \mathbb{R}^q$.

Proof. The assertion relies on the fact that there exists an isomorphism $r_0 : E \to E_0$ between the spaces (3.8) such that

$$r_0(\lambda) := \kappa_\lambda r_0 \kappa_\lambda^{-1} \in C^\infty(\mathbb{R}_+, \mathcal{L}(E, E_0)) \tag{3.10}$$

holds, where $\mathcal{L}(E, E_0)$ is endowed with the norm topology. It is then clear that

$$r^{s,\gamma}(\eta) := [\eta]^s \, r_0([\eta]) \in C^\infty(\mathbb{R}^q, \mathcal{L}(E, E_0))$$

satisfies the relations

$$r^{s,\gamma}(\lambda\eta) = \lambda^s \kappa_\lambda r^{s,\gamma}(\eta) \kappa_\lambda^{-1} \text{ for all } \lambda \geq 1, |\eta| > const$$

for a constant > 0. Thus it follows that $r^{s,\gamma}(\eta) \in S^s_{cl}(\mathbb{R}^q; E, E_0)$. So it remains to show the existence of an operator r_0 with the required property. First every operator r_0 in the pseudo-differential algebra on the open infinite stretched cone with respect to a fixed pair of weights, here γ and 0, which is of order s, has the property (3.10), cf. SCHULZE [24], Remark 2.2.52. It is not essential here that the weight shift γ is different from the interior order s, since the weights can be changed by multiplying the operators by weight shift factors k^ϱ for appropriate $\varrho \in \mathbb{R}$. So let us talk about the cone operators of order μ in the sense of [23] without weight shifts, that are a subspace of $\bigcap_{r \in \mathbb{R}} \mathcal{L}(\mathcal{K}^{r,0}(X^\wedge), \mathcal{K}^{r-\mu,0}(X^\wedge))$. We always find an elliptic element q of order $\mu = s/2$ in this class. Then qq^* is elliptic of order s and has index zero. Kernel and cokernel in the sense of Fredholm operators $\mathcal{K}^{r,0}(X^\wedge) \to \mathcal{K}^{r-s,0}(X^\wedge)$ are independent of r. Then we find a finite-dimensional operator c with kernel in $C_0^\infty(X^\wedge \times X^\wedge)$ such that $qq^* + c : \mathcal{K}^{r,0}(X^\wedge) \to \mathcal{K}^{r-s,0}(X^\wedge)$ is an isomorphism for all r. The latter property is an easy consequence of the density of $C_0^\infty(X^\wedge)$ in $\mathcal{K}^{r,0}(X^\wedge)$ for every $r \in \mathbb{R}$. Now it suffices to set $r_0 = (qq^* + c)k^{-\gamma}$. □

The operator-valued symbols $r^{s,\gamma}(\eta)$ of Theorem 3.8 are sufficient for the purposes of Section 1.3. However there exist order and weight reducing symbols with more subtle properties with respect to the scales $\mathcal{K}^{s,\gamma}(X^\wedge)$, $s,\gamma \in \mathbb{R}$:

Theorem 3.9. *For every $\mu \in \mathbb{R}$ there exists an operator-valued symbol*

$$r^\mu(\eta) \in \bigcap_{s\in\mathbb{R}} S^\mu_{cl}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

such that $r^\mu(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$ is an isomorphism for every $\eta \in \mathbb{R}^q$, $s \in \mathbb{R}$, and

$$D^\beta_\eta r^\mu(\eta) \in \bigcap_{s\in\mathbb{R}} S^{\mu-|\beta|}_{cl}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu+|\beta|,\gamma-\mu+|\beta|}(X^\wedge)) \qquad (3.11)$$

for every multi-index $\beta \in \mathbb{N}^q$.

Proof. The required $r^\mu(\eta)$ can be constructed within the class of operator-valued edge symbols in the sense of [20], [24], [5]. Let us start with local symbols in $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q$ of the form

$$t^{-\mu}(|t\tau|^2 + |\xi|^2 + |t\eta|^2 + \delta^2)^{\mu/2}$$

for a parameter $\delta \in \mathbb{R}^d$ that will be chosen below. Then, according to a Mellin quantization theorem from [22] in parameter-dependent form, with the parameter δ, there exists a function

$$\tilde{h}(z,\xi,\tilde{\eta},\delta) \in C^\infty(\mathbb{C} \times \mathbb{R}^{n+q+d}_{\xi,\tilde{\eta},\delta})$$

which is holomorphic in $z \in \mathbb{C}$ such that

$$\tilde{h}(\beta + i\tau,\xi,\tilde{\eta},\delta) \in S^\mu_{cl}(\mathbb{R}^{1+n+q+d}_{\tau,\xi,\tilde{\eta},\delta})$$

holds for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$ for arbitrary $c \le c'$, such that

$$\mathrm{op}_{t,x}((|t\tau|^2 + |\xi|^2 + |t\eta|^2 + \delta^2)^{\mu/2}) = \mathrm{op}_x \mathrm{op}^\varrho_M(\tilde{h}\big|_{\tilde{\eta}=t\eta})(\eta,\delta)$$

