Combinatorial structure of some hyperbolic buildings.

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Abstract

We construct polyhedra whose links are combinatorial one-dimensional buildings. The polyhedra of this type are interesting because of their universal coverings, which are two-dimensional buildings.

Introduction

We will call a *polyhedron* a two-dimensional complex, which is obtained from several oriented p-gons by identification of corresponding sides. Let's consider a point of the polyhedron. Let's take a sphere of a small radius at this point. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

We consider the polyhedra with the following properties: all links of all vertices are combinatorial one-dimensional buildings.

Definition. A *combinatorial one-dimensional building* such that it consists of its edges called chambers and a family of subgraphs called apartments such that:

1. Every apartment is a cycle composed of $2N$ edges for some fixed N;

2. For any two chambers there is an apartment containing both of them;

3. If two apartments have a common chamber, so there is an isomorphism of these apartments fixing their intersection pointwise.

Definition. Let $\mathcal{P}(k,m)$ a tessalation of the hyperbolic plane by a regular polygon with k sides ($k \geq 5$), with angles π/m in each vertex (with m entier, $m \geq 2$). We will call *hyperbolic building* a polygonal complex X, which can be expressed to the union of subcomplexes isomorphic to $\mathcal{P}(k,m)$ called apartments satisfying the following axioms:

(i) each point of X is an apartment;

(ii) for any two polygons of X, there is an apartment containing both of them;

(iii) for any two apartments $A_1, A_2 \in X$ containing the same polygon, there exists an isomorphism $A_1 \to A_2$ fixing $A_1 \cap A_2$.

Let $C_p, p \geq 5$ be a polyhedron whose faces are p-gons and whose links are one-dimensional buildings. We equip every face of C_p with the hyperbolic metric, such that all sides of the polygons are geodesics and all angles are $\pi/3$, sides which are identified have the same length. Then the universal covering of such polyhedron is a hyperbolic building, see [1],[6].

In the case $p = 3$, i.e. C_p is a simplex, we can also give an euclidean metric to every face. In this metric sides which are identified also have the same length and all sides of the triangles are geodesics. The universal coverings of these polyhedra with euclidean metric are euclidean buildings, see [2], [3].

The main result of our paper is the following theorem. We prove the existence of a polyhedron with a given even number of sides of every face, whose links are one-dimensional combinatorial buildings in every vertex.

Theorem. For every k, q, r there exists a polyhedron with $2k$ vertices, $2k$-gonal faces and one-dimensional spherical buildings on $q+r$ vertices as links in all vertices.

It is known (see [7], for example), that combinatorial buildings are exactly *generalized m-gons.* A *generalized m-gon* is a connected, bipartite graph of diameter m and girth $2m$, in which each vertex lies on at least two edges. A graph is *bipartite* if its set of vertices can be partitioned into two disjoint subsets such that no two vertices in the same subset lie on a common edge; the *diameter* is the maximum distance between two vertices and the *girth* is the length of a shortest circuit. The vertices of the one subset we will call black vertices and the vertices of the other subset the white ones.

Let G be a generalized m-gon on $q+r$ vertices, q black vertices and r white ones. Let $\mathcal{A}$ and $\mathcal{B}$ be two alphabets on q and r letters respectively, $\mathcal{A} = \{x_1, x_2, \dots, x_q\}$ and $\mathcal{B} = \{y_1, y_2, \dots, y_r\}$. We mark every black vertex with an element from $\mathcal{A}$ and every white vertex with an element from $\mathcal{B}$.

We will also define an incidence matrix of such graph in the following way: the first element of each line is a white vertex and all other elements are black vertices incident to this white vertex. Different lines correspond to different white vertices.

Example 1. For the smallest complete bipartite graph (generalized 2-gon) with four vertices, two black x_1, x_2 and two white y_1, y_2, the incidence matrix will be

y_1: x_1 x_2
y_2: x_1 x_2

Let's define a graph G', which is *inverse* to G, by the following way:

Graph G' has r white vetices, labelled by elements of the alphabet $\mathcal{B}$, and q black vertices, labelled by elements of the alphabet $\mathcal{A}$ and vertices y_i and x_j are incident in G' in and only if x_j and y_i are incident in G.

Remark. The graph G' can be obtained from G by changing black and white colors of vertices.

(It is easy to see, that G and G' are isomorphic as usual graphs.) Similarly, we can define the incidence matrix by "inverse" way, as correspondence of black vertices to white ones. For the graph from the example we have

x_1: y_1 y_2
x_2: y_1 y_2

Polygonal presentation.