modulo $\mathcal{S}(\mathbb{R}^{q+d}_{\eta,\delta}, L^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^n))$, for every $\varrho \in \mathbb{R}$. Here $\mathrm{op}^\varrho_M(f)u = M^{-1}_{\varrho,z\to t} f(z) M_{\varrho,t'\to z}$ is the pseudo-differential operator with symbol $f(z)$ with respect to the weighted Mellin transform $(M_\varrho u)(z) = (Mt^{-\varrho}u)(z+\varrho)$, and op_x and $\mathrm{op}_{t,x}$ indicate the pseudo-differential operators with respect to the Fourier transform in x and (t,x), respectively. Let us fix a covering $\{U_1,\dots,U_N\}$ of X by coordinate neighbourhoods, a subordinate partition of unity $\{\varphi_1,\dots,\varphi_N\}$, and a system $\{\psi_1,\dots,\psi_N\}$ of functions $\psi_j \in C^\infty_0(U_j)$ with $\varphi_j\psi_j = \varphi_j$ for all j. Then, if $\chi_j : U_j \to \mathbb{R}^n$ is a system of charts, we can form the operator families

$$a_\psi(\eta,\delta) = \sum_{j=1}^N (1 \times \chi_j^{-1})_* \varphi_j t^{-\mu} \mathrm{op}_{t,x}(|t\tau|^2 + |\xi|^2 + |t\eta|^2 + \delta^2)^{\mu/2})\varphi_j,$$

$$a^\varrho_M(\eta,\delta) = \sum_{j=1}^N (1 \times \chi_j^{-1})_* \varphi_j \mathrm{op}_x \mathrm{op}^\varrho_M(\tilde{h}\big|_{\tilde{\eta}=t\eta})(\eta,\delta)\psi_j.$$

Here $(1 \times \chi_j^{-1})_*$ is the push-forward of pseudo-differential operators under $1 \times \chi_j^{-1} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \times U_j$. Let us now fix cut-off functions $\omega(t)$, $\omega_0(t)$, $\omega_1(t)$ with $\omega\omega_0 = \omega$, $\omega\omega_1 = \omega_1$. Then, according to the known properties of operator-valued edge symbols, cf. [20], [23], [24], the (η, δ)-dependent family

$$a(\eta,\delta) := \omega(t\,[\eta])t^{-\mu}a_M^{\gamma-n/2}(\eta,\delta)\omega_0(t\,[\eta]) + (1-\omega(t\,[\eta]))a_\psi(\eta,\delta)(1-\omega_1(t\,[\eta]))$$

belongs to $S_{cl}^\mu(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ for every fixed $\delta \in \mathbb{R}$. The so-called conormal symbol of $a(\eta,\delta)$, cf. [23], [24], is a (z,δ)-dependent family of elliptic pseudo-differential operators on X, even parameter-dependent elliptic with the parameters $(\operatorname{Im} z, \delta) \in \mathbb{R}^{1+d}$, uniformly in finite intervals with respect to $\operatorname{Re} z$. Thus, for $|\delta|$ sufficiently large, by a well-known theorem on parameter-dependent ellipticity, for given fixed γ we can say that the conormal symbol is a family of isomorphisms $H^s(X) \to H^{s-\mu}(X)$ for all z with $\operatorname{Re} z = \frac{n+1}{2} - \gamma$ and for all $s \in \mathbb{R}$. Now

$$a(\eta,\delta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is a family of Fredholm operators for all $\eta \in \mathbb{R}^q \setminus \{0\}$ and $s \in \mathbb{R}$. This is a consequence of the mentioned conormal ellipticity together with the fact that for $\eta \neq 0$ the pseudo-differential operator $a(\eta,\delta)$ on $X^\wedge$ is elliptic in the usual sense and that it also satisfies the "exit ellipticity conditions" for $t \to \infty$ to ensure the existence of parametrices modulo compact operators. By inserting $[\eta]$ instead of $|\eta|$ everywhere we get a modified family $\widetilde{a}(\eta,\delta)$ which is Fredholm for all $\eta \in \mathbb{R}^q$. Another result from [24], Proposition 2.1.189, tells us that there is a smoothing Mellin operator of the form

$$\omega(t\,[\eta])t^{-\mu}\operatorname{op}_M^{\gamma-n/2}(h)\omega_0(t\,[\eta]) =: m(\eta)$$

for an $h(z) \in \mathcal{S}(\Gamma_{\frac{n+1}{2}-\gamma}, L^{-\infty}(X))$ such that $\widetilde{a}(\eta,\delta) + m(\eta)$ is of index zero. Moreover, by standard arguments similarly to the proof of the preceding Theorem 3.8, we find a symbol $g(\eta) \in S_{cl}^\mu(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\infty}(X^\wedge))$ (in particular, compact operator-valued) such that $r^\mu(\eta) := \widetilde{a}(\eta,\delta) + m(\eta) + g(\eta)$ is a family of isomorphisms. Here δ is sufficiently large and fixed. (3.11) is satisfied, since $r^\mu(\eta)$ is an edge symbol in the sense of [20], [23], [24] and edge symbols in general have this property. □

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