Definition. Suppose we have n disjoint generalized m-gons $G_1, G_2, \dots G_n$. Let P_i and L_i be the sets of black and white vertices respectively in G_i; let $P = \cup P_i, L = \cup L_i$, $P_i \cap P_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to L$.

A set $\mathcal{K}$ of k-tuples $(x_1, x_2, \dots, x_k)$, $x_i \in P$, will be called a *polygonal presentation* over P compatible with λ if

(1) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ for some $x_3, \dots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident in some G_i;

(2) $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ implies that $(x_2, x_3, \dots, x_k, x_1) \in \mathcal{K}$;

(3) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

We will call λ a *basic bijection.*

Polyhedral presentations for $n = 1$ and $k = 3$, $q = 2, 3$ were listed in [5]. They were called triangle presentations in that paper, because the case $k = 3$ was considered.

Example 2. Let's take a triangle presentation $\mathcal{T}_{A_1}$ from [5], p.160:

$$(0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,0),(5,6,1),(6,0,2).$$

We can express it as an incidence matrix of finite projective plane of order two, which corresponds to this polyhedral presentation:

0: 1 4 2
1: 3 2 5
2: 4 3 6
3: 0 4 5
4: 1 5 6
5: 0 2 6
6: 0 1 3

Construction of polyhedra.

We can associate a polyhedron K on n vertices with each polygonal presentation $\mathcal{K}$ as follows: for every cyclic k-tuple $(x_1, x_2, x_3, \dots, x_k)$ from the definition we take an oriented k-gon on the boundary of which the word $x_1 x_2 x_3 \dots x_k$ is written. To obtain the polyhedron we identify the corresponded sides of our polygons respecting orientation. We will say that the polyhedron K *corresponds* to the polygonal presentation $\mathcal{K}$.

Lemma. A polyhedron K which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \dots, G_n$ as the links.

Proof. Let's take an oriented k-gon for each cyclic k-tuple and write the k-tuple on the boundary of this k-gon. Then we identify the sides with the same label respecting orientation to obtain the polyhedron K. Each face of the polyhedron comes from one k-gon and vertices of the polyhedron come from vertices of the polygons. Look at the face of K which came from the k-tuple $(x_1, x_2, x_3, \dots, x_k)$ and take a vertex v of the polyhedron, such that the edge x_1 is entering to this vertex v and x_2 is leaving v. Let's take a sphere of a small radius with the center in v. The intersection of this sphere with x_1 will be a white vertex x_1, the intersection of this sphere with x_2 will be a black vertex x_2 and the intersection of this sphere with the face will be an edge joining x_1 and x_2. But by the definition of the polygonal presentation in the graph G_1 there is an edge joining the white vertex x_1 and black vertex x_2. Look at all edges, incident to v. On edges entering v we will put white vertices and on edges leaving v we will put black vertices. By the definition of the polygonal presentation there is an edge joining two of these vertices if and only if there is a face, where they are neighbors, and they are neighbors in a face if and only if they are neighbors in some k-tuple.

Let's look at the incidence matrix of the graph, which is a link in a vertex v of the polyhedron. It follows from the definition of the polygonal presentation, that a graph with such incidence matrix has to be G_i.

Proof of Theorem. By the Lemma, to construct the polyhedron with given links, it is sufficient to construct corresponding polygonal presentation.

Let's take $2k$ alphabets, $\mathcal{A}_i, i = 1, \ldots, 2k$, each of them with i odd consisting of q letters, $\mathcal{A}_i = \{x_1^i, \ldots, x_q^i\}$ and each of them with i even consisting of r letters, $\mathcal{A}_i = \{x_1^i, \ldots, x_r^i\}$.

Let G_1 be a generalized m-gon on $q + r$ vertices, q white and r black ones. All white vertices of G_1 will be marked by the elements of the alphabet $\mathcal{A}_1$ and all black ones of G_1 will be marked by the elements of $\mathcal{A}_2$.

Now we define ordered sets of letters and we will show, that they determine a desired polygonal presentation $\mathcal{K}$:

a $2k$-tuple $(x_j^1, x_s^2, x_j^3, x_s^4, \ldots, x_j^{2k-1}, x_s^{2k})$ belongs to $\mathcal{K}$ if and only if x_j^1 and x_s^2 are incident in G_1, where $j = 1, \ldots, q$, $s = 1, \ldots, r$, each $2k$-tuple is considered with all its cyclic permutations.

Graph G_i, where i is odd, $3 \leq i \leq 2k - 1$ is defined as follows: all white vertices of G_i will be marked by the elements of the alphabet $\mathcal{A}_i$ and all black ones of G_i will be marked by the elements of $\mathcal{A}_{i+1(mod 2k)}$; vertices x_p^i and x_t^{i+1} are incident in G_i if and only if x_p^1 and x_t^2 are incident in G_1.

Graph G_i, where i is even $2 \leq i \leq 2k$, is defined as follows: all white vertices of G_i will be marked by the elements of the alphabet $\mathcal{A}_i$ and all black ones of G_i will be marked by the elements of $\mathcal{A}_{i+1(mod 2k)}$;vertices x_p^i and x_t^{i+1} are incident in G_i if and only if x_t^1

and x_p^2 are incident in G_1. (Here and below all upper indexes will be taken modulo $2k$.)

The basic bijection λ between white and black vertices of all graphs G_i, $i = 1, \ldots, 2k$ will be defined as bijection, induced by letters of alphabets $\mathcal{A}_i$, that is the image of each black vertex x_j is the white vertex, labelled by the same letter x_j.

It is necessary to check all axioms of the polygonal presentation for an arbitrary $2k$-tuple:

(1) Let's consider an arbitrary $2k$-tuple of $\mathcal{K}$ $(x_n^t, x_p^{t+1}, \ldots, x_l^{t+2k-1(mod 2k)}, x_h^{t+2k(mod 2k)})$.

If t is odd, then $\lambda(x_n^t)$ is a white vertex x_n^t of G_t, x_p^{t+1} is a black vertex of G_t and x_n^t and x_n^{t+1} are incident in G_t by constructuion of $\mathcal{K}$. Indeed, for t odd the letter x_p^{t+1} follows after x_n^t if and only if the white vertex x_n^1 and the black one x_p^2 are incident in G_1. And vertices x_n^t and x_n^{t+1} are incident in G_t if and only if x_n^1 and x_p^2 are incident in G_1.

By similar arguments G_t for t even is isomorphic to the graph G_t which is inverse to G_1. But the graph, which is inverse to a generelized m-gon is also a generalized m-gon. So, the first property of a polygonal presentation is already checked.

(2) In our construction we take all cyclic permutations of each $2k$-tuple.

(3) Each $2k$-tuple is defined by a pair (j, s) in a unic way, where j is the index of letters of alphabets with odd numbers, and s is the index of letters of alphabets with even numbers.

So, $\mathcal{K}$ is a polygonal presentation, and by Lemma, it defines a polyhedron on $2k$ vertices with one-dimensional combinatorial buildings as links in all vertices.

Theorem is proved.

Example 3. Let's construct a polyhedron which faces are hexagons and a complete bipartite graph with two white and three black vertices as a link in each vertex of the polyhedron. We take six alphabets $\mathcal{A}_i, i = 1, ..., 6$, each of them with i odd consisting of two letters, $\mathcal{A}_i = \{x_1^i, x_2^i\}$ and each of them with i even consisting of three letters, $\mathcal{A}_i = \{x_1^i, x_2^i, x_3^i\}$.

The incidence matrix of G_i with i odd :

$x_1^i : x_1^{i+1} x_2^{i+1} x_3^{i+1}$

$x_2^i : x_1^{i+1} x_2^{i+1} x_3^{i+1}$

The incidence matrix of G_i with i even :
$x_1^i : x_1^{i+1} x_2^{i+1}$
$x_2^i : x_1^{i+1} x_2^{i+1}$
$x_3^i : x_1^{i+1} x_2^{i+1}$
The polyhedral presentation:

$$(x_1^1, x_1^2, x_1^3, x_1^4, x_1^5, x_1^6), (x_1^1, x_2^2, x_1^3, x_2^4, x_1^5, x_2^6), (x_1^1, x_3^2, x_1^3, x_3^4, x_1^5, x_3^6),$$

$$(x_2^1, x_1^2, x_2^3, x_1^4, x_2^5, x_1^6), (x_2^1, x_2^2, x_2^3, x_2^4, x_2^5, x_2^6), (x_2^1, x_3^2, x_2^3, x_3^4, x_2^5, x_3^6).$$

Each 6-tuple is considered with all its cyclic permutations.

Acknowledgments. I would like to thank Etienne Ghys for his advice to look at this subject and for the helpful discussions. I also thank Werner Ballmann for many consultations concerning euclidean and hyperbiloc buildings. This paper was written during my stay at the Max-Planck-Institut für Mathematik (MPIM), Bonn. We would like to thank the Directors and the staff of the MPIM for hospitality and for providing the excellent working conditions.

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